# FLUCTUATIONS OF SPECTRALLY NEGATIVE MARKOV ADDITIVE PROCESSES

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For spectrally negative Markov Additive Processes (MAPs) we generalize classical fluctuation identities developed in Zolotarev (1964), Takács (1967), Bingham (1975), Suprun (1976), Emery (1973), Rogers (1990) and Bertoin (1997) which concern one and two sided exit problems for spectrally negative Lévy processes. Where as for Lévy processes the latter exit problems can be characterized in terms of functions which are known as scale functions, it turns out that for MAPs, exit problems are characterized by analogous objects which we call scale matrices. Traditional techniques for Lévy processes make use of the Wiener-Hopf factorization and, more recently, Itô's excursion theory. Our approach avoids the use of these methods and appeals explicitly to martingale arguments together with suitable use of the Strong Markov Property and the fact that spectrally negative MAPs creep upwards with probability one. In particular we make heavy use of a new martingale given in Asmussen and Kella (2000). The choice of method is pragmatic as Wiener-Hopf methods and Itô excursion theory are not comfortably available in the literature for the class of MAPs we consider. The martingale approach also offers two additional benefits. The first is that when one repeats the arguments verbatim for spectrally negative Lévy processes, as a whole, one obtains a simpler set of proofs than the aforementioned existing literature offers. The second is that a Wiener-Hopf factorization falls out of the calculations, even though it is not explicitly used thereafter. **Submitted: August 10 2003.**

**1. Spectrally Negative Markov Additive Processes.** This paper presents some fundamental fluctuation identities for a special, but none the less quite general, class of Markov Additive Processes (MAP). Before entering our discussion on the subject we shall simply begin by defining the class of process we intend to work with and its properties.

Following Asmussen and Kella (2000) we consider a process $X(t)$, where $X(t) = X^{(1)}(t) + X^{(2)}(t)$, and the independent processes $X^{(1)}(t)$ and $X^{(2)}(t)$ are specified by the characteristics: $q_{ij}, G_{ij}, \sigma_i, a_i, \nu_i(dx)$ which we shall now define. Let $J(t)$ be a right-continuous, irreducible, finite state space continuous time Markov chain, with $\mathcal{I} = \{1, \ldots, N\}$, and with the intensity matrix $\mathbf{Q} = (q_{ij})$. We denote the jumps of the process $J(t)$ by $\{T_i\}$ (with $T_0 = 0$). Let $\{U_n^{(ij)}\}$ be i.i.d. random variables with distribution function $G_{ij}(\cdot)$ ($U^{(ii)} \equiv 0$). Define the jump process by

$$X^{(1)}(t) = \sum_{n\geq 1} \sum_i U_n^{(ij)} \mathbf{1}_{\{J(T_{n-1})=i,\ J(T_n)=j,\ T_n\leq t\}}.$$

For each $i \in \mathcal{I}$, let $X^i(t)$ be a Lévy process with Laplace exponent

$$\log E(\exp \alpha X^i(1)) = \psi_i(\alpha) = a_i\alpha + \frac{\sigma_i^2\alpha^2}{2} + \int_{-\infty}^0 (e^{\alpha y} - 1 - \alpha y 1_{(-1,0)}(y))\nu_i(dy),$$

*AMS 1991 subject classifications.* Primary 60J30 secondary 60K25.

*Key words and phrases.* Markov additive processes, exit problems, martingales.

where $1_{(-1,0)}(y)$ is valued 1 if $y \in (-1,0)$ and valued 0 otherwise and $\int_{-\infty}^{0}(1 \wedge |y|^2)\nu_i(dy) < \infty$. By $X^{(2)}(t)$ we denote the process which behaves in law like $X^i(t)$, when $J(t) = i$. Note that each of the measures $\nu_i$ are supported on $(-\infty, 0)$ as well as the distributions of each $U^{(ij)}$ and in this respect we say that $X$ is a *spectrally negative* MAP. We do not allow the possibility however that any of the processes $X^i$ may be a downwards subordinator. This last assumption excludes classical fluid models. Later, in Remark 9, we will see that, as far as our results are concerned, this assumption can be weakened to requiring that at least one of the processes $X^i$ is not a downwards subordinator; in which case classical fluid models are included.

Letting $\mathbf{Q} \circ \widehat{\mathbf{G}}(\alpha) = (q_{ij}\widehat{G}_{ij}(\alpha))$, where $\widehat{G}_{ij}(\alpha) = E\left(\exp(\alpha U^{(ij)})\right)$, we define *intensity matrix* of MAP $X(t)$:

$$\mathbf{F}(\alpha) = \mathbf{Q} \circ \widehat{\mathbf{G}}(\alpha) + \mathrm{diag}(\psi_1(\alpha), \ldots, \psi_N(\alpha)) \ . \tag{1}$$

Note then that $\mathbf{F}(\alpha)$ is well defined and finite at least for $\alpha \geq 0$. Within this regime of $\alpha$, Perron-Frobenius theory identifies $\mathbf{F}(\alpha)$ as having a real-valued eigenvalue with maximal absolute value which we shall label $\kappa(\alpha)$. The corresponding left and right $1 \times N$ eigenvectors we label $\mathbf{v}(\alpha)$ and $\mathbf{h}(\alpha)$ respectively. In this text we shall always write vectors in their horizontal form and use the usual $^{\mathrm{T}}$ to mean transpose. Since $\mathbf{v}(\alpha)$ and $\mathbf{h}(\alpha)$ are given up to multiplying constants, we are free to normalize them such that

$$\mathbf{v}(\alpha)\mathbf{h}(\alpha)^{\mathrm{T}} = 1 \text{ and } \pi\mathbf{h}(\alpha)^{\mathrm{T}} = 1 \ ,$$

where $\pi = \mathbf{v}(0)$ is the stationary distribution of $J$. Note also that $\mathbf{h}(0) = \mathbf{e}$, the $1 \times N$ vector consisting of a row of ones. We shall write $h_i(\alpha)$ for the $i$-th element of $\mathbf{h}(\alpha)$. The eigenvalue $\kappa(\alpha)$ is a convex function (this can also be easily verified) such that $\kappa(0) = 0$ and $\kappa'(0)$ is the asymptotic drift of $X$ in the sense that for each $i \in \mathcal{I}$ we have $\lim_{t\uparrow\infty} E(X(t)|J(0) = i, X(0) = x)/t = \kappa'(0)$. For the right inverse of $\kappa$ we shall write $\Phi$.

We shall assume the afore mentioned class of MAPs are defined on a probability space with probabilities $\{\mathbb{P}_{i,x} : i \in \mathcal{I}, x \in \mathbb{R}\}$ and right-continuous natural filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$. It can be checked that under the following Girsanov change of measure

$$\left.\frac{d\mathbb{P}^{\gamma}_{i,x}}{d\mathbb{P}_{i,x}}\right|_{\mathcal{F}_t} := e^{\gamma(X(t)-x)-\kappa(\gamma)t}\frac{h_{J(t)}(\gamma)}{h_i(\gamma)}, \text{ for } \gamma \text{ such that } \kappa(\gamma) < \infty \tag{2}$$

the process $(X, \mathbb{P}^{\gamma}_{i,x})$ is again a spectrally negative MAP whose intensity matrix $\mathbf{F}_\gamma(\alpha)$ is well defined and finite for $\alpha \geq -\gamma$. If $\mathbf{F}_\gamma(\alpha)$ has largest eigenvalue $\kappa_\gamma(\alpha)$ and associated right eigenvector $\mathbf{h}_\gamma(\alpha)$, the triple $(\mathbf{F}_\gamma(\alpha), \kappa_\gamma(\alpha), \mathbf{h}_\gamma(\alpha))$ is related to the original triple $(\mathbf{F}(\alpha), \kappa(\alpha), \mathbf{h}(\alpha))$ via

$$\mathbf{F}_\gamma(\alpha) = \Delta_{\mathbf{h}}(\gamma)^{-1}\mathbf{F}(\alpha+\gamma)\Delta_{\mathbf{h}}(\gamma) - \kappa(\gamma)\mathbf{I} \text{ and } \kappa_\gamma(\alpha) = \kappa(\alpha+\gamma) - \kappa(\gamma) \ , \tag{3}$$

where $\mathbf{I}$ is the $N \times N$ identity matrix and

$$\Delta_{\mathbf{h}}(\gamma) := \mathrm{diag}(h_1(\gamma), ..., h_N(\gamma)) \, .$$

These details and more concerning the basic characterization of MAPs can be found in Chapter II of Asmussen (2000).

**2. Time reversal.** Predominant in the forthcoming analysis will be the use of the bivariate process $(\widehat{J},\widehat{X})$, representing the process $(J,X)$ time reversed from a fixed moment in the future when $J(0)$ has the stationary distribution $\pi$. For definitiveness, we mean

$$\widehat{J}\,(s)=J((t-s)^{-})\text{ and }\widehat{X}\,(s)=X\,(t)-X\,\left((t-s)^{-}\right),\;0\leq s\leq t$$

under $\mathbb{P}_{\pi,0}=\sum_{i\in\mathcal{I}}\pi_i\mathbb{P}_{i,0}$. The characteristics of $(\widehat{J},\widehat{X})$ will be indicated by using a hat over the existing notation for the characteristics of $(J,X)$. For example $\widehat{\mathbf{F}}$, $\widehat{\mathbf{h}}$, $\widehat{\kappa}$ and so on. To relate these characteristics to the original ones, recall that the intensity matrix of $\widehat{J}$ must satisfy

$$\widehat{\mathbf{Q}}=\Delta_\pi^{-1}\mathbf{Q}^{\mathrm{T}}\Delta_\pi\;,$$

where $\Delta_\pi$ is the diagonal matrix whose entries are given by the vector $\pi$. Hence according to (1) we find that when it exists

$$\widehat{\mathbf{F}}\,(\alpha)=\Delta_\pi^{-1}\mathbf{F}\,(\alpha)^{\mathrm{T}}\,\Delta_\pi.$$

Since $(\mathbf{v}\,(\alpha)\,\mathbf{F}\,(\alpha))^{\mathrm{T}}=\mathbf{F}\,(\alpha)^{\mathrm{T}}\,\mathbf{v}\,(\alpha)^{\mathrm{T}}=\kappa\,(\alpha)\,\mathbf{v}\,(\alpha)^{\mathrm{T}}$ we have that

$$\widehat{\mathbf{F}}\,(\alpha)\,\Delta_\pi^{-1}\mathbf{v}\,(\alpha)^{\mathrm{T}}=\Delta_\pi^{-1}\mathbf{F}\,(\alpha)^{\mathrm{T}}\,\Delta_\pi\Delta_\pi^{-1}\mathbf{v}\,(\alpha)^{\mathrm{T}}=\kappa\,(\alpha)\,\Delta_\pi^{-1}\mathbf{v}\,(\alpha)^{\mathrm{T}}$$

showing that $\widehat{\kappa}\,(\alpha)\geq\kappa\,(\alpha)$. On the other hand a similar calculation reveals that

$$\widehat{\kappa}\,(\alpha)\,\Delta_\pi\widehat{\mathbf{h}}\,(\alpha)^{\mathrm{T}}=\Delta_\pi\widehat{\mathbf{F}}\,(\alpha)\,\widehat{\mathbf{h}}\,(\alpha)=\mathbf{F}\,(\alpha)^{\mathrm{T}}\,\Delta_\pi\widehat{\mathbf{h}}\,(\alpha)^{\mathrm{T}}$$

so that $\widehat{\kappa}\,(\alpha)\leq\kappa\,(\alpha)$ and hence $\widehat{\kappa}=\kappa$ and $\Delta_\pi\widehat{\mathbf{h}}\,(\alpha)^{\mathrm{T}}=\mathbf{v}\,(\alpha)^{\mathrm{T}}$.

Instead of talking about the process $(\widehat{J},\widehat{X})$ we shall talk about the process $(J,X)$ under probabilities $\{\widehat{\mathbb{P}}_{i,0}:i\in\mathcal{I}\}$ meaning the MAP whose characteristics are given by $\widehat{\mathbf{F}}$. Note also for future use, following classical time reversed path analysis, for $y\geq 0$

$$\begin{aligned}\mathbb{P}_{i,0}(-I(t)\in dy|J(t)=j)&=\mathbb{P}_{j,0}\left(\widehat{S}\,(t)-\widehat{X}(t)\in dy|\widehat{J}(t)=i\right)\\&=\widehat{\mathbb{P}}_{j,0}(S\,(t)-X(t)\in dy|J(t)=i)\;,\end{aligned}\tag{4}$$

where $I(t)=\inf_{0\leq s\leq t}X(s)$, $S(t)=\sup_{0\leq s\leq t}X(s)$ and $\widehat{S}(t)=\sup_{0\leq s\leq t}\widehat{X}(s)$. (A diagram may help to explain the last identity). See Asmussen (1989, 2000) for similar conclusions.

**3. Main results.** In this article we shall establish fundamental fluctuation identities concerning the exit times

$$\tau_a^+:=\inf\{t\geq 0:X(t)\geq a\}\text{ and }\tau_0^-:=\inf\{t\geq 0:X(t)\leq 0\}\;,$$

where $a\geq 0$. Our results can be expressed in terms of $N\times N$ matrix functions $\mathbf{W}^{(q)}:\mathbb{R}\to[0,\infty)$ and $\mathbf{Z}^{(q)}:\mathbb{R}\to[0,\infty)$ where $q\geq 0$ which we shall from now loosely refer to as *scale matricies.*

Definition 1. *For each $q \geq 0$, we say that the $N \times N$ matrix function $\mathbf{W}^{(q)}(x)$ is a scale matrix if it satisfies the following properties. For $x \leq 0$, $\mathbf{W}^{(q)}(x) = \mathbf{0}$ (the zero matrix) and otherwise it is the unique left-continuous with right-hand limits (LCRL) matrix having Laplace transform*

$$\int_0^\infty e^{-\beta x}\mathbf{W}^{(q)}(x)\,dx = (\mathbf{F}(\beta) - q\mathbf{I})^{-1}, \ \beta > \Phi(q) \tag{5}$$

*[Note that uniqueness follows from the LCRL property; cf. Th. XIII.4.2 of Feller (1971)]. When $\mathbf{W}^{(q)}$ exists, we may define a second scale matrix $\mathbf{Z}^{(q)}$ function via the relation*

$$\mathbf{Z}^{(q)}(x) = \mathbf{I}(q) + q\int_0^x \mathbf{W}^{(q)}(y)\,dy\ ,$$

*where $\mathbf{I}_{ij}(q) = \mathbb{P}_{i,0}(J(\mathbf{e}_q) = j)$.*

Remark 2. By definition of $\kappa$ the matrix $\mathbf{F}(\beta) - q\mathbf{I}$ is invertable for $0 < \beta < \Phi(q)$. To see this, note that the latter condition on $\beta$ implies that $\kappa(\beta) < q$. Within this regime of $\beta$, if $\mathbf{F}(\beta) - q\mathbf{I}$ had eigenvalue an equal to zero, it would follow that $\mathbf{F}(\beta)$ has eigenvalue equal to $q$ which contradicts the fact that the maximum eigenvalue $\kappa(\beta) < q$. Later in equation (19) we shall see that matrix $\mathbf{F}(\beta)$ is invertable for $\beta > 0$ in case of positive drift. Hence it will turn out that using Girsanov transformation (3) the matrix $\mathbf{F}(\beta) - q\mathbf{I}$ is also invertable for $\beta > \Phi(q)$. More details will follow later at the appropriate point in the text.

Remark 3. Note that an easy calculation shows that $\mathbf{I}(q) = q(q\mathbf{I} - \mathbf{Q})^{-1}$. To see this recall that the transition matrix of $J$, say $\mathbf{P}(t)$, solves the differential equation $\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$. Multiplying both sides by $q\exp\{-qt\}$ and integrating from zero to infinity (by parts on the left hand side) yields the required identity.

As much as possible, from now on, we shall prefer to work with matrix notation. For $\mathcal{F}_t$-measurable random variables $Y$ we introduce matricies of the form $\mathbf{E}_x(Y)$ to be understood as having $(i,j)$-th elements $\mathbb{E}_{i,x}(Y, J(t) = j)$. For an $\mathcal{F}_t$-measurable event, $A$, $\mathbf{P}_x(A)$ will be understood in a similar sense. For simplicity we shall follow the tradition that $\mathbf{E}(\cdot) = \mathbf{E}_0(\cdot)$ and $\mathbf{P}(\cdot) = \mathbf{P}_0(\cdot)$.

Theorem 4. *There exist scale matrices $\mathbf{W}^{(q)}$ and $\mathbf{Z}^{(q)}$ such that For each $q \geq 0$ and $x \leq a$,*

*(i)* $\mathbf{E}_x\left(e^{-q\tau_a^+}1_{(\tau_a^+ < \tau_0^-)}\right) = \mathbf{W}^{(q)}(x)\,\mathbf{W}^{(q)}(a)^{-1}$ *and*

*(ii)* $\mathbf{E}_x\left(e^{-q\tau_0^-}1_{(\tau_a^+ > \tau_0^-)}\right) = [\mathbf{Z}^{(q)}(x) - \mathbf{W}^{(q)}(x)\,\mathbf{W}^{(q)}(a)^{-1}\,\mathbf{Z}^{(q)}(a)]\mathbf{I}(q)^{-1}.$

Remark 5. In part (ii) of the above theorem, we should interpret the case of $q = 0$ in the limiting sense. That is to say, writing $\mathbf{W}$ as shorthand for $\mathbf{W}^{(0)}$, we see,

$$\mathbf{P}_x\left(\tau_a^+ > \tau_0^-\right) = \mathbf{I} - \int_0^x \mathbf{W}(y)dy\mathbf{Q} - \mathbf{W}(x)\mathbf{W}(a)^{-1}[\mathbf{I} - \int_0^a \mathbf{W}(y)dy\mathbf{Q}]. \tag{6}$$

By contrast with what one would see for spectrally negative Lévy processes, this is considerably more complicated on account of the Markov modulation.

For spectrally negative Lévy processes, it is quite straightforward to obtain one sided exit results from the analogue of the above theorem by simply taking limits as one of the barriers moves off to (plus or minus) infinity relative to the starting position. See for example the extended version of Avram et al. (2002) appearing in the thesis of Pistorius (2003). For the case of spectrally negative MAPs, the issue becomes somewhat more complicated; again this is a result of Markov modulation.

For the case of one-sided exit with upward crossing one can appeal to the Girsanov transform given in (2) however to deduce that for each $q \geq 0$ and $x \geq 0$

$$\Delta_{\mathbf{h}}(\Phi(q))^{-1}\mathbf{E}\left(e^{\Phi(q)x-q\tau_x^+}1_{(\tau_x^+<\infty)}\right)\Delta_{\mathbf{h}}(\Phi(q)) = \mathbf{I}^{\Phi(q)}(\tau_x^+)\ ,$$

where $\mathbf{I}^{\Phi(q)}_{ij}(\tau_x^+) = \mathbb{P}^{\Phi(q)}_{i,0}(J(\tau_x^+)=j, \tau_x^+<\infty) = \mathbb{P}^{\Phi(q)}_{i,0}(J(\tau_x^+)=j)$ since the drift of $(X,\mathbb{P}^{\Phi(q)})$ is given by $\kappa'_{\Phi(q)}(0) = \kappa'(\Phi(q))$ which is strictly positive by convexity of $\kappa$. Hence

$$\mathbf{E}\left(e^{-q\tau_x^+}1_{(\tau_x^+<\infty)}\right) = e^{-\Phi(q)x}\Delta_{\mathbf{h}}(\Phi(q))\mathbf{I}^{\Phi(q)}(\tau_x^+)\Delta_{\mathbf{h}}(\Phi(q))^{-1}.$$

A little thought reveals that process $\{J(\tau_x^+) : x \geq 0\}$ is again a Markov chain. Let us suppose that under $\mathbf{P}^{\Phi(q)}$ its intensity matrix is given by $\Lambda(q)$ so that $\mathbf{I}^{\Phi(q)}(\tau_x^+) = \exp(\Lambda(q)x)$.

To characterize the matrix $\Lambda(q)$, from part (i) of Theorem 4 note that

$$\mathbf{E}(e^{-q\tau_x^+}1_{(\tau_x^+<\tau_{-b}^-)}) = \mathbf{W}^{(q)}(b)\mathbf{W}^{(q)}(x+b)^{-1}\ .$$

Later in (23) we shall see that the right hand side can be simplified to

$$e^{-\Phi(q)x}\Delta_{\mathbf{h}}(\Phi(q))\mathbf{W}_{\Phi(q)}(b)\mathbf{W}_{\Phi(q)}(b+x)^{-1}\Delta_{\mathbf{h}}(\Phi(q))^{-1}\ ,$$

where $\mathbf{W}_{\Phi(q)}(x)$ is the 0-scale matrix appearing in Definition 1 when we replace $\mathbf{F}$ by $\mathbf{F}_{\Phi(q)}$. $\mathbb{P}^{\Phi(q)}$. It follows using dominated convergence that

$$\exp\{\Lambda(q)\} = \lim_{b\uparrow\infty}\mathbf{W}_{\Phi(q)}(b)\mathbf{W}_{\Phi(q)}(b+1)^{-1}.$$

Taking into account calculations similar in nature to those mentioned in Remark 3 we thus obtain the following result.

Theorem 6. *For $q \geq 0$,*

$$\mathbf{E}\left(e^{-q\tau_x^+}1_{(\tau_x^+<\infty)}\right) = \Delta_{\mathbf{h}}(\Phi(q))e^{-(\Phi(q)\mathbf{I}-\Lambda(q))x}\Delta_{\mathbf{h}}(\Phi(q))^{-1}. \tag{7}$$

Remark 7. Unfortunately, it does not seem that an explicit expression for $\Lambda(q)$ can be derived from existing literature. Indeed establishing an expression for $\Lambda(q)$ for spectrally negative MAPs is an open problem. Recent progress has been made however by Miyazawa and Takada (2003) (see also the references therein) who consider a special case of the spectrally negative MAP that we have here. That is, the case when the jump part of each of the processes $X^i$ are of bounded variation and finite mean. For this class, they prove that $\Lambda(0)$ solves $\mathbf{F}(-\Lambda(0)) = \mathbf{0}$ where we should understand the latter to mean that

$$-\Delta_{\mathbf{a}}\Lambda(0) + \Delta_{\sigma^2/2}\Lambda(0)^2 + \int_{-\infty}^{0} \Delta_\nu(dy)(e^{-\Lambda(0)y} - \mathbf{I} + \Lambda(0)y1_{[-1,0]}(y))$$
$$+\int_{-\infty}^{0} \mathbf{Q}\circ\mathbf{G}(dy)e^{-\Lambda(0)y} = \mathbf{0}\ , \tag{8}$$

where $\Delta_{\mathbf{a}}$ is the diagonal matrix with entries $a_1, ..., a_N$ along the diagonal, similarly $\Delta_{\sigma^2/2}$ is diagonal with elements $\sigma_1^2/2, ..., \sigma_N^2/2$, matrix $\Delta_\nu(\cdot)$ is diagonal with $\nu_1(\cdot), \ldots, \nu_N(\cdot)$ on the diagonal and the matrix $\mathbf{G}$ has entries $G_{ij}(\cdot)$ corresponding to the distributions of each $U_{ij}$. For discussion concerning uniqueness of the solution see Rogers (1994), Asmussen (2000) and Miyazawa and Takada (2003). We conjecture however, that in general, one may characterize $\Lambda(q)$ as a solution to the matrix equation $\mathbf{F}_{\Phi(q)}(-\Lambda(q)) = \mathbf{0}$.

We shall also obtain the following result concerning one sided exit downwards as a consequence of the proof of Theorem 4.

THEOREM 8. *For each $q > 0$ and $x \geq 0$*

$$\mathbf{E}_x\left(e^{-q\tau_0^-}1_{(\tau_0^-<\infty)}\right) = \left(\mathbf{Z}^{(q)}(x) - q\mathbf{W}^{(q)}(x)\,\widehat{\mathbf{D}}(q)^T\right)\mathbf{I}(q)^{-1}\ , \tag{9}$$

*where* $\widehat{\mathbf{D}}(q) = \Delta_{\mathbf{v}}(\Phi(q))\,(\Phi(q)\mathbf{I} - \widehat{\Lambda}(q))^{-1}\Delta_{\mathbf{v}}(\Phi(q))^{-1}$ *and* $\Delta_{\mathbf{v}}(\Phi(q))$ *is the diagonal matrix associated with the vector* $\mathbf{v}(\Phi(q))$. *Moreover,*

$$\mathbf{P}_x\left(\tau_0^- < \infty\right) = \mathbf{I} - \int_0^x \mathbf{W}(y)dy\mathbf{Q} - \mathbf{W}(x)\mathbf{D}\ , \tag{10}$$

*where*

$$\mathbf{D} = \lim_{a\uparrow\infty}\mathbf{W}(a)^{-1}[\mathbf{I} - \int_0^a \mathbf{W}(y)dy\mathbf{Q}]. \tag{11}$$

REMARK 9. The original assumption forbidding any of the processes $X^i$ to be downward subordinators may ultimately be weakened to the assumption that at least one of the processes $X^i$ is not a downward subordinator. To see why, suppose the latter holds. Consider the processes $X_n(t) := X(t) + \frac{1}{n}B(t)$ $(n = 1, 2, \ldots)$, where $B(t)$ is a Brownian motion independent of $X(t)$. Note that $X_n \to X$ in $(D[0,\infty), \mathcal{M}_2)$ Skorohod topology. Functionals sup and inf are continuous in this topology and convergence in $\mathcal{M}_2$ coincide with pointwise convergence on a dense subset of $\mathbb{R}_+$. Hence exit times converge in distribution since the limiting MAP $X(t)$ is not equal to $a$ and 0 throughout the intervals $(\tau_a^+ - \epsilon, \tau_a^+)$ and $(\tau_0^- - \epsilon, \tau_0^-)$ respectively. See also Whitt (2002), Th. 13.6.4.

**4. Discussion.** The results in Theorem 4 mirror perfectly the expressions given in equation (6) and Corollary 1 of Bertoin (1997) for spectrally negative Lévy processes. (Note a typo appears in the latter Corollary, the correct expression on the right hand side of the identity there should have a factor $q$ multiplying the second term). The results presented by Bertoin formalize and strengthen older conclusions of the same type given by Zolotarev (1964), Takács (1967), Bingham (1975), Suprun (1976) and Rogers (1990). Theorem 6 is the consequence of a direct analogue of a well established technique attributed to *Wald*. The analogous result for spectrally negative Lévy processes can also be found in Emery (1973). Theorem 8 generalizes a result of Emery (1973) again for spectrally negative Lévy processes. See also the presentation of this result in Bingham (1975) and the extended version of Avram et al. (2002) appearing in the PhD thesis of Pistorius (2003).

At first glance the striking similarity to the analogous results for spectrally negative Lévy processes is not surprising given the many similar properties of MAPs to their one dimensional counterparts. One may even be tempted to claim that the proof of these results should follow in a straightforward fashion along the lines of the *known* proofs for the case of spectrally negative Lévy processes. However, when examining these proofs, one finds heavy dependence on either complex analysis centred around the Wiener-Hopf factorization or Itô's excursion theory. In principle generalizing these proofs should be possible, however, in the absence of a comfortable excursion theory and Wiener-Hopf factorizations, this does not look like an appealing prospect. Instead we turn to a completely new approach. It turns out that we are able to produce Theorems 4 and 8 using only standard properties of martingales, the Strong Markov Property and the fact that $X$ can only creep upwards. This is partly thanks to a recently introduced martingale for such MAPs described in Asmussen and Kella (2000). The proofs are relatively straightforward, and indeed when one writes them out verbatim for the case of spectrally negative Lévy processes, as a whole, one finds relatively short and easy proofs of the existing fluctuation identities mentioned in the previous literature. In that case, issues such as matrix inversion and the involvement of terms such as $\Lambda(q)$ disappear and the Asmussen-Kella martingale takes the simpler form of the Kella-Whitt martingale; see Kella and Whitt (1992). Kyprianou and Palmowski (2003) give a full account of the proofs for the case of spectrally negative Lévy processes.

It would seem that the Asmussen-Kella martingale contains at least as much information as one would hope to gain from a Wiener-Hopf factorization. Indeed we are able to extract a Wiener-Hopf factorization from it (although we do not use the factorization explicitly thereafter).

Aside from being a natural generalization of the known results for spectrally negative Lévy processes, there are genuine reasons for wanting to know the conclusions presented in Theorems 4 and 8. In recent years classical models in the theory of financial mathematics, risk and queues have been replaced by ones involving Markov modulation and general classes of Lévy processes. In the latter case one particular class that has proved to be quite successful in this respect is spectrally negative Lévy processes, on account of the robustness of their fluctuation theory; see for example Schmidli (1999), Asmussen (2000), Asmussen and Kella (2000), Avram et al. (2002), Klüppelberg et al. (2003) and Huzak et al. (2003) to name but a few. Our results justify the reasoning that in general, one should hope to be able to work with spectrally negative MAPs in these models. In forthcoming work, we shall present some applications of these identities in the context of conditionings of MAPs in queuing and ruin models.

The rest of the paper is presented as follows. In the next section we introduce a martingale given in Asmussen and Kella (2000) and show how the existence of the scale matrices and the Theorem 4 (i) can be deduced from it. As a product of our analysis, the Wiener-Hopf factorization will emerge (although it will not be used explicitly in what follows) and this is given in Section 6. The following two sections deal with the the proof of Theorem 4 (i) and the final two sections deal with the proof of Theorem 4 (ii) and Theorem 8.

**5. Asmussen-Kella martingale.** The basis of this section is the new martingale introduced by Asmussen and Kella (2000). This martingale is closely related to other martingales of an exponential type found in Kella and Whitt (1992) and Jacod and Shiryayev (1987) for example. For our purposes we shall simply introduce the Asmussen-Kella martingale in the form that we shall use it; however the reader is urged to return to the original presentation in Asmussen and Kella (2000) in order to appreciate that it can take a more general form. The martingale in question is zero mean, vector valued and takes the form

$$\int_0^t e^{-\beta Z(u)}\mathbf{1}_{J(u)}du\cdot\mathbf{F}(\beta)+e^{-\beta Z(0)}\mathbf{1}_{J(0)}-e^{-\beta Z(t)}\mathbf{1}_{J(t)}-\beta\int_0^t\mathbf{1}_{J(u)}dS(u)\ ,$$

where $\mathbf{1}_i$ is the $1\times N$ vector whose elements are zero except at the $i$-th position where it is 1, $\beta\geq 0$ and $Z(t)=S(t)-X(t)$. From this martingale we shall deduce the following identity.

THEOREM 10. *Suppose that* $\mathbf{e}_q$ *is an exponential distribution with intensity* $q>0$ *that is independent of* $(J,X)$. *Then for* $\beta\geq 0$

$$\mathbf{E}\left(e^{\beta I(\mathbf{e}_q)}\right)^T(\mathbf{F}(\beta)-q\mathbf{I})^T=q\Delta_{\mathbf{v}}(\Phi(q))[\beta(\Phi(q)\mathbf{I}-\widehat{\Lambda}(q))^{-1}-\mathbf{I}]\Delta_{\mathbf{v}}(\Phi(q))^{-1}\ ,\tag{12}$$

*where the* $N\times N$ *matrix* $\widehat{\Lambda}(q)$ *is the intensity matrix of the process* $\{J(\tau_x^+):x\geq 0\}$ *under* $\widehat{\mathbf{P}}^{\Phi(q)}$.

*Proof.* From (4) we have that

$$\begin{aligned}\mathbb{E}_{i,0}\left(e^{\beta I(\mathbf{e}_q)}1_{(J(\mathbf{e}_q)=j)}\right)&=\mathbb{E}_{i,0}\left(e^{\beta I(\mathbf{e}_q)}|J(\mathbf{e}_q)=j\right)\mathbb{P}_{i,0}(J(\mathbf{e}_q)=j)\\&=\widehat{\mathbb{E}}_{j,0}\left(e^{-\beta Z(\mathbf{e}_q)}|J(\mathbf{e}_q)=i\right)\widehat{\mathbb{P}}_{j,0}(J(\mathbf{e}_q)=i)\,\frac{\pi_j}{\pi_i}\\&=\pi_i^{-1}\widehat{\mathbb{E}}_{j,0}\left(e^{-\beta Z(\mathbf{e}_q)}1_{(J(\mathbf{e}_q)=i)}\right)\pi_j.\end{aligned}$$

That is to say, the matrix $\widehat{\mathbf{E}}\left(e^{-\beta Z(\mathbf{e}_q)}\right)$ fulfills

$$\Delta_\pi^{-1}\mathbf{E}\left(e^{\beta I(\mathbf{e}_q)}\right)^{\mathrm{T}}\Delta_\pi=\widehat{\mathbf{E}}\left(e^{-\beta Z(\mathbf{e}_q)}\right)\ .\tag{13}$$

Now note that a simple calculation involving Fubini's Theorem shows that

$$\frac{1}{q}\widehat{\mathbb{E}}_{i,0}\left(e^{-\beta Z(\mathbf{e}_q)}\mathbf{1}_{J(\mathbf{e}_q)}\right)=\widehat{\mathbb{E}}_{i,0}\left(\int_0^{\mathbf{e}_q}e^{-\beta Z(u)}\mathbf{1}_{J(u)}du\right).$$

Making use of the Asmussen and Kella's martingale (as given above) applied to the time reversed MAP, it follows by taking expectation with respect to $\widehat{\mathbb{P}}_{i,0}$ that

$$\widehat{\mathbb{E}}_{i,0}\left(e^{-\beta Z(\mathbf{e}_q)}\mathbf{1}_{J(\mathbf{e}_q)}\right)\left(\widehat{\mathbf{F}}(\beta)-q\mathbf{I}\right)=-q\mathbf{1}_i+\beta q\widehat{\mathbb{E}}_{i,0}\left(\int_0^{\mathbf{e}_q}\mathbf{1}_{J(u)}dS(u)\right). \tag{14}$$

Now note, with the help of Fubini's Theorem and the fact that $(X,\widehat{\mathbb{P}}_{i,0})$ can only creep upwards thus allowing $\tau^+$ to be considered the inverse of $S(\cdot)$,

$$\begin{aligned}\widehat{\mathbb{E}}_{i,0}\left(\int_0^{\mathbf{e}_q}\mathbf{1}_{J(u)}dS(u)\right)&=\widehat{\mathbb{E}}_{i,0}\left(\int_0^{\infty}e^{-qu}\mathbf{1}_{J(u)}dS(u)\right)\\&=\widehat{\mathbb{E}}_{i,0}\left(\int_0^{\infty}1_{(\tau_a^+<\infty)}e^{-q\tau_a^+}\mathbf{1}_{J(\tau_a^+)}da\right).\end{aligned}$$

Further, applying a change of measure in the spirit of (2) we have

$$\widehat{\mathbb{E}}_{i,0}\left(\int_0^{\mathbf{e}_q}1_{(J(u)=j)}dS(u)\right)=\int_0^{\infty}e^{-\Phi(q)a}\widehat{\mathbb{E}}_{i,0}^{\Phi(q)}\left(1_{(\tau_a^+<\infty)}1_{(J(\tau_a^+)=j)}\right)\frac{\widehat{h}_i(\Phi(q))}{\widehat{h}_j(\Phi(q))}da. \tag{15}$$

(Recall that $\kappa=\widehat{\kappa}$ and hence $\Phi=\widehat{\Phi}$). Finally noting from (3) that $\kappa'_{\Phi(q)}(0)=\kappa'(\Phi(q))>0$, it follows that under $\widehat{\mathbb{P}}_{i,0}^{\Phi(q)}$ the process $X$ drifts to infinity thus allowing the removal of the indicator $1_{(\tau_a^+<\infty)}$ in (15). Replacing (13) and (15) in (14) and considering the latter vector equality as the $i$-th row of a matrix, we obtain

$$\Delta_\pi^{-1}\mathbf{E}\left(e^{\beta I(\mathbf{e}_q)}\right)^{\mathrm{T}}\Delta_\pi\left(\widehat{\mathbf{F}}(\beta)-q\mathbf{I}\right)=-q\mathbf{I}+\beta q\Delta_{\widehat{\mathbf{h}}}(\Phi(q))\widehat{\mathbf{M}}(q)\Delta_{\widehat{\mathbf{h}}}(\Phi(q))^{-1},$$

where

$$\widehat{\mathbf{M}}_{ij}(q)=\int_0^\infty da\cdot e^{-\Phi(q)a}\widehat{\mathbb{P}}_{i,0}^{\Phi(q)}(J(\tau_a^+)=j).$$

Note that matrix $\widehat{\mathbf{M}}(q)$ has no zero columns by the assumption that none of the Lévy processes is downward subordinator. Recalling that $\widehat{\mathbf{F}}(\alpha)=\Delta_\pi^{-1}\mathbf{F}(\alpha)^{\mathrm{T}}\Delta_\pi$ we have

$$\mathbf{E}\left(e^{\beta I(\mathbf{e}_q)}\right)^{\mathrm{T}}(\mathbf{F}(\beta)-q\mathbf{I})^{\mathrm{T}}=-q\mathbf{I}+\beta q\Delta_\pi\Delta_{\widehat{\mathbf{h}}}(\Phi(q))\widehat{\mathbf{M}}(q)\Delta_{\widehat{\mathbf{h}}}(\Phi(q))^{-1}\Delta_\pi^{-1}.$$

Further noting that $\Delta_\pi\widehat{\mathbf{h}}(\alpha)^{\mathrm{T}}=\mathbf{v}(\alpha)^{\mathrm{T}}$ implies that $\Delta_\pi\Delta_{\widehat{\mathbf{h}}}(\alpha)=\Delta_{\mathbf{v}}(\alpha)$, the diagonal matrix associated with vector $\mathbf{v}(\alpha)$, we have that

$$\mathbf{E}\left(e^{\beta I(\mathbf{e}_q)}\right)^{\mathrm{T}}(\mathbf{F}(\beta)-q\mathbf{I})^{\mathrm{T}}=-q\mathbf{I}+\beta q\Delta_{\mathbf{v}}(\Phi(q))\widehat{\mathbf{M}}(q)\Delta_{\mathbf{v}}(\Phi(q))^{-1}. \tag{16}$$

The theorem is proved once we note that $\widehat{\mathbf{M}}(q)=(\Phi(q)\mathbf{I}-\widehat{\Lambda}(q))^{-1}$ on account of an argument similar to the one given in Remark 3. $\square$

**6. Wiener-Hopf factorization.** In this section we shall make a slight diversion from the main programme and show that the equality (12) is in fact the Wiener-Hopf factorization in disguise for this class of MAP. Fix $q > 0$ and begin by noting that

$$\begin{aligned}\widehat{\mathbb{P}}_{i,0}(S(\mathbf{e}_q) > x, J(\mathbf{e}_q) = j) &= \widehat{\mathbb{E}}_{i,0}(1_{(\mathbf{e}_q > \tau_x^+, \tau_x^+ < \infty)} \widehat{\mathbb{P}}_{J(\tau_x^+), X(\tau_x^+)}(J(\mathbf{e}_q) = j)) \\ &= \widehat{\mathbb{E}}_{i,0}(e^{-q\tau_x^+} 1_{(\tau_x^+ < \infty)} \mathbf{1}_{J(\tau_x^+)} \widehat{\mathbf{I}}(q) \mathbf{1}_j^{\mathrm{T}}) \\ &= \widehat{\mathbb{E}}_{i,0}(e^{-q\tau_x^+} 1_{(\tau_x^+ < \infty)} \mathbf{1}_{J(\tau_x^+)}) \widehat{\mathbf{I}}(q) \mathbf{1}_j^{\mathrm{T}} ,\end{aligned}$$

where we have used obvious notation for $\widehat{\mathbf{I}}(q)$. That is to say

$$\widehat{\mathbf{P}}\left(S(\mathbf{e}_q) > x\right) = \widehat{\mathbf{E}}\left(e^{-q\tau_x^+} 1_{(\tau_x^+ < \infty)}\right) \widehat{\mathbf{I}}(q).$$

It follows that for $\theta > -\Phi(q)$

$$\begin{aligned}\widehat{\mathbf{E}}(e^{-\theta S(\mathbf{e}_q)}) &= \int_0^\infty e^{-\theta x} \widehat{\mathbf{P}}(S(\mathbf{e}_q) \in dx) \\ &= \int_0^\infty \theta e^{-\theta x} \widehat{\mathbf{P}}(S(\mathbf{e}_q) \leq x) dx \\ &= \widehat{\mathbf{I}}(q) - \int_0^\infty \theta e^{-\theta x} \widehat{\mathbf{P}}(S(\mathbf{e}_q) > x) dx \\ &= [\mathbf{I} - \int_0^\infty \theta e^{-\theta x} \widehat{\mathbf{E}}(e^{-q\tau_x^+} 1_{(\tau_x^+ < \infty)}) dx] \widehat{\mathbf{I}}(q) \\ &= \Delta_{\widehat{\mathbf{h}}}(\Phi(q))[\mathbf{I} - \int_0^\infty \theta e^{-((\theta + \Phi(q))\mathbf{I} - \widehat{\Lambda}(q))x} dx] \Delta_{\widehat{\mathbf{h}}}(\Phi(q))^{-1} \widehat{\mathbf{I}}(q) \\ &= \Delta_{\widehat{\mathbf{h}}}(\Phi(q))[\mathbf{I} - \theta((\theta + \Phi(q))\mathbf{I} - \widehat{\Lambda}(q))^{-1}] \Delta_{\widehat{\mathbf{h}}}(\Phi(q))^{-1} \widehat{\mathbf{I}}(q) \,, \qquad (17)\end{aligned}$$

where in the second equality we have used integration by parts and in the fifth equality we have used (7). The sixth inequality follows again by using reasoning similar to that of Remark 3.

It is also easy to check that

$$[\beta(\Phi(q)\mathbf{I} - \widehat{\Lambda}(q))^{-1} - \mathbf{I}]^{-1} = -[\mathbf{I} + \beta((\Phi(q) - \beta)\mathbf{I} - \widehat{\Lambda}(q))^{-1}]. \qquad (18)$$

It now follows from (12) that for $0 \leq \beta < \Phi(q)$

$$\Delta_\pi^{-1} \mathbf{E}\left(e^{\beta I(\mathbf{e}_q)}\right)^{\mathrm{T}} (q\mathbf{I} - \mathbf{F}(\beta))^{\mathrm{T}} \Delta_\pi = q\widehat{\mathbf{I}}(q) \widehat{\mathbf{E}}(e^{\beta S(\mathbf{e}_q)})^{-1}.$$

Recalling that $\widehat{\mathbf{Q}} = \Delta_\pi^{-1} \mathbf{Q}^T \Delta_\pi$, some straightforward algebra reveals

$$\mathbf{E}(e^{\beta I(\mathbf{e}_q)}) \frac{1}{q} (q\mathbf{I} - \mathbf{Q})[\Delta_\pi^{-1} \widehat{\mathbf{E}}(e^{\beta S(\mathbf{e}_q)})^{\mathrm{T}} \Delta_\pi] = q(q\mathbf{I} - \mathbf{F}(\beta))^{-1}$$

which is the Wiener-Hopf factorization. The right hand side is analytical in $\Upsilon := \{\beta \in \mathbf{C} : \mathbf{F}(\mathrm{Re}\ \beta) < \infty\}$. Thus by analytical extension we can in fact claim the above identity for all $\beta \in \Upsilon$.

**7. Distribution of the infimum with positive drift.** A corollary of Theorem 10 is that, when $X$ drifts to infinity, we can identify the scale matrix $\mathbf{W}(x) = \mathbf{W}^{(0)}(x)$ as, up to a multiplying invertable matrix, proportional to the matrix distribution of the all time infimum of $X$.

COROLLARY 11. *Suppose that $X$ drifts to infinity. Then there exists an LCRL matrix $\mathbf{W}^*(x)$ given by*

$$\mathbf{W}^*_{ij}(x) = \mathbb{P}_{i,x}(I(\infty) > 0, J(\infty) = j)$$

*such that for all $\beta > 0$*

$$\int_0^\infty e^{-\beta x}\mathbf{W}^*(x)\,dx = \mathbf{F}(\beta)^{-1} \times \mathbf{C}\ ,$$

*where $\mathbf{C}$ is an invertible matrix. Note that clearly by definition, $\mathbf{W}^*(x) = 0$ for $x \leq 0$.*

*Proof.* Starting with (12), we shall take the limit as $q$ tends to zero. We have

$$\mathbf{F}(\beta)\,\mathbf{E}\left(e^{\beta I(\infty)}\right) = \beta\mathbf{C}\ , \tag{19}$$

where

$$\begin{aligned}\mathbf{C} &= \lim_{q\downarrow 0}\Delta_{\mathbf{v}}^{-1}(\Phi(q))q[(\Phi(q)\mathbf{I} - \widehat{\Lambda}(q))^{-1}]^{\mathrm{T}}\Delta_{\mathbf{v}}(\Phi(q)) \\ &= \lim_{q\downarrow 0} q\widehat{\mathbf{D}}(q)^{\mathrm{T}}. \end{aligned} \tag{20}$$

Note that the matrix $\mathbf{C}$ is well-defined because the limit on the left hand side of (19) exists. Denote by $\mathrm{Adj}(\mathbf{A})$ the adjoint of a matrix $\mathbf{A}$.

Matrix $\mathbf{C}$ is invertible since

$$\begin{aligned}\det\,\mathbf{C} &= \lim_{q\downarrow 0}\ \frac{q}{\det\,(\Phi(q)\mathbf{I} - \widehat{\Lambda}(q))} \\ &= \kappa'(0)\ \lim_{q\downarrow 0}\frac{\Phi(q)}{\det\,(\Phi(q)\mathbf{I} - \widehat{\Lambda}(q))} \\ &= -\kappa'(0)\frac{1}{\mathrm{tr}\ \mathrm{Adj}\ (\widehat{\Lambda}(0))} > 0\ .\end{aligned}$$

The third equality follows from expansion of determinant in the denominator with respect to $\Phi(q)$. To see where the last inequality comes from, note that $\kappa'(0) > 0$ which is in turn a result of the assumption that $X$ drifts to infinity. Moreover, if we delete $i$-th row and $i$-th column of intensity matrix $\widehat{\Lambda}(0)$, then we obtain subintensity matrix (non-singular negative of an M-matrix), hence by Chapter 6 of Berman and Plemmons (1979) the determinant of this matrix is negative. This gives, $\mathrm{tr}\ \mathrm{Adj}\ \widehat{\Lambda}(0) < 0$.

Recalling Remark 1 we also note here that the invertibility of $\mathbf{C}$ implies directly that $\mathbf{F}(\beta)$ is invertible for each $\beta > 0$.

Now note that

$$\mathbf{E}\left(e^{\beta I(\infty)}\right) = \int_0^\infty e^{-\beta x}\mathbf{W}^*(dx) \ ,$$

where $\mathbf{W}^*_{ij}(dx) = \mathbb{P}_i(-I(\infty) \in dx, J(\infty) = j)$. Integration by parts thus yields

$$\mathbf{E}\left(e^{\beta I(\infty)}\right) = \beta\int_0^\infty e^{-\beta x}\mathbf{W}^*(x)\,dx \ ,$$

where $\mathbf{W}^*_{ij}(x) = \mathbb{P}_i(-I(\infty) < x, J(\infty) = j) = \mathbb{P}_{i,x}(I(\infty) > 0, J(\infty) = j)$ which is LCRL on account of it being a distribution. Together with (19) we have thus established the statement of the Corollary. $\square$

**8. Proof of Theorem 4 (i).** *Step 1.* Define $\mathbf{W}(x) = \mathbf{W}^*(x) \times \mathbf{C}^{-1}$. Let us keep with the assumption that $X$ drifts to infinity. With this assumption, it is clear that $X$ hits every level $a > 0$ with probability one. Consequently we have for $0 < x \leq a$

$$\begin{aligned}\mathbb{P}_{i,x}(I(\infty) > 0, J(\infty) = j) &= \sum_k \mathbb{P}_{i,x}\left(J(\tau_a^+) = k, \tau_a^+ < \tau_0^-\right)\\ &\quad\times\mathbb{P}_{i,x}(I(\infty) > 0, J(\infty) = j | J(\tau_a^+) = k, \tau_a^+ < \tau_0^-)\\ &= \sum_k \mathbb{P}_{i,x}\left(J(\tau_a^+) = k, \tau_a^+ < \tau_0^-\right)\mathbb{P}_{k,a}(I(\infty) > 0, J(\infty) = j).\end{aligned}$$

That is to say

$$\mathbf{W}(x) = \mathbf{P}_x(\tau_a^+ < \tau_0^-)\mathbf{W}(a) \ . \tag{21}$$

*Step 2.* Next note that for any $0 < x \leq a$,

$$\{\mathbf{P}_x(\tau_a^+ < \tau_0^-)\}_{ij} = \mathbb{P}_{i,x}(J(\tau_a^+) = j, \tau_a^+ < \tau_0^-)$$

is the transition matrix of the Markov chain $\{J^*(\tau_a^+) : a \geq 0\}$ of the modulating state on the upward ladder-height process of $X^*$, where $X^*$ is the process $X$ killed on exiting $(0,\infty)$. Clearly then there exists some $N \times N$ sub-stochastic intensity matrix $\Lambda^*$ such that $\mathbf{P}_x(\tau_a^+ < \tau_0^-) = \exp\{\Lambda^*(a-x)\}$ which has inverse $\exp\{-\Lambda^*(a-x)\}$.

Now suppose that there exists an $\alpha = \alpha(x_0)$ (not the zero vector) such that $\mathbf{W}(x_0)\alpha^{\mathrm{T}} = 0$. From (21) and the just established invertability, it follows that $\mathbf{W}(a)\alpha^{\mathrm{T}} = 0$ for all $a \geq x_0$. On the other hand, since for $0 < y \leq x_0$

$$\mathbf{W}(y) = \mathbf{P}_y(\tau_{x_0}^+ < \tau_0^-)\mathbf{W}(x_0) \ .$$

it follows that $\mathbf{W}(y)\alpha^{\mathrm{T}} = 0$ for all $0 \leq y \leq x_0$. In conclusion, either all of the matricies $\{\mathbf{W}(x) : x > 0\}$ are invertable or there exists a non-zero $\alpha$ (independent of the parameter $x$) which lies in the kernel of all of them. On the other hand, in the latter case, for $\beta > 0$,

$$\begin{aligned}\mathbf{0}^{\mathrm{T}} &= \int_0^\infty e^{-\beta x}\mathbf{W}(x)\alpha^{\mathrm{T}}dx \\ &= \int_0^\infty e^{-\beta x}\mathbf{W}(x)dx\alpha^{\mathrm{T}} \\ &= \mathbf{F}(\beta)^{-1}\alpha^{\mathrm{T}} ,\end{aligned}$$

where $\mathbf{0}^{\mathrm{T}}$ is the $N \times 1$ column vector with zero at each entry, which implies that $\alpha$ is the zero vector; a contradiction. It follows then that $\mathbf{W}(x)$ is invertible for all $x > 0$.

We now have from (21) that for $0 < x \leq a$

$$\mathbf{P}_x\left(\tau_a^+ < \tau_0^-\right) = \mathbf{W}(x)\,\mathbf{W}(a)^{-1}$$

and recalling that $\mathbf{W}(x) = \mathbf{0}$ for all $x \leq 0$, we see that in fact the above identity is valid for all $x \leq a$. We have thus proved part (i) of Theorem 4 for the case that $q = 0$ and $X$ drifts to infinity.

*Step 3.* Using a change of measure (2) we have for any $q > 0$ that

$$\begin{aligned}&\mathbb{E}_{i,x}\left(e^{-q\tau_a^+}1_{(\tau_a^+<\tau_0^-,J(\tau_a^+)=j)}\right) \\ &= \mathbb{E}_{i,x}\left(e^{\Phi(q)(a-x)-q\tau_a^+}1_{(\tau_a^+<\tau_0^-,J(\tau_a^+)=j)}\frac{h_j(\Phi(q))}{h_i(\Phi(q))}\right)\frac{h_i(\Phi(q))}{h_j(\Phi(q))}e^{-\Phi(q)(a-x)} \\ &= h_i(\Phi(q))\mathbb{P}_{i,x}^{\Phi(q)}\left(\tau_a^+ < \tau_0^-, J(\tau_a^+) = j\right)\frac{1}{h_j(\Phi(q))}e^{-\Phi(q)(a-x)}. \qquad (22)\end{aligned}$$

Now $\kappa'_{\Phi(q)}(0) = \kappa'(\Phi(q)) > 0$. So that under $\mathbb{P}_{i,x}^{\Phi(q)}$ the process $X$ always drifts to infinity. We may thus use the conclusions of the previous paragraph in (22) to deduce

$$\begin{aligned}\mathbf{E}_x\left(e^{-q\tau_a^+}1_{(\tau_a^+<\tau_0^-)}\right) &= [\Delta_{\mathbf{h}}(\Phi(q))\,e^{\Phi(q)x}\mathbf{W}_{\Phi(q)}(x)\,\Delta_{\mathbf{h}}(\Phi(q))^{-1}] \\ &\quad\times[\Delta_{\mathbf{h}}(\Phi(q))\,e^{\Phi(q)a}\mathbf{W}_{\Phi(q)}(a)\,\Delta_{\mathbf{h}}(\Phi(q))^{-1}]^{-1} \\ &= \mathbf{W}^{(q)}(x)\,\mathbf{W}^{(q)}(a)^{-1} ,\end{aligned}$$

where $\mathbf{W}_{\Phi(q)}(x)$ plays the role of $\mathbf{W}(x)$ for $(X, \mathbb{P}^{\Phi(q)})$ and

$$\mathbf{W}^{(q)}(x) = \Delta_{\mathbf{h}}(\Phi(q))\,e^{\Phi(q)x}\mathbf{W}_{\Phi(q)}(x)\,\Delta_{\mathbf{h}}(\Phi(q))^{-1} \qquad (23)$$

is LCRL since $\mathbf{W}_{\Phi(q)}$ is. Now note that for $\beta > \Phi(q)$, if $\mathbf{C}_{\Phi(q)}$ is the matrix defined in (20) for $(X, \mathbb{P}^{\Phi(q)})$,

$$\begin{aligned}\int_0^\infty e^{-\beta x}\mathbf{W}^{(q)}(x)\,dx &= \Delta_{\mathbf{h}}(\Phi(q))\int_0^\infty e^{-(\beta-\Phi(q))x}\mathbf{W}_{\Phi(q)}(x)\,dx\Delta_{\mathbf{h}}(\Phi(q))^{-1} \\ &= \Delta_{\mathbf{h}}(\Phi(q))\,\mathbf{F}_{\Phi(q)}(\beta-\Phi(q))^{-1}\,\Delta_{\mathbf{h}}(\Phi(q))^{-1} \\ &\quad\times\Delta_{\mathbf{h}}(\Phi(q))\,\mathbf{C}_{\Phi(q)}\Delta_{\mathbf{h}}(\Phi(q))^{-1} .\end{aligned}$$

From (3) we can check that

$$\Delta_{\mathbf{h}}(\Phi(q))\,\mathbf{F}_{\Phi(q)}(\beta-\Phi(q))^{-1}\Delta_{\mathbf{h}}(\Phi(q))^{-1}=(\mathbf{F}(\beta)-q\mathbf{I})^{-1}$$

providing $\beta-\Phi(q)>0$. [Note that it now becomes clear that $\mathbf{F}(\beta)-q\mathbf{I}$ is indeed invertible for $\beta>\Phi(q)$]. Thus defining $\mathbf{C}(q):=\Delta_{\mathbf{h}}(\Phi(q))\,\mathbf{C}_{\Phi(q)}\Delta_{\mathbf{h}}(\Phi(q))^{-1}$ we deduce that

$$\int_0^\infty e^{-\beta x}\mathbf{W}^{(q)}(x)\,dx=(\mathbf{F}(\beta)-q\mathbf{I})^{-1}\times\mathbf{C}(q).$$

The proof of the first part of Theorem 4 is complete for $q>0$ once we note that we may replace $\mathbf{W}^{(q)}(x)$ by $\mathbf{W}^{(q)}(x)\times\mathbf{C}(q)^{-1}$ and the statement in the first part of Theorem 4 is still valid.

*Step 4.* To deal with the case that $q=0$ but $X$ does not drift to infinity, one may simply take the limit as $q$ tends to zero from the previous case taking care with dominated convergence. The details are not difficult and left to the reader. □

**9. Proof of Theorem 4 (ii).** Fix $q>0$. Integration by parts together with (5) tells us that for $\beta>\Phi(q)$

$$\int_0^\infty e^{-\beta x}\mathbf{W}^{(q)}(dx)=\beta(\mathbf{F}(\beta)-q\mathbf{I})^{-1}$$

and hence together with (12) we can read off that

$$\mathbf{P}(-I(\mathbf{e}_q)\le x)=q\mathbf{W}^{(q)}(x)\,\widehat{\mathbf{D}}(q)^{\mathrm{T}}-q\int_0^x\mathbf{W}^{(q)}(y)\,dy\,,$$

where $\widehat{\mathbf{D}}(q)=\Delta_{\mathbf{v}}(\Phi(q))\,(\Phi(q)\mathbf{I}-\widehat{\Lambda}(q))^{-1}\Delta_{\mathbf{v}}(\Phi(q))^{-1}$. Note that an analogous expression was given for spectrally negative Lévy processes in Bertoin (1997). A simple manipulation yields

$$\mathbf{P}_x(I(\mathbf{e}_q)<0)=\mathbf{I}(q)+q\int_0^x\mathbf{W}^{(q)}(y)\,dy-q\mathbf{W}^{(q)}(x)\,\widehat{\mathbf{D}}(q)^{\mathrm{T}}. \tag{24}$$

For convenience, write $\tau=\tau_a^+\wedge\tau_0^-$. Note that

$$\begin{aligned}&\mathbb{P}_{i,x}(I(\mathbf{e}_q)<0,J(\mathbf{e}_q)=j|\mathcal{F}_{t\wedge\tau})\\&=e^{-q(t\wedge\tau)}\mathbb{P}_{J(t\wedge\tau),X(t\wedge\tau)}(I(\mathbf{e}_q)<0,J(\mathbf{e}_q)=j)\\&=e^{-q(t\wedge\tau)}\mathbf{1}_{J(t\wedge\tau)}\mathbf{P}_{X(t\wedge\tau)}(I(\mathbf{e}_q)<0)\,\mathbf{1}_j^{\mathrm{T}}\end{aligned}\tag{25}$$

showing that

$$e^{-q(t\wedge\tau)}\mathbf{1}_{J(t\wedge\tau)}\left[\mathbf{I}(q)+q\int_0^{X(t\wedge\tau)}\mathbf{W}^{(q)}(y)\,dy-q\mathbf{W}^{(q)}(X(t\wedge\tau))\,\widehat{\mathbf{D}}(q)^{\mathrm{T}}\right],\ t\ge0$$

is a $\mathbb{P}_{i,x}$-martingale. A similar calculation starting from

$$\mathbb{E}_{i,x}(e^{-q\tau_a^+}1_{(\tau_a^+<\tau_0^-,J(\tau_a^+)=j)}|\mathcal{F}_{t\wedge\tau})$$

shows that

$$e^{-q(t\wedge\tau)}\mathbf{1}_{J(t\wedge\tau)}\mathbf{W}^{(q)}(X(t\wedge\tau))\,\mathbf{W}^{(q)}(a)^{-1},\ t\ge0 \tag{26}$$

is also a $\mathbb{P}_{i,x}$-martingale. By linearity we may construct now the $\mathbb{P}_{i,x}$-martingale

$$e^{-q(t\wedge\tau)}\mathbf{1}_{J(t\wedge\tau)}\left[\mathbf{Z}^{(q)}\left(X(t\wedge\tau)\right)-\mathbf{W}^{(q)}\left(X(t\wedge\tau)\right)\mathbf{W}^{(q)}\left(a\right)^{-1}\mathbf{Z}^{(q)}\left(a\right)\right],\ t\geq 0. \tag{27}$$

This uniformly integrable martingale has terminal value

$$e^{-q\tau_0^-}\mathbf{1}_{J(\tau_0^-)}\left[\mathbf{I}\left(q\right)\times 1_{(\tau_0^-<\tau_a^+)}\right]=e^{-q\tau_0^-}\mathbf{1}_{J(\tau_0^-)}\mathbf{I}\left(q\right)1_{(\tau_0^-<\tau_a^+)}.$$

Defining $\mathbf{f}_{ij}(x)=\mathbb{P}_{i,x}(\mathbf{e}_q>\tau_0^-,\tau_0^-<\tau_a^+,J\left(\mathbf{e}_q\right)=j)$, a familiar calculation based on

$$\mathbb{P}_{i,x}(\mathbf{e}_q>\tau_0^-,\tau_0^-<\tau_a^+,J\left(\mathbf{e}_q\right)=j|\mathcal{F}_{t\wedge\tau})$$

gives us yet another $\mathbb{P}_{i,x}$-martingale, namely

$$e^{-q(t\wedge\tau)}\mathbf{1}_{J(t\wedge\tau)}\mathbf{f}\left(X(t\wedge\tau)\right),\ t\geq 0. \tag{28}$$

Since $\mathbf{f}(x)=\mathbf{I}\left(q\right)$ for all $x\leq 0$ and $\mathbf{f}(a)=0$ the terminal value of this latter martingale is again $e^{-q\tau_0^-}\mathbf{1}_{J(\tau_0^-)}\mathbf{I}\left(q\right)1_{(\tau_0^-<\tau_a^+)}$. The martingales (27) and (28) are positive and uniformly integrable with the same terminal value. Hence they are almost surely identical for each $t\geq 0$. It thus follows that

$$\mathbf{f}\left(x\right)=\left[\mathbf{Z}^{(q)}\left(x\right)-\mathbf{W}^{(q)}\left(x\right)\mathbf{W}^{(q)}\left(a\right)^{-1}\mathbf{Z}^{(q)}\left(a\right)\right].$$

To complete the proof note that

$$\begin{aligned}\mathbb{P}_{i,x}(\mathbf{e}_q>\tau_0^-,\tau_0^-<\tau_a^+,J\left(\mathbf{e}_q\right)=j)&=\mathbb{E}_{i,x}\left(e^{-q\tau_0^-}1_{(\tau_0^-<\tau_a^+)}\mathbb{P}_{J\left(\tau_0^-\right),X\left(\tau_0^-\right)}\left(J\left(\mathbf{e}_q\right)=j\right)\right)\\&=\mathbb{E}_{i,x}\left(e^{-q\tau_0^-}1_{(\tau_0^-<\tau_a^+)}\mathbf{1}_{J\left(\tau_0^-\right)}\mathbf{I}(q)\mathbf{1}_j^{\mathrm{T}}\right)\\&=\mathbb{E}_{i,x}\left(e^{-q\tau_0^-}1_{(\tau_0^-<\tau_a^+)}\mathbf{1}_{J\left(\tau_0^-\right)}\right)\mathbf{I}(q)\mathbf{1}_j^{\mathrm{T}}.\end{aligned}$$

That is to say $\mathbf{E}_x(e^{-q\tau_0^-}1_{(\tau_0^-<\tau_a^+)})=\mathbf{f}(x)\mathbf{I}(q)^{-1}$.

To deal with the case that $q=0$, take the limit in the previous case as $q$ tends to zero. □

**10. Proof of Theorem 8.** Using the law of total probability and first conditioning the probability on the left hand side of (24) with respect to $\mathcal{F}_{\tau_0^-}$ we have

$$\mathbf{E}_x(e^{-q\tau_0^-}1_{(\tau_0^-<\infty)})\mathbf{I}(q)=\mathbf{Z}^{(q)}(x)-q\mathbf{W}^{(q)}(x)\widehat{\mathbf{D}}(q)^{\mathrm{T}} \tag{29}$$

and hence the first result follows for $q>0$.

We may now take limits as $q\downarrow 0$ and note that the limit exists on the right hand side of (29) becasue it exists on the left hand side. Writing $\mathbf{D}=\lim_{q\downarrow 0}q\widehat{\mathbf{D}}(q)\mathbf{I}(q)^{-1}$ we now have (10). On the other hand, reconsidering (6) it becomes clear that $\mathbf{D}$ is given by (11).

**Acknowledgements.** Part of this collaborative work was carried out when the first author was visiting Risklab, ETH Zürich and the second author was a researcher at EURANDOM. Both authors would like to express respective thanks to these institutions for their hospitality and support.

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