Further probabilistic analysis of the Fisher-Kolmogorov-Petrovskii-Piscounov equation: one sided travelling-waves

J. W. Harris*, S. C. Harris† and A. E. Kyprianou‡

September 8, 2003

Abstract

Using probabilistic methods alone, we prove the classical result that the FKPP travelling wave equation with monotone solutions bounded in $[0, 1]$ and supported on $[0, \infty)$ has a unique solution with a particular asymptotic. Our methods also deliver the asymptotic in a stronger sense than earlier established. The probabilistic techniques are centred around the study of a branching Brownian motion killed at a barrier, and, as a by product of the analysis, we also offer a new result concerning the asymptotic speed of the right most particle for this process.

1 Introduction and summary

A well studied non-linear PDE is the Fisher-Kolmogorov-Petrovski-Piscounov (FKPP) equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \beta\left(u^2 - u\right) = 0 \tag{1}$$

with $u \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and given initial condition $u(0, x) := f(x)$. The FKPP equation has been much studied by both analytic techniques, as in the original papers of Fisher (1937) and Kolmogorov *et al.* (1937), as well as probabilistic methods as found in McKean (1975,1976), Bramson (1978,1983), Uchiyama (1978), Neveu (1988), Chauvin and Rouault (1988,1990), Harris (1999) and Kyprianou (2003), to name just a few.

*Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, U.K.. email: `j.w.harris@bath.ac.uk`

†Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, U.K.. email: `s.c.harris@bath.ac.uk`

‡Mathematical institute, Utrecht University, P.O.Box 80.010, 3508 TA, Utrecht, The Netherlands. email: `kyprianou@math.uu.nl`

Much attention has been given to FKPP solutions of the form $u(t,x) = f(x-\rho t)$ for $f \in C^2(\mathbb{R})$, leading to the so called FKPP travelling-wave equation

$$\frac{1}{2}f'' - \rho f' + \beta\left(f^2 - f\right) = 0 \text{ on } \mathbb{R} \tag{2}$$

with wave speed ρ. It is well known that monotone travelling-waves that connect 0 at $-\infty$ to 1 at ∞ exist and are unique (up to translation) for all speeds $\rho \geq \sqrt{2\beta}$. For $-\infty < \rho < \sqrt{2\beta}$, there exist no monotone travelling-wave solutions of speed ρ.

One of the probabilistic methods for studying equations (1) and (2) is via their connection to a branching Brownian motion (BBM) with dyadic splitting and branching rate β. See Harris (1999) and Kyprianou (2003) for more complete expositions that are also particularly relevant for the techniques used in this paper.

In this article we shall consider the class of solutions to the FKPP travelling-wave equation defined on $\mathbb{R}^+$ that satisfy $f : \mathbb{R}^+ \to [0,1]$, $f \in C^2(0,\infty)$ and

$$\begin{aligned} \frac{1}{2}f'' - \rho f' + \beta\left(f^2 - f\right) &= 0 \text{ on } (0,\infty), \\ f(0+) &= 1, \\ f(\infty) &= 0. \end{aligned} \tag{3}$$

Note that without the boundary conditions at zero and infinity, we always have that the constant functions of 0 and 1 are solutions to (3). Interestingly, solutions to (3) occur at wave speeds where there are no solutions the FKPP travelling-wave equation on $\mathbb{R}$, as is verified by the following result.

Theorem 1 *The system* (3) *has a unique solution if and only if* $-\infty < \rho < \sqrt{2\beta}$*, in which case*

$$\lim_{x\uparrow\infty} \frac{1}{x} \log f(x) = \rho - \sqrt{\rho^2 + 2\beta} < 0.$$

Further, if $\rho \geq \sqrt{2\beta}$*, there is no solution to* (3).

This theorem was given in Pinsky (1995), who himself cites Kolmogorov *et al.* (1937) and Aronson and Weinberger (1978).

In the spirit of Harris (1999) and Kyprianou (2003), we shall devote this article to a new proof of Theorem 1 using probabilistic means alone. In contrast to the probabilistic study of travelling-waves on $\mathbb{R}$, our probabilistic analysis will be concerned with a branching Brownian motion with drift $-\rho$ where particles are *killed* at the origin. Let ζ be the extinction time of the killed BBM, then we shall show that for $-\infty < \rho < \sqrt{2\beta}$, $P^x(\zeta < \infty)$ provides the unique solution to system (3). We will also establish the following stronger asymptotic.

Theorem 2 *When* $-\infty < \rho < \sqrt{2\beta}$ *the unique solution to (3) satisfies*

$$\lim_{x\uparrow\infty} e^{-(\rho-\sqrt{\rho^2+2\beta})x} f(x) = k$$

for some constant $k \in (0,\infty)$.

Inherent in the study of the one sided FKPP equation is an understanding of the asymptotic behaviour of the right most particle in the branching Brownian motion where particles are killed at the origin. As a consequence of this, at the end of this paper we also prove a new result concerning the asymptotic speed of the right-most particle in the aforementioned process on its survival set.

The paper is arranged as follows. In Section 2 we shall briefly discuss a probabilistic technique that has recently become quite popular in the branching process literature, namely, a change of measure that induces a 'spine' decomposition of the process. Thereafter in Section 3 we look at some important properties of a drifting branching Brownian motion with killing at the origin. In particular, we look at the behaviour of the right-most particle, the relationship with the survival set and survival probabilities. In Sections 4-6, we prove the union of Theorems 1 and 2 via a sequence of smaller results. That is, non-existence of travelling-waves for $\rho \geq \sqrt{2\beta}$, existence of a travelling for $-\infty < \rho < \sqrt{2\beta}$ in the form of the probability of extinction of the aforementioned branching Brownian motion with killing, uniqueness of travelling-waves for $-\infty < \rho < \sqrt{2\beta}$ and finally the asymptotic given in Theorem 2. In the final Section 7, we make use of some of the preliminary results of Section 3 to deduce a new result identifying an asymptotic speed for the right most particle in a drifting branching Brownian motion with killing at the origin.

All of the proofs are probabilistic which in the present context, for the most part, means that they appeal either to martingale arguments, spine decompositions, or fundamental properties of (branching) Brownian motion.

2 Spine decompositions for BBM

In this section we will recall certain changes of measure for a branching Brownian motion using certain fundamental additive martingales. We will discuss how the process under the new measure can be constructed by first laying down the motion of a single particle (the 'spine') as a drifting Brownian motion that gives birth at an accelerated rate along its path to new offspring which then evolve as standard branching Brownian motions. These changes of measure and their associated 'spine' constructions prove a key tool in our later analysis.

Consider a branching Brownian motion with drift $-\rho$ where $\rho \in \mathbb{R}$ and branching rate β. That is a branching process where particles diffuse independently according to a Brownian motion with drift $-\rho$ and at any moment of time undergo fission with rate β producing two particles. We shall refer to this process as a $(-\rho, \beta; \mathbb{R})$-BBM with probabilities $\{P^x : x \in \mathbb{R}\}$ so that P^x is the law of the process initiated from a single particle positioned at x. Suppose that the configuration of space at time t is given by the point process $\mathcal{X}_t^{-\rho}$ with points $\{\mathcal{Y}_u(t) : u \in \mathcal{N}_t^{-\rho}\}$ where $\mathcal{N}_t^{-\rho}$ is the set of individuals alive at time t. Associated with this process are the positive martingales

$$Z_\lambda(t) = \sum_{u \in \mathcal{N}_t^{-\rho}} e^{(\lambda+\rho)\mathcal{Y}_u(t) - (\frac{1}{2}(\lambda^2-\rho^2)+\beta)t}$$

defined for each $\lambda \in \mathbb{R}$. It is known that such martingales are uniformly integrable precisely when $|\lambda + \rho| < \sqrt{2\beta}$ and otherwise they have an almost surely zero limit. Further, when they are uniformly integrable, the limit is strictly positive with probability one. See Neveu (1987) and Kyprianou (2003) for a full account. [Note, the results for a constant drift of $\rho \in \mathbb{R}$ follow almost trivially from the $\rho = 0$ case found in these, and subsequent, references.]

When $|\lambda + \rho| < \sqrt{2\beta}$ one can define an equivalent change of measure on the probability space of the $(-\rho, \beta; \mathbb{R})$-BBM via

$$\left.\frac{d\pi_\lambda^x}{dP^x}\right|_{\mathcal{F}_t} = \frac{Z_\lambda(t)}{Z_\lambda(0)} = e^{-(\lambda+\rho)x} Z_\lambda(t).$$

Under π_λ^x the path of the $(-\rho, \beta; \mathbb{R})$-BBM can be reconstructed pathwise in the following way:

- starting from position x, the initial ancestor diffuses according to a Brownian motion with drift λ,
- at rate 2β the particle undergoes fission producing two particles,
- one of these particles is selected at random with probability one half,
- this chosen particle repeats stochastically the behaviour of their parent,
- the other particle initiates from its birth position an independent copy of a $(-\rho, \beta; \mathbb{R})$-BBM with law $P^{\cdot}$, and so on.

The selected line of descent is referred to as the *spine*. Thus, the spine moves as a Brownian motion with drift λ, giving birth at an accelerated rate 2β along its path to independent (drift $-\rho$) branching Brownian motions of birth rate β.

For detailed setup and notation for these changes of measure and *spine decompositions* in branching Brownian motion the reader should refer to Kyprianou (2003) and Chauvin and Rouault (1988).

In general, similar changes of measure for other types of branching processes have become increasingly common in the study of classical and modern branching processes, in particular the reader is referred Lyons *et al.* (1995) and Lyons (1997); Champneys *et al.* (1995), Harris and Williams (1996), Olofsson (1998), Athreya (1999), Kyprianou and Rahimzadeh Sani (2001), Biggins and Kyprianou (2002), Engländer and Kyprianou (2002) and Kuhlbusch (2002) also provide further examples of their use.

3 Killed branching Brownian motion

For the purpose of the forthcoming analysis, define the process $X^{-\rho}$ to be the part of $\mathcal{X}^{-\rho}$ which survives killing at the origin. That is to say a branching process whose motion is a Brownian motion with drift $-\rho$ killed on hitting the origin and whose particles branch at rate β. Considering $X^{-\rho}$ as a subprocess of $\mathcal{X}^{-\rho}$ we shall again work with the probabilities $\{P^x : x > 0\}$. We shall denote $\{Y_u(t) : u \in N_t^{-\rho}\}$ is the configuration of particles alive at time t where $N_t^{-\rho}$ is the number of surviving particles. In keeping with previous notation, we shall refer to this process as a $(-\rho, \beta; \mathbb{R}^+)$-BBM. Let ζ be the extinction time of the $(-\rho, \beta; \mathbb{R}^+)$-BBM so that $\{\zeta = \infty\}$ is understood to mean survival. It will turn out that all the results in Theorem 1 can be proved probabilistically by analyzing the $(-\rho, \beta; \mathbb{R}^+)$-BBM and in particular the behviour of the position of its right most particle, defined by

$$R_t = \sup\left\{Y_u(t) : u \in N_t^{-\rho}\right\}$$

on $\{\zeta = \infty\}$ and zero otherwise.

Theorem 3 *We have for all $x > 0$ and $\rho \in \mathbb{R}$,*

$$\limsup_{t\uparrow\infty} R_t = \infty \text{ on } \{\zeta = \infty\}, \ P^x\text{-a.s.}$$

Proof. Let us suppose that $Y = \{Y(t) : t \geq 0\}$ is a Brownian motion with drift $-\rho$ and probabilities $\{\mathbb{P}_{-\rho}^x : x \in \mathbb{R}\}$ and let $\tau_0 = \inf\{t \geq 0 : Y(t) = 0\}$. Note that

$$P^x(\zeta < \infty | \mathcal{F}_t) \geq \prod_{u \in N_t^{-\rho}} \mathbb{P}_{-\rho}^{Y_u(t)}(\tau_0 < \mathbf{e}_\beta) = \prod_{u \in N_t^{-\rho}} \mathbb{E}_{-\rho}^{Y_u(t)}\left(e^{-\beta\tau_0}\right)$$

where $\mathbf{e}_\beta$ is an exponential variable independent of Y having rate β. The last inequality follows on account of the fact that extinction would follow if each of the individuals alive at time t would hit the origin before splitting. Recalling that $\alpha = \rho - \sqrt{\rho^2 + 2\beta} < 0$, standard expressions for the one sided exit problem for Brownian motion (see, for example, Borodin and Salminen (1996)), imply that for all $x > 0$

$$P^x(\zeta < \infty | \mathcal{F}_t) \geq \prod_{u \in N_t^{-\rho}} e^{\alpha Y_u(t)} = \exp\left\{\alpha \sum_{u \in N_t^{-\rho}} Y_u(t)\right\}.$$

On $\{\zeta = \infty\}$ it is clear that the left hand side converges to zero and hence for all $x > 0$

$$\lim_{t\uparrow\infty} \sum_{u \in N_t^{-\rho}} Y_u(t) = \infty \text{ on } \{\zeta = \infty\} \ P^x\text{-a.s.}$$

Now let Γ_z be the event that the $(-\rho, \beta; \mathbb{R}^+)$-BBM is contained entirely in the strip $(0, z)$. For the process Y define the stopping time $\tau_z = \inf\{t \geq 0 : Y(t) = z\}$. We have for $0 < x < z$

$$P^x(\Gamma_z | \mathcal{F}_t) \leq \prod_{u \in N_t^{-\rho}} \mathbb{P}_{-\rho}^{Y_u(t)}(\tau_0 < \tau_z) \tag{4}$$

on $Y_u(t) \in (0, z)$ for $u \in N_t^{-\rho}$. This inequality follows from the fact that Γ_z implies that the spatial path of each of the lines of decent emanating from the configuration at time t must hit the origin before hitting z. First consider the case that $-\infty < \rho < 0$. In this case we can write from (4) on the event that $Y_u(t) \in (0, z)$ for each $u \in N_t^{-\rho}$

$$P^x(\Gamma_z | \mathcal{F}_t) \leq \prod_{u \in N_t^{-\rho}} e^{-|\rho| Y_u(t)} \frac{\sinh|\rho|(z - Y_u(t))}{\sinh|\rho| z} \leq \exp\{-|\rho| \sum_{u \in N_t^{-\rho}} Y_u(t)\} \to 0$$

on the event $\{\zeta = \infty\}$ as t tends to infinity . Now consider the case that $\rho > 0$. It follows by again using classical results for the two sided exit problem that on the event that $Y_u(t) \in (0, z)$ for each $u \in N_t^{-\rho}$

$$\begin{aligned} P^x(\Gamma_z | \mathcal{F}_t) &\leq \prod_{u \in N_t^{-\rho}} \left[1 - \left(\frac{e^{-\rho z}}{\sinh \rho z}\right) e^{\rho Y_u(t)} \sinh \rho Y_u(t)\right] \\ &\leq \exp\left\{-\left(\frac{e^{-\rho z}}{\sinh \rho z}\right) \sum_{u \in N_t^{-\rho}} Y_u(t)\right\} \to 0 \end{aligned}$$

where we have used the inequalities $e^x \sinh x \leq x$ and $1 - x \leq e^{-x}$. The latter exponential tends to zero on the event $\{\zeta = \infty\}$ as t tends to infinity. Finally for the case that $\rho = 0$, on $Y_u(t) \in (0, z)$ for $u \in N_t^{-\rho}$

$$P^x(\Gamma_z | \mathcal{F}_t) \leq \prod_{u \in N_t^{-\rho}} \left(1 - \frac{Y_u(t)}{z}\right) \leq \exp\left\{-\frac{1}{z} \sum_{u \in N_t^{-\rho}} Y_u(t)\right\} \to 0$$

on the event $\{\zeta = \infty\}$ as t tends to infinity.

In conclusion, for any $z > 0$, $P^x(R_t > z \text{ i.o. } | \zeta = \infty) = 1$. That is to say the statement of the theorem holds. ■

Theorem 4 *If $\rho \geq \sqrt{2\beta}$ then $P^x(\zeta < \infty) = 1$ for all $x > 0$.*

Proof. Suppose that $\mathcal{R}_t$ is the position of the right most particle in a $(-\rho, \beta; \mathbb{R})$-BBM. It is well known the 'critical' martingale $Z_{\sqrt{2\beta} - \rho}(t) \to 0$ a.s. (see Neveu (1988), Harris (1999) or Kyprianou (2003), for example), from which it is easy to deduce that

$$\lim_{t \uparrow \infty} \left\{\mathcal{R}_t - (\sqrt{2\beta} - \rho)t\right\} = -\infty \qquad a.s.$$

From our construction of the $(-\rho, \beta; \mathbb{R}^+)$-BBM, extinction of this process is guaranteed when the right most particle in the $(-\rho, \beta; \mathbb{R})$-BBM drifts to $-\infty$. Thus, when $\rho \geq \sqrt{2\beta}$, this happens with probability one. ■

Theorem 5 *If $-\infty < \rho < \sqrt{2\beta}$ then for each $x > 0$ and $\lambda \in (0, \sqrt{2\beta} - \rho)$*

(i) $E^x\left(Z_\lambda(\infty);\liminf_{t\geq 0} R_t/t \geq \lambda\right) = \pi^x_\lambda\left(\liminf_{t\geq 0} R_t/t \geq \lambda\right) \geq 1 - e^{-2\lambda x}$,

(ii) $P^x(\zeta < \infty) \in (0,1)$,

(iii) $\lim_{x\downarrow 0} P^x(\zeta < \infty) = 1$, *and*

(iv) $\lim_{x\uparrow\infty} P^x(\zeta = \infty) = \lim_{x\uparrow\infty} P^x\left(\liminf_{t\uparrow\infty} R_t/t \geq \lambda; \zeta = \infty\right) = 1$.

Proof. (i) Recall that when $|\lambda + \rho| < \sqrt{2\beta}$ (and in particular when $\lambda \in (0, \sqrt{2\beta} - \rho)$), under the measure π^x_λ (defined in section 2) a branching Brownian motion with drift $-\rho$ and branching rate β has one line of decent, the spine, which has an exceptional drift λ. The probability that this spine never meets the origin is the probability that a Brownian motion started from $x > 0$ and with drift λ has an all time infimum which is strictly positive; and this is well known to be $1 - \exp\{-2\lambda x\}$. Suppose we write $\xi = \{\xi_t : t \geq 0\}$ for the spatial path of any surviving line of decent in $X^{-\rho}$ then we have established that

$$
\begin{aligned}
& E^x\left(Z_\lambda(\infty);\zeta = \infty \text{ and } \exists \xi \text{ in } X^{-\rho} \text{ such that } \lim_{t\uparrow\infty}\frac{\xi_t}{t} = \lambda\right) \\
\geq\ & 1 - \exp\{-2\lambda x\}.
\end{aligned}
$$

Now note that

$$
\left\{\zeta = \infty \text{ and } \exists \xi \text{ in } X^{-\rho} \text{ such that } \lim_{t\uparrow\infty}\frac{\xi_t}{t} = \lambda\right\} \subseteq \{\liminf_{t\uparrow\infty} R_t/t \geq \lambda\}
$$

and hence the statement of part (i) now follows.

(ii) To prove that $P^x(\zeta < \infty) > 0$, note that there is a strictly positive probability that the initial ancestor in the process $X^{-\rho}$ hits the origin before reproducing thus resulting in extinction. To prove that $P^x(\zeta < \infty) < 1$ or equivalently $P^x(\zeta = \infty) > 0$, recall from part (i) that under π^x_λ the probability that the (λ-drifting) spine in a branching Brownian motion with does not meet the origin is strictly positive. This is implies that $E^x\left(Z_\lambda(\infty);\zeta = \infty\right) > 0$ and since $P^x(Z_\lambda(\infty) > 0) = 1$ it follows that $P^x(\zeta = \infty) > 0$.

(iii) Since extinction in a finite time is guaranteed if the original ancestor is killed before reproducing,

$$
P^x(\zeta < \infty) \geq \mathbb{P}^x_{-\rho}(\tau_0 < \mathbf{e}_\beta) = e^{-(\sqrt{\rho^2 + 2\beta} - \rho)x} \to 1
$$

as $x \to 0$. [Recall that $\tau_0 = \inf\{t \geq 0 : Y(t) = 0\}$ and $\mathbf{e}_\beta$ is exponentially distributed with parameter β and independent of the Brownian motin $(Y, \mathbb{P}^x_{-\rho}]$.

(iv) Note that $P^x\left(\liminf_{t\uparrow\infty} R_t/t \geq \lambda; \zeta = \infty\right)$ is an increasing sequence in x and therefore has a limit. Suppose this limit is not equal to one then since it was shown in part (i) of the proof that

$$
\lim_{x\uparrow\infty} E^x\left(Z_\lambda(\infty);\liminf_{t\geq 0} R_t/t \geq \lambda\right) = 1
$$

there is a contradiction since for all $x > 0$

$$P^x\left(Z_\lambda\left(\infty\right) > 0\right) = 1 \text{ and } E^x\left(Z_\lambda\left(\infty\right)\right) = 1.$$

Finally noting that $P^x(\zeta = \infty) \geq P^x\left(\liminf_{t\uparrow\infty} R_t/t \geq \lambda, \zeta = \infty\right)$ the proof is complete. ■

4 Non-existence for $\rho \geq \sqrt{2\beta}$

Theorem 6 *No travelling-waves to (3) exist for $\rho \geq \sqrt{2\beta}$.*

Proof. Now suppose that f is a solution to (3). It follows that for all $x > 0$ $\prod_{u\in N_t^{-\rho}} f(Y_u(t))$ is a martingale which converges almost surely and in $L^1(P^x)$. We have seen in Theorem 4 that if $\rho \geq \sqrt{2\beta}$, $P^x(\zeta < \infty) = 1$ for all $x > 0$ and hence

$$\lim_{t\uparrow\infty} \prod_{u\in N_t^{-\rho}} f(Y_u(t)) = 1$$

almost surely implying that $f = 1$; that is to say, there is no non-trivial solution. ■

5 Existence and uniqueness for $-\infty < \rho < \sqrt{2\beta}$

Theorem 7 *Travelling-waves to (3) exist and are unique for $-\infty < \rho < \sqrt{2\beta}$. Further, the unique solution can be represented by the extinction probability for the $(-\rho, \beta; \mathbb{R}^+)$-BBM, that is*

$$f(x) = P^x(\zeta < \infty). \tag{5}$$

Remark 8 *The representation* (5) *trivially shows that the unique solution to* (3) *is strictly monotone decreasing, although this wasn't an initial restriction. Also note, one might naively try to extended this solution to produce a travelling-wave of speed $\rho < \sqrt{2\beta}$ on the whole of $\mathbb{R}$, but such a solution would clearly fail to satisfy equation* (2) *at a single point (due to a discontinuity in the first derivative at the origin).*

Proof. Define $p(x) := P^x(\zeta < \infty)$ for $x \geq 0$. From Theorem 5, we have $p(x) \in (0, 1)$ for each $x > 0$, $\lim_{x\uparrow\infty} p(x) = 0$, $\lim_{x\downarrow 0} p(x) = 1$ and, in addition, $p(0) = 1$ because of instantaneous killing.

An application of the Branching Markov Property (cf Chauvin (1991)) together with the law of total probability gives

$$p(x) = E^x\left(P^x\left(\zeta < \infty | \mathcal{F}_t\right)\right) = E^x\left(\prod_{u\in N_t^{-\rho}} p(Y_u(t))\right) \tag{6}$$

Since this equality holds for all $x, t > 0$, one can easily deduce that $\prod_{u \in N_t^{-\rho}} p(Y_u(t))$ is a martingale which converges almost surely and in $L^1(P^x)$. Note that on $\{\zeta < \infty\}$ it is clear that the martingale limit is equal to 1 - the empty product. Note however that this martingale cannot be identically equal to 1 because its mean, $p(x)$, is strictly less than one.

An application of Kolmogorov's backwards equations (cf. Champneys *et al.* (1995) or Dynkin (1993) Theorem II.3.1) thus yields that p belongs to $C^2(0, \infty)$ and is a solution to the ODE in (3).

For uniqueness, suppose that f is a solution to (3) when $-\infty < \rho < \sqrt{2\beta}$. Again we can construct a positive martingale $M_t := \prod_{u \in N_t^{-\rho}} f(Y_u(t))$ which is bounded, hence uniformly integrable. Clearly $M_\infty = 1$ on $\{\zeta < \infty\}$. Further, since $f(+\infty) = 0$, Theorem 3 gives $\limsup_{t\uparrow\infty} R_t = \infty$ on $\{\zeta = \infty\}$ a.s. and

$$\begin{aligned} M_\infty = \lim_{t\uparrow\infty} \prod_{u \in N_t^{-\rho}} f(Y_u(t)) &= \liminf_{t\uparrow\infty} \prod_{u \in N_t^{-\rho}} f(Y_u(t)) \\ &\le \liminf_{t\uparrow\infty} f(R_t) \\ &\le f(\limsup_{t\uparrow\infty} R_t), \end{aligned}$$

we can identify the limit as $M_\infty = \mathbf{1}_{\{\zeta<\infty\}}$ a.s.. Finally,

$$f(x) = E^x(M_0) = E^x(M_\infty) = P^x(\zeta < \infty) = p(x)$$

and uniqueness follows. ■

6 Asymptotic when $-\infty < \rho < \sqrt{2\beta}$

In this section we determine the asymptotic for the solution to (3), offering proofs for both Theorem 1 and the stronger result of Theorem 2.

The following Lemma shows that the unique solution decays exponentially for sufficiently large y and it will be very useful in proving both the weak and strong asymptotic theorems:

Lemma 9 *Let f be the unique solution of the system* (3) *when $-\infty < \rho < \sqrt{2\beta}$. Let $x_0 > 0$ and define $\mu := \sqrt{\rho^2 + 2\beta(1 - f(x_0))} - \rho > 0$. Then*

$$f(y) \le \left(f(x_0)e^{\mu x_0}\right) e^{-\mu y}$$

for all $y > x_0$.

Proof. Recall that Y is a Brownian motion with drift $-\rho$ starting from $x > 0$ under $\mathbb{P}^x_{-\rho}$ and recall that for any $z \ge 0$, $\tau_z := \inf\{t : Y_t = z\}$. Itô's formula implies that,

$$M_t := f(Y_{t\wedge\tau_0}) \exp\left(\beta \int_0^{t\wedge\tau_0} (f(Y_s) - 1)\, \mathrm{d}s\right) \tag{7}$$

is a $\mathbb{P}^x_{-\rho}$-local martingale and, since $0 \le f \le 1$, it is actually a bounded martingale. Suppose that $y > x_0$. Since $\tau_{x_0} < \infty$ a.s. under $\mathbb{P}^y_{-\rho}$, the optional stopping theorem and the monotonicity of f (see remark after Theorem 7) yields

$$f(y) = \mathbb{E}^y_{-\rho}\left\{ f(x) \exp\left(\beta \int_0^{\tau_{x_0}} (f(Y_s) - 1)\,\mathrm{d}s\right)\right\} \le f(x)\mathbb{E}^y_{-\rho}\left(e^{\beta(f(x_0)-1)\tau_{x_0}}\right).$$

A well known result (see, for example, Borodin and Salminen (1996)) gives

$$\mathbb{E}^y_{-\rho}\left(e^{\beta(f(x_0)-1)\tau_{x_0}}\right) = e^{-\mu(y-x_0)},$$

where $\mu := \sqrt{\rho^2 + 2\beta(1 - f(x_0))} - \rho > 0$. Thus, inequality (6) becomes

$$f(y) \le f(x_0)e^{-\mu(y-x_0)}$$

as required. ■

As a simple corollary, we can find an exponentially decaying bound for f valid on the whole of $(0,\infty)$ that will be used for the strong asymptotic of Theorem 2.

Corollary 10 *Let f be the unique solution of the system* (3) *when $-\infty < \rho < \sqrt{2\beta}$. Given any $K > 1$, there exists a $\kappa > 0$ such that*

$$f(y) \le Ke^{-\kappa y}$$

for all $y \ge 0$.

Proof. For $K > 1$, choose $x_0 > 0$ such that $K = e^{\mu x_0}$. Note that $f(x_0) \in (0,1)$ and then set $\kappa = \sqrt{\rho^2 + 2\beta(1 - f(x_0))} - \rho > 0$. Lemma 9 says that $f(y) \le Ke^{-\kappa y}$ for all $y \ge x_0$. Also, since $0 \le f \le 1$ and for $y < x_0$ we have $Ke^{-\kappa y} > 1$, we trivially have $f(y) \le Ke^{-\kappa y}$ for all $y \le x_0$. ■

Although we shall soon prove a stronger result after some further work, we include a proof of the classical (weak) asymptotic of Theorem 1 since it is now so easily accessible.

Proof of Theorem 1, asymptotics. For any fixed $x_0 > 0$, taking logarithms in Lemma 9 immediately yields

$$\limsup_y \frac{\ln f(y)}{y} \le -\left(\sqrt{\rho^2 + 2\beta(1 - f(x_0))} - \rho\right).$$

As this is true for arbitrary $x_0 > 0$ and $f(x_0) \to 0$ as $x_0 \to \infty$, we find

$$\limsup_y \frac{\ln f(y)}{y} \le -\left(\sqrt{\rho^2 + 2\beta} - \rho\right).$$

To prove the lower bound, recall that (7) is a martingale and hence for $y > 0$, remembering that $f(x) \in [0,1]$ with $f(0) = 1$, we have

$$\begin{aligned} f(y) &= \mathbb{E}^y_{-\rho}\left\{\exp\left(\beta\int_0^{\tau_0}(f(Y_s)-1)\,\mathrm{d}s\right)\right\} \\ &\geq \mathbb{E}^y_{-\rho}\left(e^{-\beta\tau_0}\right) = e^{-\left(\sqrt{\rho^2+2\beta}-\rho\right)y}, \end{aligned} \tag{8}$$

yielding the required lower bound. ■

We now extend the analysis to prove the stronger asymptotic of Theorem 2. Crucial to the argument will be the following proposition which we shall prove at the end of this section.

Proposition 11 *With $\widetilde{\rho} := \sqrt{\rho^2+2\beta}$, $x > 0$ and $f(x)$ the unique travelling-wave at speed $-\infty < \rho < \sqrt{2\beta}$,*

$$\lim_{x\to\infty}\mathbb{E}^x_{-\widetilde{\rho}}\exp\left(\beta\int_0^{\tau_0}f(Y_s)\,\mathrm{d}s\right) < +\infty. \tag{9}$$

Proof of Theorem 2. Working with the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^x_{\lambda-\rho}}{\mathrm{d}\mathbb{P}^x_{-\rho}}\right|_{\mathcal{F}_t} = e^{\lambda(Y_t+\rho t-x)-\frac{1}{2}\lambda^2 t}$$

for $\lambda \in \mathbb{R}$ and $x > 0$ we have from (8) that

$$e^{-\lambda x}f(x) = \mathbb{E}^x_{-\rho}\left\{e^{-\lambda Y_{t\wedge\tau_0}}f(Y_{t\wedge\tau_0})e^{\beta\int_0^{t\wedge\tau_0}f(Y_s)\,\mathrm{d}s}\,e^{\lambda(Y_{t\wedge\tau_0}-x)-\beta(t\wedge\tau_0)}\right\}.$$

Choosing $\lambda = \alpha := \rho - \sqrt{\rho^2+2\beta} < 0$, so that $\beta + \rho\lambda = \frac{1}{2}\lambda^2$, and defining $v(x) := e^{-\alpha x}f(x)$ and $\widetilde{\rho} := \sqrt{\rho^2+2\beta} > 0$, this yields

$$v(x) = \mathbb{E}^x_{-\widetilde{\rho}}\left\{v(Y_{t\wedge\tau_0})\exp\left(\beta\int_0^{t\wedge\tau_0}f(Y_s)\,\mathrm{d}s\right)\right\}$$

whence

$$v(Y_{t\wedge\tau_0})\exp\left(\beta\int_0^{t\wedge\tau_0}f(Y_s)\,\mathrm{d}s\right) \tag{10}$$

is a $\mathbb{P}_{-\widetilde{\rho}}$ martingale which is positive and therefore convergent. As $\tau_0 < \infty$ a.s. $\mathbb{P}_{-\widetilde{\rho}}$ we also have $v(Y_{t\wedge\tau_0}) \to 1$ a.s. under $\mathbb{P}^x_{-\widetilde{\rho}}$, but v is not (yet) known to be a bounded function so we cannot immediately conclude that this martingale is uniformly integrable. However, using the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^x_{-\widetilde{\rho}}}{\mathrm{d}\mathbb{P}^x_{-\rho}}\right|_{\mathcal{F}_{\tau_0}} = e^{(\sqrt{\rho^2+2\beta}-\rho)x-\beta\tau_0}.$$

(which is possible because $\exp\{-(\sqrt{\rho^2+2\beta}-\rho)\,(Y_{t\wedge\tau_0}-x)-\beta\,(t\wedge\tau_0)\}$ is a uniformly integrable martingale) we may transform (8) to

$$v(x)=\mathbb{E}^x_{-\widetilde{\rho}}\left\{\exp\left(\beta\int_0^{\tau_0} f(Y_s)\,\mathrm{d}s\right)\right\} \tag{11}$$

and hence the $\mathbb{P}^x_{-\widetilde{\rho}}$ martingale in (10) is uniformly integrable. Note that from (11) it is clear that v is monotone increasing in x and hence its limit exists as x tends to infinity.

All that remains is to prove that v converges to a finite limit as x tends to infinity, which is precisely Proposition 11. Thus

$$v(x):=f(x)e^{-\alpha x}\uparrow k\in(0,\infty) \text{ as } x\to\infty.$$

Hence $f(x)$ asymptotically looks like the decaying solution of

$$\frac{1}{2}f''-\rho f'-\beta f=0,$$

that is, the linearization of equation (2) about the origin. ■

Proof of Proposition 11. Recall that $\widetilde{\rho}=\sqrt{\rho^2+2\beta}$ and for $y>0$

$$\mathbb{E}^y_{-\widetilde{\rho}}\left(e^{\gamma\tau_0}\right)=e^{(\widetilde{\rho}-\sqrt{\widetilde{\rho}^2-2\gamma})y}$$

provided that $2\gamma<\widetilde{\rho}^2$ (in particular, this holds for all $\gamma\le\beta$).

Note that for any $y>0$, since $f\in[0,1]$, we have

$$\mathbb{E}^y_{-\widetilde{\rho}}\exp\left(\beta\int_0^{\tau_0} f(Y_s)\,\mathrm{d}s\right)\le\mathbb{E}^y_{-\widetilde{\rho}}(e^{\beta\tau_0})=e^{(\sqrt{\rho^2-2\beta}-\rho)y}<\infty, \tag{12}$$

and for any $y_0>y_1>0$, the strong Markov property gives

$$\mathbb{E}^{y_0}_{-\widetilde{\rho}}\left(e^{\beta\int_0^{\tau_0}f(Y_s)\,\mathrm{d}s}\right)=\mathbb{E}^{y_0}_{-\widetilde{\rho}}\left(e^{\beta K\int_0^{\tau_{y_1}}f(Y_s)\,\mathrm{d}s}\right)\mathbb{E}^{y_1}_{-\widetilde{\rho}}\left(e^{\beta K\int_0^{\tau_0}f(Y_s)\,\mathrm{d}s}\right) \tag{13}$$

Fix any $K>1$ and recall from Corollary 10 that there then exists $\mu>0$ such that

$$f(x)\le Ke^{-\mu x}\qquad\forall x\ge 0.$$

Now fix any $d>0$. Choose a fixed $M\in\mathbb{N}$ sufficiently large such that $Ke^{-\mu y_1}<1$ where $y_1:=Md$. Then, for any $N\in\mathbb{N}$ and $y_0:=(M+N)d$, and with $S_i:=\tau_{(M+i-1)d}-\tau_{(M+i)d}$ so that the S_i are IID each distributed like the first hitting time of 0 by a Brownian motion started at d, we have

$$
\begin{aligned}
\mathbb{E}^{y_0}_{-\widetilde{\rho}} \exp\left(\beta \int_0^{\tau_{y_1}} f(Y_s)\,\mathrm{d}s\right) &\le \mathbb{E}^{y_0}_{-\widetilde{\rho}} \exp\left(\beta K \int_0^{\tau_{y_1}} e^{-\mu Y_s}\,\mathrm{d}s\right) \\
&= \mathbb{E}^{y_0}_{-\widetilde{\rho}} \exp\left(\beta K \sum_{n=1}^{N} \int_{\tau_{(N+M-n+1)d}}^{\tau_{(N+M-n)d}} e^{-\mu Y_s}\,\mathrm{d}s\right) \\
&\le \mathbb{E}^{y_0}_{-\widetilde{\rho}} \exp\left(\beta K \sum_{n=1}^{N} e^{-\mu(N+M-n)d} S_{N-n}\right) \\
&= \mathbb{E}^{y_0}_{-\widetilde{\rho}} \exp\left(\beta K e^{-\mu y_1} \sum_{k=1}^{N} e^{-\mu k\, d} S_k\right) \\
&\le \mathbb{E}^{y_0}_{-\widetilde{\rho}} \prod_{k=1}^{N} \exp(\beta e^{-\mu k\, d} S_k) \\
&= \prod_{k=1}^{N} \mathbb{E}^{y_0}_{-\widetilde{\rho}} \exp(\beta e^{-\mu k\, d} S_k) \\
&= \exp\left(d \sum_{k=1}^{N} \left\{\widetilde{\rho} - \sqrt{\widetilde{\rho}^2 - 2\beta e^{-\mu k\, d}}\right\}\right) \qquad (14)
\end{aligned}
$$

Since

$$
\begin{aligned}
\widetilde{\rho} - \sqrt{\widetilde{\rho}^2 - 2\beta e^{-\mu k\, d}} &= \sqrt{\rho^2 + 2\beta}\left(1 - \sqrt{1 - \left(\frac{2\beta}{\rho^2 + 2\beta}\right) e^{-\mu k\, d}}\right) \\
&= \left(\frac{\beta}{\sqrt{\rho^2 + 2\beta}}\right) e^{-\mu k\, d} + o(e^{-\mu k\, d}),
\end{aligned}
$$

the ratio test reveals the sum appearing in (14) is convergent when $N \to \infty$. Using this fact together with monotone convergence and equations (12) and (13) now gives the required result. ■

7 Right most particle asymptotic $-\infty < \rho < \sqrt{2\beta}$

The intention of Theorem 5 was to establish properties of the probability of extinction in order to justify it as a solution to the travelling-wave equation. However, considering parts (i) and (iii) of this same theorem there is reason to believe that like the $(-\rho, \beta; \mathbb{R})$-BBM, the $(-\rho, \beta; \mathbb{R}^+)$-BBM has a right most particle with asymptotic drift $\sqrt{2\beta} - \rho$ (but now it is necessary to specify that this happens on the survival set). This is the conclusion of the next theorem.

Theorem 12 *For all $x > 0$ we have*

$$
\lim_{t \uparrow \infty} \frac{R_t}{t} = \sqrt{2\beta} - \rho \text{ on } \{\zeta = \infty\}, \; P^x\text{-a.s.}
$$

Proof. We shall prove this theorem by establishing separately that

$$\liminf_{t\uparrow\infty}\frac{R_t}{t}\geq\sqrt{2\beta}-\rho \text{ and } \limsup_{t\uparrow\infty}\frac{R_t}{t}\leq\sqrt{2\beta}-\rho \text{ on } \{\zeta=\infty\}\ P^x\text{-a.s.}$$

Theorem 3 shows that for each $x>0$, on $\{\zeta=\infty\}$, $\limsup_{t\uparrow\infty}R_t=\infty$ P^x-almost surely and hence $\sigma_y:=\inf\{t\geq 0 : X^{-\rho}(y,\infty)>0\}$ is P^x-almost surely finite for each $y>0$ on $\{\zeta=\infty\}$. This implies that for any $\lambda>0$,

$$P^x\left(\liminf_{t\uparrow\infty}R_t/t\geq\lambda,\zeta=\infty\right)=P^x\left(\liminf_{t\uparrow\infty}R_t/t\geq\lambda,\sigma_y<\infty,\zeta=\infty\right).$$

It thus follows that for any $y>x$

$$\begin{aligned}
&P^x\left(\liminf_{t\uparrow\infty}R_t/t\geq\lambda,\zeta=\infty\right)\\
=\;&P^x(\sigma_y<\infty)\,P^x\left(\liminf_{t\uparrow\infty}R_t/t\geq\lambda,\zeta=\infty\middle|\sigma_y<\infty\right)\\
\geq\;&P^x(\sigma_y<\infty)\,P^y\left(\liminf_{t\uparrow\infty}R_t/t\geq\lambda,\zeta=\infty\right) \qquad (15)
\end{aligned}$$

where the inequality follows from the fact that at time σ_y there is one particle positioned at y which, given $\mathcal{F}_{\sigma_y}$, gives rise to an branching tree independent of other particles alive at time σ_y and further whose right most particle is bounded above by the right most particle of $X^{-\rho}$. Recalling Theorem 3, now note that as $y\to\infty$,

$$P^x(\sigma_y<\infty)\uparrow P^x\left(\limsup_{t\uparrow\infty}R_t=\infty\right)=P^x(\zeta=\infty).$$

It follows from (15) with the help of Theorem 5 (iii) that when we further insist that $\lambda\in(0,\sqrt{2\beta}-\rho)$,

$$\begin{aligned}
P^x(\zeta=\infty)&\geq P^x\left(\liminf_{t\uparrow\infty}R_t/t\geq\lambda,\zeta=\infty\right)\\
&\geq\lim_{y\uparrow\infty}P^x(\sigma_y<\infty)\,P^y\left(\liminf_{t\uparrow\infty}R_t/t\geq\lambda,\zeta=\infty\right)\\
&=P^x(\zeta=\infty).
\end{aligned}$$

We thus deduce that for any $\varepsilon>0$, P^x-almost everywhere on the event $\{\zeta=\infty\}$ we have

$$\liminf_{t\uparrow\infty}R_t/t\geq\sqrt{2\beta}-\rho-\varepsilon$$

For the limsup inequality simply note that on $\{\zeta=\infty\}$ the R_t is P^x-almost surely stochastically bounded above by $\mathcal{R}_t$, the distance of the right most particle in a $(-\rho,\beta;\mathbb{R})$-BBM. That is to say

$$\limsup_{t\uparrow\infty}R_t/t\leq\sqrt{2\beta}-\rho$$

P^x-almost everywhere on the event $\{\zeta=\infty\}$. ■

Acknowledgement 13 *This work was conducted whilst J.W.Harris was in receipt of an EPSRC research studentship. S.C.Harris and A.E.Kyprianou are pleased to acknowledge the support of the joint NWO-British Council Anglo-Dutch research grant Nr. BR 61-461.*

References

[1] Aronson, D.G. and Weinberger, H.F. (1975) Nonlinear diffusions in population genetics, combustion and nerve propagation. *Partial differential equations and related topics (J. Goldstein ed.)* Lecture notes in Math. **446** Springer-Verlag.

[2] Athreya, K. (1999) Change of Measures for Markov Chains and the LlogL Theorem for Branching Processes. *Bernoulli* **6**, 323–338.

[3] Biggins, J.D. and Kyprianou A.E. (2002) Measure change in multi-type branching. *Submitted.*

[4] Borodin, A. N. and Salminen, P. (1996) *Handbook of Brownian motion - facts and formulae*, Birkhäuser Verlag, Basel, Boston and Berlin.

[5] Bramson (1978) Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.* **31** 531–581.

[6] Bramson, M.(1983) Convergence of solutions to the Kolmogorov nonlinear diffusion equation to travelling-waves. *Mem. Amer. Math. Soc.* **44** 1–190.

[7] Champneys, A., Harris, S.C., Toland, J.F., Warren, J. and Williams, D. (1995) Algebra, analysis and probability for a coupled system of reaction-diffusion equations. *Phil. Trans. Roy. Soc. London (A)*, **350**, 69-112.

[8] Chauvin, B. (1991) Multiplicative martingales and stopping lines for branching Brownian motion. *Ann. Probab.* **30**, 1195–1205.

[9] Chauvin, B., and Rouault, A. (1988) KPP equation and supercritical branching Brownian motion in the subcritical speed area. Application to spatial trees. *Prob. Th. Rel. Fields.* **80**, 299–314.

[10] Chauvin, B., and Rouault, A. (1990) Supercritical branching Brownian motion and KPP equation in the critical speed area. *Math. Nachr.* **149**, 41–59.

[11] Dynkin, E.B. (1993) Superprocesses and partial differential equations. *Ann. Probab.*, **21**, 1185-1262.

[12] Engländer, J. and Kyprianou, A.E. (2002) Local extinction versus local exponential growth for spatial branching processes. *To appear in Ann. Probab.*

[13] Fisher, R.A. (1937) The advance of advantageous genes. *Ann. Eugenics.* **7**, 355–369.

[14] Harris, S.C. (1999) Travelling-waves for the FKPP equation via probabilistic arguments. *Proc. Roy. Soc. Edin.* **129A** 503-517.

[15] Harris, S.C. & Williams, D. (1996) Large-deviations and martingales for a typed branching diffusion : I, *Journal Asterisque*, **236**, 133–154.

[16] Kolmogorov, A., Petrovskii, I. and Piskounov, N. (1937) Étude de l'équation de la diffusion avec croissance de la quantité de la matière at son application a un problèm biologique. *Moscow Univ. Bull. Math.* **1**, 1–25.

[17] Kuhlbusch, D. (2002) Necessary and sufficient conditions for a normalized weighted branching process in a random environment to have a nondegenerate limit. *Preprint, University of Münster.*

[18] Kyprianou, A.E. (2003) Travelling-wave solutions to the K-P-P equation: alternatives to Simon Harris' probabilistic analysis. *AIHP* (to appear)

[19] Kyprianou, A.E. and Rahimsadeh Sani (2001) Martingale convergence and the functional equation in the multi-type branching random walk. *Bernoulli.* **7**, 593–604.

[20] Lyons, R. (1997) A simple path to Biggins' martingale convergence theorem. In Classical and Modern Branching Processes (K.B. Athreya and P. Jagers, eds) 84, 217–222. Springer, New York.

[21] Lyons, R., Pemantle R. and Peres, Y. (1995) Conceptual proofs of $L\log L$ criteria for mean behaviour of branching processes. *Ann. Probab.* **23**, 1125–1138.

[22] McKean, H. P. (1975) Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Comm. Pure Appl. Math.* **XXIX**, 323–331.

[23] McKean, H. P. (1976) Correction to the above. *Comm. Pure Appl. Math.*, **29**, 553–554.

[24] Neveu, J. (1988) Multiplicative martingales for spatial branching processes. In Seminar on Stochastic Processes 1987, eds E. Çinlar, K.L. Chung, R.K. Getoor. Progress in Probability and Statistics, 15, 223–241. Birkhaüser, Boston.

[25] Olofsson, P. (1998) The $x\log x$ condition for general branching processes. *J. Appl. Probab.* **35,** 537–554.

[26] Pinsky, R. (1995) K-P-P asymptotics for non-linear diffusion in a large ball with infinite boundary data and on $\mathbb{R}^d$ with infinite initial data outside a large ball. *Com. Part. Diff. Eq. 20*, 1369–1393.

[27] Uchiyama, K. (1978) The behaviour of solutions of some non-linear diffusion equations for large time. 1*J. Math. Koyoto Univ.* **18**, 453–508.