SINGULARITY EXCHANGE AT INFINITY

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Abstract. In families of polynomial functions one may encounter "singularity exchange at infinity" when singular points escape from the space and produce "virtual" singularities of the limit polynomial, which have themselves an influence on the topology. The total quantity of singularity involved in this phenomenon may not be conserved. Inspite of the fact that some of the ingredients do not behave well in deformations, we prove semi-continuity results which enable us to find rules of the exchange phenomenon.

1. Introduction

How does the topology change in a family of complex polynomials $f_s : \mathbb{C}^n \to \mathbb{C}$? One has at hand the highly developed local theory of deformations of singularities, but how to use such methods and what else is needed in order to treat our more global problem? Our study focuses on the new phenomena which occur in such a context. One of the novelties turns out to be the *singularity exchange at infinity*: as s tends to, say 0, some isolated singularities of f_s tend to infinity and disappear from $\mathbb{C}^n$, producing in the same time "virtual" singularities of f_0. Even if not situated in $\mathbb{C}^n$, those virtual singularities manifest in the topology of f_0. Let us briefly explain our approach and results.

For a given polynomial function $f_s : \mathbb{C}^n \to \mathbb{C}$ there is a well defined general fibre G_s, since the set of atypical values $\Lambda(f_s)$ is a finite set, by Verdier's result [Ve]. When specialising to f_0, the number of atypical values may vary (decrease, increase or be constant) and the topology of the general fibre may change. In order to get more insight, we focus on constant degree families and on classes of polynomials (F-class $\subset$ B-class $\subset$ W-class, cf Definition 2.1) such that the *vanishing cycles of* f_s (i.e. generators of the reduced homology of G_s) are concentrated in dimension $n-1$ and are localisable at finitely many points, in the affine space or "at infinity". In the affine space $\mathbb{C}^n$, such a point is a singular point of f_s. The sum of all affine Milnor numbers is the total Milnor number $\mu(s)$, which has an algebraic interpretation as the dimension of the quotient algebra $\mathbb{C}[x_1, \ldots, x_n]/(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$. Singularities at infinity (which are not on $\mathbb{C}^n$ but can be detected by using a compactification of $\mathbb{C}^n$) come equipped with so-called Milnor-Lê numbers and their sum is denoted by $\lambda(s)$. It is well-known that the Euler characteristic of the generic fiber G_s is $1 + (-1)^{n-1}(\mu(s) + \lambda(s))$.

The natural problem which arises, and to which we devote this paper is to understand the behaviour of the points supporting the vanishing cycles of f_s, i.e. the μ and λ-singularities, when $s \to 0$. It is well known that for singularities which tend to a μ-singular point the local balance law is *conservative*, i.e. the number of local vanishing

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cycles is conserved. In our global setting, the new phenomenon which occurs is that some μ-singularities may tend to infinity and change into λ-singularities. Our main Theorem 3.3 proves the following local balance law at some λ-singularity:

– the number of local vanishing cycles of μ and λ-singularities tending to a λ-singular point is not conserved in general, but it is *lower semi-continuous.*

In complete generality, as the homology may not be concentrated in dimension $n-1$, we however prove the

– global semi-continuity of the highest Betti number: $b_{n-1}(G_0) \leq b_{n-1}(G_s)$.

It is interesting to remark that these results are in contrast to the upper semi-continuity which holds in the case of specialising holomorphic function germs. Secondly, this result is important in finding rules of the singularity exchange at infinity. We get the following:

– Inside the F-class, a λ-singularity cannot be deformed into only μ-singularities (Corollary 5.4).
– In B-type families with constant $\mu+\lambda$, the local balance law at any λ-singularity of f_0 is conservative and atypical values cannot escape to infinity (Corollary 5.1).
– In B-type families with constant $\mu+\lambda$, the monodromy fibrations over any admissible loop (in particular, the monodromy fibrations at infinity) are isotopic in the family, whenever $n \neq 3$ (Theorem 5.5).

This shows for instance that families with constant $\mu+\lambda$ are more rigid. Another fact about singularity exchange at infinity is provided by Theorem 4.2:

– In B-type families with constant generic singularity type at infinity, λ-singularities of f_0 are locally persistent in f_s but cannot split such that more than one λ-singularity occurs in the same fibre.

In §6 we supply with a zoo of examples illustrating especially the behaviour of λ-singularities in the singularity exchange.

2. Classes of polynomials

In order to get better grip on exchange phenomena which take place in the neighbourhood of infinity, we shall work with polynomials for which the singularities at infinity are isolated, in the sense that we make precise in the following.

Let P be a deformation of f_0, i.e. $P : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}$ is a family of polynomial functions $P(x,s) = f_s(x)$ such that $f_0 = f$. Unless otherwise stated, we assume in this paper that our deformations depend holomorphically on the parameter $s \in \mathbb{C}^k$. We assume that $\deg f_s$ is independent on s, for s in some neighbourhood of 0, and we denote it by d. Following [ST2], we attach to P the following hypersurface:

$$\mathbb{Y} = \{([x:x_0], s, t) \in \mathbb{P}^n \times \mathbb{C}^k \times \mathbb{C} \mid \tilde{P}(x, x_0, s) - t x_0^d = 0\},$$

where $\tilde{P}$ denotes the homogenized of P by the variable x_0, considering s as parameter varying in a small neighbourhood of $0 \in \mathbb{C}^k$. Let $\tau : \mathbb{Y} \to \mathbb{C}$ be the projection to the t-coordinate. This extends the map P to a proper one in the sense that $\mathbb{C}^n \times \mathbb{C}^k$ is embedded in $\mathbb{Y}$ (via the graph of P) and $\tau_{|\mathbb{C}^n \times \mathbb{C}^k} = P$. Let $\sigma : \mathbb{Y} \to \mathbb{C}^k$ denote the projection to the s-coordinates.

Notations. $\mathbb{Y}_{s,*} := \mathbb{Y} \cap \sigma^{-1}(s)$, $\mathbb{Y}_{*,t} := \mathbb{Y} \cap \tau^{-1}(t)$ and $\mathbb{Y}_{s,t} := \mathbb{Y}_{s,*} \cap \tau^{-1}(t) = \mathbb{Y}_{*,t} \cap \sigma^{-1}(s)$. Note that $\mathbb{Y}_{s,t}$ is the closure in $\mathbb{P}^n$ of the affine hypersurface $f_s^{-1}(t) \subset \mathbb{C}^n$.

Let $\mathbb{Y}^\infty := \mathbb{Y} \cap \{x_0 = 0\} = \{P_d(x,s) = 0\} \times \mathbb{C}$ be the hyperplane at infinity of $\mathbb{Y}$, where P_d is the degree d homogeneous part of P in variables $x \in \mathbb{C}^n$. Remark that for any fixed s, $\mathbb{Y}^\infty_{s,t} := \mathbb{Y}_{s,t} \cap \mathbb{Y}^\infty$ does not depend on t.

For a single polynomial f, we shall use the notation $\tau : \mathbb{X} \to \mathbb{C}$ for the proper extension.

Definition 2.1. We consider the following classes of polynomials:

(i) f is a *F-type* polynomial if its compactified fibres and their restrictions to the hyperplane at infinity have at most isolated singularities.
(ii) f is a *B-type* polynomial if its compactified fibres have at most isolated singularities.
(iii) f is a *W-type* polynomial if its proper extension $\tau_{|\mathbb{X}}$ has only isolated singularities in a stratified sense ([ST1]).

We have: F-class $\subset$ B-class $\subset$ W-class. The first inclusion is clear from the definition and the second one is proved in [ST1]. In 2 variables, if f has isolated singularities in $\mathbb{C}^2$, then it is automatically of F-type. Deformations inside the F-class were introduced in [ST2] under the name $\mathcal{F}$ISI deformations. Broughton [Br] considered for the first time B-type polynomials and studied the topology of their general fibers. The W-class of polynomials appears in [ST1]. In deformations of a polynomial f_0 we usually require to stay inside the same class but we may also deform into a more special class (like B-type into F-type, Example 6.6).

The singular locus of $\mathbb{Y}$, $\operatorname{Sing}\mathbb{Y} := \{x_0 = 0, \frac{\partial P_d}{\partial x}(x,s) = 0,\ P_{d-1}(x,s) = 0, \frac{\partial P_d}{\partial s}(x,s) = 0\} \times \mathbb{C}$ is included in $\mathbb{Y}^\infty$ and is a product-space by the t-coordinate. It depends only on the degrees d and $d-1$ parts of P with respect to the variables x.

Let $\Sigma := \{x_0 = 0, \frac{\partial P_d}{\partial x}(x,s) = 0,\ P_{d-1}(x,s) = 0\} \subset \mathbb{P}^{n-1}$. If we fix s, the singular locus of $\mathbb{Y}_{s,*}$ is the analytic set $\Sigma_s \times \mathbb{C}$, where $\Sigma_s := \Sigma \cap \{\sigma = s\}$, and it is the union of the singularities at the hyperplane at infinity of the hypersurfaces $\mathbb{Y}_{s,t}$, for $t \in \mathbb{C}$.

We denote by $W_s := \{[x] \in \mathbb{P}^{n-1} \mid \frac{\partial P_d}{\partial x}(x,s) = 0\}$ the set of points at infinity where $\mathbb{Y}^\infty_{s,t}$ is singular, in other words where $\mathbb{Y}_{s,t}$ is either singular or tangent to $\{x_0 = 0\}$. It does not depend on t and we have $\Sigma_s \subset W_s$.

Remark 2.2. From the above definition and the expressions of the singular loci we have the following characterisation:

(i) f_0 is a B-type polynomial $\Leftrightarrow \dim \operatorname{Sing} f_0 \leq 0$ and $\dim \Sigma_0 \leq 0$,
(ii) f_0 is a F-type polynomial $\Leftrightarrow \dim \operatorname{Sing} f_0 \leq 0$ and $\dim W_0 \leq 0$.

Let us also remark that $\dim \Sigma_0 \leq 0$ (respectively $\dim W_0 \leq 0$) implies that $\dim \Sigma_s \leq 0$ (respectively $\dim \operatorname{Sing} f_s \leq 0$), whereas $\dim \operatorname{Sing} f_0 \leq 0$ does not imply automatically $\dim \operatorname{Sing} f_s \leq 0$ for $s \neq 0$.

3. Local semi-continuity

3.1. **Deformations in general.** It is well known that the $(n-1)$th Betti number of the Milnor fibre of a holomorphic function germ is upper semi-continuous, i.e. it does not decrease under specialisation. In case of a polynomial $f_s : \mathbb{C}^n \to \mathbb{C}$, the role of the Milnor fibre is played by the general fibre G_s of f_s. This is a Stein manifold of dimension $n-1$

and therefore it has the homotopy type of a CW complex of dimension $\leq n-1$, which is also finite, since G_s is algebraic. Moreover, the $(n-1)$th homology group with integer coefficients is free. We prove the following general specialisation result, which appears to be in contrast to the one in the local case.

Proposition 3.1. *Let $P : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}$ be any holomorphic deformation of a polynomial $f_0 := P(\cdot, 0) : \mathbb{C}^n \to \mathbb{C}$. Then the general fibre G_0 of f_0 can be naturally embedded into the general fibre G_s of f_s, for $s \neq 0$ close enough to 0. The embedding $G_0 \subset G_s$ induces an inclusion $H_{n-1}(G_0) \hookrightarrow H_{n-1}(G_s)$ which is compatible with the intersection form.*

Proof. It is enough to consider a 1-parameter family of hypersurfaces $\{f_s^{-1}(t)\}_{s\in L} \subset \mathbb{C}^n$, for fixed t, where L denotes some parametrised complex curve through 0. We denote by X_t the total space over a small neighbourhood L_ε of 0 in L. By choosing t generic enough, we may assume that $f_s^{-1}(t)$ is a generic fibre of f_s, for s in a small enough neighbourhood of 0. Let $\sigma : X_t \to L_\varepsilon$ denote the projection. Now X_t is the total space of a family of non-singular hypersurfaces. Since $\sigma^{-1}(0)$ is an affine hypersurface, by taking a large enough radius R, we get $\partial \bar{B}_{R'} \pitchfork \sigma^{-1}(0)$, for all $R' \geq R$. Moreover, the sphere $\partial \bar{B}_R$ is transversal to all nearby fibres $\sigma^{-1}(s)$, for small enough s. It follows that the projection from the pair of spaces $\sigma : (X_t \cap (B_R \times \mathbb{C}), X_t \cap (\partial \bar{B}_R \times \mathbb{C})) \to L_\varepsilon$ is a proper submersion and hence, by Ehresmann's theorem, it is a trivial fibration. By the above transversality argument, we have $B_R \cap \sigma^{-1}(0) \overset{\text{diff}}{\simeq} B_R \cap \sigma^{-1}(s)$. This shows the first claim.

The affine hypersurfaces $\sigma^{-1}(s)$ are finite cell complexes of dimension $\leq n-1$. By the classical Andreotti-Frankel [AF] argument for the distance function, the hypersurface $\sigma^{-1}(s)$ is obtained from $B_R \cap \sigma^{-1}(s)$ by adding cells of index at most $n-1$. This shows that $H_n(G_s, G_0) = 0$, so the second claim. The compatibility with the intersection form is standard. ☐

Under certain conditions we can also compare the monodromy fibrations "at infinity", see §5.2. Proposition 3.1 will actually be exploited through the following *semi-continuity of highest Betti number*, as a consequence of the inclusion of homology groups:

$$b_{n-1}(G_s) \geq b_{n-1}(G_0), \text{ for } s \neq 0 \text{ close enough to } 0. \tag{3.1}$$

3.2. **Semi-continuity at infinity.** Let P be a deformation of f_0 such that f_s is of W-type, for $s \neq 0$. After [ST1], the reduced homology of the general fiber G_s is concentrated in dimension $n-1$ and the vanishing cycles of f_s are quantified by two well defined, non-negative integers: $\mu(s) =$ the *total Milnor number*, respectively $\lambda(s) =$ *the total Milnor-Lê number at infinity*. It is shown in [ST1] that, for a W-type polynomial f_s, the general fiber G_s is a bouquet of $n-1$ spheres and that $b_{n-1}(G_s) = \mu(s) + \lambda(s)$. It is also shown in [Pa, ST1] that the vanishing cycles are localized at certain points, either in the affine space or at infinity, and which we shall call *μ-singularities* and *λ-singularities* respectively. To such a singular point $p \in \mathbb{Y}_{s,*}$ one associates its local Milnor number denoted $\mu_p(s)$ or its Milnor-Lê number $\lambda_p(s)$.

By [ST1], the atypical fibers of a W-type polynomial f_s are exactly those fibers which contain μ or λ-singularities; equivalently, those of which the Euler characteristic is different from $\chi(G_s)$. We denote by $\Lambda(f_s)$ the set of atypical values of f_s, which turns out to be exactly the image by f_s of the set of μ and λ-singularities.

Let us assume that f_0 is itself a W-type polynomial. Then the above cited facts together with our semi-continuity result (3.1) show that, for s close to 0 we have:

$$\mu(s) + \lambda(s) \geq \mu(0) + \lambda(0).$$

REMARK 3.2. The total Milnor number $\mu(s)$ is lower semi-continuous under specialization $s \to 0$. In case $\mu(s)$ decreases, we say that there is *loss of* μ *at infinity*, since this may only happen when one of the two following phenomena occur:

(a) the modulus of some critical point tends to infinity and the corresponding critical value is bounded ([ST2, Example 8.1]);
(b) the modulus of some critical value tends to infinity (Example [ST2, (8.2) and (8.3)]).

In contrast to $\mu(s)$, it turns out that $\lambda(s)$ is not semi-continuous; under specialization, it can increase or decrease (Example 6.1, 6.3). Moreover, the λ-values may behave like in (b) above, see Example 6.6.

We want to understand the behaviour of $\lambda(s)$ in more detail. In order to get more precise results, we focus on the B-class. Our main result is the following.

Theorem 3.3. (Lower semi-continuity at λ-singularities)
Let P be a constant degree one-parameter deformation inside the B-class. Then, locally at any λ-singularity $p \in \mathbb{Y}_{0,t}$ of f_0, we have:

$$\lambda_p(0) \leq \sum_i \lambda_{p_i}(s) + \sum_j \mu_{p_j}(s),$$

where p_i, resp. p_j, are the λ-singularities, resp. the μ-singularities of f_s which tend to the point p as $s \to 0$.

For the proof, we need some more notation.

Definition 3.4. Endow $\mathbb{Y}$ with the coarsest Whitney stratification $\mathcal{W}$ such that $\mathbb{C}^n \times \mathbb{C}^k \subset \mathbb{Y}$ is a stratum. Let $\Psi := (\sigma, \tau) : \mathbb{Y} \to \mathbb{C}^k \times \mathbb{C}$ be the projection. The *critical locus* $\operatorname{Crit}\Psi$ is the locus of points where the restriction of Ψ to some stratum of $\mathcal{W}$ is not a submersion.

When writing $\operatorname{Crit}\Psi$ we usually understand a small representative of the germ of $\operatorname{Crit}\Psi$ at $\mathbb{Y}_{0,*}$. It follows that $\operatorname{Crit}\Psi$ is a closed analytic set and that its affine part $\operatorname{Crit}\Psi \cap (\mathbb{C}^n \times \mathbb{C}^k)$ is the union, over $s \in \mathbb{C}^k$, of the affine critical loci of the polynomials f_s. Notice that both $\operatorname{Crit}\Psi$ and its affine part $\operatorname{Crit}\Psi \cap (\mathbb{C}^n \times \mathbb{C}^k)$ are in general not product spaces by the t-variable

REMARK 3.5. For a constant degree one-parametre deformation in the B-class, the stratification $\mathcal{W}$ has as maximal stratum in $\mathbb{P}^{n-1} \times \mathbb{C}$ the complement of the 2-surface $\Sigma \times \mathbb{C}$. At any point of this complement, all the spaces $\mathbb{Y}$, $\mathbb{Y}_{s,*}$ and $\mathbb{Y}_{s,t}$ are nonsingular. Therefore this stratum does not intersect $\operatorname{Crit}\Psi$. Since the map Ψ is submersive over a Zariski-open subset of any 2-dimensional stratum included in $\Sigma \times \mathbb{C}$, it follows that the part at infinity of $\operatorname{Crit}\Psi$ has dimension < 2. Since the affine part $\operatorname{Crit}\Psi \cap (\mathbb{C}^n \times \mathbb{C})$ has also dimension at most 1, we conclude that $\dim \operatorname{Crit}\Psi \leq 1$.

3.3. **Proof of Theorem 3.3.** By Remark 3.5 we have $\dim \mathrm{Crit}\Psi \leq 1$, but this fact does not insure that the functions σ and τ have isolated singularity with respect to $\mathcal{W}$. (It is precisely not the case in "almost all" examples.) Nevertheless, in the pencil $\sigma + \varepsilon\tau$, $\varepsilon \in \mathbb{C}$, all the functions except finitely many of them, are functions with isolated singularity at p, with respect to the stratification $\mathcal{W}$. Let us fix some ε close to zero and consider locally, in some good neighbourhood $\mathcal{B}$ of $(p, 0) \in \mathbb{Y}$, the couple of functions $\Psi_\varepsilon = (\sigma + \varepsilon\tau, \tau) \colon \mathcal{B} \to \mathbb{C}^2$.

The function $\tau_| \colon (\sigma + \varepsilon\tau)^{-1}(0) \to \mathbb{C}$ defines an isolated singularity at p. By applying the stratified Bouquet Theorem of [Ti1] we get that the general fiber of Ψ_ε—that is $\mathcal{B} \cap \Psi_\varepsilon^{-1}(s, t)$, for some $(s, t) \notin \mathrm{Disc}\,\Psi$—is homotopy equivalent to a bouquet $\bigvee S^{n-1}$; let ρ denote the number of S^{n-1} spheres in this bouquet.

On the other hand, the Milnor fiber at p of the function $\sigma + \varepsilon\tau$ is homotopy equivalent to a bouquet $\bigvee S^n$, by the same result *loc.cit.*; let ν denote the number of S^n spheres.

In the remainder, we count the vanishing cycles (I): along $(\sigma + \varepsilon\tau)^{-1}(0)$, respectively (II): along $(\sigma + \varepsilon\tau)^{-1}(u)$, for $u \neq 0$ close enough to 0, and we compare the results. The vanishing cycles are all in dimension $n - 2$. One may use Figure 1 in order to follow the computations; in this picture, the germ of the discriminant locus $\mathrm{Disc}\,\Psi$ at $\Psi(p)$ is the union of the τ-axis, σ-axis and some curves.

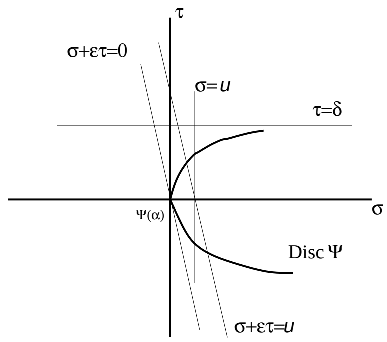

FIGURE 1. *Counting vanishing cycles.*

(I). We start with the fiber $\mathcal{B} \cap \Psi_\varepsilon^{-1}(0, \delta)$, where δ is close enough to 0. To obtain $\mathcal{B} \cap (\sigma + \varepsilon\tau)^{-1}(0)$, which is contractible, one attaches to $\mathcal{B} \cap \Psi_\varepsilon^{-1}(0, \delta)$ a certain number of $(n - 1)$ cells corresponding to the vanishing cycles at infinity, as $t \to 0$, in the family of fibers $\Psi_\varepsilon^{-1}(0, t)$. This is exactly the number ρ defined above and it is here the sum of two numbers, corresponding to the attaching in two steps, as we detail in the following. One is the number of cycles in $\mathcal{B} \cap \Psi_\varepsilon^{-1}(s, \delta)$, vanishing, as $s \to 0$, at points that tend to p when δ tends to 0; we denote this number by ξ. The other number is the number of cycles in $\mathcal{B} \cap \Psi^{-1}(0, t)$, vanishing as $t \to 0$; this number is $\lambda_p(0)$, by definition. From this one may draw the inequality: $\lambda_p(0) \leq \rho$.

(II). Here we start with the fiber $\mathcal{B} \cap \Psi_\varepsilon^{-1}(u, \delta)$, which is homeomorphic to $\mathcal{B} \cap \Psi_\varepsilon^{-1}(0, \delta)$ and to $\mathcal{B} \cap \Psi^{-1}(u, \delta)$. The Milnor fiber $\mathcal{B} \cap \{\sigma + \varepsilon\tau = u\}$ cuts the critical locus $\mathrm{Crit}\Psi$ at certain points p_k. The number of points, counted with multiplicities, is equal to the local intersection number $\mathrm{int}_p(\{\sigma + \varepsilon\tau = 0\}, \mathrm{Crit}\,\Psi)$. When walking along $\mathcal{B} \cap \{\sigma + \varepsilon\tau = u\}$, one has to add to the fiber $\mathcal{B} \cap \Psi_\varepsilon^{-1}(u, \delta)$ a number of cells corresponding to the vanishing cycles at points $\{\sigma+\varepsilon\tau = u\}\cap\{\sigma = 0\}$, which is just the number ξ defined above, and to the vanishing cycles at points $\{\sigma+\varepsilon\tau = u\}\cap\overline{\mathrm{Crit}\Psi\setminus\{\sigma = 0\}}$. The intersection number $\mathrm{int}_p(\{\sigma+\varepsilon\tau = 0\}, \overline{\mathrm{Crit}\Psi\setminus\{\sigma = 0\}})$ is less or equal to the intersection number $\mathrm{int}_p(\{\sigma = 0\}, \mathrm{Crit}\Psi)$. Now, when walking along $\mathcal{B}\cap\{\sigma = u\}$, one has to add to $\mathcal{B}\cap\Psi^{-1}(u,\delta)$ a number of cells corresponding to the vanishing cycles at points p_i and p_j, which number is, by definition, $\sum_i \lambda_{p_i}(u)+\sum_j \mu_{p_j}(u)$. We get the inequality: $\xi+\sum_i \lambda_{p_i}(u)+\sum_j \mu_{p_j}(u) \geq \rho+\nu$.

Finally, putting together the inequalities obtained at steps (I) and (II), we obtain:

$$\lambda_p(0) = \rho-\xi \leq \rho+\nu-\xi \leq \sum_i \lambda_{p_i}(u) + \sum_j \mu_{p_j}(u),$$

which proves our claim. $\square$

4. Persistence of λ-singularities

In order to get further information on the $\mu \mapsto \lambda$ exchanges we focus on two sub-classes of the B-class. In this section we define cgst-type deformations and in the next section we study deformation with constant $\mu+\lambda$.

Let us first remark that inside the B-class we have compactified fibres with only isolated singularities. The positions of these singularities depend only on s (and not on t). During deformations inside B-class these singularities can split or disappear.

Let us take some $x(0)\in\Sigma_0$. The non-splitting argument from [AC, Lê] tells us that the Milnor number of $\mathbb{Y}_{0,t}$ at $(x(0),t)$, for $t\notin\Lambda(0)$, is larger than the sum of the Milnor numbers of $\mathbb{Y}_{s,t}$ at all points $(x(s),t)$ such that $x_i(s)\to x(0)$, unless there is only one such singular point $x(s)$ and the Milnor number of $\mathbb{Y}_{s,t}$ at $(x(s),t)$ is independent on s. In the latter case we say that *the cgst assumption holds*.

Definition 4.1. We say that a constant degree deformation inside the B-class has *constant generic singularity type at infinity at some point* $x(0)\in\Sigma_0$ if the cgst assumption holds.

If the cgst assumption holds at all points in Σ_0, then we simply say *constant generic singularity type at infinity*.

Note that under the cgst assumption the highest Betti number of G_s is not necessarily constant in the B-class (see Example 6.6).

Theorem 4.2. *Let P be a constant degree deformation, inside the B-class, with constant generic singularity type at infinity. Then:*

(a) *λ-singularities of f_0 are locally persistent in f_s.*

(b) *a λ-singularity of f_0 cannot split such that two or more λ-singularities belong to the same fiber.*

Remark 4.3. Part (a) shows that λ-singularities cannot disappear when deforming f_0. The case which is not covered by part (b) can indeed occur, i.e. that some λ splits into λ's along a line $\{x(s)\}\times\mathbb{C}$, see Example 6.2.

Proof. **(a).** Let $(z,t_0)\in\Sigma_0\times\mathbb{C}$ be a λ-singularity of f_0. Let us denote by $G(y,s,t)$ the localisation of the map $\tilde{P}(x,x_0,s)-tx_0^d$ at the point $(z,0,t_0)\in\mathbb{Y}$. Let $y_0 = 0$ be the local equation of the hyperplane at infinity of $\mathbb{P}^n$. The idea is to consider the 2-parameter family of functions $G_{s,t}\colon \mathbb{C}^n \to \mathbb{C}$, where $G_{s,t}(y) = G(y,s,t)$. Then $G(y,s,t)$ is the germ of a deformation of the function $G_{0,t_0}(y)$.

We consider the germ at $(z,0,t_0)$ of the singular locus Γ of the map $(G,\sigma,\tau)\colon \mathbb{C}^n \to \mathbb{C}^3$. This is the union of the singular loci of the functions $G_{s,t}$, for varying s and t. We claim that Γ is a surface, more precisely, that every irreducible component Γ_i of Γ is a surface. We secondly claim that the projection $D \subset \mathbb{C}^3$ of Γ by the map (y_0,σ,τ) is a surface, in the sense that all its irreducible components are surfaces. Moreover, the projections $\Gamma \stackrel{(y_0,\sigma,\tau)}{\to} D$ and $D \stackrel{(s,t)}{\to} \mathbb{C}^2$ are finite (ramified) coverings.

All our claims follow from the following fact: the local Milnor number conserves in deformations of functions. The function germ G_{0,t_0} with Milnor number, say μ_0, deforms into a function $G_{s,t}$ with finitely many isolated singularities, and the total Milnor number is conserved, for any couple (s,t) close to $(0,t_0)$.

Let us now remark that the germ at $(z,0,t_0)$ of $\Sigma\times\mathbb{C}$ is a union of components of Γ and projects by (y_0,s,t) to the plane $D_0 := \{y_0 = 0\}$ of $\mathbb{C}^3$. However, the inclusion $D_0 \subset D$ cannot be an equality, by the above argument on the total "quantity of singularities" and since we have a jump $\lambda > 0$ at the point of origin $(z,0,t_0)$. So there must exist some other components of D. Every such component being a surface in $\mathbb{C}^3$, has to intersect the plane $D_0 \subset \mathbb{C}^3$ along a curve. Therefore, for every point (s',t') of such a curve, the sum of Milnor numbers of the function G on the hypersurface $\{y_0 = 0, \sigma = s', \tau = t'\}$ (where the sum is taken over the singular points that tend to the original point $(z,0,t_0)$ when $s' \to 0$) is therefore strictly higher than the one computed for a generic point of the plane D_0. Therefore our claim (a) will be proved if we prove two things:
(i). the singularities of G on the hypersurface $\{y_0 = 0, \sigma = s', \tau = t'\}$ that tend to the original point $(z,0,t_0)$ when $s' \to 0$ are included into $G = 0$, and
(ii). there exists a component $D_i \subset D$ such that $D_i \cap D_0 \neq D \cap \{s = 0\}$.

To show (i), let $g_k(y,s)$ denote the degree k part of P after localising it at p and note that $G(y,s,t) = g_d(y,s) + y_0(g_{d-1}(y,s) + \cdots) - ty_0^d$. Then observe that the set:

$$\Gamma \cap \{y_0 = 0\} = \{\frac{\partial g_d}{\partial y} = 0, g_{d-1} = 0\} \tag{4.1}$$

does not depend on the variable t and its slice by $\{\sigma = s, \tau = t\}$ consists of finitely many points. These points may fall into two types: (I). points on $\{g_d = 0\}$, and therefore on $\{G = 0\}$, and (II). points not on $\{g_d = 0\}$. We show that type II points do not actually occur. This is a consequence of our hypothesis on the constancy of generic singularity type at infinity, as follows. By choosing a generic $\hat{t}$ such that $\hat{t} \notin \Lambda(s)$ for all s, and by using the independence on t of the set (4.1), this condition implies that type II points cannot collide with type I points along the slice $\{y_0 = 0, \sigma = s, \tau = \hat{t}\}$ as $s \to 0$. By absurd, if there were collision, then there would exist a singularity in the slice $\{G = 0, y_0 = 0, \sigma = 0, \tau = \hat{t}\}$ with Milnor number higher than the generic singularity type at infinity. It then follows that:

$$\Gamma \cap \{y_0 = 0\} = \Gamma \cap \{G = y_0 = 0\} \tag{4.2}$$

which proves (i). Now observe that the equality (4.2) also proves (ii), by the a similar reason: if there were a component D_i such that $D_i \cap D_0 = D \cap \{s = 0\}$ then there would exist a singularity in the slice $\{G = 0, y_0 = 0, \sigma = 0, \tau = \hat{t}\}$ with Milnor number higher

than the generic singularity type at infinity. Notice that we have in fact proved more, namely:
(ii'). there is no component $D_i \neq D_0$ such that $D_i \cap D_0 = D \cap \{s = 0\}$.

This ends the proof of (a).

(b). Suppose that there were collision of some singularities out of which two or more λ-singularities are in the same fibre. Then there are at least two different points $z_i \neq z_j$ of Σ_s which collide as $s \to 0$. This situation is excluded by the cgst assumption (Definition 4.1). □

5. Conservation in $\mu + \lambda$ constant deformations

In §6 we comment a couple of examples where the inequality of Theorem 3.3 is strict. Nevertheless, in some situations the equality holds, as follows.

Corollary 5.1. *Let P be a constant degree deformation inside the B-class such that $\mu(s) + \lambda(s)$ is constant. Then:*

(a) *As $s \to 0$, there cannot be loss of μ or of λ with corresponding atypical values tending to infinity.*
(b) *λ is upper semi-continuous, i.e. $\lambda(s) \leq \lambda(0)$.*
(c) *there is local conservation of $\mu + \lambda$ at any λ-singularity of f_0.*

Proof. (a). If there is loss of μ or of λ, then this must necessarily be compensated by increase of λ at some singularity at infinity of f_0. But Theorem 3.3 shows that the local $\mu + \lambda$ cannot increase in the limit.
(b). is clear since $\mu(s) + \lambda(s)$ is constant and $\mu(s)$ can only decrease when $s \to 0$.
(c). Global conservation of $\mu + \lambda$ together with local semi-continuity (by Theorem 3.3) imply local conservation. □

Remark 5.2. It is interesting to point out that within the class of B-type polynomials there is no inclusion relation between the properties "constant generic singularity type" and "$\mu(s) + \lambda(s)$ constant", see Examples 6.4, 6.6. We shall see in the following that in the F-class the two conditions are equivalent because of the relation (5.2).

5.1. Rigidity in deformations with constant $\mu + \lambda$.

For B-type polynomials, we have the formula:

$$b_{n-1}(G_s) = \mu(s) + \lambda(s) = (-1)^{n-1}(\chi^{n,d} - 1) - \sum_{x \in \Sigma(s)} \mu_{x,\mathrm{gen}}(s) - (-1)^{n-1}\chi^{\infty}(s), \tag{5.1}$$

where $\chi^{n,d} = \chi(V_{gen}^{n,d}) = n + 1 - \frac{1}{d}\{1 + (-1)^n(d-1)^{n+1}\}$ is the Euler characteristic of the smooth hypersurface $V_{gen}^{n,d}$ of degree d in $\mathbb{P}^n$ and $\chi^{\infty}(s) := \chi(\{f_d(x, s) = 0\})$. We denote by $\mu_{x,\mathrm{gen}}(s)$ the Milnor number of the singularity of $\mathbb{Y}_{s,t}$ at the point $(x, t) \in \Sigma_s \times \mathbb{C}$, for a generic value of t. The change in $b_{n-1}(G_s)$ can be described in terms of change in $\mu_{x,\mathrm{gen}}(s)$ and $\chi^{\infty}(s)$. Since the latter is not necessarily semi-continuous (cf Examples 6.4–6.6), we may expect interesting exchange of data between the two types of contributions.

Proposition 5.3. *Let $\Delta\chi^{\infty}$ denote $(-1)^n(\chi^{\infty}(s) - \chi^{\infty}(0))$. Then:*

(a) *If $\Delta\chi^{\infty} < 0$ then the deformation is not cgst.*
(b) *If $\Delta\chi^{\infty} = 0$ and the deformation has constant $\mu + \lambda$ then all $\mu_{x,\mathrm{gen}}(s)$ are constant.*

(c) *If $\Delta\chi^\infty > 0$ then the deformation cannot have constant $\mu + \lambda$.*

□

For F-type polynomials, formula (5.1) takes the following form [ST2, (2.1) and (2.4)]:

$$\mu(s) + \lambda(s) = (d-1)^n - \sum_{x \in \Sigma(s)} \mu_{x,\mathrm{gen}}(s) - \sum_{x \in W(s)} \mu_x^\infty(s), \tag{5.2}$$

where $\mu_x^\infty(s)$ denotes the Milnor number of the singularity of $\mathbb{Y}_{s,t} \cap H^\infty$ at the point $(x,t) \in W_s \times \mathbb{C}$, which is independent on the value of t.

Here the change in the Betti number $b_{n-1}(G_s)$ can be described in terms of change in the $\mu_{x,\mathrm{gen}}(s)$ and change in $\mu_x^\infty(s)$. Both are semi-continuous, so they are forced to be constant in $\mu + \lambda$ constant families. Note that in the F-class we have $\Delta\chi^\infty \geq 0$.

Let us also observe that the class of F-type polynomials such that $\mu + \lambda$ is constant verifies the hypotheses of Theorem 4.2. It has been noticed by the first named author that in the deformations with constant $\mu + \lambda$ which occur in Siersma-Smeltink's lists [SS] the value of λ cannot be dropped to 0. In view if the above observation, this behaviour is now completely explained by Theorem 4.2(a), as follows:

Corollary 5.4. *Inside the F-class, a λ-singularity cannot be deformed into only μ-singularities by a constant degree deformation with constant $\mu + \lambda$.* □

5.2. **Monodromy in families with constant $\mu + \lambda$.** For some polynomial f_0, one calls *monodromy at infinity* the monodromy around a large enough disc D containing all the atypical values of f_0. The locally trivial fibration above the boundary $\partial\bar{D}$ of the disc is called *fibration at infinity.*

The global Lê-Ramanujam problem consists in showing the constancy of the monodromy fibration in a family with constant $\mu + \lambda$. Actually one can state the same problem for any *admissible loop* γ in $\mathbb{C}$, i.e. a simple loop (homeomorphic to a circle) such that it does not contain any atypical value of f_s, for all s close enough to 0.

The second named author proved this (cf [Ti2, Ti3]) in case there is no loss of μ at infinity of type 3.2(b). This hypothesis can now be removed, due to our Corollary 5.1(a). Therefore, by revisiting the statement [Ti3, Theorem 5.2], we get:

Theorem 5.5. *Let P be a constant degree deformation inside the B-class. If $\mu + \lambda$ is constant then the monodromy fibrations over any admissible loop are isotopic in the family, whenever $n \neq 3$. In particular, the monodromy fibrations at infinity are isotopic in the family, whenever $n \neq 3$.* □

6. Examples

6.1. **F-class examples; behaviour of λ.**

Example 6.1. $f_s = (xy)^3 + sxy + x$, see Figure 2(a).
This is a deformation inside the F-class, with constant $\mu + \lambda$, where λ increases. For $s \neq 0$: $\lambda = 1 + 1$ and $\mu = 1$. For $s = 0$: $\lambda = 3$ and $\mu = 0$.

EXAMPLE 6.2. $f_s = (xy)^4 + s(xy)^2 + x$, see Figure 2(b).
This deformation has constant $\mu = 0$, $\lambda(0) = 2$ at one point and $\lambda(s) = 1+1$ at two points at infinity which differ by the value of t only, namely $([0:1], s, 0)$ and $([0:1], s, -s^2/4)$.

EXAMPLE 6.3. $f_s = xy^4 + s(xy)^2 + y$, see Figure 2(c).
Here λ decreases. For $s \neq 0$: $\lambda = 2$ and $\mu = 5$. For $s = 0$: $\lambda = 1$ and $\mu = 0$.

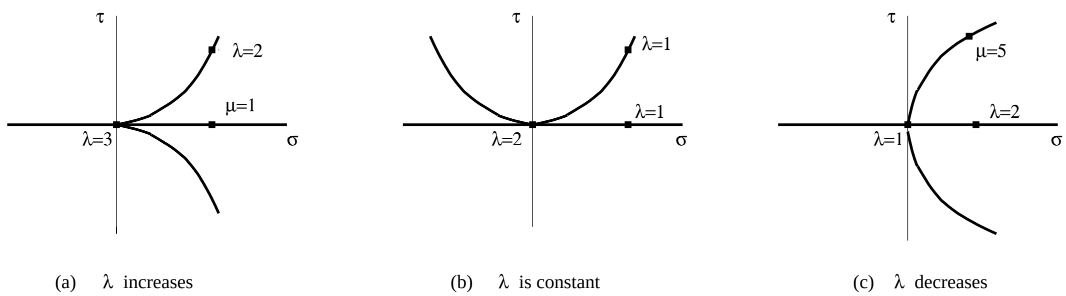

FIGURE 2. *Mixed splitting in (a) and (c); pure λ-splitting in (b).*

6.2. **B-class examples.** We use in this section formula (5.1). We pay special attention to the sign of $\Delta\chi^\infty$ and illustrate the difference between cgst-type deformations and $(\mu+\lambda)$-constant deformations.

EXAMPLE 6.4. $f_s = x^4 + sz^4 + z^3 + y$.
This is a deformation inside the B-class with constant $\mu + \lambda$, which is not cgst at infinity (Definition 4.1). We have $\lambda = \mu = 0$ for all s. Next, $\mathbb{Y}_{s,t}$ is singular only at $p := ([0:1:0], 0)$ and the singularities of $\mathbb{Y}^\infty_{0,t}$ change from a single smooth line $\{x^4 = 0\}$ with a special point p on it into the isolated point p which is a $\tilde{E}_7$ singularity of $\mathbb{Y}^\infty_{s,t}$. We use the notation $\oplus$ for the Thom-Sebastiani sum of two types of singularities in separate variables. We have:
$s = 0$: the generic type is $A_3 \oplus E_7$ with $\mu = 21$ and $\chi(\mathbb{Y}^\infty_{0,t}) = 2$.
$s \neq 0$: the generic type is $A_3 \oplus E_6$ with $\mu = 18$ and $\chi(\mathbb{Y}^\infty_{s,t}) = 5$.
The jumps of $+3$ and -3 compensate each other.

EXAMPLE 6.5. $f_s = x^4 + sz^4 + z^2y + z$.
This is a $\mu + \lambda$ constant B-type family, with two different singular points of $\mathbb{Y}_{0,t}$ at infinity, and where the change in one point interacts with the other. It is locally cgst in one point, but not in the other. We have that $\lambda = 3$ and $\mu = 0$ for all s, $\mathbb{Y}_{s,t}$ is singular in $p := ([0:1:0], 0)$ for all s (see types below) and in $q := ([1:0:0], 0)$ for $s = 0$ with type A_3. The singularities of $\mathbb{Y}^\infty_{s,t}$ change from a single smooth line $\{x^4 = 0\}$ into the isolated point p with $\tilde{E}_7$ singularity.

For the point p we have for all s the generic type $A_3 \oplus D_5$ if $t \neq 0$, which jumps to $A_3 \oplus D_6$ if $t = 0$. This causes $\lambda = 3$.

At q, the A_3 -singularity for $s = 0$ gets smoothed (independently of t) and here the deformation is not locally cgst. The change on the level of $\chi(\mathbb{Y}^\infty_{s,t})$ is from 2 to 5, so $\Delta\chi^\infty = -3$, which compensates the “vanishing” of the A_3 on $\mathbb{Y}_{s,t}$.

So the contribution in the point q jumped to the point p, due to the existence of the "channel" which consists of the line of non-isolated singularities $\{x^4 = 0\}$.

Example 6.6. $f_s = x^2y + x + z^2 + sz^3$.
This is a cgst B-type family, where $\mu + \lambda$ is not constant. Notice that f_s is F-type for all $s \neq 0$, whereas f_0 is not F-type (but still B-type). The generic type at infinity is D_4 for all s and there is a jump $D_4 \to D_5$ for $t = 0$ and all s. For $s \neq 0$ a second jump $D_4 \to D_5$ occurs for $t = c/s^2$, for some constant c.

There are no affine critical points, i.e. $\mu(s) = 0$ for all s, but $\lambda(s) = 2$ if $s \neq 0$ and $\lambda(0) = 1$. We have that $\Lambda(f_s) = \{0, c/s^2\}$ for all $s \neq 0$, and that χ^∞ changes from 3 if $s = 0$ to 2 if $s \neq 0$, so $\Delta\chi^\infty = +1$.

There is a persistent λ-singularity in the fibre over $t = 0$ and there is a branch of the critical locus $\mathrm{Crit}\Psi$ which is asymptotic to $t = \infty$.

6.3. **Cases of lower semi-continuity at λ-singularities.** In Theorem 3.3 we have an inequality which we may write in short-hand:

$$\lambda = I_{gen} - \nu \leq I_{gen} \leq I_{s=0}$$

This inequality can have two different sources:

- the number ν, which is related to the equisingularity properties of $\mathbb{Y}$,
- the nongeneric intersection number $I_{s=0}$ and its difference to the generic one I_{gen}.

So the excess in the formula is $\nu + (I_{s=0} - I_{gen})$. The following examples illustrate the different types of excess: $\nu \neq 0$, respectively $\nu = 0$ and $I_{s=0} - I_{gen} > 0$. In the latter case, the space $\mathbb{Y}$ is singular.

Example 6.7. We start with a F-type polynomial f_0 and consider a Yomdin deformation $f_0 - sx_1^d$ for sufficient general x_1. In this case the space $\mathbb{Y}$ is non-singular and the function $\sigma + \varepsilon\tau$ behaves locally as a linear function. It follows that $\nu = 0$. Moreover in this case $I_{s=0} - I_{gen}$ is positive because of the tangency of some components of the discriminant set to the s-axis. Compare [ST2, Theorem 5.4], where the local lower semi-continuity was proved in the case of Yomdin deformations.

Example 6.8. $f_s = x^2y^b + x + sxy^k$.
In the range $\frac{b}{2} < k \leq b$, this has the following data:
$s = 0$: $\lambda = b$, $\mu = 0$, $\lambda + \mu = b$;
$s \neq 0$: $\lambda = 0$, $\mu = 2k$, $\lambda + \mu = 2k$.
Both intersection numbers I_{gen} and $I_{s=0}$ are the same and equal to $2k$. We read the inequality now as: $b = 2k - \nu \leq 2k \leq 2k$. So $\nu = 2k - b$ and this is possitive in case $\frac{b}{2} < k \leq b$.

For the complementary range $1 < k < \frac{b}{2}$ we have a family with an extra λ-discriminant branch at $t = 0$. There is the following data here:
$s = 0$: $\lambda = b$, $\mu = 0$, $\lambda + \mu = b$;
$s \neq 0$: $\lambda = b - 2k$, $\mu = 2k$, $\lambda + \mu = b$.
In this range one has $\nu = 0$, $\lambda = b = I_{s=0}$, which gives equality in Theorem 3.3. This local conservation is characteristic to families with constant global $\mu + \lambda$, see Corollary 5.1(c).

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