Existence of Competitive Equilibria in Economies with a Measure Space of Consumers and Consumption Externalities*

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A solution to a long-standing open problem is given by extending Aumann's Walrasian equilibrium existence result for a pure exchange economy with a continuum of consumers [2] to the situation where preferences are allowed to depend upon the consumption profile (as well as the price vector). Our extension includes Schmeidler's existence results for a game with a continuum of players [38]. We study the existence of competitive equilibria and asymptotic competitive equilibria. The latter kind of equilibrium, which is introduced here as a natural extension of Aumann's notion, turns out to exist for a larger, more interesting class of consumption externalities. Specializations of our main results yield the original existence results of Aumann and Schmeidler.

1 Introduction

Aumann's model of a pure exchange economy with a continuum of agents [1, 2] has played an important role in economic theory. Being an idealization, it was created to perfectly capture the notion of perfect competition. In [1] Aumann proved a very elegant core equivalence theorem and in [2] he gave a Walrasian equilibrium existence result for this model (see also [26, 29, 37]). The present paper presents a solution to the long-standing open problem of extending Aumann's existence result to the situation where preferences are allowed to depend on the consumption profile and the price vector. In such a situation the exchange economy forms a complicated continuum game, so it is not surprising that our existence results should apply to Nash equilibria as well. Thus, next to generalizing [2], our existence results also turn out to generalize the two Nash equilibrium existence results for continuum games that were established by Schmeidler in his influential paper [38]. Hence, in the style of chapter 7 of [26], which is entitled *"Walras meets Nash"*, an appropriate subtitle of this paper would be *"Aumann meets Schmeidler"*. For what chapter 7 in [26] explores for economies/games with finitely many agents (this development goes back to Arrow and Debreu), the present paper does for economies/games with a continuum of agents.

To illustrate the kind of extensions studied in this paper, consider the following model of a pure exchange economy with a measure space of agents and d commodities. Let $[0,1]$ be the space of all agents (i.e. consumers); it is equipped with the Borel or Lebesgue σ-algebra and the Lebesgue measure λ. Let $\omega : [0,1] \to \mathbb{R}^d_+$ be an integrable function; each agent $t \in [0,1]$ has initial endowment $\omega(t) \in \mathbb{R}^d_+$. We suppose that in aggregate no commodity is absent from the market: $\int_0^1 \omega(t)dt \in \mathbb{R}^d_{++}$. This is a very mild, standard assumption. A *consumption profile* (alias *consumption plan*) is a (Borel) measurable function $f : [0,1] \to \mathbb{R}^d_+$ with the usual meaning: if f is the profile realized in the market, then $f(t)$ is the commodity bundle chosen by agent t, $t \in [0,1]$. Let $\mathcal{M}$ denote the set of all consumption profiles.[1] We shall now describe the consumption externality of this example: let $\tilde{g}_1, \ldots, \tilde{g}_N : [0,1]^2 \times \mathbb{R}^d_+ \to \mathbb{R}$ be N functions such that for $j = 1, \ldots, N$ the following holds:

(i) $\tilde{g}_j$ is measurable with respect to the Borel σ-algebra on $[0,1]^2 \times \mathbb{R}^d_+$,

(ii) $\tilde{g}_j(t, \tau, \cdot)$ is continuous on $\mathbb{R}^d_+$ for every $(t, \tau) \in [0,1]^2$,

(iii) $\tilde{g}_j(t, \cdot, \cdot)$ is integrably bounded on $[0,1]$ for every $t \in [0,1]$.

As usual, (iii) means that there exists a Lebesgue-integrable function $\phi_j : [0,1] \to \mathbb{R}$ such that $\sup_{x \in \mathbb{R}^d_+} |\tilde{g}_j(t, \tau, x)| \leq \phi_j(\tau)$ for every τ and $t \in [0,1]$. By (ii) and (iii) the integral

$$J_{j,t}(f) := \int_0^1 \tilde{g}_j(t, \tau, f(\tau))d\tau$$

[1] No explicit integrability restrictions are imposed on the profiles, for the integrals of their component functions are subject to the integration convention in appendix A.

is well-defined for every $f \in \mathcal{M}$. We denote $J_t(f) := (J_{1,t}(f), \ldots, J_{N,t}(f))$. Each agent $t \in [0,1]$ is supposed to have a utility function of the form

$$U_t(x, f) = V_t(x, J_t(f)).$$

Here $V_t : \mathbb{R}_+^d \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function such that $V_t(\cdot, y)$ is strongly increasing on $\mathbb{R}_+^d$ for every $y \in \mathbb{R}^N$. We also suppose that the function $V : (t, x, y) \mapsto V_t(x, y)$ on $[0,1] \times \mathbb{R}_+^d \times \mathbb{R}^N$ is jointly measurable. Somewhat more concretely, suppose that every agent $t \in [0,1]$ considers the aggregated utility of two particular target groups $E_{1,t}$ and $E_{2,t}$, being Borel subsets of $[0,1]$, as vital for her/his own consumption utility. Then $N = 2$ and agent t could employ an integral like $\int_{E_{j,t}} g_j(t, \tau, f(\tau))d\tau$ as her/his own utility evaluation of some aggregate utility enjoyed by her/his target group $E_{j,t}$ under the profile f. This means that she/he is using

$$\tilde{g}_j(t, \tau, x) := \begin{cases} g_j(t, \tau, x) & \text{if } \tau \in E_{j,t}, \\ 0 & \text{if } \tau \notin E_{j,t} \end{cases}$$

The above conditions for $\tilde{g}_j$ are then met by self-evident, similar looking conditions for the functions g_j, plus a measurability condition for the graphs of the correspondences $t \mapsto E_{j,t}$. Let $P := \{p \in \mathbb{R}_+^d : \sum_{i=1}^d p^i = 1\}$ be the simplex of price vectors; in this exchange economy the *budget set* $B_{t,p}$ of agent t, given the price vector p, is the set $\{x \in \mathbb{R}_+^d : p \cdot x \leq p \cdot \omega(t)\}$. One of the main results in this paper implies the following for this special example: there exist a price vector $p_* \in P$ and a sequence $\{f_k\}_k$ in $\mathcal{M}$ such that every consumption profile f_k is feasible for pure exchange in that it clears the market:

$$\int_0^1 f_k(t)dt = \int_0^1 \omega(t)dt.$$

At the same time, for every $\epsilon > 0$

$$\lim_{k \to \infty} \lambda(\{t \in [0,1] : U_t(f_k(t), f_k) < \sup_{x \in B_{t,p_*}} U_t(x, f_k) - \epsilon\}) = 0.$$

That is to say, $\{f_k\}_k$ gives rise to arbitrarily large sets[2], all whose agents are "ϵ-almost saturated" in their desires, given the budget constraint that is imposed by p_*. This is what we call an *asymptotic competitive equilibrium* for Aumann-type models. Such an existence result would seem to be new. Compare this to Aumann's notion of a *competitive equilibrium* in terms of the present model: it consists of a price vector $p_* \in P$ and a market-clearing consumption profile $f_* \in \mathcal{M}$ such that

$$\lambda(\{t \in [0,1] : U_t(f_*(t), f_*) < \sup_{x \in B_{t,p_*}} U_t(x, f_*)\}) = 0.$$

That is to say, almost every agent is completely saturated in her/his desires, given the price vector p_*. As examples in this paper show, under the previous conditions of our example a competitive equilibrium need not exist, but an asymptotic competitive

[2] I.e., sets with measure arbitrarily close to 1.

equilibrium does exist. Under a much more restrictive condition for the N functions $\tilde{g}_j : [0,1]^2 \times \mathbb{R}^d_+ \to \mathbb{R}$ (however, still considerably more general than in the papers by Aumann or Schmeidler [2, 38] or related references), our results actually imply the existence of a competitive equilibrium. This generalizes the original existence results in [2, 38].

The novelty of the above example is the *inclusion of consumption as an externality.* In this paper we shall also include price as an externality, but this is not new; an equilibrium existence result for an economy with a measure space of agents, featuring price as an externality, was obtained by Greenberg *et al.* [27]. They implemented Debreu's well-known idea to reformulate the market model as a game with a price-setting auctioneer. Their main result was generalized by the present author [12] for an economy with both consumption and price externalities. This economy is an abstract precursor of the relaxed economy discussed below, but with compact feasible consumption sets. However, to obtain an existence result with consumption externalities in an Aumann-type model with noncompact feasible consumption sets is a problem that is both economically more interesting and technically much more challenging. Its solution seems to have deluded several workers in this area. This can be attributed to the well-known absence of a suitable topology for the set of consumption profiles in a continuum economy, such as $\mathcal{M}$ above (topological considerations, in one form or another, are essential, because a fixed point theorem must be applied to obtain the desired existence of an equilibrium). These force the use of instruments of last resort, such as Fatou's lemma in several dimensions, that cannot do full justice to best response. This problem does *not* occur when there are only finitely many agents or when all feasible consumption sets are compact; the latter is the standard situation in games with a continuum of players, as introduced by Schmeidler [38].

To handle continuum economies $\mathcal{E}$ with consumption and price externalities, the present paper circumvents the above problem by adopting the relaxation approach to existence in continuum games from [9, 13, 17]. As shown in that work, the relaxation approach brings together a number of existence results for continuum games, such as those found in [8, 30, 31, 32, 33, 36, 38]. Relaxation has a respectable history in the calculus of varations and control theory [39], but actually, in the form of *mixing*, it was already known for games with finitely many players as early as 1713 (see [19, pp. 7-9])! Nevertheless, the rather complicated topology that is needed to deal with mixing in continuum games (i.e., the narrow topology, surveyed in appendix A), largely escaped the attention of game theorists and economists. In fact, most of the results stated in appendix A were not available before 1984. The relaxation approach to existence, which uses this topology as its principal technical tool, consists of studying a suitable – and in a certain sense easier – extension of the original existence problem, the *relaxed* existence problem. This problem, posed for a so-called relaxed exchange economy $\mathcal{RE}$, is formulated in terms of *mixed* consumption profiles (see Definition 3.1). To solve the relaxed existence problem, we use the above-mentioned narrow topology for transition probabilities (in this connection we note that mixed consumption profiles in a continuum economy are precisely transition probabilities). In contrast to virtually all other references on equilibrium existence in continuum economies, this means that one *directly* topologizes the mixed consumption profiles to apply the desired fixed point theorem. From the viewpoint of economics, the relaxed economy $\mathcal{RE}$ is an artifact. However, our use of mixing, in a situation where economies have become complicated games, would seem hardly surprising, given its historical background recalled above. After proving the existence of an equilibrium pair for $\mathcal{RE}$, we use this pair to solve the original existence problem for $\mathcal{E}$. We do this by means of purification. Because of the need for truncation, the application of relaxation in this paper is more complicated than in [9, 13, 17]. More precisely, it consists of the following four steps:

(i) Formulation of the auxiliary relaxed economy $\mathcal{RE}$.

(ii) Formation of a sequence $\{\mathcal{RE}_m\}_m$ of *truncated* relaxed exchange economies. For each $m \in \mathbb{N}$ the economy $\mathcal{E}_m$ has an "almost free disposal" relaxed equilibrium pair (q_m^*, η_m^*) (see Remark 4.1).

(iii) As a decisive advantage of the relaxation approach, a subsequence of the sequence $\{\eta_m^*\}_m$ converges in the narrow topology to a mixed consumption profile δ_* (see Lemma 4.4(ii)). Together with a suitable limit point p_* of $\{q_m^*\}_m$, this δ_* constitutes a free disposal relaxed competitive equilibrium for $\mathcal{RE}$. Moreover, when standard monotonicity conditions are added, (p_*, δ_*) becomes a relaxed competitive equilibrium pair (Lemma 4.6). This yields the auxiliary existence Theorem 3.1 for $\mathcal{RE}$.

(iv) Conversion of the [free disposal] relaxed competitive equilibrium pair (p_*, δ_*) for $\mathcal{RE}$ into a [free disposal] asymptotic equilibrium for $\mathcal{E}$ (see Theorem 2.1) or, under more stringent conditions on the consumption externality, a [free disposal] ordinary competitive equilibrium for $\mathcal{E}$ (see Theorems 2.2, 2.3). To achieve this, we apply respectively denseness and purification results (see Proposition A.5).

In the usual literature on existence of competitive equilibria in continuum economies one also truncates, but much more directly, by forming truncations of the original economy $\mathcal{E}$. In what could be seen as an analogue of step (iii), this yields a sequence of original (non-mixed) consumption profiles. The crucial difference with step (iii) above is that then there is not an analogue of Lemma 4.4(ii) available, but only a result like Fatou's lemma in several dimensions. Following [4, 6], we observe that that lemma is an immediate consequence of the principal results for the narrow topology. Thus, in the relaxation approach it makes place for the more fundamental and underlying narrow topology.

After the completion of this paper the present author became aware of recent related work by Noguchi [35] and Cornet and Topuzu [24]. They also deal with pure exchange economies with price and consumption externalities. However, their models do not extend Aumann's model, as they adopt additional convexity and quasiconcavity conditions. Our comments following Remark 2.3 below provide more details. In future work we plan to study existence when production is incorporated into the model, along the lines of [28].

2 Main existence results

We start by extending Aumann's model of a pure exchange economy $\mathcal{E}$ with a nonatomic measure space of agents. We refer to the extensive discussions of this celebrated model in [26, 29]. Several relevant examples are also presented in [26, Chapter 3]. The prominent feature of the extension presented here is that it allows for consumption externalities. This means that we allow each agent's utility to be influenced by the consumption choices of the other agents; thus, utility-contributing factors that can be observed in real-life consumption, such as envy, desires to imitate, etc. can be incorporated.

Let $(T, \mathcal{T}, \mu)$ be a nonatomic finite measure space of *agents* and let d be the number of commodities. By adopting the integration convention that is stated at the beginning of appendix A, we avoid some cumbersome integrability restrictions. Every agent $t \in T$ has a *feasible consumption set* $C_t \subset \mathbb{R}^d_+$.[3] Also, let $\omega : T \to \mathbb{R}^d_+$ be a given function; for agent $t \in T$ the commodity bundle $\omega(t) \in \mathbb{R}^d_+$ forms her/his *initial endowment*. Our assumptions are as follows:

Assumption 2.1 *The set* $C := \{(t, x) \in T \times \mathbb{R}^d_+ : x \in C_t\}$ *is* $\mathcal{T} \times \mathcal{B}(\mathbb{R}^d)$*-measurable.*

Assumption 2.2 *The function* ω *is (coordinatewise) integrable.*

Assumption 2.3 *The set* C_t *is closed and convex for every* $t \in T$*.*

As a stronger alternative to Assumption 2.3 we shall occasionally use:

Assumption 2.3' *The set* C_t *is compact and convex for every* $t \in T$*.*

Next to these primary assumptions, we shall also need to consider the following ones:

Assumption 2.4 *For every* $t \in T$, $\omega(t) \in \operatorname{int} C_t$*.*

Here $\operatorname{int} C_t$ denotes the interior of the set C_t. The latter assumption will be frequently be weakened, but at the cost of introducing an additional strong monotonicity assumption, to be formulated below.

Assumption 2.4' $\int_T \omega \, d\mu \in \mathbb{R}^d_{++}$.

For some fixed number $N \in \mathbb{N}$ let $\{\tilde{g}_1, \ldots, \tilde{g}_N\}$ be a finite collection of functions $\tilde{g}_j : T \times C \to \mathbb{R}$. These functions are required to be as follows.

Assumption 2.5 *For* $j = 1, \ldots, N$

(i) $\tilde{g}_j(t, \tau, \cdot)$ *is continuous on* C_τ *for every* $(t, \tau) \in T^2$*,*

(ii) $\tilde{g}_j$ *is* $\mathcal{T} \otimes \mathcal{T} \otimes \mathcal{B}(\mathbb{R}^d)$*-measurable,*

(iii) $\tilde{g}_j(t, \cdot, \cdot)$ *is integrably bounded on* T *for every* $t \in T$*.*

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[3] By means of an obvious translation argument the situation where there exists an integrable $\phi : T \to \mathbb{R}^d$ such that $C_t \subset \phi(t) + \mathbb{R}^d_+$ for every $t \in T$ can be reduced to the situation discussed here.

A *consumption profile* in this model is a function $f : T \to \mathbb{R}^d_+$ that is measurable with respect to $\mathcal{T}$ and $\mathcal{B}(\mathbb{R}^d)$. A consumption profile f is said to be *feasible* if $f(t) \in C_t$ for a.e. t in T (here "a.e." is short for "almost everywhere"). The set of all feasible consumption profiles will be denoted by $\mathcal{M}_C$. By (i) and (iii) in Assumption 2.5 the integral

$$J_{j,t}(f) := \int_T \tilde{g}_j(t, \tau, f(\tau))\mu(d\tau) \tag{2.1}$$

is well-defined for $j = 1, \ldots, N$ and for every $t \in T$ and $f \in \mathcal{M}_C$. We define $J_t(f) \in \mathbb{R}^N$ by $J_t(f) := (J_{1,t}(f), \ldots, J_{N,t}(f))$. The mapping $f \mapsto J_t(f)$ from $\mathcal{M}_C$ into $\mathbb{R}^N$ is called the *externality mapping* of agent t.

As already introduced in the introductory example, the set of all normalized *price vectors* is $P := \{p \in \mathbb{R}^d_+ : \sum_{i=1}^d p^i = 1\}$. For a given price vector $p \in P$, the *budget set* of agent $t \in T$ is

$$B_{t,p} := \{x \in C_t : p \cdot x \leq p \cdot \omega(t)\}.$$

Obviously, $B_{t,p}$ constitutes all the possible consumption choices, based on the market value of her/his initial endowment, that agent t can make under the price vector p. Let $V : C \times P \times \mathbb{R}^N \to \mathbb{R}$ be a given function that meets the following assumption:

Assumption 2.6 *V is $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^d \times P \times \mathbb{R}^N)$-measurable and $V(t, \cdot, \cdot, \cdot)$ is continuous on $C_t \times P \times \mathbb{R}^N$ for every $t \in T$.*

In conjunction with Assumption 2.4' we shall also need:

Assumption 2.7 *$C = T \times \mathbb{R}^d_+$ and $V(t, \cdot, p, y)$ is strongly increasing on $\mathbb{R}^d_+$ for every $(t, p, y) \in T \times P \times \mathbb{R}^N$.*

The *utility function* $U_t : C_t \times P \times \mathcal{M}_C \to \mathbb{R}$ of agent $t \in T$ is defined by

$$U_t(x, p, f) := V(t, x, p, J_t(f));$$

thus, dependence of the utility upon the profile f manifests itself (only) by way of $J_t(f)$. We shall now consider both new and classical equilibrium notions for the economy $\mathcal{E} := \langle T, \mathcal{M}_C, \omega, U \rangle$.

Definition 2.1 An *asymptotic competitive equilibrium* for $\mathcal{E}$ is a pair $(p_*, \{f_k\}_k)$, consisting of a price vector $p_* \in P$ and a sequence $\{f_k\}_k$ of consumption profiles in $\mathcal{M}_C$, such that

(i) $\int_T f_k \, d\mu = \int_T \omega \, d\mu$ for every $k \in \mathbb{N}$,

(ii) $\lim_{k\to\infty} \mu(\{t \in T : U_t(f_k(t), p_*, f_k) < \sup_{x \in B_{t,p_*}} U_t(x, p_*, f_k) - \epsilon\}) = 0$ for every $\epsilon > 0$.

If in (i) the equality sign $=$ is replaced by the coordinatewise inequality $\leq$ (for all k), then we speak of an asymptotic *free disposal* competitive equilibrium.

Our basic assumptions guarantee that the expressions in (i)-(ii) above make sense mathematically. Economically speaking, (i) ensures that each consumption profile f_k clears the market (i.e., is exchange-feasible). Also, (ii) ensures that, by implementation of f_k for large enough k, arbitrarily large sets of agents (i.e., large in terms of their μ-measure) can be made ϵ-almost-saturated in their desires, given the budget constraint imposed by the price vector p_*. This equilibrium notion is a weak form of the following one, which goes back to Aumann and Walras:

Definition 2.2 A *competitive equilibrium* (alias *Walrasian equilibrium*) is a pair $(p_*, f_*) \in P \times \mathcal{M}_C$ such that

(i) $\int_T f_* \, d\mu = \int_T \omega \, d\mu$,

(ii) $f_*(t) \in \mathrm{argmax}_{x \in B_{t,p_*}} U_t(x, p_*, f_*)$ for a.e. t in T.

If in (i) the equality sign $=$ is replaced by the coordinatewise inequality $\leq$, then we speak of a *free disposal* competitive equilibrium.

The precise connection with Definition 2.1 is as follows: $(p_*, f_*) \in P \times \mathcal{M}_C$ is a competitive equilibrium if and only if p_*, together with the constant sequence $f_*, f_*, \ldots$ constitutes an asymptotic competitive equilibrium; this affirms that the new equilibrium notion is very close in spirit to Aumann's original ideas in [1]. Clearly, (ii) in the above definition states that under the pair (p_*, f_*) almost every agent t in T is completely saturated in her/his desires, given the budget constraint imposed by p_*. As above, the economic meaning of (i) is still exchange-feasibility in the case of a competitive equilibrium, and in the case of a free disposal competitive equilibrium only the inequality sign is required in (i) to reflect exchange-feasibility. The first existence result of this paper is of a new kind:

Theorem 2.1 (i) *Under Assumptions 2.1, 2.2, 2.3', 2.4, 2.5 and 2.6 there exists an asymptotic free disposal competitive equilibrium.*
(ii) *Under Assumptions 2.1, 2.2, 2.3, 2.4', 2.5, 2.6 and 2.7 there exists an asymptotic competitive equilibrium.*

This result will be proven in section 4. We illustrate its parts (i) and (ii) respectively by the following two examples:

Example 2.1 *Consider $T := [0,1]$ with the Lebesgue σ-algebra and measure. Let $C := T \times [0,2]$ and let $\omega(t) := 2$ for all t. In this case $P = \{1\}$, so $B_{t,p} = [0,2]$ is the budget set for each agent t. Consider $U_t(x,1,f) := |x - 1 + t - \int_0^t f|$. Then $J_{1,t}(f) = \int_0^t f(\tau)d\tau$. This corresponds to $\tilde{g}_{t,1}(t,\tau,x) := x$ if $\tau \leq t$ and $:= 0$ if $\tau > t$. We shall demonstrate that in this case there does not exist a free disposal competitive equilibrium, but that there does exist an asymptotic free disposal equilibrium, as is predicted by Theorem 2.1(i), all whose assumptions are fulfilled. First, we show that the supposition that there exists a free disposal competitive equilibrium $(p_*, f_*) \in P \times \mathcal{M}_C$ leads to a contradiction. For then, by Definition 2.2(ii) we would have the* *following for a.e.* t *in* $[0,1]$*:* $f_*(t)=0$ *if* $\int_0^t f_* > t$ *and* $f_*(t)=2$ *if* $\int_0^t f_* < t$*. Define* $\psi := f_* - 1$ *and let* $\Psi(t) := \int_0^t \psi$*. Then the previous lines imply that* $\Psi(t)\psi(t) \leq 0$ *for a.e.* t *in* $[0,1]$ *(note that if* $\int_0^t f_* = t$*, then* $\Psi(t)=0$*). So the absolutely continuous function* Ψ^2 *has a nonpositive derivative. Hence, by* $\Psi(0)=0$ *this implies* $(\Psi(t))^2 = 0$ *for every* $t \in [0,1]$*. But then* $\Psi(t)=0$ *for all* t*, which implies* $\psi(t)=0$ *for a.e.* t*. So it follows that* $f_*(t)=1$ *for a.e.* t *in* $[0,1]$*, which is clearly nonsensical.*

Second, to demonstrate existence of an asymptotic free disposal equilibrium, let us consider for every $k \in \mathbb{N}$ *the set* $R_k \subset [0,1]$ *defined by*

$$R_k := (2^{-k}, 2\cdot 2^{-k}] \cup (3\cdot 2^{-k}, 4\cdot 2^{-k}] \cup \cdots \cup ((2^k-1)\cdot 2^{-k}, 1].$$

Let $f_k : [0,1] \to \{0,2\}$ *be given by* $f_k(t) := 0$ *if* $t \in R_k$ *and* $f_k(t) := 2$ *if* $t \in [0,1]\backslash R_k$*. Then* $\int_0^t f_k \in [t, t+2^{-k}]$ *is easily seen to hold for all* $t \in [0,1]$*. This immediately implies that Definition 2.1(ii) holds, and for obvious reasons part (i) of that definition holds as well.*

Example 2.2 *Let* $T := [0,1]$ *be equipped with the Lebesgue* σ*-algebra and measure. Let* $C := T \times \mathbb{R}^2_+$ *and let* $\omega(t) := (\frac{1}{2}, \frac{1}{2})$ *for al* t*. Define* $a(t,f) := (\cos\theta(t,f), \sin\theta(t,f))$*, with* $\theta(t,f) := \frac{5\pi}{4} + \frac{\pi}{4}(\int_0^t(\min(f^1,1) - \min(f^2,1)))$*. Recall here that we write* $f = (f^1, f^2)$*. Consider* $U_t(x,p,f) := |x - a(t,f)|$*, where the ordinary Euclidean norm is used. Of course, this corresponds to* $V(t,x,p,y) := |x - a'(t,y)|$*, where* $a'(t,y) := (\cos\theta'(t,y), \sin\theta'(t,y))$*, with* $\theta'(t,y) := \frac{5\pi}{4} + \frac{\pi}{4}(y^1 - y^2)$*. Similar to the previous example, but now with* $N = 2$*, we set* $\tilde{g}_{t,i}(t,\tau,x^i) := \min(x^i, 1)$ *if* $\tau \leq t$ *and* $:= 0$ *if* $\tau > t$*,* $i = 1,2$*. Observe that* $\theta'(t,y) \in [\pi, \frac{3}{2}\pi]$ *for all* $t \in T$ *and all* $y \in \mathbb{R}^2$*, so the point* $a'(t,y)$ *can only belong to the nonpositive orthant* $\mathbb{R}^2_-$ *(in fact,* $a'(t,y)$ *belongs to the quarter circle* D *that is the intersection of* $\mathbb{R}^2_-$ *and the unit circle). Therefore,* V *satisfies the strong monotonicity Assumption 2.7, and all other assumptions of Theorem 2.1(ii) are easily seen to hold as well. While this ensures the existence of an asymptotic competitive equilibrium (and we shall explicitly find one shortly), no ordinary competitive equilibrium exists in this situation. Again we demonstrate such nonexistence by a* reductio ad absurdum*. If* $(p_*, f_*) \in P \times \mathcal{M}_C$ *were a competitive equilibrium, we first of all conclude that neither* p_*^1 *nor* p_*^2 *can be zero by the strong monotonicity of* $U_t(\cdot, p_*, f_*)$*. Now consider the bisecting perpendicular of the line segment between the corner points* $((2p_*^1)^{-1}, 0)$ *and* $(0, (2p_*^2)^{-1})$ *of the budget set* B_{t,p_*}*. If this perpendicular line intersects the quarter circle* D*, defined above, in some point, then let* $(1,\beta)$ *be the polar coordinates of this intersection point, which has* $\beta \in [\pi, \frac{3}{2}\pi]$*. By Definition 2.2(ii) we have for a.e.* t *in* $[0,1]$

$$f_*(t) \in \mathrm{argmax}_{x \in B_{t,,p_*}} U_t(x,p_*,f_*) = \begin{cases} \{((2p_*^1)^{-1}, 0)\} & \textit{if } \theta(t,f_*) > \beta, \\ \{(0, (2p_*^2)^{-1})\} & \textit{if } \theta(t,f_*) < \beta, \\ \{((2p_*^1)^{-1}, 0), (0, (2p_*^2)^{-1})\} & \textit{if } \theta(t,f_*) = \beta. \end{cases} \tag{2.2}$$

This gives $(F^1(t) - F^2(t) - \gamma)(\min(f_*^1(t),1) - \min(f_*^2(t),1)) \leq 0$ *for a.e.* $t \in [0,1]$*, where we set* $F^i(t) := \int_0^t \min(f_*^i, 1)$ *and* $\gamma := \frac{4}{\pi}\beta - 5$*. So the nonnegative absolutely continuous function* $Q := (F^1 - F^2 - \gamma)^2$ *has a nonpositive derivative. If* $\gamma > 0$ *then*

$H := F^1 - F^2 - \gamma$ is negative at $t = 0$ and increases near that point. Now on T either it reaches the level zero or not. The latter case gives $f_*(t) = (0, (2p_*^2)^{-1})$ for a.e. t in T, which is impossible by $\int_T \omega^1 > 0$. The former case gives the existence of some $\alpha \in (0, 1]$ such that $H(\alpha) = 0$ and such that $H < 0$ on $[0, \alpha)$. So then $Q(\alpha) = 0$ and $Q(t) = 0$, whence $H(t) = 0$, for all $t \in [\alpha, 1]$. If $\alpha < 1$ then this implies $f_*^1(t) = f_*^2(t)$ a.e. on $[\alpha, 1]$, which is impossible in view of (2.2), and if $\alpha = 1$, then we are back in the previous case. Similarly $\gamma < 0$ leads to an impossibility, and for $\gamma = 0$ we obtain $H(0) = F^1(0) - F^2(0) = 0$, i.e., in terms of the foregoing we find $\alpha = 0$. This yields $f_*^1(t) = f_*^2(t)$ a.e. on $[0, 1]$, which is impossible. The situation where the bisecting perpendicular of the line segment between $((2p_*^1)^{-1}, 0)$ and $(0, (2p_*^2)^{-1})$ does not intersect the quarter circle D leads to f_* being a.e. equal to precisely one of the two corner points, which is evidently not possible by $\int_T \omega^i > 0$. Therefore, the desired contradiction has been reached. We conclude that a competitive equilibrium does not exist. However, according to Theorem 2.1(ii) an asymptotic competitive equilibrium exists. As a specific example, consider $p_* := (\frac{1}{2}, \frac{1}{2})$ and the sequence $\{f_k\}_k$ in $\mathcal{M}_C$ given by $f_k(t) := (1, 0)$ if $t \in R_k$ and $f_k(t) := (0, 1)$ if $t \in [0, 1]\backslash R_k$, where R_k is as in the previous Example 2.1. Then it is easy to see that $\int_0^t \min(f_k^i, 1) = \int_0^t f_k^i \in [t, t + 2^{-k}]$ for all $t \in [0, 1]$ and $i = 1, 2$. This yields $\theta(t, f_k) \in [\frac{5\pi}{4} - 2^{-k}, \frac{5\pi}{4} + 2^{-k}]$. Hence, the sequence $\{a(t, f_k)\}_k$ converges to $(-\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$. So then it is evident that Definition 2.1(ii) holds. Finally, note that $\int_0^1 f_k^i = \frac{1}{2}$ for $i = 1, 2$, so Definition 2.1(i) also holds.

Functions that are rather similar to the ones used in the examples above, have been used, among others, by Aumann *et al.* [3] in results on approximate purification of mixed strategies. In view of well-known denseness results in the presence of nonatomicity, this is no coincidence (cf. Proposition A.5). In Example 3.1 we shall come back to the rapidly oscillating sequences $\{f_k\}$ of the above two examples (known as “chattering functions” in control theory [39]) and re-interpret them in terms of relaxed [free disposal] competitive equilibrium profiles.

Under a much more restrictive consumption externality, Theorem 2.1 can be strengthened into a more classical form. For this we shall need the following stronger version of Assumption 2.5:

Assumption 2.5' *For $j = 1, \ldots, N$ there exists $\bar{g}_j : C \to \mathbb{R}$ such that*

(i) $\tilde{g}_j(t, \cdot, \cdot) = \bar{g}_j$ *for every* $t \in T$,

(ii) $\bar{g}_j(t, \cdot)$ *is continuous on* C_t *for every* $t \in T$,

(iii) $\bar{g}_j$ *is* $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^d)$*-measurable,*

(iv) $\bar{g}_j$ *is integrably bounded on* T.

This means that all agents have the *same* consumption externality. The severity of this condition prompted our previous consideration of asymptotic competitive equilibria, for whose existence we can use the much more acceptable Assumption 2.5 itself.

Theorem 2.2 *Under Assumptions 2.1, 2.2, 2.3, 2.4', 2.5', 2.6 and 2.7 there exists a competitive equilibrium.*

Evidently, Theorem 2.2 generalizes Aumann's existence result in [2] (see also [25, pp. 725-730] and Theorem 2 on p. 151 of [29]) to a situation with price and consumption externalities; together with Theorem 2.3 below, this also generalizes the main results in [27, p. 33,37]. Also, a version of these existence results for non-ordered preferences, generalizing [37], can easily be obtained by the introduction of an artificial utility function of Shafer-Sonnenschein-type (e.g., see [27, Remark 5]). We also have an existence result for a competitive equilibrium with free disposal:

Theorem 2.3 *Under Assumptions 2.1, 2.2, 2.3, 2.4, 2.5' and 2.6 there exists a free disposal competitive equilibrium.*

The proofs of these results can be found in section 4; in connection with the proof of Theorem 2.3, the following can be distilled from Remark 3.2(i):

Remark 2.1 *If in Theorem 2.3 all sets $B_{t,p}$, $p \in P$, are identical for a particular $t \in T$, then for that t the convexity condition for the set C_t in Assumption 2.3 can be omitted and in Assumption 2.4 only $\omega(t) \in C_t$ is needed.*

In connection with the recent work in [24, 35], the following extensions are of interest:

Remark 2.2 *Some extensions of the externality mapping can be made with ease, but at the cost of complicating the model. For instance, it would have been possible to allow the functions $\tilde{g}_j$ to depend upon the price vector as well, similar to the way this has been done for the utility functions U_t. By Remark 3.2(iv), Theorem 2.1 remains valid if an externality of the form*

$$J'_{j,t}(p, f) := \int_{A_{t,p}} \tilde{g}_j(t, \tau, f(\tau))\mu(d\tau)$$

is used on $P \times \mathcal{M}_C$, instead of (2.1). One should then demand in addition that $(t, p) \mapsto A_{t,p}$ maps from $T \times P$ into $\mathcal{T}$ in such a way that the following hold:
(1) $\{(t, \tau, p) \in T \times T \times P : \tau \in A_{t,p}\}$ *is $\mathcal{T} \otimes \mathcal{T} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable,*
(2) $\lim_n \int_T |1_{A_{t,p_n}} - 1_{A_{t,p_0}}| d\mu = 0$ *for every $t \in T$ and every sequence $\{p_n\}_n$ in P with $p_n \to p_0$.*
Similarly, Theorems 2.2 and 2.3 continue to hold if an externality of the form

$$J'_j(p, f) := \int_{\bar{A}_p} \bar{g}_j(\tau, f(\tau))\mu(d\tau)$$

is used, with $\bar{g}_j$ as in Assumption 2.5' and with $p \mapsto \bar{A}_p$ such that
(1) $\{(\tau, p) \in T \times P : \tau \in \bar{A}_p\}$ *is $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable,*
(2) $\lim_n \int_T |1_{\bar{A}_{p_n}} - 1_{\bar{A}_{p_0}}| d\mu = 0$ *for every sequence $\{p_n\}_n$ in P with $p_n \to p_0$.*

Remark 2.3 *A different extension of the externality mapping that can be made is as follows. Clearly, Assumption 2.5(iii) means that for every $t \in T$ there exists an integrable function $\phi_{j,t} : T \to \mathbb{R}_+$ such that*

$$|\tilde{g}_j(t, \tau, x)| \le \phi_{j,t}(\tau) \text{ for all } (\tau, x) \in C.$$

It is possible to replace this by the following, more general condition: for every $t \in T$ and $\epsilon > 0$ there exists an integrable function $\phi_{j,t,\epsilon} : T \to \mathbb{R}_+$ such that

$$|\tilde{g}_j(t, \tau, x)| \le \epsilon \sum_{i=1}^{d} x^i + \phi_{j,t,\epsilon}(\tau) \text{ for all } (\tau, x) \in C.$$

This is thanks to the fundamental role of inequalities such as $\int_T f^i \, d\mu \le \int_T \omega^i \, d\mu$ and their relaxations such as $\int_T [\int_{\mathbb{R}^d_+} x^i \delta(t)(dx)]\mu(dt) \le \int_T \omega^i d\mu$ in our proof; in particular, Remark 4.14 of [15] then plays a very important role. Details are left to the reader (but see [18]).

Noguchi [35] and Cornet and Topuzu [24] have recently studied competitive equilibrium existence in models of pure exchange economies that have externality mappings that look like the ones considered in Remarks 2.2 and 2.3. Namely, they introduce $J''_t(p, f) := \int_{A_{t,p}} f(\tau)\mu(d\tau)$; this corresponds to using a *linear* function $\tilde{g}_j(t, \tau, x) := x^j$. Although this externality mapping is of the form considered in Remark 2.2, it does not satisfy Assumption 2.5(*iii*) or its extension considered in Remark 2.3 above. Another difference is that in [24] the existence of ordinary – as opposed to asymptotic – competitive equilibria is established. In this connection it is very important to note that [24, 35] place additional convexity and quasi-concavity conditions on the consumption sets and utility functions. Together with the special linear form of their externality mappings, this makes it possible for them to work with the classical weak topology on the space of all integrable consumption profiles. In forthcoming work [18] we shall demonstrate how, by expanding the above two remarks, the relaxation approach of this paper can still be pursued to obtain existence results of the kind studied in [24, 35]. These are obtained via a purification method by which one takes expectations (i.e., barycenters) instead of the present use of nonatomicity. Thus, the situation is quite similar to what is already known for the existence of Nash equilibria in continuum games. For instance, both purification methods were employed in [9, 13, 17].

The following corollary generalizes the Nash equilibrium existence result obtained in Theorem 2 of Schmeidler's influential paper [38], which introduced games with a measure space of players. A small discrepancy regarding measurability will be clarified in the proof of Corollary 3.1 below. As is typical for games, this corollary has compact action spaces. Generalizations that reach much further can be found in [9, 30] and especially in [13, 17].

Corollary 2.1 *In addition to Assumptions 2.1, 2.2 and 2.3, suppose that there exists an integrable function $\psi : T \to \mathbb{R}_+$ such that*

$$\sup_{x \in C_t} |x| \le \psi(t) \text{ for every } t \in T. \tag{2.3}$$

Let $U' : C \times \mathbb{R}^d \to \mathbb{R}$ be such that

$$U'(\cdot, \cdot, f) \text{ is } \mathcal{T} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)\text{-measurable for every } f \in \mathcal{M}_C,$$

$$U'(t, \cdot, \cdot) \text{ is continuous on } C_t \times \mathbb{R}^d \text{ for every } t \in T.$$

Then there exists $f_* \in \mathcal{M}_C$ such that

$$f_*(t) \in \mathrm{argmax}_{x \in C_t} U'_t(x, \int_T f_* \, d\mu) = 1 \text{ for a.e. } t \text{ in } T.$$

This result remains valid if $C \subset T \times \mathbb{R}^d$ is assumed instead of $C \subset T \times \mathbb{R}^d_+$.

Proof. Let $\omega(t) := \psi(t)e$, where $e := (1, \ldots, 1)$; then evidently $B_{t,p} = C_t$ for all $p \in P$, so Remark 2.1 applies to every $t \in T$. Also, let $\bar{g}_j(t, x) := x^j$; then $J_t(f) = \int_T f$. The assumptions of Theorem 2.3 are met, so there exists a pair $(p_*, f_*) \in P \times \mathcal{M}_C$ such that $f_*(t) \in \mathrm{argmax}_{B_{t,p_*}} U'_t(\cdot, \int_T f_*)$ a.e. By $B_{t,p_*} = C_t$ we obtain the desired existence result. Because of (2.3), the final line of the statement is justified by footnote 3. QED

3 A mixed competitive equilibrium existence result

We shall now formulate the relaxed economy $\mathcal{RE}$ and an associated equilibrium existence result. To begin with, we adopt the Assumptions 2.1, 2.2, 2.3, 2.5 and 2.6 from the previous section. As a *working hypothesis* we shall also assume that the measure space $(T, \mathcal{T}, \mu)$ is separable and complete. While this is of great use for the proofs, fairly straightforward arguments (see [13, 17] for details) can be used to remove this additional hypothesis from the final results. As introduced in appendix A, let $\mathcal{R}(T; \mathbb{R}^d_+)$ be the set of all transition probabilities with respect to $(T, \mathcal{T})$ and $(\mathbb{R}^d_+, \mathcal{B}(\mathbb{R}^d_+))$. Let $\mathcal{R}_C$ be the set of all $\delta \in \mathcal{R}(T; \mathbb{R}^d_+)$ such that $\delta(t)(C_t) = 1$ for a.e. t in T; we call such transition probabilities *mixed* consumption profiles. We equip $\mathcal{R}_C$ with the relative narrow topology; cf. Definition A.3 in appendix A. Because of the separability of $(T, \mathcal{T}, \mu)$, just discussed, $\mathcal{R}(T; \mathbb{R}^d_+)$ and its subset $\mathcal{R}_C$ are semimetrizable for the narrow topology by Proposition A.2. By Assumption 2.5, the integral

$$I_{j,t}(\delta) := \int_T [\int_{\mathbb{R}^d_+} \tilde{g}_j(t, \tau, x) \delta(\tau)(dx)] \mu(d\tau) \tag{3.1}$$

is well-defined for $j = 1, \ldots, N$ for every $t \in T$ and $\delta \in \mathcal{R}_C$. First, we use this to define player t's *relaxed externality mapping*, which is $\delta \mapsto I_t(\delta) := (I_{1,t}(\delta), \ldots, I_{N,t}(\delta))$. Next, it also gives us agent t's *relaxed utility function*, which is the function $W_t : C_t \times P \times \mathcal{R}_C \to \mathbb{R}$, given by

$$W_t(x, p, \delta) := V(t, x, p, I_t(\delta)), \tag{3.2}$$

See Example 3.1 below for two concrete examples of the relaxed externality mapping and relaxed utility function.

Lemma 3.1 *(i) The function $(t,x) \mapsto W_t(x,p,\delta)$ is $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^d_+)$-measurable on C for every $(p,\delta) \in P \times \mathcal{R}_C$.*
(ii) The function $W_t : C_t \times P \times \mathcal{R}_C \to \mathbb{R}$ is continuous for every $t \in T$.

Actually, in part (i) one could even assert the product measurability of $(t,x,p,\delta) \mapsto W_t(x,p,\delta)$ (using [23, III.14], a result that continues to be applicable to our separable and semimetrizable space $\mathcal{R}_C$), but we shall not need this.

Proof. (i) In view of Assumption 2.6, it is enough to prove the measurability of $t \mapsto I_t(\delta)$ for fixed $\delta \in \mathcal{R}_C$. This follows by Assumption 2.5 from the standard measure theory in [34], referenced in appendix A.

(ii) Fix j and $t \in T$. For every $\tau \in T$ the function $\tilde{g}_j(t,\tau,\cdot) : C_\tau \to \mathbb{R}$ is continuous by Assumption 2.5(ii). Hence, by closedness of the set C_τ (Assumption 2.3) and also Assumptions 2.5(i) and (iii), the function $\hat{g}_{j,t}$, defined by $\hat{g}_{j,t} := \tilde{g}_j(t,\cdot,\cdot)$ on C and $\hat{g}_{j,t} := +\infty$ on $(T \times \mathbb{R}^d_+) \backslash C$, belongs to the set $\mathcal{G}^{bb}(T;\mathbb{R}^d_+)$ (see appendix A). In terms of (A.1), we have $I_{j,t}(\delta) = I_{\hat{g}_{j,t}}(\delta)$ for all $\delta \in \mathcal{R}_C$. Therefore, $I_{j,t}$ is lower semicontinuous on $\mathcal{R}_C$ by Definition A.3. This argument can be repeated for $-\tilde{g}_j$ instead of $\tilde{g}_j$, so we conclude that $I_{j,t}$ is continuous. Thus, by (3.2) the stated continuity follows from Assumption 2.6. QED

Definition 3.1 A *relaxed competitive equilibrium* for the relaxed economy $\mathcal{RE} := \langle T, \mathcal{R}_C, \omega, W \rangle$ is a pair $(p_*, \delta_*) \in P \times \mathcal{R}_C$ such that

(i) $\int_T [\int_{C_t} x\, \delta_*(t)(dx)] \mu(dt) = \int_T \omega \, d\mu$,

(ii) $\delta_*(t)(\mathrm{argmax}_{x \in B_{t,p_*}} W_t(x,p_*,\delta_*)) = 1$ for a.e. t in T.

If in (i) the equality sign $=$ is replaced by the coordinatewise inequality $\leq$, then we speak of a relaxed *free disposal* competitive equilibrium.

The following remark explains the use of the adjective "relaxed" in the above definition, for it shows that the original [free disposal] competitive equilibria can be identified with a subset of the larger set of relaxed [free disposal] competitive equilibria.

Remark 3.1 *Recall from appendix A that any Dirac transition probability in $\mathcal{R}^{Dirac}(T;\mathbb{R}^d_+)$ can be expressed as ϵ_f for some measurable function $f : T \to \mathbb{R}^d_+$. Then the following relationships hold respectively by the definitions of $\mathcal{R}_C$, $I_{j,t}$ and by Definition 3.1:*
(i) $\epsilon_f \in \mathcal{R}_C$ if and only if $f \in \mathcal{M}_C$,
(ii) $I_t(\epsilon_f) = J_t(f)$ for every $t \in T$,
(iii) For $p \in P$ the pair (p,ϵ_f) is a relaxed [free disposal] competitive equilibrium for $\mathcal{RE}$ if and only if (p,f) is a [free disposal] competitive equilibrium for $\mathcal{E}$.

The main result of this section, which implies the main results of section 2, is as follows. It will be proven in section 4.

Theorem 3.1 (i) *Under Assumptions 2.1, 2.2, 2.3, 2.4, 2.5, 2.6 there exists a free disposal relaxed competitive equilibrium* (p_*, δ_*).
(ii) *Moreover, if in* (i) *Assumption 2.4 is replaced by Assumptions 2.4' and 2.7 then* $p_* \in \mathbb{R}^d_{++}$ *and* (p_*, δ_*) *is actually a relaxed competitive equilibrium.*

Remark 3.2 (i) *From the observation following Lemma 4.2 below, we conclude that if all budget sets* $B_{t,p}$, $p \in P$, *are identical, then neither interiority of* $\omega(t)$ *in* C_t *nor convexity of* C_t *are needed for Theorem 3.1*(i) *to hold.*
(ii) *Theorem 3.1 remains valid when* $(T, \mathcal{T}, \mu)$ *is no longer nonatomic, for nonatomicity has not been used anywhere in this section. We shall elaborate on the role of atoms in Theorems 2.1, 2.2 and 2.3 in [18]; such a development turns out to be very similar to what was done in [13, 17] in this respect.*
(iii) *Theorem 3.1 remains valid when, instead of via (3.1)-(3.2),* W *is introduced as a general function* $W : C \times P \times \mathcal{R}_C \to \mathbb{R}$ *(with* $W_t := W(t, \cdot, \cdot, \cdot)$*), that satisfies* (i)*-*(ii) *of Lemma 3.1. This also means that certain parts of (3.1)-(3.2) can be generalized; for instance, we could allow* N *to depend upon* t, *let* $\tilde{g}_j$ *also depend on* p, *etc. Similar to [17], Theorem 3.1 continues to hold if* W *is general as above, meets* (i) *of Lemma 3.1 and if* (ii) *is replaced by requiring only upper semicontinuity of* $W_t(\cdot, \cdot, \cdot)$ *for every* $t \in T$. *However, in addition for every* $t \in T$ *the indirect utility function* $(p, \delta) \mapsto \sup_{x \in B_{t,p}} W_t(x, p, \delta)$ *must then be lower semicontinuous on* $P \times \mathcal{R}_C$. *Under these conditions the convexity condition for the set* C_t *in Assumption 2.3 can be omitted and in Assumption 2.4 only* $\omega(t) \in C_t$ *is needed for Theorem 3.1 to hold. This observation actually subsumes part* (i) *of the present remark. These observations have also some repercussions for Theorems 2.1, 2.2 and 2.3, but we prefer it to leave the details to the reader.*
(iv) *To* $J'_{j,t}(p, f)$ *as in Remark 2.2 corresponds the following relaxed externality mapping:*

$$I'_{j,t}(p, \delta) := \int_{A_{t,p}} [\int_{\mathbb{R}^d_+} \tilde{g}_j(t, \tau, x) \delta(\tau)(dx)] \mu(d\tau).$$

By the properties (1)-(2) stated in Remark 2.2, together with Assumption 2.5, $I'_{j,t}$ *is continuous on* $P \times \mathcal{R}_C$ *(apply Theorem 4.13 of [15]). So the above part* (iii) *of the present remark applies.*

Example 3.1 (i) *In Example 2.1 we had* $P = \{1\}$, $\omega \equiv 2$, $C_t \equiv [0, 2]$ *and we used* $U_t(x, 1, f) = |x - 1 + t - \int_0^t f|$. *As seen there, this corresponds to* $\tilde{g}_{t,1}(t, \tau, x) := x$ *if* $\tau \leq t$ *and* $:= 0$ *if* $\tau > t$. *By (3.1) this now gives* $I_{1,t}(\delta) = \int_0^t [\int_{[0,2]} x \, \delta(\tau)(dx)] d\tau$. *Then (3.2) gives* $W_t(x, 1, \delta) = |x - 1 + t - \int_0^t [\int_{[0,2]} x \, \delta(\tau)(dx)] d\tau|$ *for* $\delta \in \mathcal{R}_C$. *Let* $\delta_* \in \mathcal{R}_C$ *be defined as follows: for each* t *in* $[0, 1]$, $\delta_*(t) \in \mathrm{Prob}([0, 2])$ *is the probability measure that is entirely concentrated in* 0 *and* 2 *with equal probabilities* $\frac{1}{2}$. *Then* $\int_{[0,2]} x \, \delta_*(\tau)(dx) = 1$ *for all* $\tau \in [0, 1]$, *causing* $W_t(x, 1, \delta_*) = |x - 1|$. *So* $\mathrm{argmax}_{[0,2]} W_t(x, 1, \delta_*) = \{0, 2\}$*; hence, for* $p_* = 1$ *the pair* (p_*, δ_*) *is a relaxed free disposal competitive equilibrium. The connection with Example 2.1 is as follows. Let* $\{f_k\}_k$ *be the sequence obtained in Example 2.1. Then, by Example 4.4 in [15], the*

corresponding sequence $\{\epsilon_{f_k}\}_k$ *of Dirac transition probabilities converges narrowly in* $\mathcal{R}([0,1];\mathbb{R}_+)$ *to the above transition probability* δ_*.
(ii) In Example 2.2 we had $\omega \equiv (\frac{1}{2},\frac{1}{2})$ *and used* $U_t(x,p,f) = |x - a(t,f)|$. *By (3.1)-(3.2), we have* $W_t(x,p,\delta) = |x - b(t,\delta)|$, *where* $b(t,\delta) := (\cos\xi(t,\delta), \sin\xi(t,\delta))$, *with* $\xi(t,\delta) := \frac{5\pi}{4} + \frac{\pi}{4}(\int_0^t [\int_{\mathbb{R}^2_+} (\min(x^1,1) - \min(x^2,1))\delta(\tau)(d(x^1,x^2))]d\tau$. *Let* $\delta_* \in \mathcal{R}_C$ *be defined as follows: for each* t *in* $[0,1]$ $\delta_*(t) \in \mathrm{Prob}(\mathbb{R}^2_+)$ *is the probability measure that is entirely concentrated in the two points* $(1,0)$ *and* $(0,1)$, *with equal probabilities* $\frac{1}{2}$. *Then* $\int_{\mathbb{R}^2_+}(\min(x^1,1) - \min(x^2,1))\delta_*(\tau)(d(x^1,x^2)) = 0$ *for all* $\tau \in [0,1]$, *causing* $W_t(x,p,\delta_*) = |x - (-\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})|$. *So for* $p_* := (\frac{1}{2},\frac{1}{2})$ *we find* $\mathrm{argmax}_{B_{t,p_*}} W_t(x,p_*,\delta_*) = \{(1,0),(0,1)\}$. *This shows that* (p_*,δ_*) *is a relaxed competitive equilibrium. To explain the connection with Example 2.2, let* $\{f_k\}_k$ *be the sequence obtained in Example 2.2. By Example 4.4 in [15] the corresponding sequence* $\{\epsilon_{f_k}\}_k$ *converges narrowly in* $\mathcal{R}([0,1];\mathbb{R}^2_+)$ *to* δ_*, *as defined above.*

The connection between the competitive equilibria of Theorems 2.1 and Theorem 3.1, as suggested by the above examples, will be confirmed in section 5.

Corollary 3.1 *In addition to Assumptions 2.1, 2.2 and 2.3 (but not necessarily with* C_t *convex for all* t*), suppose that there exists an integrable function* $\psi : T \to \mathbb{R}_+$ *such that*

$$\sup_{x \in C_t} |x| \leq \psi(t) \text{ for every } t \in T.$$

Let $W' : C \times \mathcal{R}_C \to [-\infty,+\infty]$ *be such that*

$$W'(\cdot,\cdot,\delta) \text{ is } \mathcal{T} \otimes \mathcal{B}(\mathbb{R}^d)\text{-measurable for every } \delta \in \mathcal{R}_C,$$

$$W'(t,\cdot,\cdot) \text{ is continuous on } C_t \times \mathcal{R}_C \text{ for every } t \in T.$$

Then there exists $\delta_* \in \mathcal{R}_C$ *such that*

$$\delta_*(t)(\mathrm{argmax}_{x \in C_t} W'_t(x,\delta_*)) = 1 \text{ for a.e. } t \text{ in } T.$$

This result remains valid if $C \subset T \times \mathbb{R}^d$ *is assumed instead of* $C \subset T \times \mathbb{R}^d_+$.

Proof. Let $\omega(t) := \psi(t)e$, where $e := (1,\ldots,1)$; then evidently $B_{t,p} = C_t$ for all $p \in P$, so Remark 3.2(i) applies to all $t \in T$. With Remark 3.2(iii) in mind, we apply Theorem 3.1(i) to the function $W(t,x,p,\delta) := W'_t(x,\delta)$. Hence, there exists a pair $(p_*,\delta_*) \in P \times \mathcal{R}_C$ such that $\delta_*(t)(\mathrm{argmax}_{B_{t,p_*}} W'_t(\cdot,\delta_*)) = 1$ a.e. By $B_{t,p_*} = C_t$ we obtain what had to be proven, because, similar to the proof of Corollary 3.1, we can also invoke footnote 3. QED

The existence result of Schmeidler [38, Theorem 1] is a special case of Corollary 3.1 (recall that our previous Corollary 2.1 generalizes Theorem 2 of [38]). To see this, take all sets C_t to be equal to a fixed finite set $\{a_1,\ldots,a_m\} \subset \mathbb{R}^d_+$; then every $\delta \in \mathcal{R}_C$ can be identified with the bounded and measurable (whence integrable) vector function $t \mapsto (\delta(t)(\{a_1\}),\ldots,\delta(t)(\{a_m\}))$. So in this case the restriction of the narrow topology to $\mathcal{R}_C \cap \mathcal{R}^{Dirac}(T;\mathbb{R}^d_+)$ is immediately seen to coincide with the $\mathcal{L}^1$-weak topology, which is as used in [38]. See also [9]. A small discrepancy with [38] concerns the measurability hypotheses used for the payoff profile W', but this can be overcome by observing that the only way in which measurability of W manifests itself is via measurability of the graphs of the best response multifunctions $t \mapsto \text{argmax}_{x \in C_t} W_t(x, p, \delta)$ in the above proof of Theorem 3.1. In other words: it would have been enough to require just that, and such a condition is fulfilled in [38].

4 Proof of Theorem 3.1

Let $e \in \mathbb{R}^d_+$ be the vector given by $e := (1, \ldots, 1)$. For $m \in \mathbb{N}$ define

$$P^m := \{p \in P : p \geq \frac{1}{m} e\} = \{p \in \mathbb{R}^d_+ : \sum_{i=1}^d p^i = 1 \text{ and } p^i \geq \frac{1}{m} \text{ for } 1 \leq i \leq d\}$$

and

$$C^m_t := \{x \in C_t : e \cdot x \leq m \sum_{i=1}^d \omega^i(t)\}.$$

Note that $t \mapsto C^m_t$ is integrably bounded because of Assumption 2.2. The following elementary inclusion is important:

$$B_{t,p} \subset C^m_t \text{ for every } t \in T \text{ and } p \in P^m. \tag{4.1}$$

Since C^m_t is obviously compact, this implies that $B_{t,p}$ is also compact for every $t \in T$ and $p \in P^m$. Clearly, the set $C^m := \{(t, x) \in C : e \cdot x \leq m \sum_i \omega^i(t)\}$ is product measurable, so the corresponding set $\mathcal{R}_{C^m}$ of transition probabilities can be defined in complete analogy to $\mathcal{R}_C$ above: it is the set of all $\delta \in \mathcal{R}(T; \mathbb{R}^d_+)$ such that $\delta(t)(C^m_t) = 1$ for a.e. t in T.

Lemma 4.1 *For every $m \in \mathbb{N}$ the set $\mathcal{R}_{C^m}$ is a narrowly compact, convex and nonempty subset of $\mathcal{R}_C$.*

Proof. Define $h^m : T \times \mathbb{R}^d_+ \to [0, +\infty]$ as follows: $h^m(t, x) := 0$ if $(t, x) \in C^m$ and $h^m(t, x) := +\infty$ otherwise. Evidently, $h^m(t, \cdot)$ is inf-compact on $\mathbb{R}^d_+$ for every $t \in T$ and $I_{h^m}(\delta) = 0$ for all $\delta \in \mathcal{R}_{C^m}$. Thus, $\mathcal{R}_{C^m}$ is tight (see Definition A.4), so by Proposition A.4 it follows that $\mathcal{R}_{C^m}$ is relatively narrowly compact. But since $\mathcal{R}_{C^m}$ is precisely equal to the set $\{\delta \in \mathcal{R}(T; \mathbb{R}^d_+) : I_{h^m}(\delta) \leq 0\}$, it follows from the definition of the narrow topology that $\mathcal{R}_{C^m}$ is closed as well, because $h^m(t, \cdot)$ is also lower semicontinuous on $\mathbb{R}^d_+$ for every $t \in T$. So $\mathcal{R}_{C^m}$ is narrowly compact; also, it is trivially convex and nonempty (it contains the Dirac transition probability ϵ_ω). QED

Inspired by [12], we introduce for every $m \in \mathbb{N}$ the multifunction $F^m : P^m \times \mathcal{R}_{C^m} \to 2^{P^m \times \mathcal{R}_{C^m}}$ by defining $F^m(p, \delta)$ to be the set of all $(q, \eta) \in P^m \times \mathcal{R}_{C^m}$ for which

(i) supp $\eta(t) \subset \text{argmax}_{x \in B_{t,p}} W_t(x, p, \delta)$ for a.e. t in T,

(ii) $q \in \text{argmax}_{p' \in P^m}\, p' \cdot \int_T (\text{bar } \delta(t) - \omega(t))\mu(dt)$.

Here "supp " denotes support (see appendix A) and the *barycenter* bar $\delta(t)$ of $\delta(t) \in \text{Prob}(\mathbb{R}^d_+)$, is the vector in $\mathbb{R}^d$ whose i-th component is defined by

$$(\text{bar } \delta(t))^i := \int_{\mathbb{R}^d_+} x^i \delta(t)(dx) = \int_{C^m_t} x^i \delta(t)(dx). \tag{4.2}$$

This is just the classical first moment (expectation) of the probability measure $\delta(t)$. By compactness of the set C^m_t, this expression is a well-defined number in $\mathbb{R}_+$. Also the integral in (ii) is well-defined, because of integrable boundedness of $t \mapsto C^m_t$ and measurability of $t \mapsto \text{bar } \delta(t) : T \to \mathbb{R}^d_+$ (the latter follows by standard measure theory, referenced in appendix A).

Lemma 4.2 *For every $m \in \mathbb{N}$ the multifunction F^m has a closed graph and compact convex nonempty values.*

Proof. The proof consists of two parts.

Part 1. First, we prove that the graph of F^m is closed; this can be done by means of an argument that only involves sequences (recall that $\mathcal{R}_C$ is semimetrizable). Let $\{(p_n, \delta_n, q_n, \eta_n)\}$ converge to $(p_0, \delta_0, q_0, \eta_0)$ and suppose that $(q_n, \eta_n) \in F^m(p_n, \delta_n)$ for all $n \in \mathbb{N}$. Then by Proposition A.3 supp $\eta_0(t) \subset \text{Ls}_n\text{supp } \eta_n(t)$ for a.e. t. First, we claim that every point $x_0 \in \text{Ls}_n\text{supp } \eta_n(t)$ belongs to the set $\text{argmax}_{B_{t,p_0}} W_t(\cdot, p_0, \delta_0)$. For this it is enough to prove that for any $x \in C_t$ (actually, $x \in C^m_t$ would suffice) with $p_0 \cdot x < p_0 \cdot \omega(t)$ the inequality $W_t(x_0, p_0, \delta_0) \geq W_t(x, p_0, \delta_0)$ obtains, for clearly B_{t,p_0} is the closure of $B'_{t,p_0} := \{x' \in C_t : p_0 \cdot x' < p_0 \cdot \omega(t)\}$. This holds by convexity of C_t and nonemptiness of B'_{t,p_0} (such nonemptiness follows by Assumption 2.4). Suppose without loss of generality that x_0 is the limit of a sequence $\{x_n\}_n$ where $x_n \in \text{supp } \eta_n(t) \subset \text{argmax}_{B_{t,p_n}} W_t(\cdot, p_n, \delta_n)$ for every n. For large enough n we have $p_n \cdot x < p_n \cdot \omega(t)$, so $W_t(x, p_n, \delta_n) \leq W_t(x_n, p_n, \delta_n)$. By the continuity of W_t (see Lemma 3.1(*ii*)) this gives $W_t(x, p_0, \delta_0) \leq W_t(x_0, p_0, \delta_0)$ in the limit. So our claim has been demonstrated. It follows from the above that supp $\eta_0(t) \subset \text{argmax}_{B_{t,p_0}} W_t(\cdot, p_0, \delta_0)$ for a.e. t.

Next, we prove that $\delta \mapsto \int_T \text{bar } \delta \, d\mu$ is narrowly continuous on $\mathcal{R}_{C^m}$. Indeed, lower semicontinuity of $\delta \mapsto \int_T (\text{bar } \delta)^i$ follows by noting that $(\text{bar } \delta)^i = I_{g^m_i}(\delta)$. Here $g^m_i(t, x) := x^i$ if $(t, x) \in C^m$ and $g^m_i(t, x) := +\infty$ otherwise defines a function $g^m_i \in \mathcal{G}^{bb}(T; \mathbb{R}^d_+)$. Next, upper semicontinuity of $\delta \mapsto (\text{bar } \delta)^i$ must also be proven, which is equivalent to lower semicontinuity of $\delta \mapsto -(\text{bar } \delta)^i$. Now $-\int_T (\text{bar } \delta)^i = I_{\bar{g}^m_i}(\delta)$, where $\bar{g}^m_i(t, x) := -x^i$ if $(t, x) \in C^m$ and $\bar{g}^m_i(t, x) := +\infty$ otherwise, defines a function $\bar{g}^m_i \in \mathcal{G}^{bb}(T; \mathbb{R}^d_+)$. Note that for the latter argument it is essential that we use the truncation C^m_t of C_t, because this ensures integrable boundedness from below of $\bar{g}^m_i$. So we have demonstrated that $\delta \mapsto \int_T \text{bar } \delta$ is narrowly continuous. Hence, it follows easily that $q_0 \in \text{argmax}_{p' \in P^m} p' \cdot \int_T (\text{bar } \delta_0(t) - \omega(t))\mu(dt)$, since we were given that

$q_n \in \text{argmax}_{p' \in P^m} p' \cdot \int_T (\text{bar } \delta_n(t) - \omega(t))\mu(dt)$ for all n. This proves that (q_0, η_0) belongs to $F^m(p_0, \delta_0)$. We conclude that the graph of F^m is closed.

Part 2. In the second part of the proof we verify that $F^m(p, \delta)$ is compact and nonempty for arbitrary $p \in P^m$ and $\delta \in \mathcal{R}_{C^m}$ (of course, convexity is trivial). First, for every $t \in T$ the set $K_t := \text{argmax}_{B_{t,p}} W_t(\cdot, p, \delta) \subset C_t^m$ is nonempty and compact, for, by what was said following (4.1), this is a simple application of the Weierstrass theorem and Lemma 3.1(*ii*). By product measurability of the set C^m, the multifunction $t \mapsto C_t^m$ has a Castaing representation [23, III.7]. Hence, by Lemma 3.1 the graph of $t \mapsto K_t$ is measurable. So it follows by the von Neumann-Aumann measurable selection theorem [23, III.22] that there exists a $\mathcal{T}$-measurable function $f : T \to \mathbb{R}^d_+$ such that $f(t) \in K_t$ a.e. But then ϵ_f, the Dirac transition probability that corresponds to this measurable selection, clearly satisfies supp $\epsilon_f(t) \subset \text{argmax}_{B_{t,p}} W_t(\cdot, p, \delta)$ a.e. By (4.1) this also gives $\epsilon_f \in \mathcal{R}_{C^m}$. Finally, it follows very simply from the Weierstrass theorem that $\text{argmax}_{p' \in P^m} p' \cdot \int_T (\text{bar } \delta(t) - \omega(t))\mu(dt)$ is nonempty and compact as well. So $F^m(p, \delta)$ is nonempty. Second, the set of all $\eta \in \mathcal{R}_{C^m}$ such that supp $\eta(t) \subset K_t$ a.e., which by (4.1) coincides with the set $\{\eta \in \mathcal{R}(T; \mathbb{R}^d_+) : \text{supp } \eta(t)(K_t) = 1 \text{ a.e.}\}$, is narrowly compact. This is seen by imitating the proof of Lemma 4.1. So we conclude that $F^m(p, \delta)$ is also compact. QED

We observe that if all budget sets $B_{t,p}$, $p \in P$, are identical, then neither interiority of $\omega(t)$ in C_t nor convexity of C_t are needed for Lemma 4.2 to hold. This justifies Remark 3.2(*i*).

Lemma 4.3 *For every $m \in \mathbb{N}$ there exists $(q_m^*, \eta_m^*) \in P^m \times \mathcal{R}_{C^m}$ such that*

(i) $q_m^* \in \text{argmax}_{p \in P^m} p \cdot \int_T (\text{bar } \eta_m^*(t) - \omega(t))\mu(dt)$,

(ii) supp $\eta_m^*(t) \subset \text{argmax}_{x \in B_{t,q_m^*}} W_t(x, q_m^*, \eta_m^*)$ *for a.e. t in T.*

Proof. By Lemmas 4.1 and 4.2 the stage is set for an application of the Kakutani fixed point theorem. Here one can use the non-Hausdorff version of that result (see [13]), from which the stated result follows immediately. QED

Alternatively, to prove the above lemma one can introduce the quotient of $\mathcal{R}_{C^m}$, which is Hausdorff, by the usual μ-equivalence relation (two transition probabilities are said to be *μ-equivalent* if they coincide almost everywhere on T). Another alternative would be to use the ws-topology, quite related to the narrow topology, which is automatically Hausdorff [16].

Lemma 4.4 (*i*) *The sequence $\{a_m\}_m$ in $\mathbb{R}^d$, defined by*

$$a_m := \int_T (\text{bar } \eta_m^* - \omega) d\mu,$$

satisfies $\max_i a_m^i \leq \frac{1}{m-d} \int_T \sum_i \omega^i \, d\mu$ *for all $m \geq d+1$. Hence, it is relatively compact.*
(*ii*) *The sequence $\{\eta_m^*\}_m$ in $\mathcal{R}_C$ is tight and hence relatively compact.*

Proof. (i) Lemma 4.3(i) implies $\sup_{p\in P^m} p\cdot a_m = q_m^*\cdot a_m$. Here $q_m^*\cdot a_m\leq 0$ holds by Lemma 4.3(ii). Writing $a_m=(a_m^i)_{i=1}^d$, we can easily verify that

$$\sup_{p\in P^m} p\cdot a_m = \frac{1}{m}\sum_i a_m^i + (1-\frac{d}{m})\max_i a_m^i,$$

because of $P^m = \frac{1}{m}e + \{q\in\mathbb{R}_+^d : q\cdot e = 1-\frac{d}{m}\}$. Trivially, $a_m^i \geq -\int_T \omega^i$; thus, the above gives the desired inequality in (i). It is now easily seen that $\{a_m\}_m$ is bounded, so relative compactness of $\{a_m\}_m$ follows by the Heine-Borel theorem.

(ii) By definition (4.2) of the barycenter we have

$$\sum_i a_m^i = \int_T [\int_{\mathbb{R}_+^d} (\sum_i x^i - \sum_i \omega^i(t))\eta_m^*(t)(dx)]\mu(dt)$$

for every m. By boundedness of $\{a_m\}_m$ and integrability of ω this clearly implies

$$\sup_m \int_T [\int_{\mathbb{R}_+^d} \sum_i x^i \eta_m^*(t)(dx)]\mu(dt) < +\infty,$$

which amounts to $\sup_m I_h(\eta_m^*) < +\infty$ when we define $h(t,x) := \sum_i x^i$ on $T\times\mathbb{R}_+^d$. Because $x\mapsto \sum_i x^i$ is evidently inf-compact on $\mathbb{R}_+^d$, the claimed tightness of $\{\eta_m^*\}_m$ has been demonstrated. Therefore, relative narrow compactness of $\{\eta_m^*\}_m$ follows by Proposition A.4. QED

Remark 4.1 *Lemmas 4.3 and 4.4(i) express that (q_m^*,η_m^*) is an "almost free disposal" relaxed competitive equilibrium for the artificial economy $\mathcal{RE}_m := \langle T, \mathcal{R}_{C^m}, \omega, W\rangle$ if $m\geq d+1$.*

Lemma 4.5 (i) *The sequence $\{(a_m, q_m^*, \eta_m^*)\}_m$ contains a subsequence $\{(a_{m_j}, q_{m_j}^*, \eta_{m_j}^*)\}_j$ such that $\{a_{m_j}\}_j$ converges to a vector $a_*\in\mathbb{R}^d$ with $a_*\leq 0$, $\{q_{m_j}^*\}_j$ converges in $\mathbb{R}^d$ to a vector $p_*\in P$ and $\{\eta_{m_j}^*\}_j$ converges to a transition probability $\delta_*\in\mathcal{R}(T;\mathbb{R}_+^d)$.*
(ii) *The pair (p_*,δ_*) in part (i) constitutes a free disposal relaxed competitive equilibrium.*

Proof. (i) The existence of a subsequence that converges as stated follows immediately from Lemma 4.4 and obvious compactness of the price simplex P. Also, $a_*\leq 0$ is immediate by Lemma 4.4(i).

(ii) Fix $i\in\{1,\ldots,d\}$. We know that $a_{m_j}^i := \int_T (\text{bar }\eta_{m_j}^* - \omega)^i d\mu \to a_*^i \leq 0$ for $i=1,\ldots,d$. By (4.2) we have $a_m^i = I_{g_i}(\eta_m^*) - \int_T \omega^i\, d\mu$, where $g_i(t,x) := x^i$. Because $g_i : T\times\mathbb{R}_+^d \to \mathbb{R}$ is nonnegative, it is integrably bounded from below, and obviously $g_i(t,\cdot)$ is lower semicontinuous on $\mathbb{R}_+^d$ for every $t\in T$. Hence, $g_i\in\mathcal{G}^{bb}(T;\mathbb{R}_+^d)$, so $\lim_j I_{g_i}(\eta_{m_j}^*) \geq I_{g_i}(\delta_*)$ follows by Definition A.3. Therefore, the above yields

$$\int_T (\text{bar }\delta_*)^i d\mu = I_{g_i}(\delta_*) \leq a_*^i + \int_T \omega^i\, d\mu \leq \int_T \omega^i\, d\mu.$$

This proves that (p_*,δ_*) satisfies part (i) of Definition 3.1.

Next, by Lemma 4.3(ii) we have supp $\eta^*_{m_j}(t) \subset \mathrm{argmax}_{x\in B_{t,q^*_{m_j}}} W_t(x, q^*_{m_j}, \eta^*_{m_j})$ for every j and for a.e. t in T. By part (i), $\{\eta^*_{m_j}\}_j$ converges narrowly to δ_*. We can now mimick the first half of part 1 of the proof of Lemma 4.2. This gives supp $\delta_*(t) \subset \mathrm{argmax}_{B_{t,p_*}} W_t(\cdot, p_*, \delta_*)$ a.e. Hence, (p_*, δ_*) also satisfies part (ii) of Definition 3.1. QED

Obviously, the above Lemma 4.5 concludes the proof of Theorem 3.1(i). We now turn to the proof of part (ii) of that theorem. It is an obvious adaptation of similar strengthening of free disposal equilibrium existence results into equilibrium existence results (e.g., see [27, p. 38]). Define $\omega_k := \omega + \frac{1}{k}e$, where, as before, $e := (1, \dots, 1)$. Because now $C_t = \mathbb{R}^d_+$ by Assumption 2.7, we see that for every fixed $k \in \mathbb{N}$ Assumption 2.4 is met by ω_k. So for every $k \in \mathbb{N}$ Theorem 3.1(i), just proven, gives the existence of a pair $(p^*_k, \delta^*_k) \in P \times \mathcal{R}_C$ such that

(i) $\int_T \mathrm{bar}\ \delta^*_k \, d\mu \leq \int_T \omega_k \, d\mu$,

(ii) $\delta^*_k(t)(\mathrm{argmax}_{x\in B_{t,p^*_k}} W_t(x, p^*_k, \delta^*_k)) = 1$ for a.e. t in T.

Lemma 4.6 (i) *The sequence* $\{(p^*_k, \delta^*_k)\}_k$ *contains a subsequence* $\{(p^*_{k_j}, \delta^*_{k_j})\}_j$ *such that* $\{p^*_{k_j}\}_j$ *converges to some* $p_{**} \in P$ *and* $\{\delta^*_{k_j}\}_j$ *converges narrowly to some* δ_{**} *in* $\mathcal{R}_C$.
(ii) *The pair* (p_{**}, δ_{**}) *in part* (i) *constitutes a relaxed competitive equilibrium.*

Proof. (i) Since $\int_T \omega_k \to \int_T \omega$, part ($i$) of the preceding equilibrium statement for (p^*_k, δ^*_k) gives tightness of $\{\delta^*_k\}_k$. To see this, use again $h(t,x) := \sum_i x^i$, then $\sup_k I_h(\delta^*_k) < +\infty$, just as in the proof of Lemma 4.4(ii). So, just as in that proof and the proof of Lemma 4.5, we conclude that a subsequence $\{(p^*_{k_j}, \delta^*_{k_j})\}_j$ exists such that $\{p^*_{k_j}\}_j$ converges to some $p_{**} \in P$ and $\{\delta^*_{k_j}\}_j$ converges narrowly to some δ_{**} in $\mathcal{R}_C$.

(ii) Assumption 2.4' implies $p_{**} \cdot \int_T \omega \, d\mu > 0$; therefore $p_{**} \cdot \omega(t) > 0$ for all t in some non-null set $\tilde{A} \in \mathcal{T}$. Hence, the set $\{x \in \mathbb{R}^d_+ : p_{**} \cdot x < p_{**} \cdot \omega(t)\}$ is nonempty for $t \in \tilde{A}$, because of $C_t \equiv \mathbb{R}^d_+$. This allows us to conclude, just as in part 1 of the proof of Lemma 4.2 (where the above set is denoted $B'_{t,p_{**}}$), that

$$\delta_{**}(t)(\mathrm{argmax}_{x\in B_{t,p_{**}}} W_t(x, p_{**}, \delta_{**})) = 1 \tag{4.3}$$

holds for a.e. t in $\tilde{A}$. Thus, because of Assumption 2.7, we conclude that p_{**} belongs to $\mathbb{R}^d_{++}$. Armed with this fact, we can now conclude that $p_{**} \cdot \omega(t)$ is strictly positive for *every* $t \in T$ for which $\omega(t) \neq 0$. So it follows that (4.3) actually holds for a.e. t in T. Just as in the final part of the proof of Theorem 3.1(i), we also get

$$\int_T \mathrm{bar}\ \delta_{**} \, d\mu \leq \liminf_j \int_T \omega_{k_j} \, d\mu = \int_T \omega \, d\mu.$$

However, here we can say more, for it follows by Assumption 2.7 from (4.3), already shown to be valid a.e., that for a.e. t in T the support of $\delta_{**}(t)$ is actually concentrated on the budget line in $B_{t,p_{**}}$. In other words, we get $p_{**} \cdot \text{bar } \delta_{**}(t) = p_{**} \cdot \omega(t)$. So $p_{**} \cdot \int_T (\text{bar } \delta_{**} - \omega) d\mu = 0$. By $\int_T (\text{bar } \delta_{**} - \omega) d\mu \leq 0$ and $p_{**} \in \mathbb{R}^d_{++}$ this implies the desired equality $\int_T (\text{bar } \delta_{**} - \omega) d\mu = 0$. This proves that (p_{**}, δ_{**}) constitutes a mixed competitive equilibrium. QED

5 Proof of Theorems 2.1, 2.2 and 2.3

Theorem 3.1, just proven, will now be employed to prove the main existence results of this paper.

Proof of Theorem 2.1(i). The assumptions of Theorem 3.1(i) hold, so it follows that there exists $(p_*, \delta_*) \in P \times \mathcal{R}_C$ such that $\int_T [\int_{C_t} x\, \delta_*(t)(dx)] \mu(dt) \leq \int_T \omega$ and such that (ii) of Definition 3.1 is fulfilled. The previous inequality can be rewritten as

$$I_{g_i}(\delta_*) \leq \int_T \omega^i \, d\mu \text{ for } i = 1, \ldots, d.$$

Here $g_i : T \times \mathbb{R}^d_+ \to \mathbb{R}$ is defined by $g_i(t, x) := x^i$. Therefore, by Proposition A.5 there exists a sequence $\{f_k\}$ of measurable functions $f_k : T \to \mathbb{R}^d_+$ such that $\{\epsilon_{f_k}\}_k$ converges narrowly to δ_* and such that for every k both $\int_T f_k \leq \int_T \omega$ and $f_k(t) \in \text{argmax}_{x \in B_{t,p_*}} W_t(x, p_*, \delta_*)$ a.e. hold (here the property of supp $\delta_*(t)$ in Definition 3.1(ii) is used). By $B_{t,p_*} \subset C_t$ this implies that $\{f_k\}_k \subset \mathcal{M}_C$. Since ($i$) in Definition 2.1 is already fulfilled, it remains to prove (ii) of that definition. To this end, let

$$\hat{W}_t(\delta) := \sup_{x \in B_{t,p_*}} W_t(x, p_*, \delta).$$

Then

$$\int_T [\int_{\mathbb{R}^d_+} (\arctan(W_t(x, p_*, \delta_*) - \hat{W}_t(\delta_*))) \delta_*(t)(dx)] \mu(dt) = 0 \tag{5.4}$$

by Definition 3.1(ii); here the arctangent transformation serves to ensure boundedness, and hence integrability. Fix $t \in T$. By Lemma 3.1(ii), the function $W_t(\cdot, p_*, \cdot)$ is continuous on $C_t \times \mathcal{R}_C$. Also, the set B_{t,p_*} is compact, because of the compactness hypothesis for C_t in Assumption 2.3'. Therefore, Berge's theorem of the maximum implies that $\hat{W}_t$ is continuous. So $\ell : T \times \hat{\mathbb{N}} \times \mathbb{R}^d_+ \to \mathbb{R}$, defined by

$$\ell(t, k, x) := \begin{cases} \arctan(W_t(x, p_*, \epsilon_{f_k}) - \hat{W}_t(\epsilon_{f_k})) & \text{if } k \in \mathbb{N}, \\ \arctan(W_t(x, p_*, \delta_*) - \hat{W}_t(\delta_*)) & \text{if } k = \infty, \end{cases}$$

defines a nonpositive normal integrand $\ell \in \mathcal{G}^{bb}(T; \mathbb{R}^d_+ \times \hat{\mathbb{N}})$, which is bounded from below by the value $-\pi/2$. Applying Proposition A.1 to this ℓ, the narrow convergence of $\{\epsilon_{f_k}\}_k$ to δ_* gives

$$\liminf_k \int_T \arctan(W_t(f_k(t), p_*, \epsilon_{f_k}) - \hat{W}_t(\epsilon_{f_k})) \mu(dt) = \int_T [\int_{\mathbb{R}^d_+} \ell(t, \infty, x) \delta_*(t)(dx)] \mu(dt) = 0,$$

where the right hand side is zero by combining the above definition of $\ell(t, \infty, x)$ and (5.4). By nonpositivity of the integrands $\phi_k : t \mapsto \arctan(W_t(f_k(t), p_*, \epsilon_{f_k}) - \arctan \hat{W}_t(\epsilon_{f_k})))$, this tells us that $\{\phi_k\}_k$ converges to 0 in L^1-seminorm. *A fortiori*, it follows that $\{\phi_k\}_k$ converges to 0 in measure, as does the sequence $\{\tan \phi_k\}_k$, because the tangent function is continuous (e.g., use Exercise 20.27 in [22]). By definition of convergence in measure, this gives (ii) in Definition 2.1.

(ii) The proof of this part is a modification of the previous proof: First, note that now the assumptions of Theorem 3.1(ii) hold, so it follows that there exists $(p_*, \delta_*) \in P \times \mathcal{R}_C$ such that both (i) and (ii) of Definition 3.1 are fulfilled, and this time with $p_* \in \mathbb{R}^d_{++}$. We use the previous $g_i(t, x) := x^i$, and also $g_{d+i}(t, x) := -x^i$. Then Proposition A.5 yields existence of a sequence $\{\epsilon_{f_k}\}_k$ that converges narrowly to δ_* and such that for every k both $\int_T f_k = \int_T \omega$ and $f_k(t) \in \operatorname{argmax}_{x \in B_{t,p_*}} W_t(x, p_*, \delta_*)$ a.e. hold. Again, the continuity of $\hat{W}_t$ follows by Berge's theorem, but this time the needed compactness of B_{t,p_*} follows by $p_* \in \mathbb{R}^d_{++}$. The remainder of the proof is precisely the same.

Proof of Theorem 2.2. The assumptions of Theorem 3.1(ii) hold, so it follows that there exists $(p_*, \delta_*) \in P \times \mathcal{R}_C$ that satisfies Definition 3.1. Define $g_i(t, x) := x^i$ for $i = 1, \ldots, d$ and set $g_{d+j} := \bar{g}_j$, $j = 1, \ldots, N$ (cf. Assumption 2.5'). Then $I_{g_i}(\delta_*) < +\infty$ for $i = 1, \ldots, d + N$ by the force of Definition 3.1(i) (see also an identity in the previous proof) and by Assumption 2.5'(iv). Similar to the previous proof, it follows by Proposition A.5 that there exists a sequence $\{f_k\}_k$ of measurable functions $f_k : T \to \mathbb{R}^d_+$ such that $\{\epsilon_{f_k}\}_k$ converges narrowly to δ_* and such that for every k we have (1) $\int_T f_k^i = \int_T \omega^i$ for $i = 1, \ldots, d$, (2) $I_{\bar{g}_j}(\epsilon_{f_k}) = I_{\bar{g}_j}(\delta_*)$ for $j = 1, \ldots, N$ and (3) $f_k(t) \in \operatorname{supp} \delta_*(t) \subset \operatorname{argmax}_{x \in B_{t,p_*}} W_t(x, p_*, \delta_*)$ a.e. Define f_* to be the first function f_1 from this sequence. Then (1) implies $\int_T f_* = \int_T \omega$, (3) implies $f_*(t) \in \operatorname{argmax}_{x \in B_{t,p_*}} W_t(x, p_*, \delta_*)$ a.e. and (2) implies $J_t(f_*) = I_t(\epsilon_{f_*}) = I_t(\delta_*)$. We conclude that $f_*(t) \in \operatorname{argmax}_{x \in B_{t,p_*}} W_t(x, p_*, \epsilon_{f_*})$ a.e., which brings the proof to an end, because of Remark 3.1. QED

Proof of Theorem 2.3. The assumptions of Theorem 3.1(i) are fulfilled, so it follows that there exists $(p_*, \delta_*) \in P \times \mathcal{R}_C$ that satisfy both Definition 3.1(i), but with $\leq$ instead of $=$, and Definition 3.1(ii). The above proof of Theorem 2.2 can be repeated (only this time its (1) runs as follows: $\int_T f_*^i = I_{g_i}(\delta_*) \leq \int_T \omega^i$ for $i = 1, \ldots, d$, but the rest stays the same). QED

A The narrow topology: highlights

In this appendix we recapitulate five essential results on the narrow topology for transition probabilities (alias Young measures). This topology goes back to L.C. Young and E.J. McShane in the calculus of variations (late thirties) and, ten to fifteen years later but independently, to A. Wald and L.M. Le Cam in statistical decision theory. Via [20, 39] this led the present author in [4] to start a program of systematic extension of the main themes of classical narrow convergence of probability measures (e.g., cf. [21]) to narrow convergence of transition probabilities. This program resulted in extensive treatments, presented in [10, 14, 15]. For additional relevant expositions the reader is referred to [7, 16]; however, all the material in this appendix is taken from [5], [15] and [34].

Let $(T, \mathcal{T}, \mu)$ be a finite measure space. In this paper the following convention is used to define the integral over T of a measurable function $\psi : T \to \mathbb{R}$ that is integrably bounded from below (i.e., $\max(-\psi, 0)$ is μ-integrable). The integral $\int_T \psi d\mu$ is defined by $\int_T \psi d\mu := \int_T \max(\psi, 0) d\mu - \int_T \max(-\psi, 0) d\mu$ *even when* $\int_T \max(\psi, 0) d\mu$ *should equal* $+\infty$. For an ordinary integrable function ψ this convention contains no news, but when $\max(\psi, 0)$ is not μ-integrable, it means that we use $\int_T \psi \, d\mu := +\infty$. Let Y be a complete separable metric space; this is a special case of the Suslin spaces used in [5, 15] (of course, the present paper only uses $Y = \mathbb{R}^d_+$). The space Y is equipped with its Borel σ-algebra $\mathcal{B}(Y)$. Let $\mathrm{Prob}(Y)$ denote the set of all probability measures on $(Y, \mathcal{B}(Y))$.

Definition A.1 A *transition probability* (also known under the names of *Markov kernel* or *Young measure*) from $(T, \mathcal{T})$ into $(Y, \mathcal{B}(Y))$ is a function $\delta : T \to \mathrm{Prob}(Y)$ such that $t \mapsto \delta(t)(B)$ is $\mathcal{T}$-measurable for every $B \in \mathcal{B}(Y)$. The set of all such transition probabilities is denoted by $\mathcal{R}(T; Y)$. Special *Dirac* transition probabilities correspond to measurable functions: if $f : T \to Y$ is measurable, then ϵ_f, defined by $\epsilon_f(t)(B) := 1_B(f(t)) = 1_{f^{-1}(B)}(t)$ is easily seen to belong to $\mathcal{R}(T; Y)$. This subset of $\mathcal{R}(T; Y)$ is denoted by $\mathcal{R}^{\mathrm{Dirac}}(T; Y)$.

Definition A.2 A *normal integrand* on $T \times Y$ is a $\mathcal{T} \otimes \mathcal{B}(Y)$-measurable function $g : T \times Y \to \mathbb{R}$ such that $g(t, \cdot)$ is lower semicontinuous on Y for every $t \in T$. This function g is said to be *integrably bounded from below* if there exists an integrable $\phi : T \to \mathbb{R}$ such that $g(t, y) \geq \phi(t)$ for all $(t, y) \in T \times Y$. The set of all such normal integrands that are integrably bounded from below is denoted by $\mathcal{G}^{bb}(T; Y)$.

By the theory surrounding Fubini's theorem (see [34, III.2]) for every $\mathcal{T} \otimes \mathcal{B}(Y)$-measurable function $g : T \times Y \to (-\infty, +\infty]$ that is integrably bounded from below the integral

$$I_g(\delta) := \int_T [\int_Y g(t, y) \delta(t)(dy)] \mu(dt) \tag{A.1}$$

is well-defined in $(-\infty, +\infty]$ and, in particular, $t \mapsto \int_Y g(t, y) \delta(t)(dy)$ is $\mathcal{T}$-measurable.

Definition A.3 The *narrow topology* on $\mathcal{R}(T, Y)$ is the coarsest topology for which all mappings $\delta \mapsto I_g(\delta)$, $g \in \mathcal{G}^{bb}(T; Y)$, are lower semicontinuous.

Clearly, the narrow topology on $\mathcal{R}(T; Y)$ generalizes the usual weak topology on $\mathrm{Prob}(Y)$ (e.g., see [21]), as is seen by taking $(T, \mathcal{T}, \mu)$ to be trivial (say, a singleton). Several equivalent characterizations of the narrow topology exist; e.g., see [7, Theorem 2.3]. The following related result comes from [4] (see also [15, Theorem 4.13]), where $\hat{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ is the usual Alexandrov compactification of the set of natural numbers:

Proposition A.1 *Let $\{\delta_k\}_k$ be a sequence narrowly converging to δ_0 in $\mathcal{R}(T;Y)$ and let $\ell \in \mathcal{G}^{bb}(T;\hat{\mathbb{N}} \times Y)$. Then*

$$\liminf_k \int_T [\int_Y \ell(t,k,y)\delta_k(t)(dy)]\mu(dt) \geq \int_T [\int_Y \ell(t,\infty,y)\delta_0(t)(dy)]\mu(dt).$$

Semimetrizability of the narrow topology on $\mathcal{R}(T;Y)$ allows us to concentrate on sequences. It is guaranteed when $(T,\mathcal{T},\mu)$ is separable:

Proposition A.2 *If $(T,\mathcal{T},\mu)$ is separable then $\mathcal{R}(T;Y)$ is semimetrizable for the weak topology.*

This result follows by Theorem 2.3 of [7]. In this paper an important property of sequential narrow convergence in $\mathcal{R}(T;Y)$, in terms of the Kuratowski limes superior of the pointwise support sets, is as follows (such results were introduced in [4] – see also Theorem 4.12 of [15]). First, recall that the *support* of a probability measure $\nu \in \text{Prob}(Y)$ is defined as the smallest closed set $F \subset Y$ such that $\nu(F) = 1$; this set is denoted by supp ν.

Proposition A.3 *For every sequence $\{\delta_k\}_k$ that narrowly converges to δ_0 in $\mathcal{R}(T;Y)$*

$$\text{supp } \delta_0(t) \subset \text{Ls}_k \text{supp } \delta_k(t) \textit{ for a.e. } t \textit{ in } T$$

Recall here that the *Kuratowski limes superior* of a sequence $\{S_k\}_k$ of subsets of Y is the set $\text{Ls}_k S_k$ of all $y \in Y$ for which there exists a subsequence $\{S_{k_j}\}_j$ and corresponding points $y_{k_j} \in S_{k_j}$ such that $\{y_{k_j}\}_j$ converges to y. To obtain narrowly convergent sequences, the following generalization of Prohorov's theorem is very important. The following definition was given in [4], inspired by [20]:

Definition A.4 A subset $\mathcal{R}_0$ of $\mathcal{R}(T;Y)$ is *tight* if there exists a $\mathcal{T} \otimes \mathcal{B}(Y)$-measurable function $h : T \times Y \to [0,+\infty]$ such that for every $t \in T$ the function $h(t,\cdot)$ is inf-compact[4] and such that $\sup_{\delta \in \mathcal{R}_0} I_h(\delta) < +\infty$.

The next result, Prohorov's theorem for transition probabilities, follows by Theorem 2.2 of [7] or Theorem 4.10 of [15] (incidentally, it holds regardless of semimetrizability as established in Proposition A.2).

Proposition A.4 *Every tight subset of $\mathcal{R}(T;Y)$ is both relatively compact and relatively sequentially compact for the narrow topology.*

We remark that in the setting of this appendix, where Y is complete separable and metric, tightness is also a necessary condition for either relative compactness property in the above proposition to hold. The following denseness result, well-known in relaxed control theory in a less general form, is Corollary 3 in [5] (see also [11] and Theorem 5.6 in [15]).

[4] I.e., the set $\{y \in Y : h(t,y) \leq \beta\}$ is compact for every $\beta \in \mathbb{R}_+$.

Proposition A.5 *Suppose that* $(T, \mathcal{T}, \mu)$ *is nonatomic. For* $m \in \mathbb{N}$ *let* $\{g_1, \ldots, g_m\}$ *be a finite collection in* $\mathcal{G}^{bb}(T; Y)$. *Let* $\delta \in \mathcal{R}(T; Y)$ *be such that* $I_{g_j}(\delta) < +\infty$ *for* $j = 1, \ldots, m$. *Then there exists a sequence* $\{\epsilon_{f_k}\}_k$ *in* $\mathcal{R}^{Dirac}(T; Y)$ *that narrowly converges to* δ *such that for every* $k \in \mathbb{N}$ *the following hold:* $I_{g_j}(\epsilon_{f_k}) \leq I_{g_j}(\delta)$ *for every* $j \in \{1, \ldots, m\}$ *and* $f_k(t) \in \text{supp}\ \delta(t)$ *for a.e.* t *in* T.

Here the multifunction $\Delta : t \mapsto \text{supp}\ \delta(t)$, which has a product-measurable graph and closed values, is as required in [5, Corollary 3].

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