Autoparametric Resonance of Relaxation Oscillations

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Abstract

Stable normal mode vibrations in engineering can be undesirable and one of the possibilities for quenching these vibrations is by embedding the oscillator in an autoparametric system by coupling to a damped oscillator. We have the possibility of destabilising the undesirable vibrations by a suitable tuning and choice of coupling parameters. In the case of normal mode vibration derived from a relaxation oscillations we need low-frequency tuning of the attached oscillator. An additional feature is that to make the quenching effective we also have to deform the slow manifold by choosing appropriate coupling; this is illustrated for van der Pol relaxation. A number of numerical experiments have been done and show some interesting phenomena, such as a chaotic attractor and effective quenching.

1 Introduction

Autoparametric resonance plays an important part in nonlinear engineering while posing interesting mathematical challenges. The linear dynamics is already nontrivial whereas the nonlinear dynamics of such systems is extremely rich and largely unexplored. A general characterisation of autoparametric systems is given in Tondl, Ruijgrok, Verhulst and Nabergoj [10]. In studying autoparametric systems, the determination of stability and instability conditions of the semi-trivial solution or normal mode is always the first step. After this it is of interest to look for other periodic solutions, bifurcations and classical or chaotic limit sets.

In actual engineering problems, the loss of stability of the normal mode response depends on frequency tuning of the various components of the system, and on the interaction (the coupling) between the components. Autoparametric vibrations occur only in a limited region of the tuning parameters.

In a self-excited autoparametric system with a relaxation oscillator, to have autoparametric resonance, destabilisation of the relaxation oscillation of the system is needed. It turns out that to destabilize relaxation oscillations one needs in addition rather strong interactions of a special form. This is tied in with the necessity to perturb the slow manifold which characterises to a large extent the relaxation oscillation. The results in this paper are an

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extension of Verhulst [12].
The monograph by Tondl et al. [10] contains a survey of the literature on self-excited autoparametric systems, in particular for weak self-excitation and weak interactions; see also Schmidt and Tondl [9] and Cartmell [2].

2 Formulation of the problem

A typical formulation for autonomous systems runs as follows. Consider the one-degree-of-freedom i.e. two-dimensional system

$$\dot{x} = f(x)$$

where $f(x)$ is a smooth 2-dimensional vector field and assume that the equation has a stable periodic solution. Suppose that this corresponds with undesirable behaviour, as is for instance the case of flow-induced vibrations. Can we introduce a kind of energy absorber, mathematically speaking can we couple the equation to another system such that this periodic solution arises as an unstable normal mode in the full system? This entails the introduction of the system

$$\begin{aligned} \dot{x} &= f(x) + g(x, y), \\ \dot{y} &= h(x, y), \end{aligned} \tag{1}$$

in which y is n-dimensional, g and h are smooth vector fields, with $h(x, 0) = 0$. In most cases we assume $g(x, 0) = 0$, so that the original periodic solution corresponds with a normal mode of the coupled system. Sometimes $g(x, y)$ includes a perturbation resulting in a normal mode close to the unperturbed one. The important questions are 'what are the requirements for the coupling terms g and h to achieve effective destabilisation of the normal mode' and 'how do we choose the system parameters'.
Suppose $\phi(t) : \mathbb{R} \to \mathbb{R}^2$ is a stable T-periodic solution of the equation

$$\dot{x} = f(x) + g(x, 0).$$

We shall study the stability of this normal mode in system (1).

3 Linearisation and decoupling

We put $x = \phi(t) + u,\ y = y$ and expand to obtain the linearised system

$$\begin{aligned} \dot{u} &= \frac{\partial f}{\partial x}(\phi(t))u + \frac{\partial g}{\partial x}(\phi(t), 0)u + \frac{\partial g}{\partial y}(\phi(t), 0)y, \\ \dot{y} &= h(\phi(t), 0). \end{aligned} \tag{2}$$

With a slight abuse of notation we kept u and y for the solutions of the linear system. It is clear that the linear system is *decoupled* in the following sense. The equation for y can be studied independently with the requirement to produce instability. Subsequently in the equation for u the behaviour of y can be introduced to study the behaviour of u. If $y=0$ is unstable for the second equation, the normal mode is unstable. The instability becomes however effective for our purpose if also the solution u of the first equation is unstable.

The homogeneous part of the first equation of (2) reads

$$\dot{v} = \frac{\partial f}{\partial x}(\phi(t))v + \frac{\partial g}{\partial x}(\phi(t),0)v,$$

which is a linear equation with T-periodic coefficients. One of the solutions is $\dot{\phi}(t)$ and we can easily construct a second independent solution by d'Alembert's method. What interests us, however, are the characteristic (or Flocquet- or Lyapunov-) exponents. The exponent corresponding with $\dot{\phi}(t)$ is of course zero, as this solution is periodic. The second exponent, λ, is negative by assumption and reads

$$\lambda = \frac{1}{T}\int_0^T Tr(\frac{\partial f}{\partial x}(\phi(t)) + \frac{\partial g}{\partial x}(\phi(t),0))dt. \tag{3}$$

For a proof of these classical statements see for instance Verhulst [11]. This result can now be used to study the stability of the trivial solution of the equation for u where the inhomogeneous part $u_i(t)$ may destabilize $u=0$.
It follows from Flocquet-theory that the second independent solution is of the form $e^{-\lambda t}\psi(t)$, where $\psi(t)$ is T-periodic. With fundamental matrix $\Theta(t) = (\dot{\phi}(t), e^{-\lambda t}\psi(t))$, the inhomogeneous part of the solution for u becomes

$$u_i(t) = \Theta(t)\int_0^t \Theta^{-1}(s)\frac{\partial g}{\partial y}(\phi(s),0)y(s)ds.$$

It is clear from this expression that the growth of $y(t)$ - the instability of $y=0$ - is a necessary condition for the instability of $u=0$. Whether this condition is sufficient depends on the actual autoparametric system as we shall see in the applications.
Often the first equation of (2) is a scalar second order equation of the form

$$\ddot{u} + p(t)\dot{u} + q(t)u = F(t)y \tag{4}$$

with scalar independent (homogeneous) solutions $\dot{\phi}(t)$ and $e^{-\lambda t}\psi(t)$ and the Wronskian $e^{-\lambda t}\chi(t)$, $\chi(t)$ a T-periodic function. In this case the inhomogeneous solution of equation (4) reads

$$u_i(t) = \dot{\phi}(t)\int_0^t \frac{\psi(s)}{\chi(s)}F(s)y(s)ds - e^{-\lambda t}\psi(t)\int_0^t \frac{e^{\lambda s}}{\chi(s)}\dot{\phi}(s)F(s)y(s)ds.$$

Note that $\chi(t)$ has no zeros.

4 Weak coupling of self-excited oscillations

For reasons of comparison with the results in the sequel we make some observations about weak self-excitation and weak interaction. We shall express these by using the small, positive parameter ε.

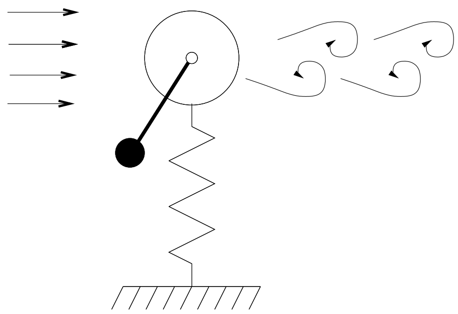

Figure 1: **Example of an autoparametric system with flow-induced vibrations**. The system consists of a single mass on a spring to which a pendulum is attached as an energy absorber. The flow excites the mass and the spring but not the pendulum.

Consider the case of flow-induced vibrations represented by the Rayleigh oscillator embedded in the autoparametric system (different from the example in the figure)

$$\begin{aligned} \ddot{x} + x &= \varepsilon(1 - \dot{x}^2)\dot{x} + \varepsilon(c_1 x^2 + c_2 xy + c_3 y^2) \\ \ddot{y} + \varepsilon\kappa\dot{y} + q^2 y &= \varepsilon y(d_1 x + d_2 y). \end{aligned} \tag{5}$$

The damping coefficient κ is positive, the frequency q and the coefficients $c_i,\ d_i$ will be chosen suitably, i.e. to provide optimal instability of the normal mode $\phi(t)$ obtained by putting $y = 0$. The T-periodic solution $\phi(t)$ corresponds with self-excited vibrations and we linearise around this normal mode, putting $x = \phi(t) + u,\ y = y$, to find:

$$\begin{aligned} \ddot{u} + u &= \varepsilon(1 - 3\dot{\phi}(t)^2)\dot{u} + \varepsilon(2c_1\phi(t)u + c_2\phi(t)y), \\ \ddot{y} + \varepsilon\kappa\dot{y} + q^2 y &= \varepsilon d_1 \phi(t) y. \end{aligned} \tag{6}$$

The equation for y is Hill's equation with damping added which can be reduced to Mathieu's equation by using that ε is small.
It is well known that we have for the periodic solution of the modified Rayleigh oscillator

$$\ddot{x} + x = \varepsilon(1 - \dot{x}^2)\dot{x} + \varepsilon c_1 x^2$$

the approximation

$$\phi(t) = 2\cos(t) + O(\varepsilon).$$

The estimate for amplitude and period is valid for all time. Inserting this into the equation for y yields

$$\ddot{y} + \varepsilon\kappa\dot{y} + (q^2 - 2\varepsilon d_1 \cos(t))y = 0.$$

In parameter space a relatively large instability domain arises on choosing $q = \frac{1}{2}$. The usual analysis (Poincaré-Lindstedt, averaging or harmonic balance) leads to the (known) requirement $|d_1| > \frac{1}{2}\kappa$ for instability of $y = 0$.

Returning to the equation for u we note that we have from equation (3) and the first equation of (6)

$$\lambda = \varepsilon \frac{1}{2\pi} \int_0^{2\pi} (1 - 3\dot{\phi}(s)^2) ds = -5\varepsilon + O(\varepsilon^2).$$

Independent solutions of the homogeneous part of the first equation of (6) are according to Flocquet theory $\dot{\phi}(t)$ and $e^{-5\varepsilon t}\psi(t) + O(\varepsilon)$ with $\psi(t)$ a T-periodic solution which can be obtained by d'Alembert's construction. The Wronskian to $\mathcal{O}(\varepsilon)$ becomes $e^{-5\varepsilon t}\chi(t)$ with again $\chi(t)$ a T-periodic function without zeros. The inhomogeneous solution of the equation for u takes the form

$$u_i(t) = \varepsilon c_2 \dot{\phi}(t) \int_0^t \frac{\psi(s)\phi(s)}{\chi(s)} y(s) ds - \varepsilon c_2 e^{-5\varepsilon t}\psi(t) \int_0^t e^{5\varepsilon s} \frac{\dot{\phi}(s)\phi(s)}{\chi(s)} y(s) ds. \qquad (7)$$

We conclude that on choosing $q = \frac{1}{2}, d_1 > \frac{1}{2}\kappa$ the solution $y = 0$ becomes unstable which destabilises the normal mode in the y-direction. On choosing $c_2 \neq 0$ the solution $u = 0$ also becomes unstable which enforces the instability of the normal mode. The parameters $c_1,\ c_3,\ d_2$ play no part at this level of approximation.
Note that Abadi [1] studied this autoparametric system in the case $c_1 = c_2 = d_2 = 0$ with emphasis on the bifurcation phenomena in the case of an unstable normal mode.

We mention that in the case of weak interaction the analysis does not change much when we replace the Rayleigh oscillator by van der Pol self-excitation. The Lyapunov exponents of the normal mode in this case are 0 and $\lambda = -\varepsilon + O(\varepsilon^2)$.

5 Interaction with relaxation oscillations

A different problem arises when we wish to quench a relaxation oscillation. We take as an example the van der Pol relaxation oscillator embedded in an autoparametric system of the form

$$\begin{aligned} \ddot{x} + x &= \mu(1 - x^2)\dot{x} + F(x, \dot{x}, y, \dot{y}), \\ \ddot{y} + \kappa\dot{y} + q^2 y &= yG(x, \dot{x}, y, \dot{y}), \end{aligned} \qquad (8)$$

where κ is again a positive damping coefficient. We assume that if the $(y, \dot{y})$-vibration is absent, F vanishes to produce a pure van der Pol relaxation oscillation: $F(x, \dot{x}, 0, 0) = 0$. The functions F and G, and the remaining parameters have to be chosen to produce instability of the (periodic relaxation) normal mode, obtained by putting $y = 0$ in the case $\mu \gg 1$. We will use results on this relaxation oscillator which were summarised and extended by Grasman [7]. We will also use results on slow manifolds in geometric singular perturbations; for an introduction see Kaper [8] and the original papers by Fenichel [3, 4, 5, 6].
Introducing $\phi(t)$ for the T_μ-periodic relaxation normal mode, putting $x = \phi(t) + u,\ y = y$ produces

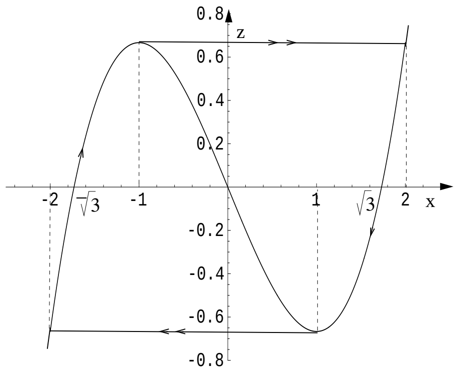

Figure 2: **The phase plane of the van der Pol relaxation oscillation**. The slow manifold is approximated by the cubic curve, fast motion is indicated by double arrows.

$$\begin{aligned} \ddot{u} + u &= \mu(1-\phi(t)^2)\dot{u} - 2\mu\phi(t)\dot{\phi}(t)u + \cdots , \\ \ddot{y} + \kappa\dot{y} + (q^2 - d_1\phi(t))y &= \cdots , \end{aligned} \tag{9}$$

where the nonlinear terms are indicated by dots. The linearised equations suffice for the stability analysis.

6 The Lyapunov exponent of relaxation

The normal mode is T_μ-periodic with the estimate (see Grasman [7])

$$T_\mu = (3 - 2\log 2)\mu + O(\frac{1}{\mu^{\frac{1}{3}}}) \text{ as } \mu \to \infty.$$

To compute the rate of attraction from the integral in (3) we use the slow-fast motion in the Liénard plane by replacing the van der Pol equation by

$$\dot{x} = \mu(z + x - \frac{1}{3}x^3), \; \mu\dot{z} = -x.$$

In the $x - z$ Liénard plane we have

$$(z + x - \frac{1}{3}x^3)\frac{dz}{dx} = -\frac{x}{\mu^2},$$

which illustrates that as μ is large dz/dx is very small except if $z = -x + \frac{1}{3}x^3$. This cubic curve in the Liénard plane corresponds with the slow manifold of the system. See Figure (2).
With this knowledge it is not difficult to obtain a first order approximation of the characteristic exponent. From equation (3) and the first equation of (9), we have

$$\lambda = \mu\frac{1}{T_\mu}\int_0^{T_\mu}(1 - \phi(t)^2)dt.$$

Integration of van der Pol's equation for the periodic solution yields

$$\int_0^{T_\mu} (\ddot{\phi}(t) + \phi(t))dt = \mu \int_0^{T_\mu} (1 - \phi^2(t))\dot{\phi}(t)dt$$

or, using the periodicity,

$$\int_0^{T_\mu} \phi(t)dt = \mu \oint (1 - x^2)dx = 0,$$

where the contour integral is taken over the limit cycle in the phase plane. With this result, using partial integration and the equation for z we have

$$\int_0^{T_\mu} \phi(t)^2 dt = -\int_0^{T_\mu} (\int \phi(t)dt)\dot{\phi}(t)dt = \mu \oint z(x)dx.$$

Integration in the Liénard plane yields the approximation

$$\lambda = \mu(1 - \frac{9}{2}\frac{\mu}{T_\mu}) + o(1) \approx -1.79\mu$$

which, as $\mu \gg 1$, corresponds with strong attraction.

7 Instability of $y = \dot{y} = 0$

The periodic coefficient $\phi(t)$ in the equation for y has a period proportional to $\mu \gg 1$, so it is natural to rescale $t = \frac{T_\mu}{2\pi}\tau$ which, after linearisation, produces

$$\frac{d^2y}{d\tau^2} + \frac{\kappa T_\mu}{2\pi}\frac{dy}{d\tau} + (\frac{q^2 {T_\mu}^2}{4\pi^2} - \frac{d_1 T_\mu^2}{4\pi^2}\phi_r(\tau))y = 0 \tag{10}$$

with $\phi_r(\tau) = \phi(\frac{T_\mu}{2\pi}\tau)$, 2π-periodic in τ. We observe that the instability behaviour of the solutions of the Flocquet equation (10) is qualitatively the same as for the damped Mathieu-equation. A consequence is that to obtain prominent instability of $y = 0$ and so destabilisation of the relaxation oscillation we have to couple to a low-frequency oscillator (y) with, using the first-order estimate for the period,

$$\kappa = O(\frac{1}{\mu}),\ q = \frac{\pi}{(3 - 2\log 2)\mu},\ d_1 = O(\frac{1}{\mu^2}) \text{ as } \mu \to \infty.$$

The actual choice of κ and d_1 depends on the amount of quenching one wants to achieve. We explore the small parameter case

$$\frac{\kappa T_\mu}{2\pi} = \frac{\kappa_0}{\mu},\ \frac{q^2 T_\mu^2}{\pi^2} = 1,\ \frac{d_1 T_\mu^2}{4\pi^2} = \frac{d}{\mu}$$

with κ_0 and d independent of μ. We have from equation (10)

$$\frac{d^2y}{d\tau^2} + \frac{\kappa_0}{\mu}\frac{dy}{d\tau} + (\frac{1}{4} - \frac{d}{\mu}\phi_r(\tau))y = 0. \tag{11}$$

To obtain the boundaries of the Flocquet-tongue we impose the periodicity conditions

$$\begin{aligned} \int_0^{2\pi} (-\kappa_0 \frac{dy(\tau)}{d\tau} + d\phi_r(\tau))y(\tau) \sin\frac{\tau}{2} &= 0, \\ \int_0^{2\pi} (-\kappa_0 \frac{dy(\tau)}{d\tau} + d\phi_r(\tau))y(\tau) \cos\frac{\tau}{2} &= 0. \end{aligned} \tag{12}$$

With the Poincaré expansion $y(\tau) = a_0 \cos\frac{\tau}{2} + b_0 \sin\frac{\tau}{2} + \frac{1}{\mu}\cdots$, we have

$$\frac{1}{2}\kappa_0 a_0 - dIb_0 = 0, \; dIa_0 - \frac{1}{2}\kappa_0 b_0 = 0,$$

where we used the symmetry of $\phi_r(\tau)$; I is a positive number given by

$$I = \int_0^{2\pi} \phi_r(\tau) \cos\tau d\tau.$$

Nontrivial solutions arise if the determinant vanishes or

$$\frac{1}{2}\kappa_0 = \pm dI.$$

We have instability of $y = 0$ if $|d| > \frac{\kappa_0}{2I}$. This is the analogue of the result obtained in section 4 for weak interaction.

8 Deformation of the slow manifold

To analyse the relaxation oscillation in the 4-dimensional problem of system (8) we assume that the interaction term F contains quadratic and cubic terms and is of the form

$$F(x, \dot{x}, y, \dot{y}) = \mu(c_1\dot{x}y + c_2 x\dot{x}y + c_3\dot{x}y^2).$$

An easy way to see that these are the leading terms of F runs as follows. Transform the time in the first equation of (8) $t \to \mu\tau$ and indicate differentiation with respect to τ with a prime:

$$\frac{1}{\mu^2}x'' + x = (1 - x^2)x' + F(x, \frac{1}{\mu}x', y, \frac{1}{\mu}y')$$

or

$$x' = v, \; \frac{1}{\mu^2}v' = (1 - x^2)v + F(x, \frac{1}{\mu}v, y, \frac{1}{\mu}y').$$

The slow manifold is obtained by putting the right-hand side of the equation for v to zero. For F to induce a significant deformation its terms have to depend on v but this produces a factor $\frac{1}{\mu}$; terms containing v^2 or vy' produce terms of order $\frac{1}{\mu^2}$ and can be omitted.
As an illustration we choose for the attached (y) oscillator $G = dx$.
We generalise the Liénard transformation $(x, \; \dot{x}) \to (x, \; z)$ to

$$\begin{aligned} \frac{1}{\mu}\dot{x} &= z + x - \frac{1}{3}x^3 + c_1 xy + \frac{1}{2}c_2 x^2 y + c_3 xy^2, \\ \dot{z} &= -\frac{1}{\mu}x - c_1 x\dot{y} - \frac{1}{2}c_2 x^2\dot{y} - 2c_3 xy\dot{y}. \end{aligned} \tag{13}$$

The slow manifold is given by

$$z = -x + \frac{1}{3}x^3 - c_1 xy - \frac{1}{2}c_2 x^2 y - c_3 xy^2.$$

It is unstable if

$$1 - x^2 + c_1 y + c_2 xy + c_3 y^2 > 0.$$

The c_3-term is semidefinite, which is important, as far as stability is concerned. So, we choose this term for our model of destabilisation of the relaxation oscillation. Replacing c_3 by c we have the system

$$\begin{aligned} \ddot{x} + x &= \mu(1 - x^2)\dot{x} + \mu c \dot{x} y^2, \\ \ddot{y} + \kappa \dot{y} + q^2 y &= dxy. \end{aligned} \tag{14}$$

In generalised Liénard variables this becomes

$$\begin{aligned} \frac{1}{\mu}\dot{x} &= z + x - \frac{1}{3}x^3 + cxy^2, \\ \dot{z} &= -\frac{1}{\mu}x - 2cxy\dot{y} \end{aligned} \tag{15}$$

with the equation for y added. The slow manifold is given by

$$z = -(1 + cy^2)x + \frac{1}{3}x^3,$$

which is unstable if $1 + cy^2 - x^2 > 0$. The slow manifold corresponds with a 3-dimensional cubic cylinder parallel to the $\dot{y}$-axis.
We are now able to illustrate the behaviour of this autoparametric system.

9 Numerical experiments

At present the dynamics of system (14) in the case of an unstable normal mode is far from clear. In anticipation of a more theoretical analysis in the near future we perform a number of numerical experiments to illustrate interesting phenomena. We choose $\mu = 10$ throughout.
The stability of the slow manifold is determined by the sign of $1 + cy^2 - x^2$. We shall take c negative, $c = -2.2$, to illustrate the effect of a growing y-oscillation. In this case $1 + cy^2 = 0$ if $y = 0.67 \cdots$. Figure 3 describes the slow manifold and the corresponding stability manifold.

1. Choose $c = -2.2,\ d = 0.03,\ \kappa = 0.075$.
   Starting the y-oscillation near the normal mode plane $y = \dot{y} = 0$, the 3-dimensional projection of the solution is rather messy but a projection on the $x-\dot{x}$ plane produces an orbit which seems to fill up a large part of the space taken by the unperturbed orbit; see Figure 4. Calculation of the Lyapunov exponents of the solution gives the result

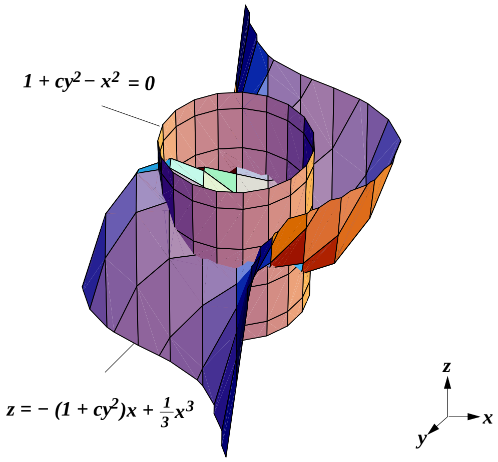

Figure 3: **The slow manifold and the stability manifold** for $c = -2.2$. The slow manifold is a 3-dimensional cubic cylinder in the complete 4-dimensional phase space. The stability of the flow on the slow manifold is determined by its position with respect to the manifold given by $1 + cy^2 - x^2 = 0$.

$\lambda_1 = 0.05205\cdots$, $\lambda_2 \approx 0$, $\lambda_3 = -0.13449\cdots$, $\lambda_4 = -13.14938\cdots$.
This leads us to a conclusion that we have a *chaotic attractor*, with the corresponding Kaplan-Yorke dimension near 2.3. Figure 5 shows the projection of the 3-dimensional Poincaré section of the orbit in $\dot{x}-y$ plane, taking $x = 0.5$ as the section (the vertical line in Figure 4). The projection fits with the previous calculation that the attractor has a dimension larger than 2.

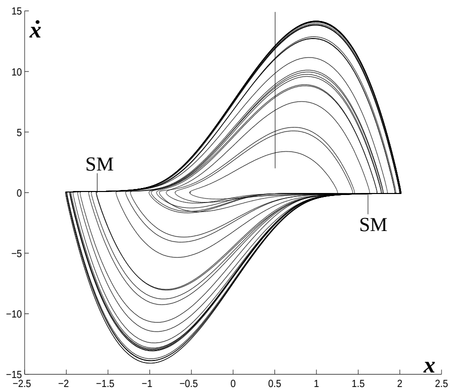

Figure 4: **A limit set of system (14)** for $\mu = 10$, $c = -2.2$, $d = 0.03$, $\kappa = 0.075$ with small starting values of the y-oscillation, projected on the $x - \dot{x}$ plane. SM is the stable part of the slow manifold. The vertical line corresponds with the Poincaré section of Figure 5

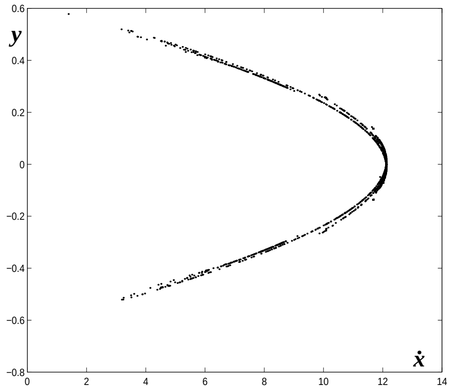

Figure 5: **The Poincaré section** of the limit set of Figure 4, projected on the $\dot{x} - y$ plane.

2. Consider the same dynamics, $c = -2.2,\ d = 0.03,\ \kappa = 0.075$, but starting at $y(0) = 3,\ \dot{y}(0) = 0.1$ we have oscillations so that $y(t)$ takes alternating values above and below $0.67\cdots$. Leaving out the transient we find a periodic limit set illustrated in Figure 6; this is a projection in 3-dimensional space. Projecting the limit set on the $x, \dot{x}$-plane we find a strongly perturbed relaxation oscillation, see Figure 7. For comparison the unperturbed relaxation oscillation (coupling $c = 0$) is indicated by dots.
   Thus, we have at least two attractors.

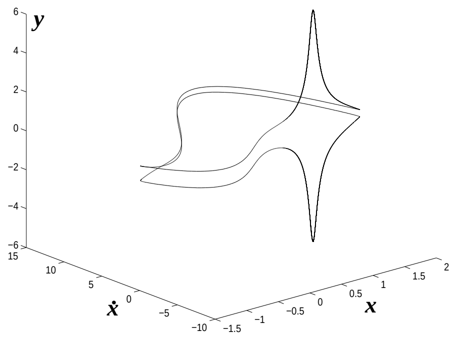

Figure 6: **A periodic limit set of system (14)** for $\mu = 10,\ c = -2.2,\ d = 0.03,\ \kappa = 0.075$ with high starting values of the y-oscillation. Transient orbits are left out. The stable part of the slow manifold is present near the extreme values of y.

3. For certain parameter values we find unbounded solutions. We discard these cases as they may correspond with a break-down of the model.

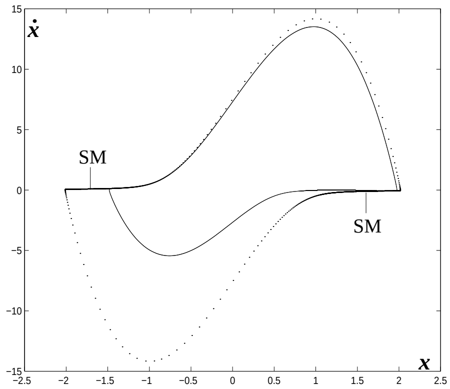

Figure 7: **A periodic limit set of system (14)** for $\mu = 10,\ c = -2.2,\ d = 0.03,\ \kappa = 0.075$ with high starting values of the y-oscillation, projected on the $x - \dot{x}$ plane. The dotted orbit corresponds with the unperturbed relaxation oscillation. In the perturbed state the slow manifolds are reduced and the limit cycle becomes asymmetric. SM is the stable part of the slow manifold.

10 A simplified model

Another possibility to clarify the dynamics is to replace system (14) by the equation

$$\ddot{x} + x = \mu(1 - x^2)\dot{x} + \mu c \dot{x} \cos^2 qt. \tag{16}$$

This equation might be illustrative for the behaviour of the relaxation oscillation in the special case when the solutions for y are $2\pi/q$-periodic. At the same time it is a model of the van der Pol relaxation oscillator with parametric excitation added.

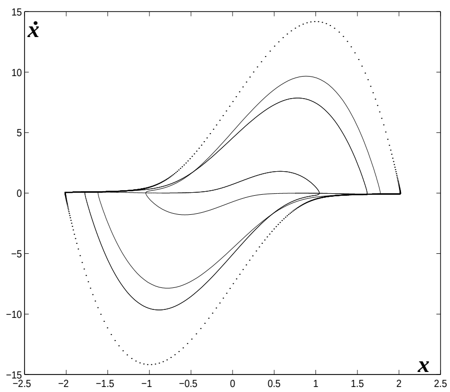

Figure 8: **A limit set of system (16)** for $\mu = 10,\ c = -16$. The limit set contains one periodic orbit. The relaxation oscillation corresponding with $c = 0$ is dotted.

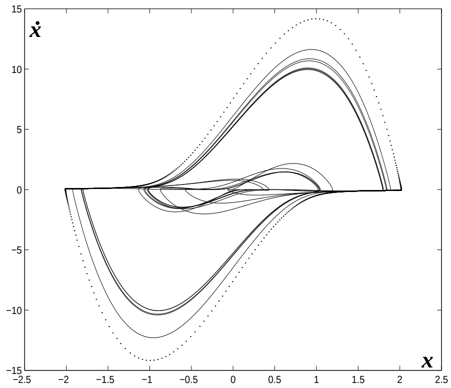

Figure 9: **A limit set of system (16)** for $\mu = 10,\ c = -18.1$. The limit set is long-periodic or aperiodic. The relaxation oscillation corresponding with $c = 0$ is dotted.

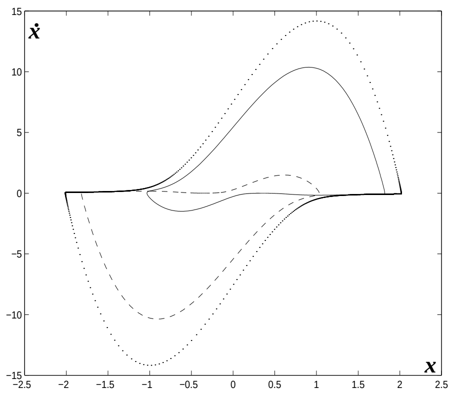

Figure 10: **A limit set of system (16)** for $\mu = 10,\ c = -19$. The limit set contains of two periodic orbits, both attracting (one is indicated by a full line and the other by dashes). The relaxation oscillation corresponding with $c = 0$ is dotted.

Interesting phenomena arise when varying c, see Figures (8-10). Choosing $c = -16$ we have periodic limiting behaviour; near $c = -18$ it is not clear whether the attractor is (long-) periodic or not periodic. This behaviour corresponds with a small window in parameter space as for $c = -19$ we have again periodic behaviour with two periodic attractors; see Figure 10. In all cases we observe quenching of the van der Pol relaxation oscillation.
If we increase c above -16 or if we decrease c below -19 the quenching is diminished.

11 Discussion

The most important conclusion is that to quench relaxation oscillations, apart from the usual tuning conditions, we have to choose the interaction such that strong deformation of the slow manifolds is possible.

In the case that the normal mode relaxation oscillation is destabilised, a number of different limit sets are possible.
The numerical experiments show some interesting results. We obtain a chaotic attractor coexisting with a stable periodic solution. The results also show the effectiveness of the coupling to deform the relaxation oscillation of the system. Related results seem to arise for the simplified model in section 10.
In the case of coupled oscillators the jumps from one part of the slow manifold to another part can become quite unpredictable and chaotic. More analysis and experiments are needed to understand qualitatively and quantitatively these phenomena.

Acknowledgement
The authors would like to thank J. M. Tuwankotta for a correction in the earlier version of this paper. T. Bakri gave technical advice for some numerical experiments in section 9.

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