Israeli options as composite exotic options

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Abstract

Introduced by Kifer (2000), Israeli options function in the same way as American options with the added feature that the writer may also choose to exercise at which time they must pay out the intrinsic option value of that moment plus a penalty. In Kyprianou (2003) explicit formulae were obtained for the value function of two classes of perpetual Israeli options. Crucial to the calculations which lead to the aforementioned formulae was the claim structure, the penalty and the perpetual nature of the option. In this article we address how to characterize the value function of the *finite expiry* versions of these Israeli options via mixtures of other exotic options by using mainly martingale arguments.

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1 Introduction

Consider the Black-Scholes market. That is, a market with a risky asset S and a riskless bond, B. The bond evolves according to the dynamic

$$dB_t = rB_t dt \text{ where } r, t \geq 0.$$

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The risky asset is written as the process $S = \{S_t : t \geq 0\}$ where

$$S_t = s\exp\{\sigma W_t + \mu t\} \text{ where } s > 0$$

is the initial value of S and $W = \{W_t : t \geq 0\}$ is a Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, P)$ satisfying the usual conditions.

Let $0 < T < \infty$. Suppose that $X = \{X_t : t \in [0,T]\}$ and $Y = \{Y_t : t \in [0,T]\}$ be two nonnegative stochastic processes defined on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with càdlàg paths (right continuous with left limits) such that with probability one $Y_t \geq X_t$ for all $t \in [0,T)$ and $Y_T = X_T$. The Israeli option, introduced by Kifer (2000), is a contract between a writer and holder at time $t = 0$ such that both have the right to exercise at any $\mathbb{F}$-stopping time before the expiry date T. If the holder exercises, then (s)he may claim the value of X at the exercise date and if the writer exercises, (s)he is obliged to pay to the holder the value of Y at the time of exercise. If neither have exercised at time T then the writer pays the holder the value $X_T = Y_T$. If both decide to claim at the same time then the lesser of the two claims is paid. But, it turns out that this marginal case has no impact on the option price as long as the payoff lies in the interval $[X_t, Y_t]$. In short, if the holder will exercise with strategy σ and the writer with strategy τ we can conclude that at any moment during the life of the contract, the holder can expect to receive $Z_{\sigma,\tau}$ where

$$Z_{s,t} = X_s \mathbf{1}_{(s\leq t)} + Y_t \mathbf{1}_{(t<s)}. \tag{1}$$

Suppose now that $\mathbb{P}_s$ is the risk-neutral measure for S under the assumption that $S_0 = s$. [Note that standard Black-Scholes theory dictates that this measure exists and is uniquely defined via a Girsanov change of measure]. We shall denote $\mathbb{E}_s$ to be expectation under $\mathbb{P}_s$. The following Theorem is Kifer's pricing result.

Theorem 1 (Kifer) *Suppose that for all $s > 0$*

$$\mathbb{E}_s\left(\sup_{0\leq t\leq T} e^{-rt} Y_t\right) < \infty. \tag{2}$$

Let $\mathcal{T}_{t,T}$ be the class of $\mathbb{F}$-stopping times valued in $[t,T]$. There is a unique no-arbitrage price process of the Israeli option under the Black-Scholes framework. It can be represented by the right continuous process $V = \{V_t : t \in [0,T]\}$ where

$$\begin{aligned} V_t &= \textit{ess-inf}_{\tau \in \mathcal{T}_{t,T}} \textit{ess-sup}_{\sigma \in \mathcal{T}_{t,T}} \mathbb{E}_s \left(e^{-r(\sigma \wedge \tau - t)} Z_{\sigma,\tau} \middle| \mathcal{F}_t \right) \\ &= \textit{ess-sup}_{\sigma \in \mathcal{T}_{t,T}} \textit{ess-inf}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_s \left(e^{-r(\sigma \wedge \tau - t)} Z_{\sigma,\tau} \middle| \mathcal{F}_t \right) \end{aligned}$$

Further, if Y has no positive jumps and X has no negative jumps, then for all $t \in [0,T]$ there exist stopping strategies

$$\sigma^*_{T-t} = \inf\{s \in [t,T] : V_s \leq X_s\} \ \textit{and} \ \tau^*_{T-t} = \inf\{s \in [t,T] : V_s \geq Y_s\}. \quad (3)$$

such that

$$\begin{aligned} V_t &= \mathbb{E}_s \left(e^{-r(\sigma^*_{T-t} \wedge \tau^*_{T-t} - t)} Z_{\sigma,\tau} \middle| \mathcal{F}_t \right) \\ &= \textit{ess-sup}_{\sigma \in \mathcal{T}_{t,T}} \mathbb{E}_s \left(e^{-r(\sigma \wedge \tau^*_{T-t} - t)} Z_{\sigma,\tau} \middle| \mathcal{F}_t \right) \\ &= \textit{ess-inf}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_s \left(e^{-r(\sigma^*_{T-t} \wedge \tau - t)} Z_{\sigma,\tau} \middle| \mathcal{F}_t \right). \quad (4) \end{aligned}$$

Cvitanić and Karatzas (1996) established a connection between the value of a Dynkin game, i.e. a stochastic stopping game as it arises in Theorem 1, and the solution of a backward stochastic differential equation with reflection. In Kyprianou (2003) it was shown that when $T = \infty$, for familiar choices of X and Y (related to the American put and Russian option claim structure) explicit formulae expressions can be achieved for V_t in terms of the process S. The two types of Israeli option considered in Kyprianou (2003) were the following.

Israeli δ-penalty put options. In this case, the holder may claim as a normal American put,

$$X_t = (K - S_t)^+.$$

The writer on the other hand will be assumed to payout the holders claim plus a constant,

$$Y_t = (K - S_t)^+ + \delta \text{ for } \delta > 0.$$

Israeli δ-penalty Russian option. The holder may exercise to take a normal Russian claim,

$$X_t = \max\left\{m, \sup_{u\in[0,t]} S_u\right\} \text{ for } m > s$$

and the writer is punished by an amount $e^{-\alpha t}\delta S_t$ for annulling the contract,

$$Y_t = \max\left\{m, \sup_{u\in[0,t]} S_u\right\} + \delta S_t \text{ for } \delta > 0.$$

In both cases, the calculations are greatly eased by the choice of X and Y and the perpetual nature of the options. For the finite expiry version of the afore mentioned calculations, no explicit formulae are possible for the same reason that there are no explicit formulae for the value function of an American put or Russian option.

In this article we establish representations of finite expiry versions of the two preceeding options via mixtures of other familiar exotic options. The method of proof relies on the classical technique of 'guess and verify'. Sections 2 – 5 deal with a detailed analysis of the Israeli δ-penalty put and Section 6 states the equivalent representation for the Israeli δ-penalty Russian option without the proof. In this latter case, the proofs follow a very similar path to those of the former Israeli option an we leave the details to the reader.

2 Israeli δ-penalty put options

It will be of help to review some facts concerning the pricing of a regular American put option (cf. Karatzas and Shreve (1991), Lamberton (1998) and Myneni (1992)). That is to say, a contract with finite expiry date T which rewards the holder with $(K - S_t)^+$ at the moment they decide to exercise and forces a payment of $(K - S_T)^+$ if they have not exercised by the time the contract expires.

Classical analysis of the American put tells us that

$$\begin{aligned} V_t &= \text{ess-sup}_{\sigma \in \mathcal{T}_{t,T}} \mathbb{E}_s \left(e^{-r(\sigma - t)} (K - S_\sigma)^+ \middle| \mathcal{F}_t \right) \\ &= v^A (S_t, T - t) \end{aligned}$$

where

$$v^A (s, u) = \sup_{\sigma \in \mathcal{T}_{0,u}} \mathbb{E}_s \left(e^{-r\sigma} (K - S_\sigma)^+ \right)$$

defined on $(s, u) \in (0, \infty) \times [0, T]$ is jointly continuous, convex and non-increasing in s and non-decreasing in u. Further, the optimal stopping strategy is given by the stopping time

$$\sigma_T^A := \inf \left\{ t \geq 0 : V_t \leq (K - S_t)^+ \right\} \tag{5}$$

so that on the event $t < \sigma_T^A$

$$V_t = \mathbb{E}_s \left(e^{-r\sigma_T^A} \left(K - S_{\sigma_T^A} \right)^+ \middle| \mathcal{F}_t \right) = \mathbb{E}_{s'} \left(e^{-r\sigma_{T-t}^A} \left(K - S_{\sigma_{T-t}^A} \right)^+ \right)$$

where $s' = S_t$ and σ_{T-t}^A has the same definition as (5) but with T replaced by $T - t$. Note that we shall use here and throughout the standard definition $\inf \emptyset = \infty$. Based on the facts above, one can show that there exists a continuous monotone decreasing curve $\varphi^A : [0, T] \to (0, K]$ with $\varphi^A (0) = K$ such that the optimal stopping strategy can otherwise to be defined as

$$\sigma_T^A = \inf \left\{ t \geq 0 : S_t \leq \varphi^A (T - t) \right\} \wedge T.$$

Finally, from the theory of optimal stopping which drives the rational behind the pricing of American options, we have that

$$\{ e^{-r(t \wedge \sigma_T^A)} v^A (S_{t \wedge \sigma_T^A}, T - (t \wedge \sigma_T^A) : t \in [0, T] \}$$

and

$$\{ e^{-rt} v^A (S_t, T - t) : t \in [0, T] \}$$

are a $\mathbb{P}_s$-martingale and a $\mathbb{P}_s$-supermartingale respectively for each $s > 0$.

3 The value function v^{IP}

Now consider what happens when we further introduce the possibility of the writer exercising forcing a payout of $Y_t = (K - S_t)^+ + \delta \mathbf{1}_{(t<T)}$, $t \in [0, T]$. By boundedness of the payoff processes condition (2) in Kifers Theorem is satisfied. In addition, as X is continuous and Y possesses no positive jumps (4) holds. Kifer's theorem together with the Markov property tells us that much like before we can write $V_t = v^{IP}(S_t, T - t)$ where

$$\begin{aligned} v^{IP}(s,u) &= \inf_{\tau \in \mathcal{T}_{0,u}} \sup_{\sigma \in \mathcal{T}_{0,u}} \mathbb{E}_s \big(\mathbf{1}_{(\sigma \leq \tau)} e^{-r\sigma} (K - S_\sigma)^+ \\ &\qquad\qquad + \mathbf{1}_{(\tau<\sigma)} e^{-r\tau} \left\{ \delta + (K - S_\tau)^+ \right\} \big) \\ &= \sup_{\sigma \in \mathcal{T}_{0,u}} \inf_{\tau \in \mathcal{T}_{0,u}} \mathbb{E}_s \big(\mathbf{1}_{(\sigma \leq \tau)} e^{-r\sigma} (K - S_\sigma)^+ + \\ &\qquad\qquad \mathbf{1}_{(\tau<\sigma)} e^{-r\tau} \left\{ \delta + (K - S_\tau)^+ \right\} \big) \end{aligned} \tag{6}$$

defined on $(s, u) \in (0, \infty) \times [0, T]$. Note that by considering strategies $\sigma = 0$ and $\tau = 0$ it can be seen that

$$(K - s)^+ \leq v^{IP}(s, u) \leq (K - s)^+ + \delta. \tag{7}$$

Our interest is in showing how the value function $v^{IP}(s, u)$ can characterized in terms of the value functions of other more familiar exotic options. However it is first necessary to understand whether the writer's rights really makes a significant difference to the case of the American put.

4 Representation of v^{IP} for large δ

Suppose that δ is very large, for example when

$$\delta > \sup_{(s,u) \in (0,\infty) \times [0,T]} v^A(s, u).$$

With such a large value of δ it would not make sense for the writer to exercise at all. For then they would be left with the responsibility of a compensation which far exceeds any amount the holder themselves would ever have claimed. We should therefore expect that in this case the saddle point in Kifer's theorem simply requires the writer to leave the decision making to the holder. That is to say, in this case, the Israeli option becomes nothing more than the American put. For smaller values of δ however, one should expect that there is rational in the writer exercising before the holder. Suppose that δ is very small and further that $S > K$. At this moment the holder can claim nothing and the writer must pay out δ. Hence rather than risking a large claim there is sense in the writer cancelling the contract and paying a small penalty. The following Lemma shows that there is a smallest δ beyond which the Israeli δ-penalty put is nothing more than an American put.

Lemma 2 *If $\delta \geq v^A(K,T)$ then $v^{IP} = v^A$, $\sigma^* = \sigma_T^A$ and $\tau^* = T$.*

Proof. As v^{IP} is varying with δ in a continuous way, it is sufficient to show the assertion for $\delta > v^A(K,T)$. Then, we have that for all $s \in (0,\infty), u \in [0,T]$

$$v^{IP}(s,u) \leq v^A(s,u) \leq (K-s)^+ + v^A(K,u) < (K-s)^+ + \delta. \tag{8}$$

Note that the first inequality is justified by considering $\tau = u$ in the definition of $v^{IP}(s,u)$. Since $V_t = v^{IP}(S_t, T-t)$, (8) implies that the optimal recall time for the seller, given by (3), is $\tau_T^* = T$. This implies $v^{IP}(s,u) = v^A(s,u)$ and $\sigma_T^* = \sigma_T^A$. ■

5 Representation of v^{IP} for small δ

Suppose now that $0 < \delta < v^A(K,T)$. That is to say at the beginning of the contract, for certain paths of S the American option is worth strictly more than the δ-penalty Israeli put; recall the bounds (7). Despite this fact, since the value function v^A is continuous and increasing in the time to expiry with $v^A(s,0) = (K-s)^+$, for all times sufficiently close to expiry the value of the American option will become uniformly in s less than the writers obligation should they decide to exercise; $v^A(s, T-t) \leq (K-s)^+ + \delta$ uniformly in s for all sufficiently large $t < T$. Suppose that a δ-penalty Israeli put has survived to almost the expiry date, say a time t'. By the Markov property the option has the same value of a fresh δ-penalty Israeli put initiated at t' with the same strike but with duration $T - t'$. Since $\delta \geq v^A(K, T - t')$, Lemma 2 tells us that the writer has no interest in exercising and the option proceeds as the tail end of an American put with strike K and expiry $T - t'$.

If we are to ask for, at which value of t' this reasoning becomes valid, a little thought yields that the clear answer is precisely at t^* for which $v^A(K, T - t^*) = \delta$ (consider the convexity of $v^A(s, u)$ in s). Note that continuity and strict monotonicity of the function $v^A(K, \cdot)$ guarantees that this value is uniquely defined. Now we know what happens at the tail end of the Israeli δ-penalty put contract, let us turn our attention to the period $[0, t^*]$.

We shall proceed by investigating the following heuristic. At first take the viewpoint of the option writer. As long as $S_t > K$ it is not rewarding to cancel the contract by paying the penalty δ. Namely, as the interest rate r is positive, it is better to wait and cancel the contract, if at all, not till S hits K. On the other hand, if $S_t < K$ we have that $e^{-rt}X_t = e^{-rt}(K - S_t)^+ = e^{-rt}(K - S_t)$. This payoff, considered as a process stopped when S hits K, is a $\mathbb{P}_s$-supermartingale. Thus the writer it doing well to wait. For this strategy also says that by waiting as long as possible the writer will reduce the penalty when taking account of discounting and can possibly even prevent to pay it. The latter argumentation is independent of the stopping strategy of the holder. On the other hand, the writer has to exercise *somewhere*. Otherwise we would have $v^{IP}(s, u) = v^A(s, u)$. But, by (3), for $s \approx K$ this would imply that recalling the option is worthwhile as $(K - s)^+ + \delta < v^A(s, u)$. That is a contradiction. Thus we might guess that the writer should exercise according to the strategy

$$\widehat{\tau} = \inf\{t \geq 0 : S_t = K\} \wedge T.$$

The holder on the other hand will reason in the same way as they would for the associated American put. That is to make a compromise between S reaching a prescribed low value and not waiting too long. Following these strategies, if neither holder nor writer takes action by time t^* the option should go on, as we have already seen, as a regular American put.

These heuristics are captured in the following exotic option whose characteristics we shall investigate and use.

5.1 The American knock-out option

Theorem 3 *Consider an American-type exotic option of duration t^* which offers the holder the right to exercise at any time claiming $(K - S_t)^+$, however, if the value of S hits K then the option is 'knocked-out' with a rebate of δ and further, if at expiry the option is still active then the holder is rewarded with an American put option with strike K and duration $T - t^*$.*

(i) *The holder of this option behaves rationally by exercising according to the stopping time*

$$\widehat{\sigma} = \inf\{t \geq 0 : \widehat{v}\,(S_t, t^* - t) = (K - S_t)^+\} \wedge t^*, \tag{9}$$

where

$$\begin{aligned}\widehat{v}\,(s,u) \;&=\; \sup_{\sigma \in \mathcal{T}_{0,u}} \mathbb{E}_s \Big(e^{-r\widehat{\tau}}\delta \mathbf{1}_{(\widehat{\tau} \leq \sigma)} + \mathbf{1}_{(\sigma < \widehat{\tau} \wedge u)} e^{-r\sigma}(K - S_\sigma)^+ \\ &\qquad\qquad + \mathbf{1}_{(\sigma = u < \widehat{\tau})} e^{-ru} v^A(S_u, T - t^*)\Big).\end{aligned} \tag{10}$$

(ii) *The discounted value of the option is given by*

$$\{e^{-r(t\wedge\widehat{\tau})}\widehat{v}\,(S_{t\wedge\widehat{\tau}}, t^* - (t \wedge \widehat{\tau})) : t \in [0, t^*]\}.$$

(iii) *The process*

$$\{e^{-r(t\wedge\widehat{\tau})}\widehat{v}\,(S_{t\wedge\widehat{\tau}}, t^* - (t \wedge \widehat{\tau})) : t \in [0, t^*]\}$$

is a $\mathbb{P}_s$-supermartingale and the process

$$\{e^{-r(t\wedge\widehat{\tau}\wedge\widehat{\sigma})}\widehat{v}\,(S_{t\wedge\widehat{\tau}\wedge\widehat{\sigma}}, t^* - (t \wedge \widehat{\tau} \wedge \widehat{\sigma})) : t \in [0, t^*]\}$$

is a $\mathbb{P}_s$-martingale.

Proof. (i) First note that the discounted claim process $\{\pi_t : t \in [0,t^*]\}$ where

$$\pi_t = \mathbf{1}_{(t<\widehat{\tau}\wedge t^*)} e^{-rt}(K-S_t)^+ + e^{-r\widehat{\tau}}\delta \mathbf{1}_{(\widehat{\tau}\leq t^* \text{ and } t\geq\widehat{\tau})} + \mathbf{1}_{(t^*<\widehat{\tau} \text{ and } t=t^*)} e^{-rt^*} v^A(S_{t^*}, T-t^*).$$

is an $\mathbb{F}$-adatped process with càdlàg paths that have no *negative* jumps and satisfies $\mathbb{E}_s\left(\sup_{t\in[0,t^*]}\pi_t\right) < \infty$. Now consider the optimal stopping problem

$$\sup_{\sigma\in\mathcal{T}_{0,t^*}} \mathbb{E}_s(\pi_\sigma). \tag{11}$$

Standard theory of American-type option pricing (cf. Shiryaev *et al.* (1995)) now tells us that this problem charaterizes the value of this option. In particular, optimal stopping strategy occurs at

$$\widetilde{\sigma} = \inf\{t \geq 0 : v_t^\pi = \pi_t\}$$

where $v^\pi = \{v_t^\pi : t \in [0,t^*]\}$ is the Snell envelope of $\{\pi_t : t \in [0,t^*]\}$. By the Strong Markov Property of S we have that on the set $\{t \leq \widehat{\tau}\}$

$$\begin{aligned} v_t^\pi &= \text{ess-sup}_{\sigma\in\mathcal{T}_{t,t^*}} \mathbb{E}_s(\pi_\sigma|\mathcal{F}_t) \\ &= e^{-rt} \sup_{\sigma\in\mathcal{T}_{0,t^*-t}} \mathbb{E}_{s'}(\pi_\sigma) \text{ where } s' = S_t \\ &= e^{-rt}\widehat{v}(S_t, t^*-t). \end{aligned}$$

Therefore, on the set $\{\widehat{\sigma} < \widehat{\tau}\}$ we have that $\widetilde{\sigma} = \widehat{\sigma}$ and on the set $\{\widehat{\sigma} \geq \widehat{\tau}\}$ we have that $\widetilde{\sigma} = \widehat{\tau}$. Thus $\widetilde{\sigma} = \widehat{\sigma} \wedge \widehat{\tau}$. As $v_t^\pi = \delta$, for $\widehat{\tau} \leq t^*$ and $t \geq \widehat{\tau}$, it follows that $\mathbb{E}_s(\pi_{\widehat{\sigma}}) = \mathbb{E}_s(\pi_{\widetilde{\sigma}})$. Thus, also $\widehat{\sigma}$ is optimal for the stopping problem (11).

(ii)–(iii) From standard theory of optimal stopping, the Snell envelope v^π is a supermartinagle and further when stopped at an opimal stopping time it forms a martingale. $e^{-r(t\wedge\widehat{\tau})}\widehat{v}(S_{t\wedge\widehat{\tau}}, t^* - t\wedge\widehat{\tau}) = v_{t\wedge\widehat{\tau}}^\pi$ implies the assertion. ■

Remark 4 The option described in the previous theorem has two different interpretations depending on the initial value s.

If $s < K$ then we can understand this option to be an American 'up-and-out' put option with reimbursement δ at the point of 'knock-out'. Further the holder is rewarded with an American put option of further duration $T - t^*$ and strike K if the option reaches its natural maturity.

If on the other hand, $s > K$ then it is not worthwhile for the holder to exercise before t^*. This gives the interpretation of our option being a European 'down-and-out' option with contingent claim $v^A(S_{t^*}, T - t^*)$ and rebate δ when the option 'knocks out'.

Without specifying on which side of K the initial value of the risky asset lies, we can say that the option in the previous theorem is the sum of the above compound American up-and-out with rebate and European down-and-out with rebate. For future reference we shall refer to this combined derivative as 'the American knock-out'.

5.2 Analytical properties of the American knock-out option

Let us progress to look at some of the analytical properties of the American knock-out option, presented as a series of lemmas, which will be of later use.

The lemmas are partial steps to establishing convexity of $\widehat{v}$ which in turn is crucial in establishing a submartingale associated with $\widehat{v}$. This submartingale serves to justify the heuristic at the beginning of this section.

Lemma 5 *There exists a function* $f : (0, \infty) \times [0, t^*] \to \mathbb{R}$ *which is convex in its first variable such that*

$$f(s, u) \leq \widehat{v}(s, u) \text{ and } f(K, u) = \widehat{v}(K, u) = \delta. \tag{12}$$

Proof. Recall that for $u \in [0, t^*]$, $v^A(K, T - t^* + u) \geq \delta$, where v^A is the value function of the corresponding standard American put. Define

$$f(s, u) = v^A(s, T - t^* + u) + \delta - v^A(K, T - t^* + u)$$

We have of course that $f(K, u) = \widehat{v}(K, u)$ and convexity follows from the fact that v^A is known to be convex.

Write σ^A as short hand for $\sigma^A_{T-t^*+u}$, the optimal exercising time for an American put with maturity $T - t^* + u$, and recall that $\widehat{\tau}$ is the first hitting time of K. We have

$$
\begin{aligned}
&\widehat{v}(s,u) \\
&\geq \mathbb{E}_s\left(e^{-r(\sigma^A\wedge\widehat{\tau}\wedge u)}\widehat{v}(S_{\sigma^A\wedge\widehat{\tau}\wedge t}, u-\sigma^A\wedge\widehat{\tau}\wedge u)\right) \\
&\geq \mathbb{E}_s e^{-r(\sigma^A\wedge\widehat{\tau}\wedge u)}\left(\mathbf{1}_{(\widehat{\tau}<\sigma^A\wedge u)}\delta + \mathbf{1}_{(\widehat{\tau}\geq\sigma^A\wedge u)}v^A(S_{\sigma^A\wedge u}, T-t^*+u-\sigma^A\wedge u)\right) \\
&= \mathbb{E}_s\left(e^{-r(\sigma^A\wedge\widehat{\tau}\wedge u)}v^A(S_{\sigma^A\wedge\widehat{\tau}\wedge u}, T-t^*+u-\sigma^A\wedge\widehat{\tau}\wedge u)\right) \\
&\quad -\mathbb{E}_s\left(e^{-r\widehat{\tau}}\mathbf{1}_{(\widehat{\tau}<\sigma^A\wedge u)}\left(v^A(K, T-t^*+u-\widehat{\tau})-\delta\right)\right) \\
&\geq \mathbb{E}_s\left(e^{-r(\sigma^A\wedge\widehat{\tau}\wedge u)}v^A(S_{\sigma^A\wedge\widehat{\tau}\wedge u}, T-t^*+u-\sigma^A\wedge\widehat{\tau}\wedge u)\right) \\
&\quad -\mathbb{E}_s\left(\mathbf{1}_{(\widehat{\tau}<\sigma^A\wedge u)}\left(v^A(K, T-t^*+u)-\delta\right)\right) \\
&\geq \mathbb{E}_s\left(e^{-r(\sigma^A\wedge\widehat{\tau}\wedge u)}v^A(S_{\sigma^A\wedge\widehat{\tau}\wedge u}, T-t^*+u-\sigma^A\wedge\widehat{\tau}\wedge u)\right) \\
&\quad -\left(v^A(K, T-t^*+u)-\delta\right) \\
&= v^A(s, T-t^*+u)+\delta-v^A(K, T-t^*+u) \\
&= f(s,u).
\end{aligned}
$$

The first inequality is due to the supermartingale property stated in Theorem 3. The second inequality can be derived by an ω-wise comparison. (Note at $\sigma^A \wedge u$ in the third line we have $v^A(S_{\sigma^A\wedge u}, T-t^*+u-\sigma^A\wedge u)$ which is equal to $\widehat{v}(S_u, 0)$ in the case that $u \leq \sigma^A$ and equal to $(K - S_{\sigma^A})^+ \leq \widehat{v}(S_{\sigma^A}, u - \sigma^A)$ when $u > \sigma^A$). The second equality comes from the martingale property of the American put. ■

Lemma 6 *We have that for all $u \geq 0$ and $s > 0$, $\widehat{v}(s,u) > 0$ and*

$$(K-s)^+ \leq \widehat{v}\,(s,u) \leq (K-s)^+ + \delta. \tag{13}$$

Further, for each $s > 0$ the function $\widehat{v}(s,\cdot)$ is monotone increasing and continuous and for each $u \in [0,t^*]$ the function $\widehat{v}(\cdot,u)$ is monotone decreasing and continuous; and hence $\widehat{v}$ is jointly continuous.

Proof. The lower bounds follow by considering the stopping times $\sigma = u$ and $\sigma = 0$ in the expression given for $\widehat{v}$ in (10).

For the upper bound we make a case differentiation. For initial price $s \geq K$ the assertion is trivial as the (discounted) payoff by the American knock-out cannot exceed δ. For $s < K$ we bring to mind that the stopped process $(e^{-rt}(K-S_t)^+)^{\widehat{\tau}}_{t\in[0,u]} = (e^{-rt}(K-S_t))^{\widehat{\tau}}_{t\in[0,u]}$, where $\widehat{\tau} = \inf\{t \geq 0 : S_t = K\}$, is a supermartingale. Therefore from the definition of $\widehat{v}(s,u)$ we have

$$\begin{aligned}\widehat{v}(s,u) &\leq \sup_{\sigma\in\mathcal{T}_{0,u}} \mathbb{E}_s\left(e^{-r(\sigma\wedge\widehat{\tau})}(K-S_{\sigma\wedge\widehat{\tau}})^+ + \delta\right)\\ &= (K-s)^+ + \delta.\end{aligned}$$

For the inequality above, recall that at $\sigma = u$ the option is switched to an American put with duration $T-t^*$. Its value becomes $v^A(S_u, T-t^*) \leq (K - S_u)^+ + v^A(K, T-t^*) = (K-S_u)^+ + \delta$.

Let us now show the monotonicity of $\widehat{v}(s,\cdot)$. As a partial step we shall show that

$$v^A(s, T-t^*) \leq \widehat{v}(s,u) \text{ for all } s > 0 \text{ and } u \in [0,t^*]. \tag{14}$$

By the dynamic programming principle, the value of the standard American put option coincides with the value of an American put which is knocked out when S_t hits K, paying then the amout $v^A(K, T-t^*-\widehat{\tau})$. We thus obtain for $u \in [0,t^*]$ that

$$\begin{aligned}&v^A(s,T-t^*)\\ &\quad= \sup_{\sigma\in\mathcal{T}_{0,T-t^*}} \mathbb{E}_s\left(e^{-r\sigma}(K-S_\sigma)^+\right)\\ &\quad= \sup_{\sigma\in\mathcal{T}_{0,T-t^*}} \mathbb{E}_s\left(\mathbf{1}_{(\widehat{\tau}\leq\sigma\wedge u)}e^{-r\widehat{\tau}}v^A(K,T-t^*-\widehat{\tau}) + \mathbf{1}_{(\widehat{\tau}>\sigma\wedge u)}e^{-r\sigma}(K-S_\sigma)^+\right).\end{aligned}\tag{15}$$

Further, since by the definition of t^*, an American option with remaining term less than $T-t^*$ is less than δ, it follows from the right hand side above that

$$\begin{aligned}&v^A(s,T-t^*)\\ &\quad\leq \sup_{\sigma\in\mathcal{T}_{0,T-t^*}} \mathbb{E}_s\left(\mathbf{1}_{(\widehat{\tau}\leq\sigma\wedge u)}e^{-r\widehat{\tau}}\delta + \mathbf{1}_{(\widehat{\tau}>\sigma\wedge u)}e^{-r\sigma}(K-S_\sigma)^+\right)\\ &\quad\leq \sup_{\sigma\in\mathcal{T}_{0,T-t^*+u}} \mathbb{E}_s\left(\mathbf{1}_{(\widehat{\tau}\leq\sigma\wedge u)}e^{-r\widehat{\tau}}\delta + \mathbf{1}_{(\widehat{\tau}>\sigma\wedge u)}e^{-r\sigma}(K-S_\sigma)^+\right).\end{aligned}\tag{16}$$

Finally we note that by considering the American knock-out option with expiry u as having ultimate expiry time $T - t^* + u$ by taking into account the rebated American option of length $T - t^*$ issued at time t^*, we may simply identify the right hand side above as $\widehat{v}(s, u)$.

Now, suppose that $0 \le u_1 \le u_2 \le t^*$. We compare American knock-out options with remaining times u_1 and u_2, respectively. Until u_1 the payoffs coincide. At time u_1 the first option is switched to an American put and the second is still a knock-out option with remaining time $u_2 - u_1$. We use (14) with $u = u_2 - u_1$ and obtain

$$\begin{aligned}
&\widehat{v}(s, u_1) \\
&= \sup_{\sigma \in \mathcal{T}_{0,u_1}} \mathbb{E}_s \Big(\mathbf{1}_{(\widehat{\tau} \le \sigma)} e^{-r\widehat{\tau}} \delta + \mathbf{1}_{(\sigma < \widehat{\tau} \wedge u_1)} e^{-r\sigma} (K - S_\sigma)^+ \\
&\qquad\qquad + \mathbf{1}_{(\sigma = u_1 < \widehat{\tau})} e^{-ru_1} v^A(S_{u_1}, T - t^*) \Big) \\
&\le \sup_{\sigma \in \mathcal{T}_{0,u_1}} \mathbb{E}_s \Big(\mathbf{1}_{(\widehat{\tau} \le \sigma)} e^{-r\widehat{\tau}} \delta + \mathbf{1}_{(\sigma < \widehat{\tau} \wedge u_1)} e^{-r\sigma} (K - S_\sigma)^+ \\
&\qquad\qquad + \mathbf{1}_{(\sigma = u_1 < \widehat{\tau})} e^{-ru_1} \widehat{v}(S_{u_1}, u_2 - u_1) \Big) \\
&= \widehat{v}(s, u_2). \qquad (17)
\end{aligned}$$

This is the required monotonicity in u.

For continuity in u, have again a look at (16). As we have $v^A(K, T - t^* - u) \to \delta$ for $u \to 0$, the second line in (16) becomes an approximation for the last line of (15) as $u \to 0$. For $u < T - t^*$ the last line in (16) coincides with

$$\begin{aligned}
\sup_{\sigma \in \mathcal{T}_{0,T-t^*}} \mathbb{E}_s \Big(&\mathbf{1}_{(\widehat{\tau} \le \sigma \wedge u)} e^{-r\widehat{\tau}} \delta + \mathbf{1}_{(\widehat{\tau} \wedge T - t^* > \sigma \wedge u)} e^{-r\sigma} (K - S_\sigma)^+ \\
&+ \mathbf{1}_{(\sigma = T - t^* > \widehat{\tau})} e^{-r(T - t^*)} v^A(S_{T-t^*}, u) \Big).
\end{aligned}$$

As $v^A(s, u) \to (K - s)^+$ for $u \to 0$, uniformly in $s \in (0, \infty)$, also the third line in (16) becomes an approximation for the line before as $u \to 0$. We obtain

$$\widehat{v}(s, u) \to v^A(s, T - t^*), \quad u \to 0, \text{ uniformly in } s \in \mathbb{R}_+. \qquad (18)$$

The asymptotic (18) together with an inspection of (17) reveals that $\widehat{v}(s, u_1) - \widehat{v}(s, u_2) \to 0$ for $u_2 - u_1 \to 0$.

For monotonicity in s, let $0 \le s_1 < s_2 \le K$ and write

$$\widehat{\tau}_{s_2} = \inf\{t \ge 0 : s_2 S_t = K\}.$$

By (12) and the monotonicity of v^A we have that for every $u \in [0, t^*]$

$$\widehat{v}\left(\frac{s_1}{s_2}K, u\right) \ \ge \ v^A\left(\frac{s_1}{s_2}K, u\right) + \delta - v^A(K, u) \ge \delta. \tag{19}$$

It now follows that (19) and the monotonicity of $s \mapsto K - s$ and $s \mapsto v^A(s, u)$ imply that

$$\begin{aligned}
&\widehat{v}(s_2, u)\\
&\quad = \sup_{\sigma \in \mathcal{T}_{0,u}} \mathbb{E}_1 \Big(\mathbf{1}_{(\widehat{\tau}_{s_2} \le \sigma)} e^{-r\widehat{\tau}_{s_2}} \delta + \mathbf{1}_{(\sigma < \widehat{\tau}_{s_2} \wedge u)} e^{-r\sigma} (K - s_2 S_\sigma)^+ \\
&\qquad\qquad + \mathbf{1}_{(\sigma = u < \widehat{\tau}_{s_2})} e^{-ru} v^A(s_2 S_u, T - t^*) \Big) \\
&\quad \le \sup_{\sigma \in \mathcal{T}_{0,u}} \mathbb{E}_1 \left(\mathbf{1}_{(\widehat{\tau}_{s_2} \le \sigma)} e^{-r\widehat{\tau}_{s_2}} \widehat{v}\left(\frac{s_1}{s_2}K, u - \widehat{\tau}_{s_2}\right) + \mathbf{1}_{(\sigma < \widehat{\tau}_{s_2} \wedge u)} e^{-r\sigma} (K - s_1 S_\sigma)^+ \right.\\
&\qquad\qquad \left. + \mathbf{1}_{(\sigma = u < \widehat{\tau}_{s_2})} e^{-ru} v^A(s_1 S_u, T - t^*) \right) \\
&\quad = \widehat{v}(s_1, u).
\end{aligned} \tag{20}$$

The last equality follows from the dynamic programming principle (Note that $\widehat{\tau}_{s_2} \le \inf\{t \ge 0 : s_1 S_t = K\}$). By (13) we have $\widehat{v}(K s_1/s_2, u) \le (K - K s_1/s_2)^+ + \delta$ and therefore

$$\widehat{v}\left(\frac{s_1}{s_2}K, u\right) \to \delta, \quad s_1 \to s_2 > 0. \tag{21}$$

The limiting relation (21) and an inspection of (20) reveals continuity in s.

On $[K, \infty)$ the proof of monotonicity and continuity is similar, but easier. It makes again use of the fact that $\widehat{v}(s, u) \ge v^A(s, u) + \delta - v^A(K, u) \to \delta$, $s \to K$, uniformly in $u \in [0, t^*]$. The complete proof is left to the reader. ■

The monotonicity properties of $\widehat{v}$ and its lower bounds together with the fact that $\widehat{v}(K, u) = \delta$, this implies that there exists an open set $\mathcal{C}$ taking the form

$$\mathcal{C} = (K, \infty) \times (0, t^*) \cup \{(s, u) \in (0, K) \times (0, t^*) : s > b(u)\}$$

where $b : (0, t^*] \to [0, K)$, given by

$$b(u) = \sup\{s \ge 0 : \widehat{v}(s, u) = (K - s)\}$$

(with the convention that $\sup\emptyset = 0$) is monotone decreasing, satisfying $\lim_{u\downarrow 0} b(u) \leq \varphi^A(T-t^*)$ such that the optimal stopping time $\widehat{\tau}\wedge\widehat{\sigma}$ corresponds to

$$\tau^{\mathcal{C}} := \inf\{t > 0 : (S_t, t^* - t) \notin \mathcal{C}\}$$

Lemma 7 *The value function $\widehat{v}(s,u)$ is twice continuously differentiable in s and once continuously differentiable in u on the continuation region $\mathcal{C}$ with*

$$\frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \widehat{v}}{\partial s^2} + rs\frac{\partial \widehat{v}}{\partial s} - r\widehat{v} - \frac{\partial \widehat{v}}{\partial u} = 0 \text{ in } \mathcal{C}.$$

Proof. We recall a technique used in Karatzas and Shreve (1991), p90. That is to say, construct the parabolic Dirichlet problem

$$\begin{aligned} \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s} - rV - \frac{\partial V}{\partial u} &= 0 \text{ in } \mathcal{R} \\ V &= \widehat{v} \text{ on } \partial^0\mathcal{R} \end{aligned}$$

where $\mathcal{R}$ is the open rectangle $(s_1, s_2)\times(u_1,u_2) \subset \mathcal{C}$ with parabolic boundary

$$\partial^0\mathcal{R} = \partial\mathcal{R} - ((s_1,s_2)\times\{u_2\}).$$

On account of the fact that $\widehat{v}$ is joinly continuous in u and s, classical theory of boundary value problems dictates that the above Dirichlet problem has a unique solution which is $C^{2,1}$ in $\mathcal{R}$ (cf. Friedman (1976)). By part (iii) of Theorem 3 we have that

$$\{e^{-rt}\widehat{v}\,(S_t, t^* - t) : t \in [t^* - u_2, \tau^{\mathcal{R}}]\}$$

is a uniformly integrable martingale where $\tau^{\mathcal{R}} = \inf\{t \geq t^* - u_2 : (S_t, t^* - t) \notin \mathcal{R}\}$. On the other hand, stochastic representation tells us also that

$$\{e^{-rt}V\,(S_t, t^* - t) : t \in [t^* - u_2, \tau^{\mathcal{R}}]\}$$

is also a uniformly integrable martingale. Since both have the same terminal value, we are forced to conclude they are the same martingale and hence $V = \widehat{v}$ in $\mathcal{R}$. Since $\mathcal{R}$ may be placed anywhere in $\mathcal{C}$ the theorem is proved. ■

Lemma 8 *For each $u \in [0, t^*]$ the function $\widehat{v}(\cdot, u)$ is convex on $(0, \infty)$.*

Proof. Let

$$L = \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + rs\frac{\partial}{\partial s} - r - \frac{\partial}{\partial u}$$

and recall that $L\widehat{v} = 0$ on $\mathcal{C}$ (in particular $\widehat{v}$ is smooth on $\mathcal{C}$). From Lemma 6 we have that $\widehat{v}$ is decreasing in its first variable and increasing in its second variable. Hence it follows that $\partial\widehat{v}/\partial s \leq 0$ and $\partial\widehat{v}/\partial u \geq 0$ on $\mathcal{C}$. These latter two observations together with the fact that $\widehat{v} \geq 0$ and $L\widehat{v} = 0$ leads to the conclusion that $\partial^2\widehat{v}/\partial s^2 \geq 0$ on $\mathcal{C}$.

Since $\widehat{v}$ is jointly continuous and bounded below by a convex function f (cf. Lemma 5) having the property that $f(K, u) = \widehat{v}(K, u)$ it follows from the conclusion of the previous paragraph that $\widehat{v}(\cdot, u)$ is convex on $(b(u), \infty)$. As $\widehat{v}(s, u) \geq (K - s)^+$, when $\widehat{v}$ joins the function $(K - s)^+$ it does so with an increasing gradient in s. It now follows that $\widehat{v}(\cdot, u)$ is convex on $(0, \infty)$. ■

Lemma 9 *Let $t' \in [0, t^*]$ and*

$$\widehat{\sigma}_{t'} := \inf\{t \geq t' : \widehat{v}(S_t, t^* - t) = (K - S_t)^+\} \wedge t^*.$$

We have that

$$\{e^{-r(t\wedge\widehat{\sigma}_{t'})}\widehat{v}\left(S_{t\wedge\widehat{\sigma}_{t'}}, t^* - (t \wedge \widehat{\sigma}_{t'})\right) : t \in [t', t^*]\}$$

is a $\mathbb{P}_s$-submartingale.

Proof. Recall again that $L\widehat{v} = 0$ on $\mathcal{C}$. Using a modern version of Itô's formula as given in Peskir (2002) we may deduce that on $t \in [t', \widehat{\sigma}_{t'}]$

$$\begin{aligned} e^{rt}d[e^{-rt}\widehat{v}(S_t, t^* - t)] &= L\widehat{v}(S_t, t^* - t)dt + dM_t \\ &+ \left\{\frac{\partial\widehat{v}}{\partial s}(K^+, t^* - t) - \frac{\partial\widehat{v}}{\partial s}(K^-, t^* - t)\right\} dL_t^K \quad (22) \end{aligned}$$

where L^K is local time of S at level K and M is a martingale. Note, in Peskir (2002) one requires that $\widehat{v}$ has continuous derivatives up to and including the point $s = K$ from both sides. However, careful inspection of his proof reveals that it suffices that in fact $L\widehat{v}$ and $\partial\widehat{v}/\partial s$ are continuous upto and including $s = K$ from both sides; a fact that Peskir takes advantage of in his note on the American option Peskir (2002). Due to the fact that $L\widehat{v} = 0$ on $\mathcal{C}$ and since, by the previous Lemma, $\widehat{v}$ is convex, these requirements are satisfied.

Since $\widehat{v}$ is convex in s, we know that the local time term in (22) is monotone increasing and hence the result follows. ■

5.3 The δ-penalty Israeli put is a composite exotic option

Now we are ready to show what we have already alluded to. Namely that the δ-penalty Israeli put option of length T is nothing more than the American knock-out option with expiry t^* followed through to the expiration of the rebated American put of length $T - t^*$ if appropriate.

Theorem 10 *Suppose that* $\delta < v^A(K,T)$ *and define*

$$t^* = \sup\{t \geq 0 : v^A(K, T-t) = \delta\}.$$

The δ*-penalty Israeli put value function* v^{IP} *is given by*

$$v^{IP}(s,u) = \begin{cases} v^A(s,u) & \text{for } (s,u) \in (0,\infty) \times [0, T-t^*] \\ \widehat{v}(s, u - T + t^*) & \text{for } (s,u) \in (0,\infty) \times [T-t^*, T]. \end{cases} \tag{23}$$

Further, with

$$\begin{aligned} \mathcal{S}^{IP} &= \{(s,u) : s \geq \varphi^A(T-u),\ u \in [0, T-t^*]\} \\ &\quad \cup \{(s,u) : s \geq \widehat{\varphi}(u + t^* - T),\ u \in (T-t^*, T]\} \end{aligned}$$

the optimal stopping strategy of the holder is given by

$$\sigma^{IP} = \inf\{t \geq 0 : (S_t, T-t) \in \mathcal{S}^{IP}\} \wedge T$$

and the optimal stopping strategy of the writer is

$$\tau^{IP} = \inf\{t \in [0, t^*] : S_t = K\} \wedge T.$$

Proof. Let us define a new function $v(s,u)$ which will be equal to the right hand side of (23). Already from the definitions of v^{IP} and $\widehat{v}$ in (6) and (10), respectively, it becomes evident that $v \geq v^{IP}$ as v corresponds to the value in case of a certain recall strategy of the seller, namely τ^{IP}, whereas for v^{IP} we take the infimum over all stopping times τ. All that we need is to prove that $v \leq v^{IP}$; then v is the solution to the saddle point problem (6).

It turns out that the submartingale properties associated with $\widehat{v}$ will be crucial for the proof. As the value of an American put is a martingale up to the optimal exercise time it follows from Lemma 9 that

$$\{e^{-r(t\wedge\sigma^{IP})}v\left(S_{t\wedge\sigma^{IP}}, T-(t\wedge\sigma^{IP})\right) : t \geq 0\} \text{ is a } \mathbb{P}_s\text{-submartingale.}$$

We can perform a calculation similar in nature to the calculations in Kyprianou (2003). To this end, define

$$\sigma_t^{IP} = \inf\{q \geq 0 : (S_q, T-t-q) \in \mathcal{S}^{IP}\} \wedge (T-t)$$

and

$$\tau_t^{IP} = \begin{cases} \inf\{q \in [0, t^*-t] : S_q = K\} \wedge (T-t) & \text{if } t \leq t^* \\ T-t & \text{if } t > t^*. \end{cases}$$

That is when $s' = S_t$

$$\begin{aligned} & v\left(s', T-t\right) \\ = & \inf_{\tau\in\mathcal{T}_{0,T-t}} \mathbb{E}_{s'}\left(e^{-r(\tau\wedge\sigma_t^{IP})}v\left(S_{\tau\wedge\sigma_t^{IP}}, T-t-(\tau\wedge\sigma_t^{IP})\right)\right) \\ \leq & \inf_{\tau\in\mathcal{T}_{0,T-t}} \mathbb{E}_{s'}\left(e^{-r(\tau\wedge\sigma_t^{IP})}\left\{\mathbf{1}_{(\sigma_t^{IP}\leq\tau)}(K-S_{\sigma_t^{IP}})^+ + \mathbf{1}_{(\sigma_t^{IP}>\tau)}[(K-S_\tau)^+ + \delta]\right\}\right) \\ \leq & \sup_{\sigma\in\mathcal{T}_{0,T-t}}\inf_{\tau\in\mathcal{T}_{0,T-t}} \mathbb{E}_{s'}\left(e^{-r(\tau\wedge\sigma)}\left\{\mathbf{1}_{(\sigma\leq\tau)}(K-S_\sigma)^+ + \mathbf{1}_{(\sigma>\tau)}[(K-S_\tau)^+ + \delta]\right\}\right) \\ = & v^{IP}\left(s', T-t\right) \end{aligned}$$

where the first equality holds by Lemma 9 and the corresponding martingale property for the American put. In the first inequality we have used that $v\left(S_{\sigma_t^{IP}}, T-t-\sigma^{IP}\right) = (K-S_{\sigma_t^{IP}})^+$ and the fact that $(K-s)^+ + \delta$ is an upper bound for v. The latter follows from (13) and the estimation $v^A(s,u) \leq$

$(K-s)^+ + v^A(K,u) \leq (K-s)^+ + \delta$ for American puts with duration u less than $T - t^*$. ■

Remark 11 *From this proof and Theorem 3 (iii) we saw that*

$$\{e^{-r(t\wedge\sigma^{IP}\wedge\tau^{IP})} v^{IP}\left(S_{t\wedge\sigma^{IP}\wedge\tau^{IP}}, T - t\wedge\sigma^{IP}\wedge\tau^{IP}\right) : t\in[0,T]\}$$

is a martingale. This is the martingale which the holder should hedge in order to replicate the option.

6 Israeli δ-penalty Russian option

6.1 Reviewing the Russian option

Recall that the Russian option with expiry $T < \infty$ is an American-type option with contingent claim of the form

$$\max\left\{m, \sup_{u\in[0,t]} S_u\right\} \text{ for } m > 0.$$

Introduced by Shepp and Shiryayev (1993, 1995) as being an option where one has 'reduced regret' because a minimum payout of m is guaranteed, this option can be considered to be something like an American-type lookback option. Again classical optimal stopping arguments for American-type options tell us that the value of this option is given by the process

$$V_t = \text{ess-sup}_{\sigma\in\mathcal{T}_{t,T}}\mathbb{E}_s\left(e^{-r(\sigma-t)}\max\left\{m, \sup_{u\in[0,\sigma]} S_u\right\}\middle|\mathcal{F}_t\right).$$

Following the lead of Shepp and Shiryayev, we use the fact that $e^{-rt}S_t$ is an exponential martingale to make a change of measure via

$$\left.\frac{d\mathbf{P}_s}{d\mathbb{P}_s}\right|_{\mathcal{F}_t} = \frac{e^{-rt}S_t}{s}.$$

Note that under $\mathbf{P}$. the underlying Brownian motion W has a new positive drift $(\sigma/2 + r/\sigma)$. Suppose that $-T_0$ is some arbitrary moment in the past before the contract was initiated ($T_0 > 0$). Define $\overline{S}_t = \sup_{u\in[-T_0,t]} S_u$ and assume that $\overline{S}_0$

is $\mathcal{F}_0$ measurable. With a slight abuse of notation, we can adapt the definition of the measure $\mathbf{P}_s(\cdot)$ to $\mathbf{P}_{m/s}(\cdot) = \mathbf{P}\left(\cdot|\overline{S}_0 = m, S_0 = s\right)$. In that case the value process of the option can be written more neatly as

$$\left\{S_t v^R\left(\Psi_t, T-t\right) : t \in [0,T]\right\} \tag{24}$$

where $\Psi = \{\Psi_t = \overline{S}_t/S_t : t \in [0,T]\}$ and

$$v^R\left(\psi, u\right) = \sup_{\sigma \in \mathcal{T}_{0,u}} \mathbf{E}_\psi\left(\Psi_\sigma\right).$$

Recent characterizations of finite expiry Russian options were given in Peskir (2003) and Duistermaat et al. (2003). Below we give a summary of the main stochastic and analytical features of this option.

Theorem 12 *For each $u \in [0,T]$ the univariate $v^R(\cdot,u)$ is convex, for each $\psi \geq 1$ the univariate $v^R(\psi,\cdot)$ is non-decreasing and the bivariate $v^R(\cdot,\cdot)$ is continuous. Further there exists a mapping $\varphi^R : [0,T] \to [1,\infty)$ which is continuous, monotone decreasing with $\varphi^A(0) = 1$ such that the optimal stopping time in (24) is given by*

$$\sigma_T^R = \inf\{t \geq 0 : \Psi_t \geq \varphi^R(T-t)\} \wedge T.$$

The processes

$$\{v^R(\Psi_{t\wedge\sigma_T^R}, T-(t\wedge\sigma_T^R)) : t \in [0,T]\}$$

and

$$\{v^R\left(\Psi_t, T-t\right) : t \in [0,T]\}$$

are a $\mathbf{P}_\psi$-martingale and $\mathbf{P}_\psi$-supermartingale respectively for each $\psi \geq 1$.

The heuristic behind this exercise strategy is as follows. The holder would like to wait as long as possible to exercise in order to observe as large a past maximum in the value of the risky asset as possible. However, because of the exponential discounting, waiting too long can also go against the holders interests. The compromise is to exercise when the current value of the stock gets too far from the previous maximum. For in such a case it will take too much time to return to the previous maximum. The more time that is left, the more tolerant the holder is and they will allow the stock to wander before exercising it, thus explaining the monotonicity in φ^R.

6.2 The value function v^{IR}

Like the Russian option, we can make a similar transformation to the value process of the Israeli δ-penalty Russian option using the Markov property and a change of measure to establish a value function for the Israeli δ-penalty Russian. That is we can write

$$V_t = S_t v^{IR}\left(\Psi_t, T-t\right)$$

where

$$\begin{aligned} v^{IR}\left(\psi, u\right) &= \inf_{\tau\in\mathcal{T}_{0,u}} \sup_{\sigma\in\mathcal{T}_{0,u}} \mathbf{E}_\psi\left(\mathbf{1}_{(\sigma\leq\tau)}\Psi_\sigma + \mathbf{1}_{(\sigma>\tau)}\left(\Psi_\tau+\delta\right)\right) \\ &= \sup_{\sigma\in\mathcal{T}_{0,u}} \inf_{\tau\in\mathcal{T}_{0,u}} \mathbf{E}_\psi\left(\mathbf{1}_{(\sigma\leq\tau)}\Psi_\sigma + \mathbf{1}_{(\sigma>\tau)}\left(\Psi_\tau+\delta\right)\right). \end{aligned}$$

The issue of pricing the Israeli δ-penalty Russian option thus boils down to characterizing the above stochastic saddle point problem.

Kyprianou (2003) shows for the perpetual version of this option the rational place for the writer to exercise is precisely when the value of the risky asset hits the prescribed 'initial supremum' m. This stops the claim increasing in value and is the latest moment at which such a blockage can occur reducing the required capital to hedge. Irrespective of this, the holder on the other hand should follow the same reasoning as for the regular Russian option and will exercise when the value of the risky asset drifts too far from the previous supremum. The exact quantification of 'too far' being dependent on the remaining time.

We have also experienced a phenomenon for the Israeli δ-penalty put that when δ is too large, the writer has no interest in exercising at all. Using the same heuristic we can reason that the same should be true here. Again, the logical cut-off value of δ would be $v^R(1,T)-1$ where we mean that v^R is the value of a Russian option with the same parameter m.

We may also appeal to our previous experience to establish what will happen when $\delta < v^R(1,T) - 1$. Suppose now that t^* is the smallest t such that $\delta \geq v^R(1, T-t) - 1$. Just like the Israeli δ-penalty put, we can reason that if the option is still live at this time, then on account of the Markov property, in the remaining time, the Israeli δ-penalty Russian behaves like a Russian with parameter m and duration $T - t^*$. In the time interval $[0, t^*]$ we can construct a 'knock-out' Russian option which knocks out when S reaches m and which we can show is an equivalent characterization of the Israeli δ-penalty Russian.

The following result captures the above ideas more formally. The majority of the proofs follow the arguments given in the previous sections for the Israeli δ-penalty put option with straightforward if not obvious adjustments. Therefore we refrain from producing them here and leave them as an exercise for the reader.

Theorem 13 *Suppose that m and T are strictly positive.*

(i) If $\delta \geq v^R(1,T) - 1$ then the value of the Israeli δ-penalty Russian option is precisely that of the regular Russian option with parameters m and T.

(ii) Suppose that $\delta < v^R(1,T) - 1$ and define

$$t^* = sup\{t \geq 0 : v^R(1, T-t) = \delta\}.$$

Then the value of the Israeli δ-penalty Russian option is the same as that of the following compounded exotic option. A contract in which the holder has the right to exercise at any time in $t \in [0, t^*]$ claiming $\sup_{u \in [0,t]} S_u \vee m$ (with $m > s$). The option is 'knocked-out' as soon as S hits m and otherwise, at expiry date t^*, the holder is awarded with a regular Russian option with parameters S_{t^*}, m, and $T - t^*$.

7 Conclusion

We have shown that for certain familiar choices of contingent claims, the Israeli option is equivalent to the composition of other known exotic options. That is to say the stochastic saddle point in Kifer's pricing formula of Israeli options is semi-explicitly identifiable thus giving a basis for further research of these options. Indeed with further work, one should be able to show that the given composite exotic options characterize uniquely the solution to a free boundary problem as one sees for American put and Russian options.

In related work, the reader is also refered to Kühn and Kyprianou (2003) where the value of a more general class of finite expiry Israeli options are characterized via a pathwise pricing formulae.

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