Pricing Israeli options: a pathwise approach

C. Kühn* and A. E. Kyprianou†

Abstract

An Israeli option (also referred to as game option or recall option) generalizes an American option by also allowing the seller to cancel the option prematurely, but at the expense of some penalty. Kifer (2000) shows that in the classical Black-Scholes market such contracts have unique no-arbitrage prices. In Kyprianou (2002) and Kühn and Kyprianou (2003) characterizations were obtained for the price of two classes of Israeli options. For the general case, we give a dual resp. pathwise pricing formula similar to Rogers (2002). It offers alternative ways of numerically simulating the price for more complicated claim structures and leads to candidate hedging strategies for the option.

Keywords: Israeli options, convertible bonds, Dynkin games, hedging strategies, Monte Carlo simulation.

JEL Classification: G13, C73.

Mathematics Subject Classification (2000): 91A15, 60G40, 60G48.

1 Introduction

There are few examples of derivatives having the feature that they can be both exercised by the holder prematurely and recalled by the writer. The most promi-

*Address: Frankfurt MathFinance Institute, Johann Wolfgang Goethe-Universität, 60054 Frankfurt a.M., Germany, e-mail: ckuehn@math.uni-frankfurt.de

†Address: Utrecht University, P.O.Box 80.010, 3500 TA, Utrecht, The Netherlands, e-mail: kyprianou@math.uu.nl

nent example are convertible bonds. The holder can convert them into a predetermined number of stocks of the issuing firm, and the issuer can recall them, paying some compensation to the holder. These contracts were approached in the economic literature for the first time by Brennan and Schwartz (1977) and Ingersoll (1977a), (1977b). Optimal conversion and call policies were derived. These approaches are limited however to Markovian claim structures. The first general analysis of such kind of derivatives, using no-arbitrage arguments in conjunction with game theory, was made by Kifer (2000). For a practical example of a convertible callable bond see McConnell and Schwartz (1986), for an overview of recent progress in handling convertible bonds see Sirbu, Pikovsky, and Shreve (2002) and the references therein.

Let us introduce Kifer's model. Fix some finite time horizon $T \in (0, \infty)$. Suppose that $X = \{X_t : t \in [0, T]\}$ and $Y = \{Y_t : t \in [0, T]\}$ are two stochastic processes, defined on some filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, P\right)$, with values in $\mathbb{R}_+ \cup \{+\infty\}$ and càdlàg paths (right continuous with left limits). We assume that $Y_t \geq X_t$ for all $t \in [0, T)$ and $Y_T = X_T$. The filtered probability space satisfies the usual conditions of right-continuity and completeness.

The Israeli option, as introduced by Kifer (2000), is a contract between a writer and holder at time $t = 0$ such that both have the right to exercise at any time before the expiry date T. If the holder exercises, then (s)he may claim the value of X at the exercise date and if the writer exercise, (s)he is obliged to pay to the holder the value of Y at the time of exercise. If neither have exercised at time T then the writer pays the holder the amount $X_T = Y_T$. If both decide to claim at the same time then the lesser of the two claims is paid. But, it turns out that this marginal case has no impact on the option price as long as the payoff lies in the interval $[X_t, Y_t]$. In short, if the holder will exercise with stopping time σ and the writer with stopping time τ we can conclude that the holder receives at time $\sigma \wedge \tau$ the amount

$$Z_{\sigma,\tau} = X_\sigma \mathbf{1}_{(\sigma \leq \tau)} + Y_\tau \mathbf{1}_{(\tau < \sigma)}. \tag{1.1}$$

In a complete market, with a *unique* risk-neutral measure $\mathbb{P} \sim P$, Kifer obtained a unique no-arbitrage price for such contracts.

Suppose that the financial market consists of one riskless and d risky assets, i.e. $S = (S^0, S^1, \dots, S^d)$, where S^i, $i = 0, \dots, d$, are positive locally bounded semimartingales. To simplify notations we assume w.l.o.g. that $S^0 = 1$. Put differently, we work with discounted values with respect to the numéraire S^0.

Assume that the market is complete, i.e. there is a *unique* equivalent measure $\mathbb{P}$ under which S is a local martingale. This is the situation in the Black-Scholes model. We shall denote $\mathbb{E}$ to be expectation under $\mathbb{P}$. The following Theorem is a slight generalization of Kifer's pricing result. Kifer stated it under the Black-Scholes model and a slightly stronger integrability assumption. However, the arguments are based on martingale representation which holds for complete markets in general, see Theorem 2.1 resp. Theorem 3.3. in Kramkov (1996) applied to the case that the set of equivalent martingale measures is a singleton. For a proof of Theorem 1.1 see Kifer (2000) in conjunction with Step 1 of the proof of Theorem 2.6 in this paper.

Theorem 1.1 *Suppose that*

$$\mathbb{E}\left(\sup_{t\in[0,T]} X_t \right) < \infty. \tag{1.2}$$

Let $\mathcal{T}_{t,T}$ be the class of $\mathbb{F}$-stopping times valued in $[t,T]$. There is a unique no-arbitrage price process of the Israeli option. It can be represented by the right continuous process $V = \{V_t : t \in [0,T]\}$ where

$$\begin{aligned} V_t &= \textit{ess-inf}_{\tau\in\mathcal{T}_{t,T}}\, \textit{ess-sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}(\, Z_{\sigma,\tau} \,|\, \mathcal{F}_t) \\ &= \textit{ess-sup}_{\sigma\in\mathcal{T}_{t,T}}\, \textit{ess-inf}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}(\, Z_{\sigma,\tau} \,|\, \mathcal{F}_t)\,, \end{aligned} \tag{1.3}$$

i.e. V_t is the dynamic value of a Dynkin game. Further, if Y has no positive jumps and X has no negative jumps then optimal stopping strategies exist and are given by

$$\sigma_t^* = \inf\{s \in [t,T] : V_s \le X_s\} \quad \textit{and} \quad \tau_t^* = \inf\{s \in [t,T] : V_s \ge Y_s\} \tag{1.4}$$

for all $t \in [0,T]$.

Remark 1.2 *In incomplete markets no-arbitrage arguments alone are not sufficient to determine unique derivative prices. An established approach to price derivatives in incomplete markets is by utility (indifference) arguments. For American and Israeli options this was done in Kallsen and Kühn (2003). It turns out that the "fair price" of an Israeli option is again the value of a Dynkin game. The unique equivalent martingale measure $\mathbb{P}$ in (1.3) is replaced by a well-chosen so-called neutral pricing measure $P^* \sim P$ which plays a crucial role in utility maximization, see Kallsen and Kühn (2003). Therefore, the results of this paper can be used in the same manner to simulate (utility based) option prices in incomplete markets.*

Although in special cases the optimal stopping problem in (1.3) can be solved explicitly (see Kühn and Kyprianou (2003) for the Israeli put option and the Israeli-Russian option), in general, analytical methods are virtually out of the question. Clearly one should not expect to be in any a better position than when posing the same question for American claims. Consider for example, for large $d \in \mathbb{N}$, an American option with contingent claim of the form

$$\left(K - \min\left\{S_t^1, ..., S_t^d\right\}\right)^+ .$$

Trying to characterize free boundary problems for such an option can also be a problem because of the high dimensionality. For such cases, Rogers proposes to work via a dual pricing formula which inspires a different outlook when it comes to simulation.

The goal of this paper is to produce a dual representation of the price of an Israeli option which could in the same way be used for Monte Carlo simulation as Rogers' contribution on American options. American claims are covered in this paper by setting $Y_t = \infty$ for $t \in [0, T)$ since we only need an integrability assumption for the lower bound X. We made the following observation which can help when it comes to simulation. Let us interpret the stopping game in (1.3) no longer as a stochastic but as a *deterministic* stopping game, by choosing the optimal exercise strategies for each path $\omega \in \Omega$ separately. This represents the hypothetical situation where all information about future price movements of the underlyings are available at the very beginning. Instead of the payoff function $Z_{s,t}$ as defined in (1.1) we consider $Z_{s,t}(\omega) - M^*_{s\wedge t}(\omega)$ where M^* is a martingale which will be characterized later on. In the complete market M^* corresponds to the gain process generated by a hedging strategy. It turns out that the values of these stopping games coincide P-a.s. with V_0, the price of the Israeli option, see Theorem 2.9. Therefore, with the right martingale M^* it is possible to attain the exact option price by the simulation of a single sample path. Together with Theorem 2.12, which provides estimations in the case of a simulation with a wrong martingale $M \neq M^*$, this is the main contribution of this paper.

Cvitanić and Karatzas (1996) established a connection between the value of a Dynkin game, i.e. a stochastic stopping game as it arises in Theorem 1.1, and the solution of a backward stochastic differential equation with reflection. In this framework they also construced a corresponding deterministic stopping game, i.e. a game played path by path. However, their pathwise obtained values of the deterministic stopping games are still stochastic and coincides only in expectation with the value of the stochastic stopping game.

2 Representations of the Israeli option price

Throughout the whole paper we assume that X satisfies the integrability condition (1.2).

Remark 2.1 *We do not need that* $\sup_{t\in[0,T]} X_t \in L^p(\mathbb{P})$ *for some* $p > 1$ *as it is assumed in Rogers (2002). In addition, we state our duality result, Theorem 2.12, for all martingales vanishing at zero, not assuming that the pathwise supremum of the martingale is integrable.*

Firstly we need a couple of notations. They are rather voluminous. Primarily, this is the price for capturing the case of discontinuous payoff processes.

Definition 2.2 *A stochastic process* $U = (U_t)_{t\in[0,T]}$ *is said to be of class (D) if* $\{U_\tau \mid \tau \in \mathcal{T}_{0,T}\}$ *is uniformly integrable.*

Definition 2.3 *Denote by* $\mathcal{M}_0$ *the set of martingales vanishing at zero.*

In continuation of the usual notations for stopped processes we make the following definition.

Definition 2.4 *Let* U *be a stochastic process,* τ *a stopping time, and* D *an* $\mathcal{F}_{\tau-}$*-measurable subset of* Ω *(*$\mathcal{F}_{\tau-}$ *is the* σ*-algebra generated by* $\mathcal{F}_0$ *and by the sets of the form* $A \cap \{t < \tau\}$*, where* $t \in (0,T]$ *and* $A \in \mathcal{F}_t$*). We denote by* $U^{\tau,D}$ *the process* U *stopped at* $\tau-$*, if the event* D *occurs, and at* τ*, if* D *does not occur, i.e.*

$$U_t^{\tau,D} = U_t \mathbf{1}_{(t<\tau)} + U_{\tau-}\mathbf{1}_{\left(t\geq\tau \text{ and } \omega\in D\right)} + U_\tau \mathbf{1}_{\left(t\geq\tau \text{ and } \omega\notin D\right)}.$$

Remark 2.5 *If* U *is a martingale or predictable, then the respective property also holds for* $U^{\tau,D}$*. This can be derived by using e.g. Proposition I.2.10 in Jacod and Shiryayev (2003).*

Define the stopping times

$$\tau^\varepsilon = \inf\{s \geq 0 \mid V_s \geq Y_s - \varepsilon\}, \quad \varepsilon > 0, \quad \text{and} \quad \widehat{\tau} = \lim_{\varepsilon\to 0} \tau^\varepsilon. \tag{2.1}$$

The limit is well defined as τ^ε is monotone in ε. If Y is lower semicontinuous from the left (i.e. it has no positive jumps), then $\widehat{\tau} = \tau^*$. Analoguously we set

$$\sigma^\varepsilon = \inf\{s \geq 0 \mid V_s \leq X_s + \varepsilon\}, \quad \varepsilon > 0, \quad \text{and} \quad \widehat{\sigma} = \lim_{\varepsilon\to 0} \sigma^\varepsilon.$$

Let g be the American claim given by the (discounted) payoff process

$$g_s = X_s \mathbf{1}_{(s<\widehat{\tau})} + Y_{\widehat{\tau}-}\mathbf{1}_{\left(\widehat{\tau}\leq s \text{ and } \tau^\varepsilon<\widehat{\tau}\ \forall\varepsilon>0\right)} + Y_{\widehat{\tau}}\mathbf{1}_{\left(\widehat{\tau}\leq s \text{ and } \exists\varepsilon>0 \text{ s.t. } \tau^\varepsilon=\widehat{\tau}\right)}. \tag{2.2}$$

g is the payoff process the buyer is faced with – given the optimal exercise strategy of the seller. Analogously we define the payoff process the seller is faced with by

$$h_t = X_{\widehat{\sigma}-}\mathbf{1}_{\left(\widehat{\sigma}\leq t \text{ and } \sigma^\varepsilon<\widehat{\sigma}\ \forall\varepsilon>0\right)} + X_{\widehat{\sigma}}\mathbf{1}_{\left(\widehat{\sigma}\leq t \text{ and } \exists\varepsilon>0 \text{ s.t. } \sigma^\varepsilon=\widehat{\sigma}\right)} + Y_t\mathbf{1}_{(t<\widehat{\sigma})}. \tag{2.3}$$

Let $D_1 = \{\tau^\varepsilon < \widehat{\tau}, \quad \forall\varepsilon > 0\}$ and $D_2 = \{\sigma^\varepsilon < \widehat{\sigma}, \quad \forall\varepsilon > 0\}$.

Theorem 2.6 *The process $V^{\widehat{\tau},D_1}$ is a supermartingale, the process $V^{\widehat{\sigma},D_2}$ is a submartingale, and the process $\left(V^{\widehat{\tau},D_1}\right)^{\widehat{\sigma},D_2} = \left(V^{\widehat{\sigma},D_2}\right)^{\widehat{\tau},D_1}$ is a martingale. It follows that $V^{\widehat{\sigma}\vee\widehat{\tau}}$ enjoys a canonical decomposition of the form*

$$V^{\widehat{\sigma}\vee\widehat{\tau}} = V_0 + M^* + A - B, \tag{2.4}$$

where

(i) *the process $M^* = \{M_t^* : t \in [0,T]\}$ belongs to $\mathcal{M}_0$,*

(ii) *$A = \{A_t : t \in [0,T]\}$ is predictable and non-decreasing such that $A^{\widehat{\tau},D_1} = 0$.*

(iii) *$B = \{B_t : t \in [0,T]\}$ is predictable and non-decreasing such that $B^{\widehat{\sigma},D_2} = 0$.*

In addition, the Snell envelope of g

$$V_t' := \textit{ess-sup}_{\sigma\in\mathcal{T}_{t,T}}\mathbb{E}(\, g_\sigma \,|\, \mathcal{F}_t) \,,$$

exists as an element of class (D) so that it possesses a Doob-Meyer decomposition. Analogously, the lower *Snell envelope of h*

$$V_t'' := \textit{ess-inf}_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}(\, h_\tau \,|\, \mathcal{F}_t) \,.$$

exists as an element of class (D) and possesses a Doob-Meyer decomposition.

We have $V_0 = V_0' = V_0''$. The process $(M^*)^{\widehat{\tau},D_1}$ coincides with the martingale part in the Doob-Meyer decomposition of V' and $(M^*)^{\widehat{\sigma},D_2}$ coincides with the martingale part in the Doob-Meyer decomposition of V''.

Proof. *Step 1:* Let us show that under assumption (1.2) the Dynkin game possesses an equilibrium point, i.e. the process V in (1.3) exists. This was proven by Lepeltier and Maingueneau (1984) (henceforth LM) for bounded payoff processes X and Y. However, the existence of an equilibrium point holds also under the weaker assumption (1.2), as it is shown in Kallsen and Kühn (2003). To see this assume in the first instance that Y is bounded by a $\mathbb{P}$-martingale M. W.l.o.g. $P(M_T > 0) = 1$. By applying the statements of LM to the processes

$\widetilde{X}_t := \frac{X_t}{M_t}$ and $\widetilde{Y}_t := \frac{Y_t}{M_t}$ (which have values in $[0,1]$), taking the measure $\widetilde{\mathbb{P}}$ defined by $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \frac{M_T}{M_0}$, we obtain the existence of a value process $\widetilde{V}$. On the other hand, we have for $U \in \{X, Y\}$, $t \in [0,T]$, $\sigma \in \mathcal{T}_{t,T}$

$$\widetilde{\mathbb{E}}\left(\widetilde{U}_\sigma \middle| \mathcal{F}_t\right) = \frac{1}{M_t}\mathbb{E}\left(M_T \widetilde{U}_\sigma \middle| \mathcal{F}_t\right) = \frac{1}{M_t}\mathbb{E}(\, U_\sigma | \, \mathcal{F}_t).$$

This implies that the value process V for the payoff processes X, Y and the measure $\mathbb{P}$ also exists with $V_t = M_t \widetilde{V}_t$. Thus, the following implication holds

(2.5) $\quad Y$ is bounded by a martingale $\Longrightarrow$ value process V for (X,Y) exists.

Now, assume only (1.2), instead of $Y \leq M$, which immediately implies that $X \leq \bar{M}$, where $\bar{M}$ is the martingale $t \mapsto \mathbb{E}\left(\sup_{s\in[0,T]} X_s | \mathcal{F}_t\right)$. Instead of Y consider the upper bound $\bar{Y} := Y \wedge (1 + \bar{M})$ and let $\bar{V}$ be the corresponding value process for the payoff processes $\bar{X} = X$ and $\bar{Y}$. As $X \leq \bar{Y}$ and $X_T = \bar{Y}_T$ the value process of this Dynkin game exists by (2.5). Let us show that the value process V for X and Y also exists and coincides with $\bar{V}$. We have

$$\bar{V}_t \leq \text{ess-sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}(\, X_\sigma | \, \mathcal{F}_t) \leq \bar{M}_t. \tag{2.6}$$

Consider now the ε-optimal recall time of the option seller given by $\bar{\tau}_t^\varepsilon := \inf\{s \geq t \mid \bar{V}_s \geq \bar{Y}_s - \varepsilon\}$. By right-continuity we have $\bar{V}_{\bar{\tau}_t^\varepsilon} \geq \bar{Y}_{\bar{\tau}_t^\varepsilon} - \varepsilon$. Together with (2.6) this implies that $\bar{Y}_{\bar{\tau}_t^\varepsilon} < 1 + \bar{M}_{\bar{\tau}_t^\varepsilon}$ for all $\varepsilon \in (0,1)$. Therefore, by definition of $\bar{Y}$,

$$\bar{Y}_{\bar{\tau}_t^\varepsilon} = Y_{\bar{\tau}_t^\varepsilon}, \quad \forall \varepsilon \in (0,1). \tag{2.7}$$

With this we obtain

$$\begin{aligned}
&\text{ess-sup}_{\sigma\in\mathcal{T}_{t,T}} \text{ess-inf}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left(X_\sigma \mathbf{1}_{(\sigma\leq\tau)} + Y_\tau \mathbf{1}_{(\tau<\sigma)} \middle| \mathcal{F}_t\right) \\
&\geq \bar{V}_t \\
&= \lim_{\varepsilon\to 0} \text{ess-sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}\left(X_\sigma \mathbf{1}_{(\sigma\leq\bar{\tau}_t^\varepsilon)} + \bar{Y}_{\bar{\tau}_t^\varepsilon} \mathbf{1}_{(\bar{\tau}_t^\varepsilon<\sigma)} \middle| \mathcal{F}_t\right) \\
&= \lim_{\varepsilon\to 0} \text{ess-sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}\left(X_\sigma \mathbf{1}_{(\sigma\leq\bar{\tau}_t^\varepsilon)} + Y_{\bar{\tau}_t^\varepsilon} \mathbf{1}_{(\bar{\tau}_t^\varepsilon<\sigma)} \middle| \mathcal{F}_t\right) \\
&\geq \text{ess-inf}_{\tau\in\mathcal{T}_{t,T}} \text{ess-sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}\left(X_\sigma \mathbf{1}_{(\sigma\leq\tau)} + Y_\tau \mathbf{1}_{(\tau<\sigma)} \middle| \mathcal{F}_t\right).
\end{aligned} \tag{2.8}$$

The first inequality is by $Y \geq \bar{Y}$. The first equality is by Theorem 11 in LM applied to X and $\bar{Y}$ and the second equality by (2.7). By (2.8) V exists and coincides with $\bar{V}$.

Step 2: Assumption (1.2) implies that

$$V_t \leq \mathbb{E}\left(\sup_{s\in[t,T]} X_s | \mathcal{F}_t \right) \leq \bar{M}_t, \tag{2.9}$$

where $\bar{M}$ is defined in Step 1. In addition we have

$$\begin{aligned} V_t' &\leq \text{ess-sup}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left(X_\sigma \,|\, \mathcal{F}_t\right) + \mathbb{E}\left(V_{\widehat{\tau}-} \mathbf{1}_{(\tau^\varepsilon < \widehat{\tau}\ \forall\varepsilon>0)} + V_{\widehat{\tau}} \mathbf{1}_{\left(\exists\varepsilon>0 \text{ s.t. } \tau^\varepsilon = \widehat{\tau}\right)} \middle| \mathcal{F}_t \right) \\ &\overset{(2.9)}{\leq} \bar{M}_t + \bar{M}_t^{\widehat{\tau},D_1} \end{aligned}$$

and

$$V_t'' \leq \mathbb{E}(Y_T|\mathcal{F}_t) + \mathbb{E}\left(\sup_{s\in[0,T]} X_s | \mathcal{F}_t \right) \leq 2\bar{M}_t,$$

where the second inequality follows because $Y_T = X_T$. As $\bar{M}$ and $\bar{M}^{\widehat{\tau},D_1}$ are martingales they are processes of class (D), cf. e.g. Proposition I.1.46 in Jacod and Shiryaev (2003). Therefore, V, V', and V'' are processes of class (D), which is of later use for the Doob-Meyer decomposition.

Step 3: Let us show that $V^{\widehat{\tau},D_1} = V'$. Fix a $t \in [0,T]$. On the event $\{\widehat{\tau} \leq t\}$ the assertion is obvious. In the following calculations, we assume that we are working on the event $\{\widehat{\tau} > t\}$. By Theorem 11 in LM, we have for $\varepsilon > 0$

$$\text{ess-sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}\left(Z_{\sigma,\tau_t^\varepsilon} \,|\, \mathcal{F}_t\right) \leq V_t + \varepsilon, \tag{2.10}$$

where $\tau_t^\varepsilon = \inf\{s \geq t \mid V_s \geq Y_s - \varepsilon\}$. Let $\delta > 0$. Again, by Theorem 11 in LM, we can choose an δ-optimal stopping strategy σ^δ such that $\mathbb{E}(g_{\sigma^\delta} | \mathcal{F}_t) \geq$ $\text{ess-sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}(g_\sigma | \mathcal{F}_t) - \delta$. Let $\varepsilon_n \downarrow 0$ as $n \uparrow \infty$. Possibly after re-definition of $Z_{s,t}$ for $s = t$ (which does not change the value process V), we have pointwise convergence of $Z_{\sigma^\delta,\tau_t^{\varepsilon_n}}$ to g_{σ^δ} as $n \to \infty$ (note that $\tau_t^{\varepsilon_n} = \tau^{\varepsilon_n}$ for sufficiently small ε_n). By (2.9) the sequence $\left(Z_{\sigma^\delta,\tau_t^{\varepsilon_n}}\right)_{n\in\mathbb{N}}$ is uniformly integrable so that we have convergence of $\mathbb{E}\left(Z_{\sigma^\delta,\tau_t^{\varepsilon_n}} \middle| \mathcal{F}_t\right)$ to $\mathbb{E}(g_{\sigma^\delta} | \mathcal{F}_t)$ in $L^1(\mathbb{P})$. This implies

P-a.s. convergence on a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$, w.l.o.g. $n_k = k$. Together with (2.10) applied to $\varepsilon = \varepsilon_{n_k}$ we obtain

$$\begin{aligned} V_t + \varepsilon_k &\geq \text{ess-sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}\left(Z_{\sigma,\tau_t^{\varepsilon_k}} \,\middle|\, \mathcal{F}_t \right) \\ &\geq \mathbb{E}\left(Z_{\sigma^\delta,\tau_t^{\varepsilon_k}} \,\middle|\, \mathcal{F}_t \right) \\ &\stackrel{k\to\infty}{\longrightarrow} \mathbb{E}(\, g_{\sigma^\delta} \,|\, \mathcal{F}_t) \\ &\geq \text{ess-sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}(\, g_\sigma \,|\, \mathcal{F}_t) - \delta. \end{aligned}$$

As $\delta > 0$ was arbitrary this implies $V_t \geq \text{ess-sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}(\, g_\sigma \,|\, \mathcal{F}_t)$. For the opposite estimation we use that, again by Theorem 11 in LM,

$$V_t - \varepsilon \leq \text{ess-inf}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left(Z_{\sigma_t^\varepsilon,\tau} \,\middle|\, \mathcal{F}_t \right), \tag{2.11}$$

where $\sigma_t^\varepsilon = \inf\{s \geq t \mid V_s \leq X_s + \varepsilon\}$. On a subsequence $(n_k)_{k\in\mathbb{N}}$ we have again pointwise convergence of $\mathbb{E}\left(Z_{\sigma_t^\varepsilon,\tau^{\varepsilon_{n_k}}} \,\middle|\, \mathcal{F}_t \right)$ to $\mathbb{E}\left(g_{\sigma_t^\varepsilon} \,\middle|\, \mathcal{F}_t \right)$, w.l.o.g. $n_k = k$. We obtain

$$\begin{aligned} V_t - \varepsilon &\leq \text{ess-inf}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left(Z_{\sigma_t^\varepsilon,\tau} \,\middle|\, \mathcal{F}_t \right) \\ &\leq \mathbb{E}\left(Z_{\sigma_t^\varepsilon,\tau^{\varepsilon_k}} \,\middle|\, \mathcal{F}_t \right) \\ &\stackrel{k\to\infty}{\longrightarrow} \mathbb{E}\left(g_{\sigma_t^\varepsilon} \,\middle|\, \mathcal{F}_t \right) \\ &\leq \text{ess-sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}(\, g_\sigma \,|\, \mathcal{F}_t), \quad k\to\infty. \end{aligned}$$

Altogether we arrive at

$$V^{\widehat{\tau},D_1} = V'. \tag{2.12}$$

Using similar reasoning one can conclude that

$$V^{\widehat{\sigma},D_2} = V''. \tag{2.13}$$

As V' is a supermartingale and V'' is a submartingale it follows that $V^{\widehat{\tau}\vee\widehat{\sigma}}$ is a special semimartingale, i.e. it possesses a unique decomposition

$$V = V_0 + M^* + A - B,$$

with the properties as stated in the Theorem. The last assertion of the theorem follows then from (2.12), (2.13), and Remark 2.5. ∎

Remark 2.7 *Theorem 2.6 provides a canonical decomposition of the process V only up to $\widehat{\sigma} \vee \widehat{\tau}$. For the whole process such a decomposition need not exist, as V is in general not a semimartingale. For example, V can be deterministic and possess infinite variation on $[0,T]$. This is in contrast to American claims where V is a supermartingale and hence a semimartingale.*

Lemma 2.8 *Let $L = (L_t)_{t\in[0,T]}$ be a positive process such that its Snell-envelope V^L exists as an element of class (D) and let $V^L = V_0^L + M^L + A^L$ be the Doob-Meyer decomposition of the Snell envelope. Then we have*

$$V_0^L \quad = \quad \sup_{s\in[0,T]} \left(L_s - M_s^L\right), \quad P\text{-}a.s.$$

Proof. By $L \leq V^L \leq V_0^L + M^L$, the left-hand side is obviously not smaller than the right-hand side. On the other hand for each $\varepsilon > 0$ define the $[0,T]$-valued stopping time $\tau^\varepsilon = \inf\{s \geq 0 \mid V_s^L \leq L_s + \varepsilon\}$. By equation (24) in Fakeev (1970) we have $A_{\tau^\varepsilon}^L = 0$ and therefore $L_{\tau^\varepsilon} - M_{\tau^\varepsilon}^L \geq V_{\tau^\varepsilon}^L - M_{\tau^\varepsilon}^L - \varepsilon = V_0^L - \varepsilon$. ∎

Theorem 2.9 *We have*

$$\begin{aligned} V_0 &= \sup_{s\in[0,T]} \inf_{t\in[0,T]} \left(Z_{s,t} - M^*_{s\wedge t}\right) \\ &= \inf_{t\in[0,T]} \sup_{s\in[0,T]} \left(Z_{s,t} - M^*_{s\wedge t}\right), P\text{-}a.s, \end{aligned}$$

where M^* is the martingale part in (2.4).

Proof. Consider the American contingent claim g defined in (2.2). By Theorem 2.6 its Snell envelope V' possesses a Doob-Meyer decomposition $V' = V_0' + M' + A'$ with $V_0' = V_0$ and $M' = (M^*)^{\widehat{\tau}, D_1}$. By this and Lemma 2.8 we obtain

$$\begin{aligned} V_0 &= \sup_{\sigma\in\mathcal{T}_{0,T}} \mathbb{E}(g_\sigma) \\ &= \sup_{s\in[0,T]} \left(g_s - M_s'\right) \\ &\geq \inf_{t\in[0,T]} \sup_{s\in[0,T]} \left(Z_{s,t} - M^*_{s\wedge t}\right), \quad P\text{-a.s.} \end{aligned} \tag{2.14}$$

To assure oneself of the last inequality choose $t(\omega) = \tau^{\varepsilon}(\omega)$ for all $\varepsilon > 0$.

Again, by Theorem 2.6 we know that the lower Snell envelope V'' of the process h defined in (2.3) has a Doob-Meyer decomposition $V'' = U_0'' + M'' - B''$ with $V_0'' = V_0$ and $M'' = (M^*)^{\bar{\sigma}, D_2}$. We obtain as in (2.14)

$$\begin{aligned} V_0 &= \inf_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}(h_\tau) \\ &= \inf_{t \in [0,T]} (h_t - M_t'') \\ &\leq \sup_{s \in [0,T]} \inf_{t \in [0,T]} (Z_{s,t} - M^*_{s \wedge t}), \quad P\text{-a.s.} \end{aligned} \tag{2.15}$$

(2.14) and (2.15) imply the assertion. ■

Theorem 2.10 *Let $M \in \mathcal{M}_0$ and assume that there exists a $v' \in \mathbb{R}$ such that*

$$\begin{aligned} v' &= \sup_{s \in [0,T]} \inf_{t \in [0,T]} (Z_{s,t} - M_{s \wedge t}) \\ &= \inf_{t \in [0,T]} \sup_{s \in [0,T]} (Z_{s,t} - M_{s \wedge t}), \quad P\text{-a.s.} \end{aligned} \tag{2.16}$$

Then $v' = V_0$ (The supremum and the infimum in (2.16) can ever be interchanged).

Remark 2.11 *Theorem 2.9 removes the involvement with stopping strategies and instead transfers the essence of the pricing problem to a clever choice of a martingale M. In complete markets the choice of a $M \in \mathcal{M}_0$ corresponds to a hedging strategy. With the right martingale $M = M^*$ the random variable*

$$\sup_{s \in [0,T]} \inf_{t \in [0,T]} (Z_{s,t} - M_{s \wedge t}) \tag{2.17}$$

degenerates to a real number and the option price can be attained by the simulation of a single sample path.

¿From a practical point of view it remains of course the problem to approximate M^*. Arguably, one has not made things any easier. Theorem 2.10 helps at least to identify the right M. It says that M is the "right" martingale if and only if the random variable (2.17) is a constant. This can be "checked" by a Monte Carlo simulation. One way to approximate M^* could be to choose a finite basis of martingales in $\mathcal{M}_0$ and minimize the empirical variance of the

random variable (2.17) over linear combinations of this basis. Rogers argues that with sensible choices of basis one can do quite well in this respect. For an effective computation of the dual upper bound we refer the reader to Kolodko and Schoenmakers (2003). For a general overview on Monte Carlo methods for American options we refer to Glasserman (2003) and the references therein.

Proof of Theorem 2.10. The interchangeability of supremum and infimum in (2.16) follows from LM. Assume that

$$\inf_{t\in[0,T]} \sup_{s\in[0,T]} (Z_{s,t} - M_{s\wedge t}) = v', \quad P\text{-a.s.}$$

for some $v' \in \mathbb{R}$ with $v' \neq V_0$. W.l.o.g. $v' > V_0$. We have of course that

$$\sup_{s\in[0,T]} \left(g_s - M_s^{\widehat{\tau},D_1}\right) > V_0,$$

where g is defined in (2.2). By $X_s \leq V_s \quad \forall s \in [0,T]$, and $Y_T^{\widehat{\tau},D_1} = V_T^{\widehat{\tau},D_1}$ we have that $g \leq V^{\widehat{\tau},D_1}$. Remember that V has the decomposition $V = V_0 + M^* + A - B$ with $A^{\widehat{\tau},D_1} = 0$. This implies

$$\begin{aligned} \sup_{s\in[0,T]} \left(V_0 + (M^*)_s^{\widehat{\tau},D_1} - M_s^{\widehat{\tau},D_1}\right) &\geq \sup_{s\in[0,T]} \left(V_s^{\widehat{\tau},D_1} - M_s^{\widehat{\tau},D_1}\right) \\ &\geq \sup_{s\in[0,T]} \left(g_s - M_s^{\widehat{\tau},D_1}\right) > V_0, \quad P\text{-a.s.} \end{aligned}$$

But this is a contradiction to the fact that $(M^*)^{\widehat{\tau},D_1} - M^{\widehat{\tau},D_1}$ is a martingale, cf. Remark 2.5. ■

Theorem 2.12 *We have*

$$\text{(2.18)} \qquad \begin{aligned} V_0 &= \inf_{M\in\mathcal{M}_0} \inf_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}\left(\sup_{s\in[0,T]} (Z_{s,\tau} - M_{s\wedge\tau})\right) \\ &= \sup_{M\in\mathcal{M}_0} \sup_{\sigma\in\mathcal{T}_{0,T}} \mathbb{E}\left(\inf_{t\in[0,T]} (Z_{\sigma,t} - M_{\sigma\wedge t})\right). \end{aligned}$$

Further, the infimum as well as the supremum are achieved when M is chosen to be M^*.

Remark 2.13 *By choosing an arbitrary pair* $(\tau, M) \in \mathcal{T}_{0,T} \times \mathcal{M}_0$ *and simulating* $\mathbb{E}\left(\sup_{s\in[0,T]}(Z_{s,\tau} - M_{s\wedge\tau})\right)$ *we can get an upper bound for* V_0 *and by simulating* $\mathbb{E}\left(\inf_{t\in[0,T]}(Z_{\sigma,t} - M_{\sigma\wedge t})\right)$ *with an arbitrary pair* $(\sigma, M) \in \mathcal{T}_{0,T} \times \mathcal{M}_0$ *we obtain a lower bound for* V_0. *Of course, to obtain tight estimations we have to find both a "good" stopping time and a "good" martingale. This corresponds to the fact that for Israeli options a hedging strategy consists of a trading strategy (in the underlyings) and a stopping time.*

Proof of Theorem 2.12. By symmetry we have only to show (2.18). For any $(\tau, M) \in \mathcal{T}_{0,T} \times \mathcal{M}_0$ we have that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{s\in[0,T]}(Z_{s,\tau} - M_{s\wedge\tau})\right)\\
&\geq \sup_{\sigma\in\mathcal{T}_{0,T}} \mathbb{E}(Z_{\sigma,\tau} - M_{\sigma\wedge\tau})\\
&= \sup_{\sigma\in\mathcal{T}_{0,T}} \mathbb{E}(Z_{\sigma,\tau}).
\end{aligned}$$

Taking the infimum over all $\tau \in \mathcal{T}_{0,T}$ this implies that the right-hand side of (2.18) is at least as big as V_0.

To prove the other direction we take for each $\varepsilon > 0$ the stopping time τ^ε as defined in (2.1) and the martingale part M^ε of the Snell envelope for the American claim with buying back time τ^ε. We obtain

$$\begin{aligned}
&\mathbb{E}\left(\sup_{s\in[0,T]}(Z_{s,\tau^\varepsilon} - M^\varepsilon_{s\wedge\tau^\varepsilon})\right)\\
&= \sup_{\sigma\in\mathcal{T}_{0,T}} \mathbb{E}(Z_{\sigma,\tau^\varepsilon})\\
&\leq V_0 + \varepsilon,
\end{aligned}$$

where the equality holds by Lemma 2.8 and the inequality by Theorem 11 of LM. ∎

References

[1] Brennan, M.J. and Schwartz, E.S. (1977) Convertible bonds: valuation and optimal strategies for call and conversion. *Journal of Finance* **32,** 1699–1715.

[2] Cvitanić and Karatzas (1996) Backward stochastic differential equations with reflection and Dynkin games. *Annals of Probability* **24** 2024–2056.

[3] Fakeev, A. (1970) Optimal Stopping Rules for Stochastic Processes with Continuous Parameters. *Theory of Probability and its Applications* **15,** 324–331.

[4] Glasserman, P. (2003) *Monte Carlo Methods in Financial Engineering.* Springer.

[5] Ingersoll, J.E. (1977a) A contingent-claims valuation of convertible securities. *Journal of Financial Economics* **4,** 289–322.

[6] Ingersoll, J.E. (1977b) An examination of corporate call policies on convertible securities. *Journal of Finance* **32,** 463–478.

[7] Jacod, J. and Shiryayev, A.N. (2003) *Limit Theorems for Stochastic Processes.* Springer. 2.Auflage.

[8] Kallsen, J. and Kühn, C. (2003) Pricing Derivatives of American and Game Type in Incomplete Markets. *Finance & Stochastics. To appear.*

[9] Kifer, Y. (2000) Game options. *Finance & Stochastics* **4,** 443–463.

[10] Kolodko, A. and Schoenmakers, J. (2003) An Efficient Dual Monte Carlo Upper Bound for Bermudan Style Derivatives. *Preprint.*

[11] Kramkov, D. (1996) Optional decomposition of supermartingales and hedging contingent claims in incomplete markets. *Probability Theory and Related Fields* **105,** 459–479.

[12] Kühn, C. and Kyprianou, A.E. (2003) Israeli options as composite exotic options. *Preprint.*

[13] Kyprianou, A.E. (2002) Some calculations for Israeli options. *Finance & Stochastics. To appear.*

[14] Lepeltier, J. and Maingueneau, M. (1984) Le jeu de Dynkin en théorie générale sans l'hypothèse de Mokobodski. *Stochastics* **13**, 25–44.

[15] McConnell, J.J. and Schwartz, E.S. (1986) LYON Taming. *Journal of Finance* **41** 561–576.

[16] Rogers, L.C.G (2002) Monte-Carlo valuation of American options. *Mathematical Finance* **12** 271–286.

[17] Sirbu,M. and Pikovsky, I. and Shreve, S. (2002) Perpetual Convertible Bonds. *Preprint.*