

# MEASURES OF MAXIMAL ENTROPY FOR RANDOM $\beta$ -EXPANSIONS

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ABSTRACT. Let  $\beta > 1$  be a non-integer. We consider  $\beta$ -expansions of the form  $\sum_{i=1}^{\infty} \frac{d_i}{\beta^i}$ , where the digits  $(d_i)_{i \geq 1}$  are generated by means of a random map  $K_\beta$  defined on  $\{0, 1\}^{\mathbb{N}} \times [0, \lfloor \beta \rfloor / (\beta - 1)]$ . We show that  $K_\beta$  has a unique measure  $\nu_\beta$  of maximal entropy  $\log(1 + \lfloor \beta \rfloor)$ . Under this measure, the digits  $(d_i)_{i \geq 1}$  form a uniform Bernoulli process, and the projection of this measure on the second coordinate is an infinite convolution of Bernoulli measures. In case 1 has a finite greedy expansion with positive coefficients, the measure of maximal entropy is Markov. We also discuss the uniqueness of  $\beta$ -expansions.

## 1. INTRODUCTION

Let  $\beta > 1$  be a non-integer. There are two well-known expansions of numbers  $x$  in  $[0, \frac{\lfloor \beta \rfloor}{\beta - 1}]$  of the form,

$$x = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i},$$

with  $a_i \in \{0, 1, \dots, \lfloor \beta \rfloor\}$ . The largest in lexicographical order is the *greedy expansion*; [P], [R1], [R2], and the smallest is the *lazy expansion*; [JS], [EJK], [DK1]. The greedy expansion is obtained by iterating the transformation  $T_\beta$  defined on  $[0, \lfloor \beta \rfloor / (\beta - 1)]$  by

$$T_\beta(x) = \begin{cases} \beta x \pmod{1}, & 0 \leq x < 1, \\ \beta x - \lfloor \beta \rfloor, & 1 \leq x \leq \lfloor \beta \rfloor / (\beta - 1). \end{cases}$$

The lazy expansion is obtained by iterating the map  $S_\beta : [0, \lfloor \beta \rfloor / (\beta - 1)] \rightarrow [0, \lfloor \beta \rfloor / (\beta - 1)]$ , defined by

$$S_\beta(x) = \beta x - d \quad \text{for } x \in \Delta(d),$$

where

$$\Delta(0) = \left[ 0, \frac{\lfloor \beta \rfloor}{\beta(\beta - 1)} \right],$$

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and

$$\begin{aligned}\Delta(d) &= \left( \frac{\lfloor \beta \rfloor}{\beta - 1} - \frac{\lfloor \beta \rfloor - d + 1}{\beta}, \frac{\lfloor \beta \rfloor}{\beta - 1} - \frac{\lfloor \beta \rfloor - d}{\beta} \right) \\ &= \left( \frac{\lfloor \beta \rfloor}{\beta(\beta - 1)} + \frac{d - 1}{\beta}, \frac{\lfloor \beta \rfloor}{\beta(\beta - 1)} + \frac{d}{\beta} \right), \quad d \in \{1, 2, \dots, \lfloor \beta \rfloor\}.\end{aligned}$$

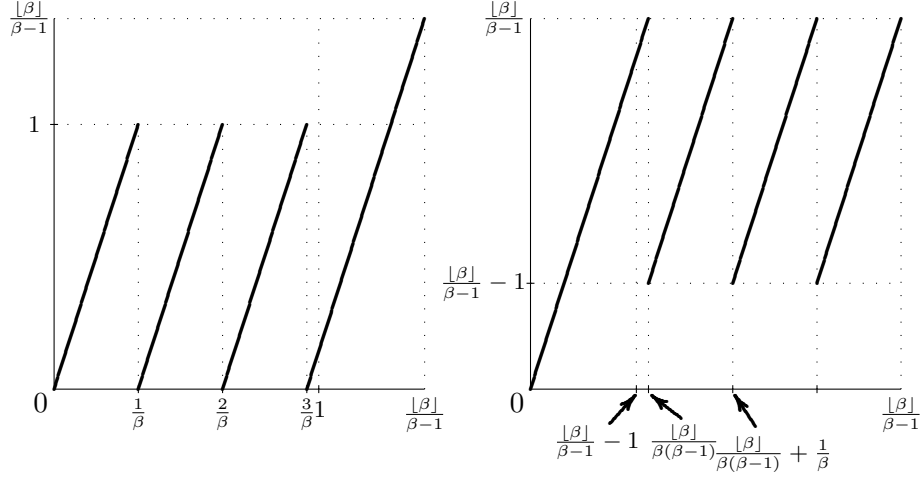


FIGURE 1. The greedy map  $T_\beta$  (left), and the lazy map  $S_\beta$  (right). Here  $\beta = \pi$

We denote by  $\mu_\beta$  the extended *Parry* measure (see [P],[G]) on  $[0, \lfloor \beta \rfloor / (\beta - 1)]$  which is absolutely continuous with respect to Lebesgue measure, and with density

$$h_\beta(x) = \begin{cases} \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \frac{1}{\beta^n} 1_{[0, T_\beta^n(1)]}(x) & 0 \leq x < 1, \\ 0 & 1 \leq x \leq \lfloor \beta \rfloor / (\beta - 1), \end{cases}$$

where  $F(\beta) = \int_0^1 (\sum_{x < T_\beta^n(1)} \frac{1}{\beta^n}) dx$  is a normalizing constant.

Define  $\ell : [0, \lfloor \beta \rfloor / (\beta - 1)] \rightarrow [0, \lfloor \beta \rfloor / (\beta - 1)]$  by

$$\ell(x) = \frac{\lfloor \beta \rfloor}{\beta - 1} - x,$$

and consider the *lazy* measure  $\rho_\beta$  defined on  $[0, \lfloor \beta \rfloor / (\beta - 1)]$  by  $\rho_\beta(A) = \mu_\beta(\ell(A))$  for every measurable set  $A$ . It is easy to see ([DK1]) that  $\ell$  is a measurable isomorphism between  $([0, \lfloor \beta \rfloor / (\beta - 1)], \mu_\beta, T_\beta)$  and  $([0, \lfloor \beta \rfloor / (\beta - 1)], \rho_\beta, S_\beta)$ .

In order to produce other expansions in a dynamical way, a new  $\beta$ -transformation  $K_\beta$  was introduced in [DK2]. The expansions generated by iterating this map are random mixtures of greedy and lazy expansions. This is done as follows. Super impose the greedy map and the corresponding lazy

map on  $[0, \lfloor \beta \rfloor / (\beta - 1)]$ , one then gets  $\lfloor \beta \rfloor$  overlapping regions of the form

$$S_k = \left[ \frac{k}{\beta}, \frac{\lfloor \beta \rfloor}{\beta(\beta - 1)} + \frac{k - 1}{\beta} \right], \quad k = 1, \dots, \lfloor \beta \rfloor,$$

which one refers to as *switch regions*. On  $S_k$ , the greedy map assigns the digit  $k$ , while the lazy map assigns the digit  $k - 1$ . Outside these switch regions both maps are identical, and hence they assign the same digits. Now, define a new random expansion in base  $\beta$  by randomizing the choice of the map used in the switch regions. So, whenever  $x$  belongs to a switch region flip a coin to decide which map will be applied to  $x$ , and hence which digit will be assigned. To be more precise, partition the interval  $[0, \lfloor \beta \rfloor / (\beta - 1)]$  into switch regions  $S_k$  and *equality regions*  $E_k$ , where

$$E_k = \left( \frac{\lfloor \beta \rfloor}{\beta(\beta - 1)} + \frac{k - 1}{\beta}, \frac{k + 1}{\beta} \right), \quad k = 1, \dots, \lfloor \beta \rfloor - 1,$$

$$E_0 = \left[ 0, \frac{1}{\beta} \right) \quad \text{and} \quad E_{\lfloor \beta \rfloor} = \left( \frac{\lfloor \beta \rfloor}{\beta(\beta - 1)} + \frac{\lfloor \beta \rfloor - 1}{\beta}, \frac{\lfloor \beta \rfloor}{\beta - 1} \right].$$

Let

$$S = \bigcup_{k=1}^{\lfloor \beta \rfloor} S_k, \quad \text{and} \quad E = \bigcup_{k=0}^{\lfloor \beta \rfloor} E_k,$$

and consider  $\Omega = \{0, 1\}^{\mathbb{N}}$  with product  $\sigma$ -algebra  $\mathcal{A}$ . Let  $\sigma : \Omega \rightarrow \Omega$  be the left shift, and define  $K_\beta : \Omega \times [0, \lfloor \beta \rfloor / (\beta - 1)] \rightarrow \Omega \times [0, \lfloor \beta \rfloor / (\beta - 1)]$  by

$$(1) \quad K_\beta(\omega, x) = \begin{cases} (\omega, \beta x - k) & x \in E_k, \quad k = 0, 1, \dots, \lfloor \beta \rfloor, \\ (\sigma(\omega), \beta x - k) & x \in S_k \text{ and } \omega_1 = 1, \quad k = 1, \dots, \lfloor \beta \rfloor, \\ (\sigma(\omega), \beta x - k + 1) & x \in S_k \text{ and } \omega_1 = 0, \quad k = 1, \dots, \lfloor \beta \rfloor. \end{cases}$$

The elements of  $\Omega$  represent the coin tosses ('heads'=1 and 'tails'=0) used every time the orbit hits a switch region. Let

$$d_1 = d_1(\omega, x) = \begin{cases} k & \text{if } x \in E_k, \quad k = 0, 1, \dots, \lfloor \beta \rfloor, \\ & \text{or } (\omega, x) \in \{\omega_1 = 1\} \times S_k, \quad k = 1, 2, \dots, \lfloor \beta \rfloor, \\ k - 1 & \text{if } (\omega, x) \in \{\omega_1 = 0\} \times S_k, \quad k = 1, 2, \dots, \lfloor \beta \rfloor, \end{cases}$$

then

$$K_\beta(\omega, x) = \begin{cases} (\omega, \beta x - d_1) & \text{if } x \in E, \\ (\sigma(\omega), \beta x - d_1) & \text{if } x \in S. \end{cases}$$

Set  $d_n = d_n(\omega, x) = d_1(K_\beta^{n-1}(\omega, x))$ , and let  $\pi_2 : \Omega \times [0, \lfloor \beta \rfloor / (\beta - 1)] \rightarrow [0, \lfloor \beta \rfloor / (\beta - 1)]$  be the canonical projection onto the second coordinate. Then

$$\pi_2(K_\beta^n(\omega, x)) = \beta^n x - \beta^{n-1} d_1 - \dots - \beta d_{n-1} - d_n,$$

and rewriting yields

$$x = \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \dots + \frac{d_n}{\beta^n} + \frac{\pi_2(K_\beta^n(\omega, x))}{\beta^n}.$$

Since  $\pi_2(K_\beta^n(\omega, x)) \in [0, \lfloor \beta \rfloor / (\beta - 1)]$ , it follows that

$$x - \sum_{i=1}^n \frac{d_i}{\beta^i} = \frac{\pi_2(K_\beta^n(\omega, x))}{\beta^n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This shows that for all  $\omega \in \Omega$  and for all  $x \in [0, \lfloor \beta \rfloor / (\beta - 1)]$  one has that

$$x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} = \sum_{i=1}^{\infty} \frac{d_i(\omega, x)}{\beta^i}.$$

The random procedure just described shows that with each  $\omega \in \Omega$  corresponds an algorithm that produces expansions in base  $\beta$ . Further, if we identify the point  $(\omega, x)$  with  $(\omega, (d_1(\omega, x), d_2(\omega, x), \dots))$ , then the action of  $K_\beta$  on the second coordinate corresponds to the left shift.

In [DK2], the dynamical properties of the map  $K_\beta$  were studied for  $\beta$  satisfying  $\beta^2 = n\beta + k$  (with  $1 \leq k \leq n$ ) and  $\beta^n = \beta^{n-1} + \dots + \beta + 1$ . It was shown that for these values of  $\beta$ , the underlying random  $\beta$ -transformation is isomorphic to a mixing Markov chain. However, the invariant measure considered is not the measure of maximal entropy (see Section 4, Remarks 6(3)). In this paper, we study the dynamical properties of  $K_\beta$  for any  $\beta > 1$ . In Section 2, we show that the map  $K_\beta$  captures all possible expansions in base  $\beta$  which are lexicographically ordered by the natural lexicographical ordering on  $\Omega$ . We also discuss briefly unique expansions.

In Section 3, we prove that the maximal entropy of  $K_\beta$  is  $\log(1 + \lfloor \beta \rfloor)$ . Further,  $K_\beta$  has a unique measure  $\nu_\beta$  of maximal entropy under which the random digits  $(d_i)$ , generated by the map  $K_\beta$ , form a uniform Bernoulli process. Moreover, the projection of the measure  $\nu_\beta$  on the second coordinate is an infinite convolution of Bernoulli measures. In Section 4, we show that if 1 has a finite greedy expansion of the form  $1 = b_1/\beta + b_2/\beta^2 + \dots + b_n/\beta^n$  with  $b_i \geq 1$  for  $i = 1, 2, \dots, n$ , then the measure  $\nu_\beta$  is Markov, and the underlying Markov chain is explicitly given.

## 2. BASIC PROPERTIES OF RANDOM $\beta$ -TRANSFORMATIONS

Let  $<_{lex}$  and  $\leq_{lex}$  denote the lexicographical ordering on both  $\Omega$ , and  $\{0, 1, \dots, \lfloor \beta \rfloor\}^{\mathbb{N}}$ . For each  $x \in [0, \frac{\lfloor \beta \rfloor}{\beta - 1}]$ , consider the set

$$D_x = \{(d_1(\omega, x), d_2(\omega, x), \dots) : \omega \in \Omega\}.$$

We now show that the elements of  $D_x$  are ordered by the lexicographical ordering on  $\Omega$ .

**Theorem 1.** *Suppose  $\omega, \omega' \in \Omega$  are such that  $\omega <_{lex} \omega'$ , then*

$$(d_1(\omega, x), d_2(\omega, x), \dots) \leq_{lex} (d_1(\omega', x), d_2(\omega', x), \dots).$$

**Proof.** Let  $i$  be the first index where  $\omega$  and  $\omega'$  differ. Since  $\omega <_{lex} \omega'$ , then  $\omega_i = 0$  and  $\omega'_i = 1$ . Notice that  $\pi_2(K_\beta^j(\omega, x)) = \pi_2(K_\beta^j(\omega', x))$  for  $j = 0, \dots, r_i$ , where  $r_i \geq 0$  is the time of the  $i^{th}$  visit to the region  $\Omega \times S$  of the orbit of  $(\omega, x)$  under  $K_\beta$ . Then,  $d_j(\omega, x) = d_j(\omega', x)$  for all  $j = 0, \dots, r_i$ .

If  $r_i = \infty$ , then  $d_j(\omega, x) = d_j(\omega', x)$  for all  $j$ .

If  $r_i < \infty$ , then  $K_\beta^{r_i}(\omega, x), K_\beta^{r_i}(\omega', x) \in \Omega \times S$ . Since  $\omega_i = 0$  and  $\omega'_i = 1$ , it follows that  $d_{r_i+1}(\omega', x) = d_{r_i+1}(\omega, x) + 1$ . Hence,

$$(d_1(\omega, x), d_2(\omega, x), \dots) <_{lex} (d_1(\omega', x), d_2(\omega', x), \dots).$$

□

The next theorem shows that the map  $K_\beta$  captures all algorithms producing  $\beta$ -expansions, i.e., for all  $x \in [0, \lfloor \beta \rfloor / (\beta - 1)]$ , any representation of  $x$  of the form  $x = \sum_{i=1}^{\infty} a_i / \beta^i$  with  $a_i \in \{0, 1, \dots, \lfloor \beta \rfloor\}$  can be generated by means of the map  $K_\beta$  by choosing an appropriate  $\omega \in \Omega$ .

**Theorem 2.** *Let  $x \in [0, \lfloor \beta \rfloor / (\beta - 1)]$ , and let  $x = \sum_{i=1}^{\infty} a_i / \beta^i$  with  $a_i \in \{0, 1, \dots, \lfloor \beta \rfloor\}$  be a representation of  $x$  in base  $\beta$ . Then there exists an  $\omega \in \Omega$  such that  $a_i = d_i(\omega, x)$ .*

For the proof we need the following lemma.

**Lemma 1.** *For  $x \in [0, \lfloor \beta \rfloor / (\beta - 1)]$ , one has*

- (i) *If  $x \in E_j$  for some  $j \in \{0, \dots, \lfloor \beta \rfloor\}$ , then  $a_1 = j$ .*
- (ii) *If  $x \in S_j$  for some  $j \in \{1, \dots, \lfloor \beta \rfloor\}$ , then  $a_1 \in \{j - 1, j\}$ .*

**Proof.** The proof is by contradiction.

- (i) Suppose  $a_1 \neq j$ . If  $a_1 \leq j - 1$ , then  $j \geq 1$  and  $x = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i} \leq \frac{j-1}{\beta} + \sum_{i=2}^{\infty} \frac{\lfloor \beta \rfloor}{\beta^i} = \frac{j-1}{\beta} + \frac{\lfloor \beta \rfloor}{\beta(\beta-1)}$ . If  $a_1 \geq j + 1$ , then  $j \leq \lfloor \beta \rfloor - 1$  and  $x \geq \frac{j+1}{\beta}$ . In both cases  $x \notin E_j$ .
- (ii) Suppose  $a_1 \notin \{j-1, j\}$ . If  $a_1 \leq j-2$ , then  $j \geq 2$  and  $x \leq \frac{j-2}{\beta} + \frac{\lfloor \beta \rfloor}{\beta(\beta-1)}$ . If  $a_1 \geq j+1$ , then  $j \leq \lfloor \beta \rfloor - 1$  and  $x \geq \frac{j+1}{\beta}$ . In both cases  $x \notin S_j$ . □

**Proof of Theorem 2:** Define the numbers  $\{x_n : n \in \mathbb{N}\}$  by  $x_n = \sum_{i=1}^{\infty} \frac{a_{i+n-1}}{\beta^i}$ . Notice that  $x_1 = x$ . Furthermore, we define a set  $\{\ell_n(x) : n \in \mathbb{N}\}$  that keeps track of the number of times we flip a coin. More precisely,

$$\ell_n(x) = \sum_{i=1}^n \mathbf{1}_S(x_i).$$

We use induction on the number of digits already determined.

- If  $x \in E_j$ , then  $\ell_1(x) = 0$  and by Lemma 1,  $a_1 = j$ . We set  $\Omega_1 = \Omega$ .
- If  $x \in S_j$ , then  $\ell_1(x) = 1$  and by Lemma 1,  $a_1 \in \{j - 1, j\}$ .
  - If  $a_1 = j - 1$ , we set  $\Omega_1 = \{\omega \in \Omega : \omega_1 = 0\}$ .
  - If  $a_1 = j$ , we set  $\Omega_1 = \{\omega \in \Omega : \omega_1 = 1\}$ .

It follows that  $\Omega_1$  is a cylinder of length  $\ell_1(x)$  and for all  $\omega \in \Omega_1$ ,  $d_1(\omega, x) = a_1$ . By a cylinder of length 0 we mean of course the whole space  $\Omega$ . Suppose we have obtained  $\Omega_n \subseteq \dots \subseteq \Omega_1$ , so that  $\Omega_n$  is a cylinder of length  $\ell_n(x)$  and for all  $\omega \in \Omega_n$ ,  $d_1(\omega, x) = a_1, \dots, d_n(\omega, x) = a_n$ . Notice that for all  $\omega \in \Omega_n$ ,  $x_{n+1} = \pi_2(K_\beta^n(\omega, x))$ .

- If  $x_{n+1} \in E_j$ , then  $\ell_{n+1}(x) = \ell_n(x)$  and for all  $\omega \in \Omega_n$ ,  $d_{n+1}(\omega, x) = d_1(K_\beta^n(\omega, x)) = j = a_{n+1}$ , by Lemma 1. We set  $\Omega_{n+1} = \Omega_n$ .

- If  $x_{n+1} \in S_j$ , then  $\ell_{n+1}(x) = \ell_n(x) + 1$  and  $b_{n+1} \in \{j-1, j\}$  by Lemma 1.
  - If  $a_{n+1} = j-1$ , we set  $\Omega_{n+1} = \{\omega \in \Omega_n : \omega_{\ell_{n+1}} = 0\}$ . Then, for all  $\omega \in \Omega_{n+1}$ ,  $d_{n+1}(\omega, x) = d_1(K_\beta^n(\omega, x)) = j-1 = a_{n+1}$ .
  - If  $a_{n+1} = j$ , we set  $\Omega_{n+1} = \{\omega \in \Omega_n : \omega_{\ell_{n+1}} = 1\}$ . Then, for all  $\omega \in \Omega_{n+1}$ ,  $d_{n+1}(\omega, x) = d_1(K_\beta^n(\omega, x)) = j = a_{n+1}$ .

In all cases we see that  $\Omega_{n+1}$  is a cylinder of length  $\ell_{n+1}(x)$ , and for all  $\omega \in \Omega_{n+1}$ ,  $d_1(\omega, x) = a_1, \dots, d_{n+1}(\omega, x) = a_{n+1}$ .

If the map  $K_\beta$  hits the switch regions infinitely many times, then  $\ell_n(x) \rightarrow \infty$  and, as is well known,  $\bigcap \Omega_n$  consists of a single point. If this happens only finitely many times, then the set  $\{\ell_n(x) : n \in \mathbb{N}\}$  is finite and  $\bigcap \Omega_n$  is exactly a cylinder set. In both cases  $\bigcap \Omega_n$  is non-empty and  $\omega \in \bigcap \Omega_n$  satisfies  $d_j(\omega, x) = a_j, j \geq 1$ .  $\square$

**Remark 1.** Theorems 1 and 2 give another proof of the fact that among all possible  $\beta$ -expansions of a point  $x \in [0, \lfloor \beta \rfloor / (\beta - 1)]$ , the greedy expansion is the largest in lexicographical order (it corresponds to the largest element  $(1, 1, \dots)$  of  $\Omega$ ), and the lazy is the smallest (it corresponds to the smallest element  $(0, 0, \dots)$  of  $\Omega$ ). Furthermore, from the proof of Theorem 2, one sees that  $x$  has a unique representation in base  $\beta$  of the form

$$x = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i}$$

with  $a_i \in \{0, 1, \dots, \lfloor \beta \rfloor\}$  if and only if  $a_i = d_i(\omega, x)$  for all  $i \geq 1$ , and all  $\omega \in \Omega$ . Equivalently, the greedy expansion of  $x$  is the only representation of  $x$  in base  $\beta$  if and only if  $x_n \in E$  for all  $n \geq 1$ . In this case, we have for all  $n \geq 1$ ,  $x_n = T_\beta^{n-1}x = S_\beta^{n-1}x$ .

Remark 1 gives in fact a characterization of unique expansion in terms of the greedy expansion. Namely, if  $x$  has an infinite greedy expansion of the form  $x = a_1/\beta + a_2/\beta^2 + \dots$ , then  $x$  has a unique expansion in base  $\beta$  if and only if  $T_\beta^n x \in E_{a_{n+1}}$  for all  $n \geq 0$ . We would like to give other characterizations. Although some of the results are already known (see [KL]), we give simple proofs for completeness. We first observe that  $1 \in S_{\lfloor \beta \rfloor} \cup E_{\lfloor \beta \rfloor}$ , and  $1 \in E_{\lfloor \beta \rfloor}$  if and only if  $\frac{\lfloor \beta \rfloor}{\beta-1} - 1 \in E_0$ . The following proposition gives a characterization of the case  $1 \in E_{\lfloor \beta \rfloor}$  using the greedy expansion of 1.

**Proposition 1.** *Suppose 1 has a finite or infinite greedy expansion of the form  $1 = b_1/\beta + b_2/\beta^2 + \dots$ .*

- (i) *If  $b_i = 0$  for all  $i \geq 3$ , then  $1 \in E_{b_1}$  if and only if  $b_2 \geq 2$ . Moreover, if  $b_2 = 1$ , then  $1 = \frac{\lfloor \beta \rfloor}{\beta-1} - \frac{1}{\beta}$ .*
- (ii) *If  $b_i \geq 1$  for some  $i \geq 3$ , then  $1 \in E_{b_1}$  if and only if  $b_2 \geq 1$ .*

**Proof.** First observe that  $\lfloor \beta \rfloor = b_1$ , and that  $1 = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \frac{1}{\beta^2} T_\beta^2 1$ . The latter implies that  $\beta^2 - b_1 \beta = b_2 + T_\beta^2 1$ . Now, by definition  $1 \in E_{b_1}$  if and only if  $1 > \frac{b_1}{\beta-1} - \frac{1}{\beta}$ , or equivalently  $\beta^2 - b_1 \beta > 1$ .

In case (i), we have  $T_\beta^2 1 = 0$  which implies that  $\beta^2 - b_1\beta = b_2$ . Hence,  $1 \in E_{b_1}$  if and only if  $b_2 \geq 2$ . If  $b_2 = 1$ , then  $\beta^2 - b_1\beta = 1$ ; equivalently,  $1 = \frac{|\beta|}{\beta-1} - \frac{1}{\beta}$ .

In case (ii), we have that  $0 < T_\beta^2 1 < 1$ . Hence,  $\beta^2 - b_1\beta = b_2 + T_\beta^2 1 > 1$  if and only if  $b_2 \geq 1$ .  $\square$

Before we proceed to the characterization of the uniqueness of the  $\beta$ -expansion of  $x$ , we need the following simple lemma.

**Lemma 2.** *Suppose  $x$  has a finite or infinite greedy expansion of the form  $x = a_1/\beta + a_2/\beta^2 + \dots$ . If  $a_{n+1} \geq 1$ , then  $T_\beta^n x \in E_{a_{n+1}}$  if and only if  $T_\beta^{n+1} x > \frac{|\beta|}{\beta-1} - 1$ .*

**Proof.** Notice that  $T_\beta^n x = \frac{a_{n+1}}{\beta} + \frac{1}{\beta} T_\beta^{n+1} x \in S_{a_{n+1}} \cup E_{a_{n+1}}$ . Thus,  $T_\beta^n x \in E_{a_{n+1}}$  if and only if  $T_\beta^n x > \frac{|\beta|}{\beta(\beta-1)} + \frac{a_{n+1}-1}{\beta}$ . Rewriting one gets that  $T_\beta^n x \in E_{a_{n+1}}$  if and only if  $T_\beta^{n+1} x > \frac{|\beta|}{\beta-1} - 1$ .  $\square$

Note that if  $a_{n+1} = 0$ , then  $T_\beta^n x \in E_0$ .

The following theorem is an immediate consequence of the above lemma. We remark that this theorem was obtained independently for the case  $x = 1$ , and via other methods in [KL, Theorem 3.1].

**Theorem 3.** *Suppose  $x$  has an infinite greedy expansion of the form  $x = a_1/\beta + a_2/\beta^2 + \dots$ . Then,  $x$  has a unique expansion in base  $\beta$  if and only if for all  $n \geq 0$  with  $a_{n+1} \geq 1$ , we have  $T_\beta^{n+1} x > \frac{|\beta|}{\beta-1} - 1$ .*

**Corollary 1.** *Suppose  $x$  has an infinite greedy expansion of the form  $x = a_1/\beta + a_2/\beta^2 + \dots$ , with  $a_i \geq 1$  for all  $i \geq 1$ . Then,  $x$  has a unique  $\beta$ -expansion.*

**Proof.** Observe that for all  $n \geq 0$ ,  $T_\beta^n x \geq \frac{1}{\beta-1}$  and that  $\frac{1}{\beta-1} > \frac{|\beta|}{\beta-1} - 1$ . The result follows from Theorem 3.  $\square$

**Corollary 2.** *If 1 has a unique  $\beta$ -expansion, then there exists a  $k \geq 0$  such that in the greedy expansion of 1, every block of consecutive zeros consists of at most  $k$  terms.*

**Proof.** Let  $1 = b_1/\beta + b_2/\beta^2 + \dots$  be the greedy expansion. By uniqueness  $1 \in E_{b_1}$ , so  $\frac{b_1}{\beta-1} - 1 < \frac{1}{\beta}$ . Hence, there exists a  $k$  such that  $\frac{1}{\beta^{k+1}} < \frac{b_1}{\beta-1} - 1 < \frac{1}{\beta^k}$ . If  $b_{i-1}b_i \dots b_j$  is a block with  $b_{i-1} \geq 1$ ,  $b_i = \dots = b_j = 0$  and  $j-i+1 \geq k+1$ , then  $T_\beta^{i-1} 1 < \frac{1}{\beta^{k+1}} < \frac{b_1}{\beta-1} - 1$ , contradicting Theorem 3.  $\square$

Another immediate corollary of Theorem 3, and Proposition 1 is the following.

**Corollary 3.** *Suppose 1 has an infinite greedy expansion of the form  $1 = b_1/\beta + b_2/\beta^2 + \dots$ , with  $b_2 \geq 1$ . Let  $k \geq 1$  be the unique integer such that  $\frac{1}{\beta^{k+1}} \leq \frac{b_1}{\beta-1} - 1 < \frac{1}{\beta^k}$ . If in the greedy expansion of 1 every block of consecutive zeros contains at most  $k-1$  terms, then 1 has a unique  $\beta$ -expansion.*

3. MEASURES OF MAXIMAL ENTROPY FOR RANDOM  $\beta$ -EXPANSIONS

In this section we show that the map  $K_\beta$  on  $\Omega \times [0, \frac{\lfloor \beta \rfloor}{\beta-1}]$  can be essentially identified with the left shift on  $\{0, \dots, \lfloor \beta \rfloor\}^{\mathbb{N}}$ . This will enable us to prove that  $K_\beta$  has a unique measure of maximal entropy.

Let  $D = \{0, \dots, \lfloor \beta \rfloor\}^{\mathbb{N}}$  be equipped with the product  $\sigma$ -algebra  $\mathcal{D}$ , and the uniform product measure  $\mathbb{P}$ . Let  $\sigma'$  be the left shift on  $D$ . On the set  $\Omega \times [0, \lfloor \beta \rfloor / (\beta - 1)]$  we consider the product  $\sigma$ -algebra  $\mathcal{A} \times \mathcal{B}$ , where  $\mathcal{B}$  is the Borel  $\sigma$ -algebra on  $[0, \lfloor \beta \rfloor / (\beta - 1)]$ , and  $\mathcal{A}$  the product  $\sigma$ -algebra on  $\Omega$ . Define the function  $\varphi : \Omega \times [0, \frac{\lfloor \beta \rfloor}{\beta-1}] \rightarrow D$  by

$$\varphi(\omega, x) = (d_1(\omega, x), d_2(\omega, x), \dots).$$

It is easily seen that  $\varphi$  is measurable, and  $\varphi \circ K_\beta = \sigma' \circ \varphi$ . Furthermore, Theorem 2 implies that  $\varphi$  is surjective. We will now show that  $\varphi$  restricted to an appropriate  $K_\beta$ -invariant subset is in fact invertible.

Let

$$Z = \{(\omega, x) \in \Omega \times [0, \frac{\lfloor \beta \rfloor}{\beta-1}] : K_\beta^n(\omega, x) \in \Omega \times S \text{ infinitely often}\},$$

and

$$D' = \{(a_1, a_2, \dots) \in D : \sum_{i=1}^{\infty} \frac{a_{j+i-1}}{\beta^i} \in S \text{ for infinitely many } j\}.$$

Then,  $\varphi(Z) = D'$ ,  $K_\beta^{-1}(Z) = Z$  and  $(\sigma')^{-1}(D') = D'$ . Let  $\varphi'$  be the restriction of the map  $\varphi$  to  $Z$ .

**Lemma 3.** *The map  $\varphi' : Z \rightarrow D'$  is a bimeasurable bijection.*

**Proof.** For any sequence  $(a_1, a_2, \dots) \in D'$ , define recursively

$$\begin{aligned} r_1 &= \min\{j \geq 1 : \sum_{l=1}^{\infty} \frac{a_{j+l-1}}{\beta^l} \in S\}, \\ r_i &= \min\{j > r_{i-1} : \sum_{l=1}^{\infty} \frac{a_{j+l-1}}{\beta^l} \in S\}. \end{aligned}$$

If  $\sum_{l=1}^{\infty} \frac{a_{r_i+l-1}}{\beta^l} \in S_j$  then, according to Lemma 1,  $a_{r_i} \in \{j-1, j\}$ . If  $a_{r_i} = j-1$ , let  $\omega_i = 0$ , otherwise let  $\omega_i = 1$ . Define  $(\varphi')^{-1} : D' \rightarrow Z$  by

$$(\varphi')^{-1}((a_1, a_2, \dots)) = (\omega, \sum_{i=1}^{\infty} \frac{a_i}{\beta^i}).$$

It is easily checked that  $(\varphi')^{-1}$  is measurable, and is the inverse of  $\varphi'$ .  $\square$

**Lemma 4.**  $\mathbb{P}(D') = 1$ .

**Proof.** For any sequence  $(a_1, a_2, \dots) \in D$ , define for  $m \geq 1$ ,

$$x_m = \frac{1}{\beta} + \frac{a_1}{\beta^{m+1}} + \frac{a_2}{\beta^{m+2}} + \dots$$



Clearly  $x_m \geq 1/\beta$ . On the other hand,

$$x_m \leq \frac{1}{\beta} + \sum_{i=1}^{\infty} \frac{\lfloor \beta \rfloor}{\beta^{m+i}} = \frac{1}{\beta} \left( 1 + \frac{1}{\beta^{m-1}(\beta-1)} \right).$$

Since  $1 + \frac{1}{\beta^{m-1}(\beta-1)} \downarrow 1$  as  $m \rightarrow \infty$ , there exists an integer  $N > 0$  such that for all  $m \geq N$ ,

$$\frac{1}{\beta} \leq x_m \leq \frac{\lfloor \beta \rfloor}{\beta(\beta-1)},$$

i.e.,  $x_m \in S_1$  for all  $m \geq N$ .

Let

$$D'' = \{(a_1, a_2, \dots) \in D : a_j a_{j+1} \dots a_{j+N-1} = 1 \underbrace{00 \dots 0}_{(N-1)\text{-times}} \text{ for infinitely many } j\}.$$

From the above, we conclude that  $D'' \subseteq D'$ . Clearly  $\mathbb{P}(D'') = 1$ , hence  $\mathbb{P}(D') = 1$ .  $\square$

Now, consider the  $K_\beta$ -invariant measure  $\nu_\beta$  defined on  $\mathcal{A} \times \mathcal{B}$  by  $\nu_\beta(A) = \mathbb{P}(\varphi(Z \cap A))$ . The following theorem is a simple consequence of Lemmas 3 and 4.

**Theorem 4.** *Let  $\beta > 1$  be a non-integer. Then the map  $\varphi : (\Omega \times [0, \frac{\lfloor \beta \rfloor}{\beta-1}], \mathcal{A} \times \mathcal{B}, \nu_\beta, K_\beta) \rightarrow (D, \mathcal{D}, \mathbb{P}, \sigma')$  is a measurable isomorphism.*

**Remark 2.** The above theorem implies that  $h_{\nu_\beta}(K_\beta) = \log(1 + \lfloor \beta \rfloor)$ . Further, since  $\mathbb{P}$  is the unique measure of maximal entropy on  $D$ , we see that  $\nu_\beta$  is the only  $K_\beta$ -invariant measure with support  $Z$  and maximal entropy  $\log(1 + \lfloor \beta \rfloor)$  i.e., any other  $K_\beta$ -invariant measure with support  $Z$  has entropy strictly less than  $\log(1 + \lfloor \beta \rfloor)$ . We now investigate the entropy of  $K_\beta$ -invariant measures  $\mu$  for which  $\mu(Z^c) > 0$ .

**Lemma 5.** *Let  $\mu$  be a  $K_\beta$ -invariant measure for which  $\mu(Z^c) > 0$ . Then,  $h_\mu(K_\beta) < \log(1 + \lfloor \beta \rfloor)$ .*

**Proof.** Since  $Z$  and  $Z^c$  are  $K_\beta$ -invariant, there exist  $0 \leq \alpha < 1$  and  $K_\beta$ -invariant measures  $\mu_1, \mu_2$  concentrated on  $Z$  and  $Z^c$  respectively, such that  $\mu = \alpha\mu_1 + (1 - \alpha)\mu_2$ . Then,  $h_\mu(K_\beta) = \alpha h_{\mu_1}(K_\beta) + (1 - \alpha)h_{\mu_2}(K_\beta)$ . From Remark 2, we have  $h_{\mu_1}(K_\beta) \leq \log(1 + \lfloor \beta \rfloor)$ . We now show that  $h_{\mu_2}(K_\beta) < \log(1 + \lfloor \beta \rfloor)$ . To this end, let

$$E = \{x \in [0, \frac{\lfloor \beta \rfloor}{\beta-1}] : x \text{ has a unique } \beta\text{-expansion}\}.$$

Then,  $\Omega \times E \subseteq K_\beta^{-1}(\Omega \times E)$ , and  $\bigcup_{i=0}^{\infty} K_\beta^{-i}(\Omega \times E) = Z^c$ . From the above we see that  $\mu_2(\Omega \times E) = 1$ , hence it is enough to study the entropy of the map  $K_\beta$  restricted to the set  $\Omega \times E$ . On this set the map  $K_\beta$  has the form  $I_\Omega \times T_\beta$ , where  $I_\Omega$  is the identity map on  $\Omega$ , and  $T_\beta$  the greedy map restricted to the set  $E$ . On  $E$  we consider the Borel  $\sigma$ -algebra  $E \cap \mathcal{B}$ . Notice that  $\mu_2 \circ \pi_2^{-1}$  is a  $T_\beta$ -invariant measure with support  $E$ , hence  $h_{\mu_2 \circ \pi_2^{-1}}(T_\beta) \leq \log \beta$ .

Let  $\mathcal{F}$  and  $\mathcal{E}$  be any two measurable partitions of  $\Omega$  and  $E$  respectively. For any  $n \geq 1$ ,

$$\bigvee_{i=0}^{n-1} K_\beta^{-i}(\mathcal{F} \times \mathcal{E}) = \bigvee_{i=0}^{n-1} (I_\Omega \times T_\beta)^{-i}(\mathcal{F} \times \mathcal{E}) = \mathcal{F} \times \bigvee_{i=0}^{n-1} T_\beta^{-i} \mathcal{E},$$

modulo sets of  $\mu_2$  measure 0. Hence,

$$\begin{aligned} H_{\mu_2}(\Omega \times \bigvee_{i=0}^{n-1} T_\beta^{-i} \mathcal{E}) &\leq H_{\mu_2}(\bigvee_{i=0}^{n-1} K_\beta^{-i}(\mathcal{F} \times \mathcal{E})) \\ &\leq H_{\mu_2}(\mathcal{F} \times E) + H_{\mu_2}(\Omega \times \bigvee_{i=0}^{n-1} T_\beta^{-i} \mathcal{E}). \end{aligned}$$

Now, dividing by  $n$  and taking the limit as  $n \rightarrow \infty$ , we get

$$h_{\mu_2}(K_\beta, \mathcal{F} \times \mathcal{E}) = h_{\mu_2}(K_\beta, \Omega \times \mathcal{E}) = h_{\mu_2 \circ \pi_2^{-1}}(T_\beta, \mathcal{E}) \leq \log \beta.$$

Since  $\mathcal{F}$  and  $\mathcal{E}$  are arbitrary partitions, we have

$$h_{\mu_2}(K_\beta) \leq \log \beta < \log(1 + \lfloor \beta \rfloor).$$

Therefore,  $h_\mu(K_\beta) < \log(1 + \lfloor \beta \rfloor)$ .  $\square$

From Remark 2, and Lemma 5 we arrive at the following theorem.

**Theorem 5.** *The measure  $\nu_\beta$  is the unique  $K_\beta$ -invariant measure of maximal entropy.*

An interesting consequence of the above theorems is that if  $\beta, \beta' > 1$  are non-integers, then

$$\lfloor \beta \rfloor = \lfloor \beta' \rfloor \text{ if and only if } (K_\beta, \nu_\beta) \text{ is isomorphic to } (K_{\beta'}, \nu_{\beta'}).$$

We investigate the projection of the measure  $\nu_\beta$  on the second coordinate. As before, we let  $\pi_2 : \Omega \times [0, \frac{\lfloor \beta \rfloor}{\beta-1}] \rightarrow [0, \frac{\lfloor \beta \rfloor}{\beta-1}]$  be the natural projection  $\pi_2(\omega, x) = x$ , we are interested in identifying the measure  $\nu_\beta \circ \pi_2^{-1}$  defined on  $[0, \frac{\lfloor \beta \rfloor}{\beta-1}]$ . To do so, we consider the purely discontinuous measures  $\{\delta_i\}_{i \geq 1}$  defined on  $\mathbb{R}$  as follows:

$$\delta_i(\{0\}) = \frac{1}{\lfloor \beta \rfloor + 1}, \dots, \delta_i(\{\lfloor \beta \rfloor \beta^{-i}\}) = \frac{1}{\lfloor \beta \rfloor + 1}.$$

So  $\delta_i$  is concentrated on the set

$$\{0, \beta^{-i}, \dots, \lfloor \beta \rfloor \beta^{-i}\}.$$

Let  $\delta$  be the corresponding infinite Bernoulli convolution,

$$\delta = \lim_{n \rightarrow \infty} \delta_1 * \dots * \delta_n.$$

**Theorem 6.**  $\nu_\beta \circ \pi_2^{-1} = \delta$ .

**Proof.** Let  $h : D \rightarrow [0, \frac{\lfloor \beta \rfloor}{\beta-1}]$  be given by  $h(y) = \sum_{i=1}^{\infty} \frac{y_i}{\beta^i}$ , where  $y = (y_1, y_2, \dots)$ . Then,  $\pi_2 = h \circ \varphi$ , and  $\delta = \mathbb{P} \circ h^{-1}$ . Furthermore, for each Borel

set  $B \in \mathcal{B}$ , we have  $\varphi(\pi_2^{-1}(B) \cap Z) = h^{-1}(B) \cap D'$ . Since  $\mathbb{P}(D') = 1$ , it follows that

$$\mathbb{P}(h^{-1}(B)) = \mathbb{P}(h^{-1}(B) \cap D') = \nu_\beta(\pi_2^{-1}(B)).$$

Hence,  $\nu_\beta \circ \pi_2^{-1} = \delta$ .  $\square$

If  $\beta \in (1, 2)$  then  $\delta$  is an Erdős measure on  $[0, \frac{1}{\beta-1}]$ , and lots of things are already known. For example, if  $\beta$  is a Pisot number, then  $\delta$  is singular with respect to Lebesgue measure; [E1], [E2], [S]. Further, for almost all  $\beta \in (1, 2)$  the measure  $\delta$  is equivalent to Lebesgue measure; [So], [MS]. There are many generalizations of these results to the case of an arbitrary digit set (see [PSS] for more references and results).

#### 4. FINITE GREEDY EXPANSION OF 1 WITH POSITIVE COEFFICIENTS, AND THE MARKOV PROPERTY OF THE RANDOM $\beta$ -EXPANSIONS

We now assume that the greedy expansion of 1 in base  $\beta$  satisfies  $1 = b_1/\beta + b_2/\beta^2 + \dots + b_n/\beta^n$  with  $b_i \geq 1$ , and  $n \geq 2$  (notice that  $\lfloor \beta \rfloor = b_1$ ). We show that in this case the dynamics of  $K_\beta$  can be identified with a subshift of finite type with an irreducible adjacency matrix. As a result the unique measure of maximal entropy  $\nu_\beta$  obtained in the previous section is Markov.

The analysis of the case  $\beta^2 = b_1\beta + 1$  needs some adjustments. For this reason, we assume here that  $\beta^2 \neq b_1\beta + 1$ , and refer the reader to section 4 (Remarks 6(2)) for the appropriate modifications needed for the case  $\beta^2 = b_1\beta + 1$ .

We begin by a proposition that is an immediate consequence of Proposition 1 and Lemma 2, and which plays a crucial role in finding the Markov partition describing the dynamics of the map  $K_\beta$ , as defined in section 1.

**Proposition 2.** *Suppose 1 has a finite greedy expansion of the form  $1 = b_1/\beta + b_2/\beta^2 + \dots + b_n/\beta^n$ . If  $b_j \geq 1$  for  $1 \leq j \leq n$ , then*

- (i)  $T_\beta^i 1 = S_\beta^i 1 \in E_{b_{i+1}}$ ,  $i = 0, 1, \dots, n-2$ ,
- (ii)  $T_\beta^{n-1} 1 = S_\beta^{n-1} 1 = \frac{b_n}{\beta} \in S_{b_n}$ ,  $T_\beta^n 1 = 0$ , and  $S_\beta^n 1 = 1$ .
- (iii)  $T_\beta^i(\frac{b_1}{\beta-1} - 1) = S_\beta^i(\frac{b_1}{\beta-1} - 1) \in E_{b_1 - b_{i+1}}$ ,  $0 \leq i \leq n-2$ ,
- (iv)  $T_\beta^{n-1}(\frac{b_1}{\beta-1} - 1) = S_\beta^{n-1}(\frac{b_1}{\beta-1} - 1) = \frac{b_1}{\beta(\beta-1)} + \frac{b_1 - b_n}{\beta} \in S_{b_1 - b_n + 1}$ ,  
 $T_\beta^n(\frac{b_1}{\beta-1} - 1) = \frac{b_1}{\beta-1} - 1$ , and  $S_\beta^n(\frac{b_1}{\beta-1} - 1) = \frac{b_1}{\beta-1}$ .

Moreover, by Proposition 1 and Lemma 2, one has that  $T_\beta^i 1 = S_\beta^i 1 > \frac{b_1}{\beta-1} - 1$ , and  $T_\beta^i(\frac{b_1}{\beta-1} - 1) = S_\beta^i(\frac{b_1}{\beta-1} - 1) < 1$  for all  $i = 1, 2, \dots, n-1$ .

To find the Markov chain behind the map  $K_\beta$ , one starts by refining the partition

$$\mathcal{E} = \{E_0, S_1, E_1, S_2, \dots, S_{b_1}, E_{b_1}\}$$

of  $[0, \frac{b_1}{\beta-1}]$ , using the orbits of 1 and  $\frac{b_1}{\beta-1} - 1$  under the transformation  $T_\beta$ . We place the end-points of  $\mathcal{E}$  together with  $T_\beta^i 1$ ,  $T_\beta^i(\frac{b_1}{\beta-1} - 1)$ ,  $i = 0, 1, \dots, n-2$ ,

in increasing order. We use these points to form a new partition  $\mathcal{C}$  which is a refinement of  $\mathcal{E}$ , consisting of intervals. We write  $\mathcal{C}$  as

$$\mathcal{C} = \{C_0, C_1, \dots, C_L\}.$$

We choose  $\mathcal{C}$  to satisfy the following. For  $0 \leq i \leq n-2$ ,

- $T_\beta^i 1 \in C_j$  if and only if  $T_\beta^i 1$  is a left end-point of  $C_j$ ,
- $T_\beta^i(\frac{b_1}{\beta-1} - 1) \in C_j$  if and only if  $T_\beta^i(\frac{b_1}{\beta-1} - 1)$  is a right end-point of  $C_j$ .

Recall that the map  $\ell : [0, \lfloor \beta \rfloor / (\beta - 1)] \rightarrow [0, \lfloor \beta \rfloor / (\beta - 1)]$  defined by  $\ell(x) = \frac{\lfloor \beta \rfloor}{\beta-1} - x$ , satisfies  $T_\beta \circ \ell = \ell \circ S_\beta$ . Thus, if  $x \in E_i$  for some  $i$ , then  $T_\beta x = S_\beta x$  and  $T_\beta \ell(x) = \ell T_\beta(x)$ . From the dynamics of  $K_\beta$  on this refinement, one reads the following properties of  $\mathcal{C}$ .

- p1-**  $C_0 = [0, \frac{b_1}{\beta-1} - 1]$  and  $C_L = [1, \frac{b_1}{\beta-1}]$ .
- p2-** For  $i = 0, 1, \dots, b_1$ ,  $E_i$  can be written as a finite disjoint union of the form  $E_i = \cup_{j \in M_i} C_j$  with  $M_0, M_1, \dots, M_{b_1}$  disjoint subsets of  $\{0, 1, \dots, L\}$ . Further, the number of elements in  $M_i$  equals the number of elements in  $M_{b_1-i}$ .
- p3-** For each  $S_i$  there corresponds exactly one  $j \in \{0, 1, \dots, L\} \setminus \cup_{k=0}^{b_1} M_k$  such that  $S_i = C_j$ . This is possible since the  $T_\beta$ -orbits of 1 and  $\frac{b_1}{\beta-1} - 1$  never hit the interior of  $\cup_{i=1}^{b_1} S_i$ .
- p4-** If  $C_j \subset E_i$ , then  $T_\beta(C_j) = S_\beta(C_j)$  is a finite disjoint union of elements of  $\mathcal{C}$ , say  $T_\beta(C_j) = C_{i_1} \cup \dots \cup C_{i_\ell}$ . Since  $\ell(C_j) = C_{L-j} \subset E_{b_1-i}$ , it follows that  $T_\beta(C_{L-j}) = C_{L-i_1} \cup \dots \cup C_{L-i_\ell}$ .
- p5-** If  $C_j = S_i$ , then  $T_\beta(C_j) = C_0$  and  $S_\beta(C_j) = C_L$ .

Define the partition  $\mathcal{P}$  of  $\Omega \times [0, \frac{b_1}{\beta-1}]$  by

$$\mathcal{P} = \{\Omega \times C_j : j \in \cup_{k=0}^{b_1} M_k\} \cup \{\{\omega_1 = i\} \times S_j : i = 0, 1, j = 1, \dots, b_1\}$$

From **p4** and **p5** we conclude that  $\mathcal{P}$  is a Markov partition underlying the map  $K_\beta$ .

To define the underlying subshift of finite type associated with the map  $K_\beta$ , we consider the  $(L+1) \times (L+1)$  matrix  $A = (a_{i,j})$  with entries in  $\{0, 1\}$  defined by

$$(2) \quad a_{i,j} = \begin{cases} 1 & \text{if } i \in \cup_{k=0}^{b_1} M_k \text{ and } C_j \subset T_\beta(C_i), \\ 0 & \text{if } i \in \cup_{k=0}^{b_1} M_k \text{ and } \lambda(C_i \cap T_\beta^{-1}C_j) = 0, \\ 1 & \text{if } i \in \{0, \dots, L\} \setminus \cup_{k=0}^{b_1} M_k \text{ and } j = 0, L, \\ 0 & \text{if } i \in \{0, \dots, L\} \setminus \cup_{k=0}^{b_1} M_k \text{ and } j \neq 0, L. \end{cases}$$

**Remark 3.** Because of our assumption  $\beta^2 \neq b_1\beta + 1$ , we have that  $\lambda(C_i \cap T_\beta^{-1}C_j) = 0$  if and only if  $C_i \cap T_\beta^{-1}C_j = \emptyset$ . However, for the analysis of the case  $\beta^2 = b_1\beta + 1$ , we need the definition of the matrix  $A$ , as given in equation (2).

Let  $Y$  denote the topological Markov chain (or the subshift of finite type) determined by the matrix  $A$ . That is,  $Y = \{y = (y_i) \in \{0, 1, \dots, L\}^{\mathbb{N}} :$

$a_{y_i y_{i+1}} = 1\}$ . We let  $\sigma_Y$  be the left shift on  $Y$ . For ease of notation, we denote by  $s_1, s_2, \dots, s_{b_1}$  the states  $j \in L \setminus \bigcup_{k=0}^{b_1} M_k$  corresponding to the switch regions  $S_1, S_2, \dots, S_{b_1}$  respectively.

For each  $y \in Y$ , we associate a sequence  $(e_i) \in \{0, 1, \dots, b_1\}^{\mathbb{N}}$  and a point  $x \in \left[0, \frac{b_1}{\beta-1}\right]$  as follows. Let

$$(3) \quad e_j = \begin{cases} i & \text{if } y_j \in M_i, \\ i & \text{if } y_j = s_i \text{ and } y_{j+1} = 0, \\ i-1 & \text{if } y_j = s_i \text{ and } y_{j+1} = L. \end{cases}$$

Now set

$$(4) \quad x = \sum_{j=1}^{\infty} \frac{e_j}{\beta^j}.$$

Our aim is to define a map  $\psi : Y \rightarrow \Omega \times \left[0, \frac{b_1}{\beta-1}\right]$  that commutes the actions of  $K_\beta$  and  $\sigma_Y$ . Given  $y \in Y$ , equations (3) and (4) describe what the second coordinate of  $\psi$  should be. In order to be able to associate an  $\omega \in \Omega$ , one needs that  $y_i \in \{s_1, \dots, s_{b_1}\}$  infinitely often. For this reason it is not possible to define  $\psi$  on all of  $Y$ , but only on an invariant subset. To be more precise, let

$$Y' = \{y = (y_1, y_2, \dots) \in Y : y_i \in \{s_1, \dots, s_{b_1}\} \text{ for infinitely many } i\}.$$

Define  $\psi : Y' \rightarrow \Omega \times \left[0, \frac{b_1}{\beta-1}\right]$  as follows. Let  $y = (y_1, y_2, \dots) \in Y'$ , and define  $x$  as given in (4). To define a point  $\omega \in \Omega$  corresponding to  $y$ , we first locate the indices  $n_i = n_i(y)$  where the realization  $y$  of the Markov chain is in state  $s_r$  for some  $r \in \{1, \dots, b_1\}$ . That is, let  $n_1 < n_2 < \dots$  be the indices such that  $y_{n_i} = s_r$  for some  $r = 1, \dots, b_1$ .

Define

$$\omega_j = \begin{cases} 1 & \text{if } y_{n_{j+1}} = 0, \\ 0 & \text{if } y_{n_{j+1}} = L. \end{cases}$$

Now set  $\psi(y) = (\omega, x)$ .

The following two lemmas reflect the fact that the dynamics of  $K_\beta$  is essentially the same as that of the Markov chain  $Y$ . These lemmas are generalizations of the Lemmas 1 and 2 in [DK2], and the proofs are slight modifications of the arguments in there.

**Lemma 6.** *Let  $y \in Y'$  be such that  $\psi(y) = (\omega, x)$ . Then,*

- (i)  $y_1 = k$  for some  $k \in \bigcup_{i=0}^{b_1} M_i \Rightarrow x \in C_k$ .
- (ii)  $y_1 = s_i, y_2 = 0 \Rightarrow x \in S_i$  and  $\omega_1 = 1$  for  $i = 1, \dots, b_1$ .
- (iii)  $y_1 = s_i, y_2 = L \Rightarrow x \in S_i$  and  $\omega_1 = 0$  for  $i = 1, \dots, b_1$ .

**Lemma 7.** *For  $y \in Y'$ , we have*

$$\psi \circ \sigma_Y(y) = K_\beta \circ \psi(y).$$

**Remark 4.** From Lemmas 6 and 7 we have the following. If  $y \in Y'$  with  $\psi(y) = (\omega, x)$ , then for any  $i \geq 1$  and any  $k \in \{0, 1, \dots, L\}$

$$y_i = k \Rightarrow \pi_2(K_\beta^{i-1}(\omega, x)) \in C_k.$$

Having defined the map  $\psi$  with the above properties, we now consider the measure  $Q$  of maximal entropy on  $Y$ . This measure is unique since the adjacency matrix  $A = (a_{i,j})$ , as defined in (2), is irreducible [W, Theorem 8.10]. In order to describe  $Q$  explicitly, we first study the matrix  $A$ .

From the dynamics of  $K_\beta$  as well as properties **p1-p5** one easily sees that  $A$  has the following properties:

- (i)  $a_{i,j} = a_{L-i,L-j}$  for all  $i, j = 0, 1, \dots, L$ ,
- (ii)  $\sum_{i=0}^L a_{i,j} = b_1 + 1$  for all  $j = 0, 1, \dots, L$ .

By induction one can easily show that if  $A^k = (a_{i,j}^{(k)})$ , then  $A^k$  satisfies

- (iii)  $a_{i,j}^{(k)} = a_{L-i,L-j}^{(k)}$  for all  $i, j = 0, 1, \dots, L$ ,
- (iv)  $\sum_{i=0}^L a_{i,j}^{(k)} = (b_1 + 1)^k$  for all  $j = 0, 1, \dots, L$ .

Since  $A$  is an irreducible, nonnegative integral matrix, we calculate the topological entropy  $h(Y)$  of  $Y$  by the formula

$$h(Y) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |B_n(Y)|,$$

where  $B_n(Y)$  denotes the collection of blocks of length  $n$  in the shift space  $Y$ . According to property (v) above  $|B_n(Y)| = \sum_{i,j} a_{i,j}^{(n)} = (L+1)(b_1+1)^n$ . Hence  $h(Y) = \log(b_1+1)$ . It follows that the Perron eigenvalue  $\lambda_A$  equals  $b_1+1$  (i.e. the largest positive eigenvalue of the matrix  $A$ ). To determine the measure of maximal entropy we need to find a positive left eigenvector  $u$  and a positive right eigenvector  $v$ . According to property (ii) above a left eigenvector is given by  $u = (1, 1, \dots, 1)$ . For the right eigenvector, we choose a positive vector  $v$  satisfying  $\sum_{i=0}^L v_i = 1$ . Using the technique developed by Parry, the measure  $Q$  of maximal entropy is the Markov measure generated by the transition matrix  $P = (p_{i,j})$ , where  $p_{i,j} = a_{i,j} \frac{v_j}{(b_1+1)v_i}$ , and stationary distribution  $p = v$ . We equip the space  $Y$  with the  $\sigma$ -algebra  $\mathcal{G}$  generated by the cylinders. We have the following theorem.

**Theorem 7.** *The map*

$$\psi : (Y, \mathcal{G}, Q, \sigma_Y) \rightarrow \left( \Omega \times \left[ 0, \frac{b_1}{\beta-1} \right], \mathcal{A} \times \mathcal{B}, Q \circ \psi^{-1}, K_\beta \right)$$

*is a measurable isomorphism.*

**Proof.** First note that  $\psi$  is only defined almost surely. From Lemma 7 we have that  $\psi$  commutes the actions of  $K_\beta$  and  $\sigma_Y$ . Furthermore, it is easily checked that  $\psi : Y' \rightarrow Z$  is a bimeasurable bijection. The inverse  $\psi^{-1} : Z \rightarrow Y'$  is given by  $\psi^{-1}(\omega, x) = y$ , where  $y_i = k$  if  $\pi_2(K_\beta^{i-1}(\omega, x)) \in C_k$ .  $\square$

**Remark 5.** The proof of the above theorem shows that  $Q \circ \psi^{-1}$  is a  $K_\beta$ -invariant measure on  $\Omega \times \left[ 0, \frac{b_1}{\beta-1} \right]$  with support  $Z$ , and of maximal entropy  $\log(1 + \lfloor \beta \rfloor)$ . By Theorem 5 it follows that  $Q \circ \psi^{-1} = \nu_\beta$ . In Theorem 6, the projection of this measure on the second coordinate was identified as an infinite convolution of Bernoulli measures.

Let  $\pi_1 : \Omega \times [0, \lfloor \beta \rfloor / (\beta - 1)] \rightarrow \Omega$  be the canonical projection onto the first coordinate. Consider the measure  $Q' = \nu_\beta \circ \pi_1^{-1}$  on  $\Omega$ . Then,  $Q' = Q \circ \alpha^{-1}$ , where  $\alpha = \pi_1 \circ \psi : Y' \rightarrow \Omega$ .

**Theorem 8.** *The measure  $Q'$  is the uniform Bernoulli measure on  $\{0, 1\}^{\mathbb{N}}$ .*

**Proof.** Define the stopping times  $(T_i)_{i \geq 1}$  on  $Y'$  recursively as follows:

$$\begin{aligned} T_1 &= \min\{m \geq 2 : y_{m-1} \in \{s_1, \dots, s_{b_1}\}\}, \\ T_i &= \min\{m > T_{i-1} : y_{m-1} \in \{s_1, \dots, s_{b_1}\}\}, \quad i \geq 2. \end{aligned}$$

An application of the Strong Markov Property yields that the stopped process  $y_{T_1}, y_{T_2}, \dots$  is also a Markov chain with state space  $\{0, L\}$  and transition probabilities given by  $q_{ij} = \frac{1}{2}$  for  $i, j \in \{0, L\}$ . Therefore, if  $j_1, \dots, j_l \in \{0, L\}$ , then

$$Q(\{y_{T_1} = j_1, \dots, y_{T_l} = j_l\}) = \frac{1}{2^l}.$$

Define  $\chi : \{0, L\} \rightarrow \{0, 1\}$  by  $\chi(0) = 1, \chi(L) = 0$ . It follows that

$$Q'(\{\omega_1 = \chi(j_1), \dots, \omega_l = \chi(j_l)\}) = Q(\{y_{T_1} = j_1, \dots, y_{T_l} = j_l\}) = \frac{1}{2^l}.$$

□

**Remarks 6:**

(1) If 1 has a finite greedy expansion  $1 = b_1/\beta + \dots + b_n/\beta^n$  with some of the coefficients  $b_i$  equal zero, then one is able to find examples of such  $\beta$ 's where the map  $K_\beta$  has an underlying Markov partition similar to the one described above i.e, determined by the random orbits of 1 and  $\frac{b_1}{\beta-1} - 1$ . On the other hand, one is also able to find examples where  $K_\beta$  has no such Markov partition. For example, for  $n \geq 2$ , let  $\beta_n \in (1, 2)$  be the unique solution to the equation

$$\beta^n = \beta^{n-1} + 1,$$

then 1 has a greedy expansion  $1 = \frac{1}{\beta} + \frac{1}{\beta^n}$ . For  $n = 2, 3, 4, 5$ , it is not hard to see that  $K_\beta$  has a natural underlying Markov partition (one might need to divide the switch regions as well). However for  $n$  sufficiently large this is not the case. For in [EK] it was shown that for *each*  $\beta$ , sufficiently close to 1, there exists a sequence  $(\epsilon_i)$  of zeroes and ones satisfying  $\sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i} = 1$ , and containing all possible finite variations of the digits 0 and 1. Now, it is easy to check that  $\beta_n \downarrow 1$  as  $n \rightarrow \infty$ . Hence, if  $\beta_n$  is sufficiently close to 1, then by Theorem 1 there is an  $\omega \in \Omega$ , such that  $\epsilon_i = d_i(\omega, 1)$  for each  $i$ . Since each block of zeroes and ones appears in  $(d_i(\omega, 1))_{i \geq 1}$  this implies that

$$\overline{\{\pi_2(K_\beta^n(\omega, 1)) : n \geq 0\}} = [0, \frac{1}{\beta_n - 1}].$$

Hence, there is no underlying Markov partition (determined by the random orbits of 1 and  $\frac{1}{\beta_n-1} - 1$ ) for the map  $K_\beta$ .

Notice that  $\beta_5$  is the smallest Pisot number. One might conjecture that for  $\beta \in (1, \beta_5)$ , one cannot construct a Markov partition similar to the one described in this section.

(2) We consider now the case  $\beta^2 = b_1\beta + 1$ . In this case we let  $\mathcal{C} = \mathcal{E}$ , since 1 and  $\frac{b_1}{\beta-1} - 1$  are already endpoints of intervals in  $\mathcal{E}$ . For ease of notation, we denote the alphabet of  $Y$  by  $\{e_0, s_1, e_1, s_1, \dots, s_{b_1}, e_{b_1}\}$ . In this case we have for any  $1 \leq i \leq b_1$ ,

$$T_\beta(S_i) = \overline{E}_0 = [0, 1/\beta], \text{ and } S_\beta(S_i) = \overline{E}_{b_1} = [1, \frac{b_1}{\beta-1}].$$

As a result, Lemmas 6 and 7 do not hold for elements in  $Y'$  corresponding to endpoints of elements of  $\mathcal{E}$ . To be precise, we define for  $1 \leq i \leq b_1$ , the sequences  $x^{(i)}, y^{(i)}, q^{(i)}$  and  $r^{(i)}$  as follows.

- Let  $x^{(i)} = (s_i, e_{b_1}, s_1, e_{b_1}, s_1, \dots)$ , then  $\psi(x^{(i)}) = (\omega^{(0)}, \frac{i}{\beta})$  where  $\omega^{(0)} = (0, 0, 0, \dots)$ . We have that for  $m \geq 1$ ,  $x_{2m+1}^{(i)} = s_1$ , while for  $j \geq 2$ ,  $\pi_2(K_\beta^{(j)}(\omega^{(0)}, \frac{i}{\beta})) = \frac{b_1}{\beta-1}$ .
- Let  $y^{(i)} = (e_i, s_1, e_{b_1}, s_1, e_{b_1}, \dots)$ , then  $\psi(y^{(i)}) = (\omega^{(0)}, \frac{b_1}{\beta(\beta-1)} + \frac{i-1}{\beta})$ . We have that for  $m \geq 1$ ,  $y_{2m}^{(i)} = s_1$ , while for  $j \geq 1$ ,  $\pi_2(K_\beta^{(j)}(\omega^{(0)}, \frac{b_1}{\beta(\beta-1)} + \frac{i-1}{\beta})) = \frac{b_1}{\beta-1}$ .
- Let  $q^{(i)} = (e_{i-1}, s_{b_1}, e_0, s_{b_1}, e_0, \dots)$ , then  $\psi(q^{(i)}) = (\omega^{(1)}, \frac{i}{\beta})$  where  $\omega^{(1)} = (1, 1, 1, \dots)$ . We have that for  $m \geq 1$ ,  $q_{2m}^{(i)} = s_{b_1}$ , while for  $j \geq 1$ ,  $\pi_2(K_\beta^{(j)}(\omega^{(1)}, \frac{i}{\beta})) = 0$ .
- Let  $r^{(i)} = (s_i, e_0, s_{b_1}, e_0, s_{b_1}, \dots)$ , then  $\psi(r^{(i)}) = (\omega^{(1)}, \frac{b_1}{\beta(\beta-1)} + \frac{i-1}{\beta})$ . We have that for  $m \geq 1$ ,  $r_{2m+1}^{(i)} = s_{b_1}$ , while for  $j \geq 2$ ,  $\pi_2(K_\beta^{(j)}(\omega^{(1)}, \frac{b_1}{\beta(\beta-1)} + \frac{i-1}{\beta})) = 0$ .

Except for these points, the analysis used in the previous section remains valid. So, the only modification needed is the removal of a set of measure zero from the domain of  $Y'$ , namely all points whose orbit under  $\sigma_Y$  eventually equals  $x^{(i)}, y^{(i)}, q^{(i)}$  or  $r^{(i)}$  for some  $i = 1, 2, \dots, b_1$ .

(3) Suppose in the switch regions we decide to flip a biased coin, with  $0 < \mathbf{P}(\text{Heads}) = p < 1$ , in order to decide whether to use the greedy or the lazy map.

The measure of maximal entropy discussed in this section does not reflect this fact. A natural invariant measure that preserves this property is obtained by considering the Markov measure  $Q_\lambda$  on  $Y$  with transition probabilities  $p_{i,j}$  given by,

$$p_{ij} = \begin{cases} \lambda(C_i \cap T_\beta^{-1}C_j)/\lambda(C_i) & \text{if } i \in \cup_{k=0}^{b_1} M_k \\ p & \text{if } i \in \{0, 1, \dots, L\} \setminus \cup_{k=0}^{b_1} M_k \text{ and } j = 0, \\ 1 - p & \text{if } i \in \{0, 1, \dots, L\} \setminus \cup_{k=0}^{b_1} M_k \text{ and } j = L, \end{cases}$$

(here  $\lambda$  denotes Lebesgue measure), and initial distribution the corresponding stationary distribution (see [DK2]). Another interesting feature is that the projection of  $Q_\lambda \circ \psi^{-1}$  on the second coordinate for  $p = 1$  is the Parry measure  $\mu_\beta$ , and for  $p = 0$  is the lazy measure  $\rho_\beta$  (see section 1).



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