ERGODIC PROPERTIES OF SIGNED BINARY EXPANSIONS

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ABSTRACT. Every integer n has a unique *signed binary expansions* of the form

$$n = \sum_{i=0}^{k-1} a_i 2^i,$$

satisfying $a_i \times a_{i+1} = 0$. Based on the signed binary expansion a compactification K of $\mathbb{Z}$ is introduced. On K there are two natural operations: the classical shift map σ and the odometer τ, which extends to K the addition of 1. In this paper the dynamical properties of this set K are studied both under the shift σ and the odometer τ.

1. INTRODUCTION

1.1. **Signed binary expansions.** Given any non-zero integer n, one can expand n in infinitely many different sums —*'signed binary expansions'*— of the form

$$n = \sum_{i=0}^{k-1} a_i 2^i, \tag{1}$$

where the digits a_i are 0, 1 or -1 and $a_{k-1} \neq 0$. Motivated by automatic computing, Booth in [Boo] used signed binary expansions to introduce a new process for multiplying integers.
In general, a representation (1) is called *non-adjacent* or *separated* if

$$\forall\, i \in \mathbb{N}, \quad a_i \times a_{i+1} = 0\,, \tag{2}$$

where $\times$ represents the ordinary product (we set $a_i = 0$ for all indices $i \geq k$). Reitwiesner [R] showed, among other things, that such a representation (1) with constraints (2) is unique. We will call this representation the separated signed binary expansion of n, say SSB expansion of n for short, and set $SSB(n) = a_0 \cdots a_{k-1}$, using the classical notation for words of length k on the alphabet $\mathcal{A} = \{-1, 0, 1\}$.
It is proved in [Gü-Pa] that among all binary representations (1) of any integer n, constraints (2) lead to a representation which has the least number of nonzeros terms. In fact Güntzer and Paul in [Gü-Pa] introduced the notion of SSB expansion in connection to jump interpolation search trees. Another reason to study SSB expansions concerns the computation of a power x^n in a group where the inverse x^{-1} is easily computed; the exponentiation is then optimized by finding expansions (1) such that the *Hamming weight* $w(n) := \sum_{i=0}^{k-1} |a_i|$ is minimal. This strategy has been exploited in [Je-Mi], and has several applications in elliptic cryptosystems [Ko-Ts, Mo-Ol]. A recent detailed combinatorial study of this binary expansion can be found in [Bo].

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1.2. **Outline of this paper.** In Section 2, we first introduce the lazy algorithm which generates the digits a_i in (1) under the constraint (2) and yields a constructive proof of the existence and uniqueness of the separated signed binary expansion.
In the next two subsections we study further the classical transducer which writes the SSB expansion $a_0 \cdots a_{k-1}$ of any non negative integer n by reading its usual binary representation. We also introduce a greedy algorithm which computes directly from n its SSB digits a_i in reverse order. In Subsection 2.4 an *SSB-compactification* K of $\mathbb{Z}$ is introduced. In fact, using the notation of infinite words,

$$K = \{x_0x_1x_2 \cdots \in \{-1, 0, 1\}^{\mathbb{N}};\ \forall\, i \in \mathbb{N}, x_i \times x_{i+1} = 0\}.$$

On K there are two natural operations: the classical shift map σ and the odometer τ, which extends to K the addition of 1. In this paper the dynamical properties of this set K are studied both under the shift σ and the odometer τ. Due to (2) the shift (K, σ) is a topological mixing Markov chain; the well known Parry measure ν_K—giving the maximal entropy of (K, σ) (the topological one)—can be easily obtained: it is the Markov probability measure ν_K determined by the transition probability matrix

$$P := \begin{pmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{pmatrix},$$

with initial distribution $(1/6, 2/3, 1/6)$ (which is the stationary distribution). The shift can be identified with the map $S : [-2/3, 2/3) \to [-2/3, 2/3)$, piecewise linear on $[-2/3, -1/3)$, $[-1/3, 1/3)$ and $[1/3, 2/3)$ and given by $S(x) = 2x \pmod{\mathbb{Z}}$. The ergodic properties of S are studied in Subsection 2.4.

In Section 3 the operation of adding 1 to an integer, using the SSB expansion, is extended to K in a natural manner, defining a $\mathbb{Z}$-action τ called the SSB-odometer. It is then shown that the topological dynamical system (K, τ) is uniquely ergodic and homeomorphic to the usual dyadic odometer. The τ-invariant measure on K can be viewed as a Markov measure μ which is nothing but the pull-back of the Haar measure after identifying K with the compact group of dyadic integers $\mathbb{Z}_2$. The measure μ has also transition matrix P, but now with initial distribution the vector $(1/4, 1/2, 1/4)$.

In Section 4, we study additive block functions $f : \mathbb{N}$ (or $\mathbb{Z}$) $\to \mathbb{R}$, which are natural generalizations of the classical *sum–of–digit function* $s(n)$, or the Hamming weight $w(n)$. We show that the function Δf given by $\Delta f(n) = f(n+1) - f(n)$ can be extended to a well-defined and continuous function on $K \setminus \{\alpha, \beta\}$, where $\alpha = (010101\ldots)$, and $\beta = (101010\ldots)$. This extension is also denoted by Δf. To study properties of block functions we use the notion of a cocycle with respect to τ. More precisely, for any map $F : K \to A$, where A is any abelian group, we consider the cocycle associated with F. This is the map

$$(n, x) \mapsto F(n, x) \qquad (3)$$

defined on $\mathbb{Z} \times K$ by $F(0, x) = 0_A$, $F(n, x) = \sum_{k=0}^{n-1} F(\tau^k x)$, and $F(-n, x) = \sum_{k=1}^{n} F(\tau^{-k} x)$ for $n \geq 1$. Note that $F(1, x) = F(x)$ and for m, $n \in \mathbb{Z}$, one has $F(n+m, x) = F(n, x) + F(m, \tau^n x)$, which is called the *cocycle identity*. We use the same notation for a map and its associated cocycle, the distinction will be clear from the context. In case of a block function f, the cocyle associated to Δf with respect to τ will be simply called the Δ-cocycle of f.

We show that Δf can be continuously extended if and only if Δf is a τ-coboundary, *i.e.*, $\Delta f = g \circ \tau - g$ ($\mu - a.e.$) for a Borel map g. If Δf is not continuous, then the skew product transformation $(x, g) \mapsto (\tau x, g + \Delta f(x))$ defined on $K \times G$ is ergodic, here G is the closed subgroup of $\mathbb{R}$ generated by $\{\Delta(f + f^*)(n) : n \in \mathbb{Z}\}$, where f^* is a suitable τ-coboundary related to f (see Subsection 4.2). We end this section with some examples, and a straightforward generalization by replacing $\mathbb{R}$ with any locally compact metrizable abelian group. As a by-product, it is shown that the Hamming weight function and the classical sum-of-digits function are statistically independent.

In the final section, K is equipped with the stationary Markov measure ν_p given by the transition matrix

$$\Pi(p) = \begin{pmatrix} 0 & 1 & 0 \\ p_{-1} & p_0 & p_1 \\ 0 & 1 & 0 \end{pmatrix},$$

with $p_{-1}p_0p_1 \neq 0$, and it is proved that there is no σ-finite τ-invariant measure equivalent to ν_p, unless $p = (1/4, 1/2, 1/4)$.

2. The signed separated binary (SSB) representation

2.1. **The lazy algorithm.** A natural way to produce expansions (1) with a minimal number of non zero digits is to compute consecutive digits $c_0, c_1, \ldots$ using the following strategy.

- If n is even, replace n by $T(n) = n/2$ and set $c_0 = 0$;
- If n is odd, choose $c_0 \in \{-1, 1\}$ such that $n \equiv c_0 (\operatorname{mod} 4)$, and replace n by $T(n) = \frac{n - c_0}{2}$.

This leads to the maps $c : \mathbb{Z} \to \{-1, 0, 1\}$ and $T : \mathbb{Z} \to \mathbb{Z}$ given by

$$c(n) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ 1 & \text{if } n \equiv 1 \,(\operatorname{mod} 4), \\ -1 & \text{if } n \equiv -1 \,(\operatorname{mod} 4), \end{cases}$$

and

$$T(n) := \frac{n - c(n)}{2}.$$

Due to these definitions, one has

$$n = c(n) + 2T(n),$$

and by iterating

$$n = c_0(n) + c_1(n)2 + \cdots + c_{k-1}(n)2^{k-1} + T^k(n)2^k, \tag{4}$$

where $c_j(n) = c(T^j(n))$. By convention, $T^0(n) = n$. It is easy to see that for $n \in \mathbb{Z} \setminus \{0\}$ one has $|T(n)| < |n|$. Since $T(0) = 0$, there exists for any $n \in \mathbb{Z}$ a unique non-negative integer h (called the *height* of n) such that $T^h(n) = 0$, but $T^j(n) \neq 0$ for $0 \leq j \leq h - 1$, where we set the height of 0 equal to 0. With these definitions, for $n \neq 0$ of height h one has for $i \in \{0, 1, \ldots, h - 1\}$

$$c_i \in \{-1, 0, 1\}, \qquad c_i \neq 0 \;\Rightarrow\; c_{i+1} = 0,$$

and

$$c_{h-1} = \operatorname{sgn}(n),$$

where $\mathrm{sgn}(n) = n/|n|$ if $n \neq 0$, and $\mathrm{sgn}(0) = 0$. Hence, the map T describes the (lazy) algorithm to compute the SSB expansion of any rational integer n, and if h is the height of $n \neq 0$, formula (4) with $k = h$ will be identified with the following infinite string on the alphabet $\{-1, 0, 1\}$:

$$J(n) = c_0(n) \cdots c_{h-1}(n) 0^\omega,$$

where in general v^ω denotes the infinite string obtain by concatenating the word v with itself infinitely often. In case $n = 0$, we simply have $J(0) = 0^\omega$.

2.2. **From the usual binary expansion to the SSB one.** A simple way to transform any usual binary expansion $n = \sum_{i=0}^{k-1} e_i 2^i$ (represented by the infinite binary string $e_0 \cdots e_{k-1} 0^\omega$) into the corresponding SSB expansion has been given in [Gü-Pa] (see also [Prod]), using rewriting rules which lead to a simple transducer given in Figure 1:

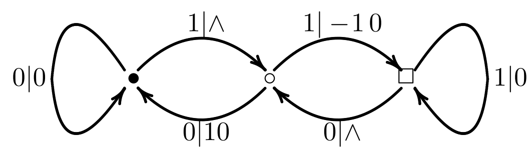

Figure 1. The SSB transducer T_{SSB}

The transducer, denoted by T_{SSB}, reads the successive inputs $e_0, e_1, e_2, \ldots$ starting from the black state $\bullet$, following a unique labelled path $\bullet \xrightarrow{e_1} s_1 \ldots \xrightarrow{e_k} s_k \ldots$ and writes at the same time the output string of blocks $w_1 \ldots w_k \ldots$, where each labelled arc $s_{j-1} \xrightarrow{e_j} s_j$ corresponds to a block w_j such that $e_j | w_j$ is the label of a unique arc from x_{j-1} to x_j. Here and after, we have set $w_j = \wedge$ to point out that w_j is the empty block $\wedge$. Notice that after the input e_{k-1}, (and reading at least 2 more digits which are both 0) the path goes back to the black state where it stays forever.
In the sequel, given any binary string (finite or infinite) $b = b_0 \cdots b_{k-1} \cdots$ we shall denote by $T_{SSB}(b)$ the output string one gets when the input b is read by the transducer.
For example, 91 corresponds to the binary string 11011010^ω which gives the sequence of states $(\bullet \circ \Box \circ \Box\Box \circ \Box \circ \bullet \bullet \bullet \ldots)$ and the output sequence (writing from left to right) $\wedge(-10) \wedge (-10)0 \wedge (-10) \wedge (10)000\ldots$ corresponding to the SSB expansion $91 = -1 - 4 - 32 + 128$. Hence $T_{SSB}(1101101) = (-)10(-1)00(-1)0$ but $T_{SSB}(11011010^\omega) = (-1)0(-1)00(-1)010^\omega$.

From the transducer, we easily see that if $2^{k-1} \leq n < 2^k$, the *height* of n is k or $k+1$. We are interested to find all integers of the same height. To this end, let us introduce for any $h \in \mathbb{N}$ the following integers:

$$d_h = \begin{cases} \dfrac{2 \cdot 2^h - 1}{3} & \text{if } h \text{ is odd } (= 2m+1), \\[2ex] \dfrac{2 \cdot 2^h - 2}{3} & \text{if } h \text{ is even } (= 2m), \end{cases}$$

and denote by $h(n)$ the height of n. Notice that $d_0 = 0$.

Proposition 1. *For $h \in \mathbb{N}$ one has*

(a) $T(d_{h+1}) = d_h$, *and for* $n \neq 0$

$$h(n) = h \quad \Leftrightarrow \quad |n| \in \{d_{h-1}+1, d_{h-1}+2, \ldots, d_h\}.$$

(b) *The signed dyadic expansion of* d_h *is given by*

$$J(d_h) = \begin{cases} (10)^m 10^\omega & \text{if } h = 2m+1, \\ (01)^m 0^\omega & \text{if } h = 2m. \end{cases}$$

Proof. Property (b) is an immediate consequence of the following two observations

$$d_{2m+1} = \frac{4^{m+1}-1}{3} = 1 + 4 + \cdots + 4^m = 2^0 + 22 + 2^4 + \cdots + 2^{2m},$$

and

$$d_{2m} = 2\left(\frac{4^m - 1}{3}\right) = 2(1 + \cdots + 4^{m-1}) = 2^1 + 2^3 + \cdots + 2^{2m-1}.$$

To prove property (a), let $n = c_0 + c_1 \cdot 2 + \cdots + c_{h-1}2^{h-1}$ be the SSB expansion of n. Suppose that $c_{h-1} = 1$, so that if $h = 2m+1$ we get

$$n \leq 1 + 4 + \cdots + 4^m = \frac{4^{m+1}-1}{3} = d_{2m+1} = d_h$$

and

$$n \geq 4^m - 4^{m-1} - \cdots - 1 = 4^m - \frac{4^m - 1}{3} = \frac{2 \cdot 4^m - 2}{3} + 1 = d_{h-1} + 1.$$

Now, for $h = 2m$ ($m > 0$) we get

$$\begin{aligned} n &\leq 0 + 2 + 0 + 2 \cdot 4 + \cdots + 2 \cdot 4^{m-1} \\ &= 2(1 + 4 + \cdots + 4^{m-1}) = 2 \cdot \frac{4^m - 1}{3} = d_{2m} = d_h, \end{aligned}$$

and

$$\begin{aligned} n &\geq 2 \cdot 4^{m-1} - 2 \cdot 4^{m-2} - \cdots - 2 \\ &= 2 \cdot 4^{m-1} - 2\left(\frac{4^{m-1}-1}{3}\right) \\ &= \frac{4^m - 1}{3} + 1 = d_{2m-1} + 1 = d_{h-1} + 1. \end{aligned}$$

It remains to show that if $n \in \{d_{h-1}+1, \ldots, d_h\}$, then $h(n) = h$. By considering the signed binary expansion of such numbers n, the above estimate shows that $h(n) = h$. The case where $c_{h-1} = -1$ is similar. □

2.3. **A greedy algorithm.** For a given integer $n \neq 0$, let h be the height of n such that $d_{h-1} + 1 \leq |n| \leq d_h$. Proposition 1 and a straightforward computation leads to

$$J(d_{h-1}+1) = \begin{cases} ((-1)0)^m 10^\omega & \text{if } h = 2m+1, \\ (0(-1))^{m-1} 010^\omega & \text{if } h = 2m, \end{cases}$$

and furthermore

$$\frac{d_{h-1} + d_h}{2} = 2^{h-1} - \frac{1}{2}.$$

This shows that the sequence d_h can be recursively defined. Notice that

$$\{\{0\}, [-d_h, -d_{h-1}), (d_{h-1}, d_h]; h \geq 1\}$$

is a partition of $\mathbb{Z}$ such that each atom consists of all integers of the same height and parity. Hence, if n is given then the height is easily determined. Suppose n has height h, then the SSB expansion of n has the form $n = c_0 + c_1 \cdot 2 + \cdots + c_{h-1} \cdot 2^{h-1}$, with $c_{h-1} = \mathrm{sgn}(n)$. Let $G_1(n) = n - \mathrm{sgn}(n) \cdot 2^{h-1}$, then $G_1(n) = c_0 + c_1 \cdot 2 + \cdots + c_{h-2} \cdot 2^{h-2}$ and $c_{h-2} = \mathrm{sgn}(G_1(n))$. Continuing in this manner, just by knowing n one can calculate its SSB digits in reverse order.
This suggests the following greedy algorithm to calculate the SSB digits of n.

SSB-Greedy Algorithm (n)
Input n (rational integer);
Output $(S(n), H(n), G(n))$.
begin
$h := 0$;
`if` $n \neq 0$ `then`
  `begin`
    `repeat` $h := h + 1$ `until` $d_{h-1} < |n| \leq d_h$;
    $S :=$ `sign`$(n))$,
    $H := h - 1$,
    $G := n -$ `sign`$(n) 2^{h-1}$;
  `endbegin;`
`else` $S := 0$, $H := 0$, $G := 0$;
`endif;`
endbegin.

By iterating the SSB-Greedy Algorithm successively with the first input $n = n_0 \neq 0$ we derive the sequence

$$(S_0, H_0, G_1), \ldots, (S_{\ell-1}, H_{\ell-1}, G_\ell) \tag{5}$$

with $S_0 = S(n)$ $(= \mathrm{sign}(n))$, $H_0 = H(n)$ $(= h(n) - 1)$, $G_1 = G(n)$ and after r iterations $(1 \leq r \leq \ell)$, $S_{r-1} = S(G^{r-1}(n))$, $H_{r-1} = H(G^{r-1}(n))$ $(= h(G^{r-1}(n)) - 1)$ and $G_r = G^r(n)$. Moreover, ℓ is the least positive integer m such that $G^m(n) = 0$, hence $G_\ell = 0$. If we start with $n = 0$, the SSB-Greedy Algorithm gives the output $(0, 0, 0)$ $(\ell = 1)$. According to (4), the sequence (5) leads to the SSB expansion

$$n = \sum_{0 \leq i < \ell} S_i 2^{H_i} .$$

2.4. **SSB-expansions of numbers in [-2/3, 2/3).** Consider the following set K of infinite strings on the alphabet $\mathcal{A} = \{-1, 0, 1\}$:

$$K := \left\{ x_0 x_1 x_2 \cdots \in \mathcal{A}^{\mathbb{N}};\ \forall i \geq 0, x_i \neq 0 \Rightarrow x_{i+1} = 0 \right\} .$$

Clearly K is a nonempty compact subset of $\mathcal{A}^{\mathbb{N}}$ endowed with the weak topology, so two points are close if they agree on a sufficiently large beginning block. The set K can be viewed as a straightforward compactification of $\mathbb{Z}$ in connection with the separated signed binary expansion.
To each word $a_0 \cdots a_{k-1} \in \mathcal{A}^k$ of length k, we associate the so-called cylinder set

$$[a_0 \cdots a_{k-1}] = \{x \in \mathcal{A}^{\mathbb{N}};\ x_0 \cdots x_{k_1} = a_0 \cdots a_{k-1}\}.$$

We denote by W_k the set of words $a_0 \cdots a_{k-1} \in \mathcal{A}^k$ such that $[a_0 \cdots a_{k-1}] \cap K \neq \emptyset$ or equivalently $a_1 \times a_{i+1} = 0$ for all i satisfying $0 \leq i < k - 1$. Such words are said to be admissible. For the sake of simplicity, the notation $[a_0 \cdots a_{k-1}]$ will be also used to denote a cylinder set in K, where $a_0 \cdots a_{k-1} \in W_k$.

Let σ be the usual shift map on $\mathcal{A}^{\mathbb{N}}$, then $\sigma(K) = K$. In this section we will study the shift σ on K. To do so, we first identify K, up to a countable set, with an interval of $\mathbb{R}$. By considering the usual binary expansion of real numbers, it is natural to introduce the map $\Psi : K \to \mathbb{R}$ given by

$$\Psi(x) = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \,. \tag{6}$$

Lemma 1. *The map Ψ has the following properties.*

(i) The map Ψ is continuous.

(ii) $\Psi(K) = [-\frac{2}{3}, \frac{2}{3}]$, $\Psi([-1]) = [-\frac{2}{3}, -\frac{1}{3}]$, $\Psi([0]) = [-\frac{1}{3}, \frac{1}{3}]$, $\Psi([1]) = [\frac{1}{3}, \frac{2}{3}]$.

(iii) For any $a_0 \cdots a_{k-1} \in W_k$ with $k \geq 2$, if $a_{k-1} \neq 0$

$$\Psi([a_0 \cdots a_{k-1}]) = [\Psi(a_0 \cdots a_{k-1} 0^\omega) - \frac{1}{3.2^k}, \Psi(a_0 \cdots a_{k-1} 0^\omega) + \frac{1}{3.2^k}], \tag{7}$$

and if $a_{k-1} = 0$

$$\Psi([a_0 \cdots a_{k-2} 0]) = [\Psi(a_0 \cdots a_{k-2} 0^\omega) - \frac{1}{3.2^{k-1}}, \Psi(a_0 \cdots a_{k-2} 0^\omega) + \frac{1}{3.2^{k-1}}]. \tag{8}$$

Proof. Clearly Ψ is continuous, and $\min \Psi(K) = \Psi((-10)^\omega) = -\frac{2}{3}$, $\max \Psi(K) = \Psi((10)^\omega) = \frac{2}{3}$. For each admissible word $a = a_0 \cdots a_{h-1}$ let $\tilde{a} = a_{h-1} \cdots a_0$ be the reverse word of a. Clearly $\tilde{a}$ is also admissible. Using Proposition 1, one see that for any integer $n \in [-d_h, d_h] \cap \mathbb{Z}$ the height is at most h, so there exists $a_0 \cdots a_{h-1} \in W_h$ such that $J(n) = \tilde{a} 0^\omega$ and consequently $\Psi(a_0 \cdots a_{h-1} 0^\omega) = \frac{n}{2^h}$. In particular,

$$\Psi(W_h 0^\omega) = \{\frac{n}{2^h} \, ; \, -d_h \leq n \leq d_h\}$$

and

$$-\frac{2}{3} + \frac{3 + (-1)^h}{6.2^h} \leq \Psi(a_0 \cdots a_{h-1} 0^\omega) \leq \frac{2}{3} - \frac{3 + (-1)^h}{6.2^h}.$$

Since $\bigcup_{h \in \mathbb{N}} \Psi(W_h 0^\omega)$ is a dense subset of $[-\frac{2}{3}, \frac{2}{3}]$, by continuity of Ψ it follows that $\Psi(K) = [-\frac{2}{3}, \frac{2}{3}]$. Now, since $\min \Psi([0]) = \Psi((0(-1))^\omega) = -\frac{1}{3}$ and $\max \Psi([0]) = \Psi((01)^\omega) = \frac{1}{3}$, the above shows that $\Psi([0]) = [-\frac{1}{3}, \frac{1}{3}]$. Similarly for the other two cases of (ii).

To show (iii), consider any $y = a_0 \ldots a_{k-1} x_0 x_1 \cdots \in K$. Then $x = x_0 x_1 \ldots$ belongs to K, and

$$\Psi(y) = \Psi(a_0 \ldots a_{k-1} 0^\omega) + \frac{1}{2^k} \Psi(x).$$

If $a_{k-1} \neq 0$, then x is any element of $[0]$. Using (ii), one has

$$\begin{aligned} \Psi([a_0 ... a_{k-1}]) &= [\Psi(a_0 ... a_{k-1} (0(-1))^\omega), \Psi(a_0 ... a_{k-1} (01)^\omega)] \\ &= \Psi(a_0 ... a_{k-1} 0^\omega) + \frac{1}{2^k} [-\frac{1}{3}, \frac{1}{3}], \end{aligned}$$

i.e., (7) follows. On the other hand, if $a_{k-1} = 0$, then x can be any element of K, hence

$$\begin{aligned} \Psi([a_0 ... a_{k-2} 0]) &= [\Psi(a_0 ... a_{k-2} (0(-1))^\omega), \Psi(a_0 ... a_{k-2} (01)^\omega)] \\ &= \Psi(a_0 ... a_{k-2} 0^\omega) + \frac{1}{2^{k-1}} [-\frac{1}{3}, \frac{1}{3}], \end{aligned}$$

and (8) follows. □

Lemma 1 shows that every $t \in [-\frac{2}{3}, \frac{2}{3}]$ has a *separated signed binary expansion* (SSB) of the form

$$t = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}}, \tag{9}$$

where $x_k \in \{-1, 0, 1\}$, and $x_k \times x_{k+1} = 0$ $(k \geq 0))$, *i.e.*, $x = x_0 x_1 x_2 \cdots \in K$ and $\Psi(x) = t$. From the proof of the above lemma, it is straightforward to see that for each $k \geq 1$, and any two distinct elements $a_0 \dots a_{k-1}$, $b_0 \dots b_{k-1}$ in W_k, the intervals $\Psi[a_0 \dots a_{k-1}]$ and $\Psi[b_0 \dots b_{k-1}]$ are either disjoint or intersect at one of the endpoints. These endpoints have exactly two SSB-expansions, one ending with the infinite string $(01)^\omega$, the other ending with $(0(-1))^\omega$. More precisely,

$$\Psi(a_0 \cdots a_{k-1}(0(-1))^\omega) = \begin{cases} \Psi(a_0 \cdots a_{k-2} 0(-1)(01)^\omega) \text{ if } a_{k-1} = 0, \\ \Psi(a_0 \cdots a_{k-2}(01)(01)^\omega) \text{ if } a_{k-1} = 1, \end{cases} \quad (a_{k-1} \neq -1).$$

Notice that the endpoints of the intervals $\Psi[a_0 ... a_{k-1}]$, except for $-2/3$ and $2/3$, have exactly two SBB-expansions.
There is a dynamical way of directly generating an SSB-expansion of points $[-\frac{2}{3}, \frac{2}{3})$. Consider the map $S : [-\frac{2}{3}, \frac{2}{3}) \to [-\frac{2}{3}, \frac{2}{3})$, defined by

$$S(t) = 2t - a(t), \tag{10}$$

where

$$a(t) = \begin{cases} -1 & \text{if } -2/3 \leq t < -1/3 \\ 0 & \text{if } -1/3 \leq t < 1/3 \\ 1 & \text{if } \quad 1/3 \leq t < 2/3, \end{cases} \tag{11}$$

see also Figure 2. Setting $a_k = a_k(t) := a\left(S^k(t)\right)$, for $k \geq 0$, it follows from (10)

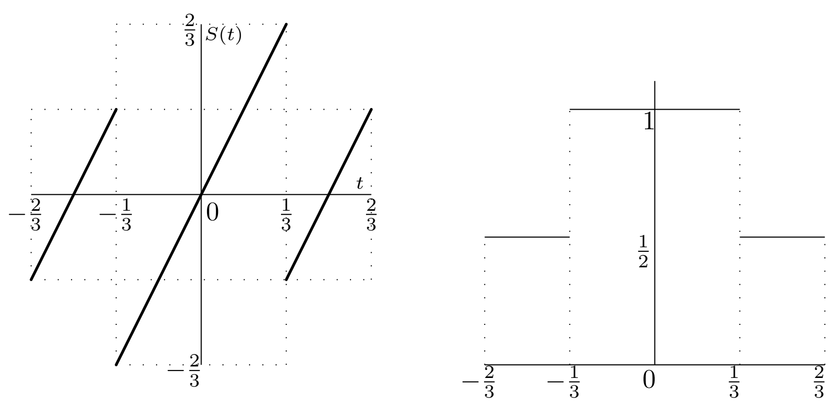

FIGURE 2. Graph of S and density ρ

that

$$\begin{aligned} t &= \frac{a_0}{2} + \frac{1}{2}S(t) = \frac{a_0}{2} + \frac{1}{2}\left(\frac{a_1}{2} + \frac{1}{2}S^2(t)\right) \\ &= \cdots = \frac{a_0}{2} + \frac{a_1}{4} + \cdots + \frac{a_k}{2^{k+1}} + \frac{1}{2^{k+1}}S^{k+1}(t). \end{aligned}$$

Since $\left|S^k(t)\right| \leq \frac{2}{3}$, the infinite series $\sum_{k=0}^{\infty} a_k/2^{k+1}$ converges to t.
In the expansion

$$t = \sum_{k=0}^{\infty} a_k/2^{k+1} \tag{12}$$

obtained from S, the digits a_k satisfy (2). We call this expansion the *canonical* SSB-expansion of t, referred to as the CSSB-expansion of t. For example

$$\frac{1}{3} = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{1}{2.4^k}.$$

Let I_a be the interval $[-2/3,-1/3)$, $[-1/3,1/3)$ or $[1/3,2/3)$ according to whether $a=-1$, 0 or $+1$, and more generally, for any $a_0\cdots a_{k-1} \in \mathcal{A}^k$, set

$$I_{a_0\cdots a_{k-1}} = \bigcap_{0\leq j<k} S^{-j} I_{a_j}.$$

By construction, $I_{a_0\cdots a_{k-1}} \neq \emptyset$ if and only if $a_0\cdots a_{k-1} \in W_k$, and in that case, $I_{a_0\cdots a_{k-1}}$ is the set of t in $[-2/3,2/3)$ such that $S^j(t) \in I_{a_j}$ (and so $a_j(t)=a_j$) for $0 \leq j < k$. Using Lemma 1 we can easily prove

Lemma 2. *For any $x = x_0x_1x_2\cdots \in K$ and any $k \geq 1$,*

$$\Psi([x_0\cdots x_{k-1}]) = \overline{I_{x_0\cdots x_{k-1}}}$$

The above shows that the SSB-expansion of points in $[-\frac{2}{3},\frac{2}{3})$ generated by S is the one which does not end with the infinite string $(01)^\omega$. In fact S has one cycle of order 2, namely $\{-\frac{2}{3},-\frac{1}{3}\}$. A point $t \in (-\frac{2}{3},\frac{2}{3})$ has two SSB-expansions if and only if there exists an integer $k \geq 0$ such that $S^k(t) = -\frac{1}{3}$. Further, the CSSB-expension of t is the one which ends with $(0(-1))^\omega$. The other points t have an unique SBB-expansion which is canonical. In particular $-2/3 = \Psi(((-1)0)^\omega)$, but $2/3 = \Psi(10^\omega)$, this SSB-expansion being unique and not canonical.

We introduce the set K_0 of infinite strings in K which do not end with $(01)^\omega$ and summarize the above results in the first part of following proposition; the second part is a simple fact and is left to the reader.

Proposition 2. *The map Ψ is Borel measurable, one-to-one and onto the interval $[-2/3,2/3)$. For any $t \in [-2/3,2/3)$, the CSSB-expansion* (12) *of t corresponds to the unique point $x \in K_0$ such that $\Psi(x) = t$. Moreover, extending S at $\frac{2}{3}$ by $S(\frac{2}{3}) = \frac{1}{3}$, then Ψ is a conjugation between σ and S, namely*

$$\Psi\circ\sigma = S\circ\Psi\,.$$

The map S is non-singular with respect to λ, where λ denotes normalized Lebesgue measure on $[-\frac{2}{3},\frac{2}{3})$. In order to find an S-invariant probability measure of density ρ with respect to λ on $([-2/3,2/3),\mathcal{B})$ (where $\mathcal{B}$ denotes the σ-algebra of Borel sets of $[-2/3,2/3)$), we introduce (a version of) the natural extension of S. Let

$$\Omega = \left([-\tfrac{2}{3},-\tfrac{1}{3})\times[-\tfrac{1}{3},\tfrac{1}{3}]\right) \cup \left([-\tfrac{1}{3},\tfrac{1}{3})\times[-\tfrac{2}{3},\tfrac{2}{3}]\right) \cup \left([\tfrac{1}{3},\tfrac{2}{3})\times[-\tfrac{1}{3},\tfrac{1}{3}]\right),$$

and let $\tilde{S} : \Omega \to \Omega$ be defined by

$$\tilde{S}(x,y) = \left(S(x), \frac{a(x)}{2} + \frac{y}{2} \right),$$

see Figure 3. Note that on Ω the normalized Lebesgue measure λ_Ω is $\mathcal{S}$-invariant. It

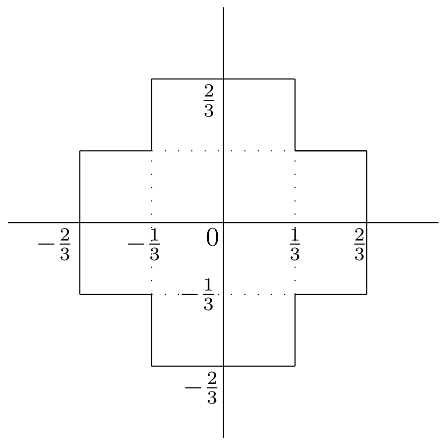

FIGURE 3. Domain of definition of the Natural Extension $\tilde{S}$

is easy to show that the dynamical system $(\Omega, \mathcal{B}_\Omega, \lambda_\Omega, \tilde{S})$, where $\mathcal{B}_\Omega$ is the collection of Borel sets of Ω, is isomorphic to a two-sided mixing Markov chain with state space $\{-1, 0, 1\}$, transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{pmatrix},$$

and initial distribution the stationary distribution $(\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$. This implies that $(\Omega, \mathcal{B}_\Omega, \lambda_\Omega, \tilde{S})$ is isomorphic to a Bernoulli shift. Projecting on the first coordinate, yields that

$$([-2/3, 2/3), \mathcal{B}, \nu, S)$$

is a mixing Markov chain, where ν is the first marginal probability measure on $[-2/3, 2/3)$. A straightforward computation shows that ν has density function $\rho(x) = \frac{d\nu}{d\lambda}(x)$ given by

$$\rho(x) = \begin{cases} 1/2 & \text{if } -2/3 \le x < -1/3 \\ 1 & \text{if } -1/3 \le x < 1/3 \\ 1/2 & \text{if } \quad 1/3 \le x < 2/3, \end{cases}$$

see also Figure 2. From this we see that the measure $\bar{\nu}$ given by $\bar{\nu}(A) = \nu(\Psi(A))$ is S-invariant. On the other hand, (K, σ) is a topological Markov chain with state space $\{-1, 0, 1\}$, and adjacency matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

This adjacency matrix is irreducible, and $\bar{\nu}$ is the unique shift-invariant measure of maximal entropy (the Parry measure).

3. The separated Binary odometer

3.1. **Construction and dynamical properties.** Using the SSB expansion of integers, we would like to extend the operation of adding 1 on $\mathbb{Z}$ to a map $\tau : K \to K$. To this end, recall that we have embedded $\mathbb{Z}$ into K via the map

$$J\,:\,n \mapsto J(n) = c_0 c_1 \cdots c_{k-1} 0^\omega,$$

where $c_{k-1} = \mathrm{sgn}(n)$ and $n = c_0 + c_1 \cdot 2 + \cdots + c_{k-1} \cdot 2^{k-1}$ is the SSB expansion of n. Given $x \in K$, we set for any $m \in \mathbb{N}$:

$$x[m] \,=\, x_0 + x_1 2 + \cdots + x_{m-1} 2^{m-1}.$$

A natural way to extend the operation of adding 1's to K is to set

$$\tau(x) \,:=\, \lim_{m\to\infty} J\left(x[m] + 1\right). \tag{13}$$

If x corresponds to an integer n, we have $\tau(J(n)) = J(n+1)$, and the limit in (13) exists. However, it is not obvious that this limit exists for any $x \in K$.

We have the following theorem.

Theorem 1. *The limit in* (13) *always exists and defines an homeomorphism* $\tau : K \to K$.

Proof. We study the definition of τ in detail. For any $x = x_0 x_1 x_2 \cdots \in K$, using $2^k + 2^{k+1} = -2^k + 2^{k+2}$ and $2^k - 2^{k+1} = -2^k$, we get

$$\tau(x) = \begin{cases} 00x_2x_3\cdots & \text{if } x = (-1)0x_2x_3\ldots, \\ (-10)^m 10 x_{2m+2}x_{2m+3}\cdots & \text{if } x = (01)^m 00 x_{2m+2}x_{2m+3}\cdots, \\ (-10)^m(-1)0x_{2m+2}\cdots & \text{if } x = (01)^m 0(-1)x_{2m+2}\cdots, \\ (0(-1))^m 010 x_{2m+3}\cdots & \text{if } x = (10)^m 100 x_{2m+3}\cdots, \\ (0(-1))^m 0(-1)0x_{2m+3}\cdots & \text{if } x = (10)^m 10(-1)x_{2m+3}\cdots \end{cases} \tag{14}$$

And finally

$$\tau((01)^\omega) = ((-1)0)^\omega, \quad \tau((10)^\omega) = (0(-1))^\omega,$$

proving that τ is well defined by (13) and moreover, if $x_0 x_1 \cdots x_k = y_0 y_1 \cdots y_k$ ($k \geq 2$) then

$$\tau(x)_0 \tau(x)_1 \cdots \tau(x)_{k-2} = \tau(y)_0 \tau(y)_1 \cdots \tau(y)_{k-2}$$

implying that τ is continuous. The bijectivity of τ can be obtained easily from formula (14) (or from the next proof) and continuity of τ^{-1} follows from compactness of K. □

We want to show that (K, τ) is homeomorphic to the translation by 1 on the compact group $\mathbb{Z}_2$ of 2-adic integers. To this end we define $\mathbb{Z}_2$ as the limit of the inverse system $\{c_{m,n} : \mathbb{Z}/2^m \to \mathbb{Z}/2^n\}$ where $m \geq n$ and $c_{m,n}$ are the canonical maps.
We also identify $\mathbb{Z}_2$ to the compact infinite product space $\{0,1\}^{\mathbb{N}}$ by means of the classical dyadic Hensel expansion.

Lemma 3. *For all integers* $m \geq 1$ *the map* $\varphi_m : K \to \mathbb{Z}/2^m\mathbb{Z}$ *defined by*

$$\varphi_m(x) = x[m] \pmod{2^m}$$

is a continuous epimorphism from (K, τ) *to* $(\mathbb{Z}/2^m\mathbb{Z}, \tau_m)$.

Proof. We denote by τ_m the translation $\xi \mapsto \xi + 1$ on the group $\mathbb{Z}/2^m\mathbb{Z}$. It is clear that φ_m is a continuous map from K onto $\mathbb{Z}/2^m\mathbb{Z}$. Let x be an element of K distinct of $(01)^\omega$ or $(10)^\omega$. By the above construction (see (14)), there exists an integer $N \geq 0$ such that for all $n \geq N$, $\tau(x)[n] = x[n] + 1$. Reducing mod 2^m $(m < n)$, we get $\varphi_m(\tau(x)) = \tau_m(\varphi_m(x))$. If $x = (01)^\omega$, a straightforward computation shows $x[2n] + 1 = \tau(x)[2n] + 2^{2n}$ and $x[2n+1] + 1 = \tau(x)[2n+1] - 2^{2n+1}$. The other case is similar and consequently $\varphi_m \circ \tau = \tau_m \circ \varphi_m$ in all cases. □

From Lemma 3 the family of epimorphismes φ_m defines a continuous epimorphism (a factor map) $\varphi : K \to \mathbb{Z}_2$ such that

$$\varphi(\tau(x)) = \varphi(x) + 1$$

for all $x \in K$.

Theorem 2. *(K, τ) is homeomorphic by φ to the translation by 1 on the compact group $\mathbb{Z}_2$. In particular (K, τ) is uniquely ergodic with invariant probability measure $\mu = \mu_2 \circ \varphi$ where μ_2 is the standard Haar measure on $\mathbb{Z}_2$.*

Proof. It remains to show that φ is injective. Let $x \neq y$ in K. There exists $n \geq 0$ such that $x_i = y_i$ for all indices $i < n$ but $x_n \neq y_n$. If $x_n = 0$ or $y_n = 0$ then $\varphi_{n+1}(x) \neq \varphi_{n+1}(y)$. Otherwise $x_n = -y_n$ $(\neq 0)$. In that case $\varphi_{n+1}(x) = \varphi_{n+1}(y)$ but $x_{n+1} = y_{n+1} = 0$ so that $\varphi_{n+2}(x) \neq \varphi_{n+2}(y)$. Hence $\varphi(x) \neq \varphi(y)$. □

Corollary 1. *The inverse map $\varphi^{-1} : \mathbb{Z}_2 \to K$ can be computed by the transducer T_{SSB}:*

$$\forall\, y \in \mathbb{Z}_2\,;\ \varphi^{-1}(y) = T_{SSB}(y).$$

Proof. Both φ^{-1} and $T_{SSB}(\cdot)$ are continuous and coincide on all infinite binary strings corresponding to binary expansions of natural numbers. □

Remark. For any integers k and n, $n \geq 1$, set

$$\psi_{k,n}(x) \;=\; e^{2\pi i \frac{k \cdot x[n+1]}{2^n}},$$

and observe that for all $\ell \geq n + 1$:

$$\psi_{k,n}(x) \;=\; e^{2\pi i \frac{k \cdot x[\ell]}{2^n}}.$$

As an easy consequence of the definition of τ we have

$$\psi_{k,n}(\tau x) \;=\; e^{2\pi i \frac{k}{2^n}}\, \psi_{k,n}(x), \tag{15}$$

showing that the maps $\psi_{k,n}$ are continuous eigenfunctions for τ with corresponding eigenvalues $e^{2\pi i k/2^n}$ and by Theorem 2, all eigenfunctions of the dynamical system (K, τ, μ) are given by the family $\psi_{k,n}$.

3.2. **Computation of the Haar measure viewed on K.** In the previous section we have shown that there exists a continuous factor map $\varphi : K \to \mathbb{Z}_2$ such that $\varphi(\tau x) = \varphi(x) + 1$. On $\mathbb{Z}_2$ we have the Haar measure μ_2, which is the $(1/2, 1/2)$-Bernoulli measure,

and on K we consider the measure $\mu = \mu_2 \circ \varphi$. We show that μ is a Markov measure on K. From the transducer of Section 2.2 (see Figure 1), one sees that

$$\varphi([0]) = [0], \quad \varphi([-1]) = \varphi([-10] = [11] \quad \text{and} \quad \varphi([1]) = \varphi([10]) = [10].$$

Therefore

$$\mu([0]) = 1/2, \quad \mu([-1]) = \mu([-10]) = 1/4 \quad \text{and} \quad \mu([1]) = \mu([10]) = 1/4.$$

Now, for any cylinder $[c_0c_1\cdots c_n]$ in K, let

$$B(c_0,\dots,c_n)=\varphi([c_0\cdots c_n])=\{y\in\mathbb{Z}_2:\varphi^{-1}(y)_i=c_i, i=0,1,\dots,n\}$$

so that $\mu([c_0\cdots c_n])=\mu_2(B(c_0,\dots,c_n))$. We want to find a relationship between $\mu_2(B(c_0,\dots,c_n))$ and $\mu_2(B(c_0,\dots,c_n,a))$ where $c_0\cdots c_n a\in E_{n+1}$.
First we claim that $B(c_0,\dots,c_n)$ is always a cylinder set $[y_0\cdots y_m]$ in $\mathbb{Z}_2$ with

$$c_n=0\ \Rightarrow\ n=m. \tag{16}$$

From above $B(0)=[0]$, $B(1)=B(1,0)=[10]$, $B(-1)=B(-1,0)=[11]$ and also $B(0,0)=[00]$. Assume that $B(c_0,\dots,c_n)=[y_0\cdots y_m]$ $(n\geq 2)$. By construction $T_{SSB}([y_0\cdots y_m])=[c_0\cdots c_n]$ and we can assert that the state s where the transducer is staying after reading the word $y_0\cdots y_m$ belongs to $\{\bullet,\square\}$ (otherwise the output is already $c_0\cdots c_n$ after reading $y_0\cdots y_{m-1}$ and then $[y_0\cdots y_{m-1}]\subset B(c_0,\dots,c_n)$, a contradiction).
Now observe that if $c_n\in\{-1,1\}$, then $a=0$. This implies that $B(c_0,\dots,c_n)=B(c_0,\dots,c_n,0)=[y_0\cdots y_m]$ and by recurrence assumption, one gets $n+1=m$. Finally, for $c_n=0$, considering all possible cases depending on the value of s, one has

$$B(c_0,\dots,c_{n-1},0,0)=\begin{cases}[y_0\dots y_m0] & \text{if}\quad s=\bullet\\[2ex] [y_0\dots y_m1] & \text{if}\quad s=\square\end{cases}$$

$$B(c_0,\dots,c_{n-1},0,1)=\begin{cases}[y_0\dots y_m10] & \text{if}\quad s=\bullet\\[2ex] [y_0\dots y_m00] & \text{if}\quad s=\square\end{cases}$$

and

$$B(c_0,\dots,c_{n-1},0,-1)=\begin{cases}[y_0\dots y_m11] & \text{if}\quad s=\bullet\\[2ex] [y_0\dots y_m01] & \text{if}\quad s=\square.\end{cases}$$

This ends the prove of our claim by induction on n and in addition, shows that

$$\mu([c_0\dots c_{n-1}00])=\frac{1}{2}\mu([c_0\dots c_{n-1}0])$$
$$\mu([c_0\dots c_{n-1}01])=\frac{1}{4}\mu([c_0\dots c_{n-1}0])$$
$$\mu([c_0\dots c_{n-1}0(-1)])=\frac{1}{4}\mu([c_0\dots c_{n-1}0]).$$

We have proved the following theorem.

Theorem 3. *Let $\mathcal{M}$ be the Markov chain with state space $\mathcal{A}:=\{-1,0,1\}$ and probability transition matrix (with index-set $\mathcal{A}$)*

$$P:=\begin{pmatrix}0 & 1 & 0\\ 1/4 & 1/2 & 1/4\\ 0 & 1 & 0\end{pmatrix}.$$

Then μ $(=\mu_2\circ\varphi)$ is the Markov measure which corresponds to the initial probability vector $p:=(1/4,1/2,1/4)$. In other words

$$\mu([c_0\cdots c_n])=p_{c_0}\prod_{i=0}^{n-1}P_{c_ic_{i+1}} \tag{17}$$

Remark. The Markov chain $\mathcal{M}$ is irreducible and aperiodic with stationary distribution ν_K given by initial probability vector $\pi := (1/6, 2/3, 1/6)$. Hence μ is not the stationary measure. Since ν_K is also defined by formula (17) after replacing p_{c_0} by π_{c_0}, we readily get

$$\frac{d\nu_K}{d\mu}(x) = \begin{cases} 4/3 & \text{if } x \in [0] \\ 2/3 & \text{otherwise.} \end{cases}$$

Passing to the interval $[-2/3, 2/3]$ with Ψ one gets

$$\mu \circ \Psi = \lambda_{[-2/3,2/3]}$$

where λ is the normalized Lebesgue measure on $[-2/3, 2/3)$. Let $T : [-2/3, 2/3) \to [-2/3, 2/3)$ be the map defined by $T(t) = \tau(x)$ if x corresponds to the SSB-expansion of t, so that $\Psi \circ \tau = T \circ \Psi$ and T is right continuous. From (14) one sees that T is piecewise linear. More precisely,

$$T(t) = \begin{cases} t + \dfrac{1}{2} & \text{if } t \in [\frac{-2}{3}, \frac{-1}{3}), \\ t - 1 + \dfrac{3}{2^{2m+1}} & \text{if } t \in [\frac{1}{3} - \frac{1}{2^{2m+1}}, \frac{1}{3} - \frac{1}{3.2^{2m+1}}), \\ t - 1 + \dfrac{3}{2^{2m+2}} & \text{if } t \in [\frac{1}{3} - \frac{1}{3.2^{2m-1}}, \frac{1}{3} - \frac{1}{4.2^{2m-1}}), \\ t - 1 + \dfrac{3}{2^{2m+2}} & \text{if } t \in [\frac{2}{3} - \frac{1}{2^{2m+2}}, \frac{2}{3} - \frac{1}{3.2^{2m+2}}), \\ t - 1 + \dfrac{3}{2^{2m+3}} & \text{if } t \in [\frac{2}{3} - \frac{1}{3.2^{2m}}, \frac{2}{3} - \frac{1}{4.2^{2m}}), \end{cases}$$

and in particular $T(1/3) = \Psi(\tau(10((-1)0)^\omega) = -7/24$, $\lim_{t \to 2/3} T(t) = -1/3$. The map T is similar to the adding-machine on $[0, 1)$ introduced by von Neumann in [voN], which corresponds to the usual dyadic expansion. Figure 4 shows partially the graph of T on the first five intervals ($m = 0$):

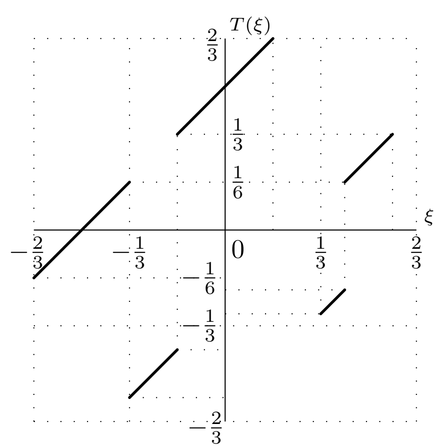

FIGURE 4. Partial graph of T

4. Block functions and associated cocycles

It is an old arithmetical tradition to study the statistical and harmonic properties of the sum-of-digits function

$$s(n) = \sum_j c_i(n).$$

and the Hamming weight $w(n)$

$$w(n) = \sum_j |c_i(n)|.$$

These are examples of additive 1-block maps, defined in general as follows.

Definition 1. *A map $f : \mathbb{Z} \to \mathbb{R}$ is said to be an additive k-block map if there exists a map $\tilde{f} : \{-1, 0, 1\}^k \to \mathbb{R}$, such that $\tilde{f}(0^k) = 0$ and for any integer $n \in \mathbb{Z}$, with SSB digits $c_j(n)$ given by* (4)*, one has*

$$f(n) = \sum_{j \geq 0} \tilde{f}(c_j \cdots c_{j+k-1}). \tag{18}$$

4.1. Additive k-block maps.

For any function $f : \mathbb{Z} \to \mathbb{R}$, let

$$\Delta f(n) = f(n+1) - f(n).$$

In this section we are seeking functions $f : \mathbb{Z} \to \mathbb{R}$ such that for $x \in K$ the limit

$$\lim_{n\to\infty} \Delta f(x[n])$$

exists almost everywhere according to the Haar measure μ on K. The family of additive k-block maps defined above has this property.
One can extend $\tilde{f}$ in Definition 4.1 to a function on K, by simply setting $\tilde{f}(x) = \tilde{f}(x_0 \cdots x_{k-1})$, hence

$$f(n) = \sum_{j \geq 0} \tilde{f}(\sigma^j J(n)).$$

Note that, for $x \notin \{(01)^\omega, (10)^\omega\}$, and n sufficiently large, one has

$$(\tau x)[n] = x[n] + 1.$$

Hence, if f is an additive k-block map on $\mathbb{Z}$, then

$$\Delta f(x) = \lim_{n\to\infty} \big(f((\tau x)[n]) - f(x[n])\big)$$

is well-defined for $x \in K \setminus \{(01)^\omega, (10)^\omega\}$; in fact, $\Delta f(x)$ can be written as a finite sum. To see this, let $m(x)$ be the least integer m such that $\tau(x)_j = x_j$ if $j > m$. Then

$$\forall\, \ell \geq m(x), \quad \Delta f(x) = \sum_{i=0}^{\ell} (\tilde{f}(\sigma^i \tau x) - \tilde{f}(\sigma^i x)). \tag{19}$$

Theorem 4. *Let f be an additive k-block map. Then Δf is continuous at all points $x \in K \setminus \{(01)^\omega, (10)^\omega\}$ and*

$$\int_K \Delta f(x) d\mu(x) = 0\,. \tag{20}$$

Proof. Using the fact that the function $m(\cdot)$ has the constant value $s = m(x)$ on the cylinder set $[x_0 \cdots x_s]$, the first part of the theorem follows from the continuity of τ, σ, $\tilde{f}$ (extended to K) and (19). For any integer $M \geq 1$ let Ω_M the set of $x \in K$ such that $m(x) \leq K$. Obviously $\Omega_M \subset \Omega_{M'}$ if $M < M'$ and from (14), Ω_M is a finite union of cylinder sets such that $\bigcup_{M\geq 1} \Omega_M = K \setminus \{(01)^\omega, (10)^\omega\}$. Put $g_M(x) = \sum_{0\leq i\leq M}(\tilde{f}(\sigma^i\tau x) - \tilde{f}(\sigma^i x))$. The measure μ being invariant under τ, one gets $\int_K g_M(x)d\mu(x) = 0$. Now, the sequence of maps $g_M(\cdot)$ (which are continuous on K) converges simply to Δf on $K \setminus \{(01)^\omega, (10)^\omega\}$. In fact, $g_M(x) = \Delta f(x)$ as soon as $x \in \Omega_L$ (L fixed) with $M \geq L$. For all $x \in (K \setminus \{(01)^\omega, (10)^\omega\})$, the inequality

$$|\Delta f(x) - g_M(x)| \leq 2||\tilde{f}||_\infty (m(x) - M)\mathbf{1}_{K\setminus\Omega_M}(x)$$

holds, and consequently, it is enough to prove that the Borel map $m(\cdot)$ is μ−integrable, the Lebesgue dominated convergence theorem doing the rest for establishing (20). From Theorem 3 and constraints (2), the inequality

$$\mu([c_0 \cdots c_{n-1}] \leq \frac{1}{2^n}$$

follows for any $c_0 \cdots c_{n-1} \in W_n$. Formula (14) giving τ, shows that $m(x) = 1$ on $[-1]$, $m(x) = 2s + 1$ on $A_s = [(01)^s 00]$, $m(x) = 2s + 2$ on $B_s = [(01)^s 0(-1)]$ and $B'_s = [(10)^s 100]$, and finally $m(x) = 2s + 3$ on $C_s = [(10)^s 10(-1)]$. But

$$K \setminus \{(01)^\omega, (10)^\omega\} = [-1] \cup \bigcup_{s\geq 0}(A_s \cup B_s \cup B'_s \cup C_s),$$

this union being disjoint. That implies

$$\begin{aligned}
\int_K m(x)dx = {} & \frac{1}{4} + \sum_{s=0}^{\infty}(2s+1)\mu(A_s) \\
& + \sum_{s=0}^{\infty}(2s+2)(\mu(B_s) + \mu(B'_s)) \\
& + \sum_{s=0}^{\infty}(2s+3)\mu(C_s) \\
\leq {} & \frac{1}{4} + \sum_{s=0}^{\infty}(2s+1)2^{-2s-2} \\
& + \sum_{s=0}^{\infty}(2s+2)(2^{-2s-2} + 2^{-2s-3}) \\
& + \sum_{s=0}^{\infty}(2s+3)2^{-2s-3} \\
< {} & +\infty,
\end{aligned}$$

as expected. □

We now investigate the case where Δf can be extended to a continuous function on K.

Theorem 5. *Let $f : \mathbb{Z} \to \mathbb{R}$ be an additive k-block map, and suppose that Δf can be continuously extended to the points $\alpha := (01)^\omega$ and $\beta := (10)^\omega$. Then*

$$\tilde{f}(\alpha) + \tilde{f}(\beta) = \tilde{f}(\tau\alpha) + \tilde{f}(\tau\beta). \tag{21}$$

Further, the continuous extension of Δf, still denoted by Δf, depends on at most the first $2k$ coordinates, and

$$\Delta f(x) = \begin{cases} \Delta f(\alpha) & \text{for } x \in [(01)^{\lfloor \frac{k+1}{2} \rfloor}], \\ \Delta f(\beta) & \text{for } x \in [(10)^{\lfloor \frac{k+1}{2} \rfloor}]. \end{cases}$$

In particular $(\Delta f) \circ \tau^{2^{2k+1}} = \Delta f$. Finally,

$$\begin{aligned} \Delta f(\alpha) &= \tilde{f}(\tau\alpha) - \tilde{f}(\alpha) + \Delta f(\beta), \\ \Delta f(\beta) &= \tilde{f}(\tau\beta) - \tilde{f}(\beta) + \Delta f(\alpha). \end{aligned}$$

Proof. Notice that for $m \geq 1$ and $x \in [(01)^m] \cup [(10)^m]$, formula (14) implies the important relation $\tau(\sigma(x) = \sigma(\tau(x))$ Consequently, for any $m > \lfloor \frac{k+1}{2} \rfloor$ and any sequence $c_0c_1c_2 \cdots \in K$, we have

$$\begin{aligned} \Delta f((01)^m 0 c_0 c_1 c_2 \cdots) &= \tilde{f}(\tau\alpha) - \tilde{f}(\alpha) + \Delta f\big((10)^m c_0 c_1 c_2 \cdots\big) \\ &= \tilde{f}(\tau\alpha) - \tilde{f}(\alpha) + \tilde{f}(\tau\beta) - \tilde{f}(\beta) \\ &\qquad + \Delta f\left((01)^{m-1} 0 c_0 c_1 c_2 \cdots\right). \end{aligned} \tag{22}$$

Suppose Δf is continuous at α and β. Taking limits as $m \to \infty$, and using the continuity of Δf at α, one gets (21). After $m - \lfloor \frac{k+1}{2} \rfloor$ iterations of (22), and using (21), we find that

$$\Delta f\big((01)^m 0 c_0 c_1 \dots\big) = \Delta f\big((01)^{\lfloor \frac{k+1}{2} \rfloor} 0 c_0 c_1 \dots\big).$$

Similarly,

$$\Delta f\big((10)^m c_0 c_1 \dots\big) = \Delta f\big((10)^{\lfloor \frac{k+1}{2} \rfloor} c_0 c_1 \dots\big).$$

Taking the limit as $m \to \infty$, one gets for any sequence $c_0c_1c_2 \cdots \in K$,

$$\Delta f(\alpha) = \Delta f\big((01)^{\lfloor \frac{k+1}{2} \rfloor} 0 c_0 c_1 \dots\big)$$

and

$$\Delta f(\beta) = \Delta f\big((10)^{\lfloor \frac{k+1}{2} \rfloor} c_0 c_1 \dots\big).$$

This shows that on $[(01)^{\lfloor \frac{k+1}{2} \rfloor}]$, Δf has the constant value $\Delta f(\alpha)$ and on $[(10)^{\lfloor \frac{k+1}{2} \rfloor}]$ it has the constant value $\Delta f(\beta)$. For $2p < k$, we now consider Δf on each cylinder of the form $[(01)^p 00]$, $[(01)^p 0(-1)]$, $[(10)^p 0]$ and $[(10)^p(-1)]$. In view of (14), it appears that Δf depends on at most the first $2p+1+k$ coordinates, but $2p+1+k$ is at most $2k$ for k odd, and at most $2k-1$ if k is even. On the cylinders $[00]$ and $[(-1)0]$, Δf depends only on the first k coordinates and finally, on $[0(-1)]$, it depends on at most the first $k+1$ coordinates.

Since for any $n \geq 2$ and $x \in K$, $\left(\tau^{2^n} x\right)_i = x_i$ for $0 \leq i \leq n-2$, one has that Δf is invariant under $\tau^{2^{2k+1}}$ if k is odd, and invariant under $\tau^{2^{2k}}$ if k is even.

Finally, from

$$\Delta f((01)^m 0 c_0 c_1 \dots) = \tilde{f}(\tau\alpha) - \tilde{f}(\alpha) + \Delta f((01)^m c_0 c_1 \dots)$$

and

$$\Delta f((10)^m 0 c_0 c_1 \dots) = \tilde{f}(\tau\alpha) - \tilde{f}(\alpha) + \Delta f((01)^{m-1} 0 c_0 c_1 \dots),$$

one has by taking limits that

$$\Delta f(\alpha) = \tilde{f}(\tau\alpha) - \tilde{f}(\alpha) + \Delta f(\beta)$$

and

$$\Delta f(\beta) = \tilde{f}(\tau\beta) - \tilde{f}(\beta) + \Delta f(\alpha).$$

□

Lemma 4. *Let $F : K \to \mathbb{R}$ in $L^1(K, \mu)$ such that $\int_K F(x)d\mu(x) = 0$ and assume there exists an integer $s > 0$ such that*

$$F \circ \tau^{2^s} = F \quad (\mu - a.e.)$$

(in particular F depends only on the first $s+1$ variables). Then, there exists a continuous map $g : K \to \mathbb{R}$ such that $g \circ \tau^{2^s} = g$ and

$$F = g \circ \tau - \tau \quad (\mu - a.e.).$$

Proof. Let $g(x) = F(2^s, x) = \sum_{i=0}^{2^s-1} F(\tau^i x)$, then $g(x) = g(\tau x)$. Hence by ergodicity of τ, the function g must be a constant a.e. with respect to μ. But

$$\int_K g(x)d\mu(x) = 2^s \int_K F(x)d\mu(x) = 0.$$

Hence, $g = 0$ a.e. with respect to μ. Let $h(x) = \frac{1}{2^s} \sum_{p=0}^{2^s-1} F(p, x)$. Using the cocycle identity, and the fact that $F(2^s, x) = 0$, one has

$$h(\tau x) = h(x) - F(x).$$

Hence F is a coboundary. Since τ is continuous, and for each $0 \le p \le 2^s - 1$ the function $F(p, x)$ is continuous, it follows that h is continuous. Further, by the cocycle identity, the invariance of F under τ^{2^s}, and the fact that $F(2^s, x) = 0$, we have $h(x) = h(\tau^{2^s} x)$. Hence, h depends on at most the first $s+1$ coordinates. This proves the lemma. □

We remark that by using character theory one can give another short proof of the above Lemma. For any non trivial character $\psi_{k,n}$, the invariance of μ under τ gives the relation

$$(1 - e^{2\pi i k 2^{s-n}}) \int_K F\overline{\psi_{k,n}} d\mu = 0\,,$$

which implies $\int_K F\overline{\psi_{k,n}} d\mu = 0$ if $n > s$. Consequently, F is a linear combination of characters $\psi_{k,n}$ with $0 < n \le s$. If $n > 0$ (and $0 \le k < 2^n$) then, $\psi_{k,n} = \frac{1}{e^{2\pi i k 2^{-n}} - 1}(\psi_{k,n} \circ \tau - \psi_{k,n})$. Therefore F has the form $g \circ \tau - g$ where $g : K \to \mathbb{R}$ is continuous and satisfies $g \circ \tau^{2^s} = g$, as required.

Corollary 2. *Let $f : \mathbb{Z} \to \mathbb{R}$ be an additive k-block map, and suppose that Δf can be continuously extended to K. There exists a continuous map $g : K \to \mathbb{R}$ depending on at most the first $2k+1$ coordinates, such that*

$$\Delta f = g \circ \tau - g.$$

Proof. The result follows from Lemma 4, Theorem 4 and Theorem 5. □

Remark. For $k = 1$, a straightforward computation shows that if Δf can be extend by continuity, then $\tilde{f}$ is constant (and $\Delta f = 0$). For $k = 2$, relations *(i)* and *(ii)* allow us to determine explicitly $\tilde{f}$. Up to an additive constant, $\tilde{f}$ is given by the following table where a and b are any real numbers. Notice that in that case, $(\Delta f) \circ \tau^4 = \Delta f$. (See Table 4.1)

c_0c_1	$\tilde{f}(c_0c_1)$	$\Delta f(c_0c_1\cdots)$
0 0	0	$-b$
$(-1)0$	a	$-a$
$0(-1)$	$-a$	a
0 1	b	a
1 0	$-b$	b

TABLE 1. Continuous Δ-cocycles generated by 2-block maps

4.2. **Skew product with Δ-cocycles.** Let f be an additive k-block map generated by the function $\tilde{f}$. If f cannot be continuously extended at α and β, we set $\Delta f(\alpha) = \Delta f\left((01)^k 0^\omega\right)$ and $\Delta f(\beta) = \Delta f\left((10)^k 0^\omega\right)$. Define the map $f^* : K \to \mathbb{R}$ as follows

$$f^*(x) = \tilde{f}(0x_0x_1\ldots x_{k-2}) + \tilde{f}(00x_0x_1\ldots x_{k-3}) + \cdots + \tilde{f}(0^{k-1}x_0).$$

If $\tilde{f}$ is a 1-block map, then $f^* = 0$. Set $\Delta f^*(x) = f^*(\tau x) - f^*(x)$, and let $G = G(\tilde{f})$ be the closed subgroup of $\mathbb{R}$ generated by $\{\Delta(f+f^*)(n) : n \in \mathbb{Z}\}$. Let λ_G be the Haar measure on G, and consider the skew product $\tau_{\tilde{f}} : K \times G \to K \times G$, defined by

$$\tau_{\tilde{f}}(x,\gamma) = (\tau x, \Delta(f+f^*)(x) + \gamma)\,.$$

It is easy to see that $\tau_{\tilde{f}}$ is measure preserving w.r.t. the product measure $\mu \otimes \lambda_G$. Using the cocycle notation, notice that

$$\tau_{\tilde{f}}^n(x,\gamma) = (\tau^n x, \Delta(f+f^*)(n,x) + \gamma).$$

We are interested in characterizing additive Δ-cocycles such that $\tau_{\tilde{f}}$ is ergodic. Using the tools developed by K. Schmidt, the question of ergodicity of $\tau_{\tilde{f}}$ can be completely answered by studying the set of essential values $E(\Delta(f+f^*))$ of $\Delta(f+f^*)$. The set $E(\Delta(f+f^*))$ consists of all elements $\gamma \in G$ with the property that for every neighborhood V_γ of γ and for every Borel subset B of K of positive μ-measure

$$\mu\Big(\bigcup_{n\in\mathbb{Z}} B \cap \tau^{-n}B \cap \big\{x \in K : \Delta(f+f^*)(n,x) \in V_\gamma\big\}\Big) > 0.$$

Notice that since Δf^* is a τ-coboundary, we have

$$E(\Delta(f)) = E(\Delta(f+f^*)),$$

where $E(\Delta(f))$ is the set of essential values of Δf.

Remark. For any $z \in K$, any $n \geq 0$ and $h \geq 1$, since the first k-coordinates of z and $\tau^{2^{h+k}} z$ are the same,

$$\Delta f^*(2^{h+k}, x) = \sum_{i=0}^{2^{h+k}-1} \Delta f^*(\tau^i z) = f^*(\tau^{2^{h+k}} z) - f^*(z) = 0.$$

Lemma 5. *For any k-block function f, $E(\tilde{f}) = G(\tilde{f})$. Moreover, Δf can be extended continuously if and only if $G(\tilde{f}) = \{0\}$.*

Proof. Let γ be any value of $\Delta(f+f^*)$. Clearly, if we prove that $\gamma \in E(\Delta(f+f^*))$, then $E(\Delta(f+f^*)) = G$. To this end, let e be an integer with SSB-expansion $e = e_0 \dots e_\ell 0^\omega$ such that $\gamma = \Delta(f+f^*)(e)$. For any non empty cylinder $C = [x_0 \cdots x_{h-1}]$ in K, consider the cylinder

$$C_e = \{y \in C\,, y_h \cdots y_{h+k-1} = 0^k,\ y_{h+k} \cdots y_{h+2k+\ell+1} = e_0 \dots e_\ell 0^{k+1}\}.$$

Then, $\mu(C_e) > 0$. Since $\tau^{-2^{h+1}} C = C$, we see that $C_e \subset C \cap \tau^{-2^{h+k}} C = C$. Now, for any $y \in C_e$, we have $\Delta f^*(2^{h+k}, x) = 0$. Thus,

$$\begin{aligned}
\Delta(f+f^*)(n,y) &= \sum_{i=0}^{2^{h+k}-1} \Delta f(\tau^i y) \\
&= \sum_{i=0}^{m_{2^{h+k}}(z)} \tilde{f}\ (\sigma^i \tau^{2^{h+k}} y) - \tilde{f}(\sigma^i y),
\end{aligned}$$

where $m_{2^{h+k}}(z) = \max\Big(m(z), \dots, m(\tau^{2^{h+k}-1}(z))\Big)$. By definition of C_e, one has $m_{2^{h+k}}(z) \le h+k+l+2$, and since the first $h+k$ coordinates of z and $\tau^{2^{h+k}}(z)$ are the same, one has

$$\begin{aligned}
\Delta(f+f^*)(2^{h+k}, y) &= \sum_{i=h+1}^{h+k+\ell+2} \tilde{f}(\sigma^i \tau^{2^{h+k}} y) - \tilde{f}(\sigma^i y) \\
&= \sum_{i=1}^{k-1} \tilde{f}(0^{k-i} \tau e) - \tilde{f}(0^{k-i} e) \\
&\quad + \sum_{i=0}^{\ell+2} \tilde{f}(\sigma^i \tau e) - \tilde{f}(\sigma^i e) \\
&= \Delta f^*(e) + \Delta f(e) = \gamma.
\end{aligned}$$

Let ρ_γ be the map defined on the Borel σ-algebra $\mathcal{B}$ of K by

$$\rho_\gamma(B) = \mu\Big(\bigcup_{n\in\mathbb{Z}} \tau^{-n}(B) \cap B \cap \{x \in K\ :\ \Delta(f+f^*)(n,x) = \gamma\}\Big)$$

for $B \in \mathcal{B}$. Then ρ_γ is subadditive, and the above computations show that for all cylinder sets C,

$$\rho_\gamma(C) \ge \mu(C_e) \ge \left(\frac{1}{4}\right)^{2k+\ell+2} \mu(C).$$

Hence this inequality also holds for all finite disjoint unions of cylinders, i.e., for all elements of the generating algebra $\mathcal{B}_0$ of $\mathcal{B}$. Setting

$$\mathcal{C} = \{B \in \mathcal{B} :\ \rho_\gamma(B) \ge (\tfrac{1}{4})^{2k+\ell+2} \mu(B)\},$$

then it is easily checked that $\mathcal{C}$ is a monotone class containing $\mathcal{B}_0$, hence $\mathcal{C} = \mathcal{B}$. Therefore γ belongs to $E(\Delta(f+f^*)$.

Now suppose that Δf is continuous on K, then by Corollary 2 there is a continuous map $g : K \to \mathbb{R}$ such that

$$\Delta f(x) = g(\tau x) - g(x).$$

Thus Δf and $\Delta(f+f^*)$ are coboundaries, implying that $E(\Delta(f+f^*)) = E(\Delta f) = \{0\}$ (see [Sch]) and from the first part of the lemma $G(\tilde{f}) = \{0\}$. Conversely, if $G(\tilde{f}) = \{0\}$ then $\Delta f = -\Delta f^*$. Hence Δf is continuous. □

The following theorem is a simple corollary of the above lemma, see [Sch].

Theorem 6. *If Δf is not continuous, then the skew product $(K \times G, \tau_{\tilde{f}}, \mu \otimes \lambda_G)$ is not trivial (i.e., $G \neq \{0\}$) and ergodic.*

Remark. Since $\Delta f^* = g \circ \tau - g$, with g continuous, $I_g : K \times G \to K \times \mathbb{R}$ defined by $I_g(x, \gamma) = (x, \gamma - g(x))$ is a continuous metrical isomorphism between $(X \times G, \mu \otimes \lambda_G, \tau_{\tilde{f}})$ and $(X \times \mathbb{R}, (\mu \otimes \lambda_G) \circ I_g^{-1}, \tau'_{\tilde{f}})$ where $\tau'_{\tilde{f}}$ is given by

$$\tau'_{\tilde{f}}(x, \gamma) = (\tau x, \Delta f(x) + \gamma)$$

and the support of $(\mu \otimes \lambda_G) \circ I_g^{-1}$ is exactly $I_g(K \times G)$. Recall that g is constant on cylinder sets of length $k+1$.

4.3. **Examples.** **(1)** For additive 1-block maps f, one has $f^* = 0$ and by the above remark $\tau_{\tilde{f}}$ is ergodic if and only if $\tilde{f}$ is not constant. If $r, s \in R$ and

$$\tilde{f}(x) = \begin{cases} r & \text{if } x_0 = 1 \\ s & \text{if } x_0 = -1 \\ 0 & \text{if } x_0 = 0, \end{cases}$$

then $G(\tilde{f}) = E(\Delta f) = \overline{r\mathbb{Z} + s\mathbb{Z}}$. Hence, all possible closed subgroups of $\mathbb{R}$ can be realized by the group $G(\tilde{f})$. For instance, if $r = s \neq 0$, then $G(\tilde{f}) = r\mathbb{Z}$ and if r, s are rationally independent, then $G(\tilde{f}) = \mathbb{R}$. Two 1-block maps are of particular interest. The first one is the *Hamming weight function* w, given by

$$w(n) = \sum_{i=0}^{\infty} |c_i| = \#\text{ nonzero SSB digits of } n$$

if $n = c_0 c_1 \dots c_m 0^\omega$ is the SSB-expansion of n. From formula (14) one easily sees that

$$\Delta w(n) = w(n+1) - w(n) \in \{-1, 0, 1\};$$

more precisely $\Delta w = -1$ on $[-1]$ and $[(10)^m 10(-1)]$, $\Delta w = 1$ on $[(01)^m 00]$, and $\Delta w = 0$ on $[(01)^m 0(-1)] \cup [(01)^m 100]$. Extending Δw on K, one has that Δw is generated by the 1-block map $\tilde{w} : K \to \mathbb{R}$, defined by $\tilde{w}(x) = |x_0|$. Δw is not continuous at $\alpha = (01)^\omega$ and $\beta = (10)^\omega$ and $G(\tilde{w}) = E(\Delta w) = \mathbb{Z}$. The second 1-block map is the sum-of-digits s defined by

$$s(n) = \sum_{i=0}^{\infty} c_i.$$

Formula (14) shows that $\Delta s = 1$ on $[-1]$, $\Delta s = -2m$ on $[(01)^m 0(-1)] \cup [(10)^m 100]$, $\Delta s = -2m+1$ on $[(01)^m 00]$ and $\Delta s = -2m-1$ on $[(10)^m 10(-1)]$. Obviously, Δs is not continuous at α and β and $G(\tilde{w}) = E(\Delta w) = \mathbb{Z}$.

(2) Table 1 gives all 2-block maps which are coboundaries. Theorem 8 (in the next section) gives examples of 2-block maps which are not coboundaries.

4.4. **Generalization.** In the definition of k-block functions we may replace $\mathbb{R}$ by any locally compact metrizable separable abelian group A. The definition of essential values remain unchanged and the tools developed in [Sch] can be used. In this general setting, Theorem 5 remains unchanged but Corollary 5 cannot be applied except for particular group like $A = \mathbb{R}^s$. The first part of Lemma 5 still holds, that is to say

$$G(\tilde{f}) = E(\Delta f) \tag{23}$$

and as a consequence, if Δf is a coboundary, then $G(\tilde{f}) = \{0_A\}$, hence $\Delta f = -\Delta f^*$ on $\mathbb{Z}$, so that Δf extends continuously on K. Now, Theorem 6 is true in full generality. It is worth to notice that if $\psi : A \to A'$ is a group homomorphism, then $\psi \circ f$ is the k-block function associated to $= \psi \circ \tilde{f}$ and formula (23) yields

$$G(\psi \circ \tilde{f}) = E(\Delta(\psi \circ f)) = \psi(G(\tilde{f})).$$

Using the fact that Δf is always μ-continuous (i.e. the set of discontinuity points is μ negligible, see [Ku-Nie]), Theorem 6 in [Li2] and classical results on generic points for skew products (see [Li1] for example) give the following Theorem.

Theorem 7. *With the above notations, if A is compact, then $\tau_{\tilde{f}}$ is uniquely ergodic and all points are generic; in particular, for any bounded $\mu \otimes \lambda_G$-continuous map $\varphi : K \times G \to \mathbb{C}$, one has*

$$\lim_N \frac{1}{N} \sum_{0 \le n < N} \varphi(J(n), f(n)) = \int_{K \times G} \varphi \, d(\mu \otimes \lambda_G).$$

To end this section, let us consider the $\mathbb{Z}^2$-valued 1-block functions $(w, s) : n \mapsto (w(n), s(n))$. The above computations show that $G((\tilde{w}, \tilde{s})) = \mathbb{Z}^2$ hence the skew product $\tau_{(\tilde{w},\tilde{s})}$ is ergodic and, for any irrational numbers η, ξ, the sequence

$$n \mapsto (w(n)\eta, s(n)\xi)$$

is uniformly distributed mod $\mathbb{Z}^2$ (and in fact, well distributed in the sense of [Ku-Nie]). Analogous results hold if η or ξ are rational. The details are left to the reader.

5. σ-finite invariant measures

In this section we use the Markov chain structure of K to build Markov shift invariant measures ν' for which τ is non-singular, and there is no τ-invariant measure which is equivalent to ν', except if $\nu' = \nu$. Such a result is a natural generalization of a theorem of Arnold [Arn]. To do that we study some Maharam transformations. For any given positive probability vector $p = (p_{-1}, p_0, p_1)$, consider the Markov chain with state space $S = \{-1, 0, 1\}$ and transition matrix

$$\Pi(p) := \begin{pmatrix} 0 & 1 & 0 \\ p_{-1} & p_0 & p_1 \\ 0 & 1 & 0 \end{pmatrix}$$

which we denote by Π if the reference to p is unambiguous. The stationary Markov measure associated to Π has initial distribution $q = (\frac{p_{-1}}{2-p_0}, \frac{1}{2-p_0}, \frac{p_1}{2-p_0})$, which we denote by ν_p. From the classical theory of Markov chains, we see that $(K, \sigma, \mathcal{B}, \nu_p)$ is ergodic and more precisely is exact *i.e.*, the tail σ-algebra $\bigcap_{n \ge 1} \sigma^{-n}\mathcal{B}$ is trivial.

Theorem 8. *The measure* ν_p *is non-atomic. Moreover, let* $f : \mathbb{Z} \to \mathbb{R}$ *be the additive 2-block function defined by* $\tilde{f} : K \to \mathbb{R}$, *with* $\tilde{f}(0(-1)) = \log p_{-1}$, $\tilde{f}(01) = \log p_1$ *and* $\tilde{f}$ *is zero otherwise. Let* $g : K \to \mathbb{R}$ *defined by* $g(x) = |x_0|$. *Then*

$$\log\left(\frac{d\nu_p \circ \tau}{d\nu_p}(x)\right) = \Delta(f + f^*)(x) - (g(\tau x) - g(x)) \log p_0. \tag{24}$$

More explicitly,

$$g(\tau x) - g(x) = \begin{cases} 1 & \text{if} \quad x_0 = 0 \\ -1 & \text{otherwise,} \end{cases}$$

and

$$\frac{d\nu_p \circ \tau}{d\nu_p}(x) = \begin{cases} \frac{p_0}{p_{-1}} & \text{if} \quad x \in [-1]; \\ \frac{p_1}{p_0}\left(\frac{p_{-1}}{p_1}\right)^m & \text{if} \quad x \in [(01)^m 00]; \\ p_0\left(\frac{p_{-1}}{p_1}\right)^m & \text{if} \quad x \in [(01)^m 0(-1)]; \\ \frac{1}{p_0}\left(\frac{p_{-1}}{p_1}\right)^m & \text{if} \quad x \in [(10)^m 100]; \\ \frac{p_0}{p_1}\left(\frac{p_{-1}}{p_1}\right)^m & \text{if} \quad x \in [(10)^m 10(-1)]; \end{cases} \tag{25}$$

with $m \geq 0$.

Proof. The proof that ν_p is non-atomic is left to the reader (recall that the coordinates of p are positive). Formula (25) follows from (14), and the fact that the Radon-Nikodym derivative is constant on cylinders of the form $[-1]$, $[(01)^m 00]$, $[(01)^m 0(-1)]$, $[(10)^m 100]$, $[(01)^m 10\ -1]$. Now, formula (24) follows from the definition of $\tilde{f}$ and formula (25). □

Remark. If one coordinate of the probability vector p is 0, then the calculations in Theorem 8 show that the derivative $\frac{d\nu_p \circ \tau}{d\nu_p}$ takes the value 0 on at least one cylinder set of positive ν_p measure, and the value ∞ on at least one cylinder of positive ν_p measure. In other words, the positivity of p is necessary and sufficient for $\nu_p \circ \tau$ to be equivalent to ν_p, and so τ is non-singular with respect to ν_p.

Proposition 3. *The map* τ *is an invertible, conservative, ergodic non-singular transformation of* (K, ν_p).

Proof. Due to the above results, we only need to prove the ergodicity. Let B be a Borel set of K such that $\tau^{-1}(B)$ equals B up to a set of ν_p-measure 0. Without loss of generality, we may assume that $\tau^{-1}(B) = B$. Hence, for any $b = b_0 b_1 b_2 \cdots$ in B, we have $\tau^{-b[n]} b = 0^n b_n b_{n+1} b_{n+2} \cdots \in B$. Now for any $x_0 \cdots x_n \in \mathcal{A}^n$ such that $x_0 \ldots x_{n-1} b_n b_{n+1} b_{n+2} \cdots$ is an element of K, one has that $x = \tau^{-b[n]+x[n]} b$, and hence $x \in B$. Consequently, for any integer $n \geq 0$, we have $B = \sigma^{-n}(\sigma^n(B))$. In particular, B is measurable with respect to the tail σ-algebra $\bigcap_{n \geq 1} \sigma^{-n} \mathcal{B}$ which is $\{\emptyset, K\}$ mod ν_p by exactness of $(K, \sigma, \mathcal{B}, \nu_p)$. Therefore, $\nu_p(B)$ is 0 or 1. □

Theorem 9. *The Markov measure* ν_p *admits no* τ*-invariant equivalent measure unless* $p = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$.

Proof. Suppose ρ is a τ-invariant measure on K which is equivalent to ν_p. Then there exists a measurable function h which is positive ν_p a.e. such that

$$\frac{d\nu_p \circ \tau}{d\nu_p} = \frac{h \circ \tau}{h} \quad \nu_p\text{-a.e.}$$

From formula (24), one sees that $\Delta(f)$ is a coboundary, and by a result of K. Schmidt this is equivalent to saying that the group of essential values of $\Delta(f)$ is reduced to $\{0\}$. By Lemma 5, Δf is continuous. Using Table 1 (or Theorem 5), we derive that $\log p_{-1} = \log p_1 = 2 \log p_0$, hence $p_1 = p_{-1} = p_0^2$, but $p_{-1} + p_1 + p_0 = 1$ (and $p_0 \geq 0$) so that $p_0 = \frac{1}{2}$ and $p_{-1} = p_1 = \frac{1}{4}$. $\square$

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