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The uniform saturation property for a singularly perturbed reaction-diffusion equation

Rob Stevenson[⋆]

Department of Mathematics, Utrecht University, P.O. Box 80.010, NL-3508 TA Utrecht, The Netherlands. e-mail: stevenson@math.uu.nl

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Summary Although adaptive finite element methods for solving elliptic problems often work well in practice, they are usually not proven to converge. For Poisson like problems, an exception is given by the method of Dörfler ([8]), that was later improved by Morin, Nochetto and Siebert ([11]). In this paper we extend these methods by constructing an adaptive finite element method for a singularly perturbed reaction-diffusion equation that, in energy norm, converges uniformly in the size of the reaction term. Moreover, in this algorithm the arising Galerkin systems are solved only inexactly, so that, generally, the number of arithmetic operations is equivalent to the number of triangles in the final partition.

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1 Introduction

For solving elliptic boundary value problems with a solution that has singularities, the use of adaptive finite element methods has the potential advantage of a strong reduction of the computational cost compared to non-adaptive methods. Although adaptive methods that

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are found in the literature indeed often exhibit such a reduction, they are usually not proven to converge. For Poisson type problems, an exception is given by the method of Dörfler in [8], that was later improved by Morin, Nochetto and Siebert in [11]. These methods are based on an adaptive refinement strategy that guarantees the so-called saturation property, saying that the difference between the solutions on two successive partitions is greater than some multiple of the error in the solution on the first partition. Exploiting Galerkin orthogonality, linear convergence then easily follows.

In this paper, we consider reaction-diffusion problems. Such problems arise, e.g., in the modeling of thin plates, in the numerical treatment of reaction processes, and as a result of an implicit time discretization of parabolic or hyperbolic equations of second order. As a first step towards a uniformly convergent adaptive method, efficient and reliable a posteriori error estimators, uniformly in the reaction term, were developed in [15,1], and for anisotropic meshes, in [10]. Considering piecewise linear approximations on isotropic meshes in two dimensions, we extend the results from [11] by constructing an adaptive refinement strategy that guarantees the saturation property uniformly in the reaction term, and with that leads to a uniformly convergent adaptive method.

We consider two variants of the adaptive method. With the first variant, to control what is called being "data oscillation" in [11], the right-hand side of the equation is approximated by piecewise constants. The resulting method is uniformly linearly convergent, however, for a strongly dominant reaction term it can happen that the overall rate is limited by the rate of convergence of these approximations for the right-hand side. What is more, in the setting of time dependent problems, it is not very attractive to be forced to replace the right-hand side, which is dominated by the piecewise linear approximation on the previous time step, by a piecewise constant approximation. Therefore, in the second variant, we allow the right-hand side of the equation to be approximated by piecewise linears. In this variant, the method can be shown to be uniformly convergent, however, generally only when, depending on the reaction term, the initial mesh is sufficiently fine.

A novel aspect of our adaptive method is that we allow that the Galerkin systems are solved only inexactly. In general, the number of arithmetic operations is of the order of the number of triangles in the final partition.

This paper is organized as follows. In Sect. 2, we describe the boundary value problem under consideration. In Sect. 3, we define the type of (nonconforming) finite element partitions that we will consider. The a posteriori error estimator is discussed in Sect. 4. In Sect. 5, we prove the uniform saturation property, assuming for the moment that the right-hand side is piecewise constant with respect to the current mesh. Considering general right-hand sides, that inside the algorithm are approximated by piecewise constants, and allowing inexact solutions of the Galerkin systems, in Sect. 6 we construct a uniformly convergent adaptive finite element method. In Sect. 7, we show that with piecewise linear approximations for the right-hand side, uniform convergence is guaranteed when, depending on the reaction term, the initial mesh is sufficiently fine. Finally, in Sect. 8 we report on numerical experiments carried out with both variants of the algorithm.

In order to avoid the repeated use of generic but unspecified constants, in this paper by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on, here in particular the size of the reaction term or the underlying finite element partition. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2 Boundary value problem

Let Ω be a polygonal bounded domain in $\mathbb{R}^2$. We consider the following model reaction-diffusion problem in variational form: Given $f \in H^{-1}(\Omega)\,(= H_0^1(\Omega)')$, find $u \in H_0^1(\Omega)$ such that

$$a(u,w) := \tfrac{1}{1+\kappa^2} \int_\Omega [\nabla u \cdot \nabla w + \kappa^2 uw] = f(w), \quad (w \in H_0^1(\Omega)), \tag{2.1}$$

where κ is a scalar, constant over the domain. Defining $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ by $(Lu)(w) = a(u,w)$, (2.1) can be rewritten as

$$Lu = f.$$

For $f \in L_2(\Omega)$, we interpret $f(w)$ as $\int_\Omega fw$.

We will measure the error of any approximation for u in the energy norm

$$\|w\|_E := a(w,w)^{\frac{1}{2}}, \quad (w \in H_0^1(\Omega)),$$

with dual norm $\|f\|_{E'} := \sup_{0 \neq w \in H_0^1(\Omega)} \frac{|f(w)|}{\|w\|_E}$, $(f \in H^{-1}(\Omega))$. Equipped with these norms, L is an isomorphism between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

For $\Sigma \subset \Omega$, we define $\|w\|_{E,\Sigma} = (\frac{1}{1+\kappa^2} \int_\Sigma [|\nabla w|^2 + \kappa^2 |w|^2])^{\frac{1}{2}}$.

Remark 2.1 Instead of the commonly used formulations of the reaction-diffusion equation as $(-\triangle + \kappa^2 I)u = \tilde{f}$ or $-(\kappa^{-2}\triangle + I)u = \hat{f}$, we have chosen for the variational formulation of $\frac{1}{1+\kappa^2}(-\triangle + \kappa^2 I)u = f$, in order both to include the Poisson problem and to have a problem that is well-scaled in the sense that for given $f \in L_2(\Omega)$, $\|f\|_{E'} \eqsim 1$ uniformly in κ. Indeed, by the Poincaré-Friedrichs inequality we have $\|\cdot\|_{L_2(\Omega)} \lesssim \|\cdot\|_E$ on $H_0^1(\Omega)$, and so $\|f\|_{E'} \lesssim \|f\|_{L_2(\Omega)}$. On the other hand, since $H_0^1(\Omega)$ is dense in $L_2(\Omega)$, there exists a $w \in H_0^1(\Omega)$ with $\langle f, w\rangle_{L_2(\Omega)} \geq \frac{1}{2}\|w\|^2_{L_2(\Omega)}$, so that from $\|w\|_E \leq \|w\|_{H^1(\Omega)}$ we obtain $\|f\|_{E'} \geq \frac{1}{2}\frac{\|w\|^2_{L_2(\Omega)}}{\|w\|_{H^1(\Omega)}}$.

3 Finite element approximation

We will use P to denote a *partition* of Ω, defined as a collection of closed triangles, also called elements, such that $\bar{\Omega} = \cup_{\triangle \in P} \triangle$, and for any pair of different $\triangle, \tilde{\triangle} \in P$, the intersection $\triangle \cap \tilde{\triangle}$ is either empty, or a joint vertex, or a complete edge of either $\triangle$ or $\tilde{\triangle}$. A partition P is called *conforming* when for any pair of different $\triangle, \tilde{\triangle} \in P$, $\triangle \cap \tilde{\triangle}$ is either empty, or a joint vertex, or a complete edge of *both* $\triangle$ and $\tilde{\triangle}$. A partition $\tilde{P}$ is called a *refinement* of P when any $\triangle \in P$ is the union of one or more triangles from $\tilde{P}$. A vertex v of a $\triangle \in P$ is called a *non-hanging* vertex of the partition when it is a vertex of all $\tilde{\triangle} \in P$ that contain v, and otherwise it is called a hanging vertex. With $\bar{V}_P$ or V_P we will denote the set of all *non-hanging* vertices of P or all *non-hanging, interior* vertices of P respectively. An edge e of a $\triangle \in P$ is called *an edge of* P when it does not contain hanging vertices in its interior. The set of all interior edges of P will be denoted as E_P.

Let P_0 be some fixed conforming partition of Ω. From now on, throughout this paper, we will consider only partitions P that are refinements of this P_0 by a sequence of so-called *red-refinements*, which is the replacement of a triangle $\triangle$ by 4 subtriangles $\triangle_1, \ldots, \triangle_4$ created by connecting the midpoints of the edges. We will refer to $\triangle$ as being the parent of its children $\triangle_1, \ldots, \triangle_4$. Clearly, any $\triangle \in P$ is *similar* to a triangle from P_0, showing that the triangles from any P are uniformly *shape regular*, only dependent on P_0. Except in the uniform refinement case, these red-refinements create hanging nodes. So our partitions are generally *nonconforming*.

Two different $\triangle, \tilde{\triangle} \in P$ are called *neighbours* when $\triangle \cap \tilde{\triangle} \neq \emptyset$, so also if they share only a vertex. The generation $\mathrm{gen}(\triangle)$ of a $\triangle \in P$ is defined as the number of red-refinements that are needed to create $\triangle$ starting from a triangle in P_0, which has generation zero. We will call a partition P *admissible* when the generations of all neighbours in P differ at most 1. Usually, we will use the symbols P^a, $\tilde{P}^a$ etc. to denote admissible partitions.

We will assume that a partition P is stored as a quadtree, having as roots all $\triangle_0 \in P_0$ and as leaves all $\triangle \in P$, and where the other nodes correspond to all triangles that were created as intermediate steps to get from the roots to the leaves by a sequence of red-refinements. The total number of nodes in this tree is less than or equal to $\frac{4}{3}\#P$. Given P, for $0 \leq i \leq n := \max_{\triangle \in P} \mathrm{gen}(\triangle)$, we define the partition P_i as the set of leaves of the tree that is defined by removing all nodes $\triangle$, from the tree associated to P, which have $\mathrm{gen}(\triangle) > i$. Besides the tree structure, we assume that additional information is stored that allows to find, for any $0 \leq i \leq n$, all neighbours in P_i of any $\triangle \in P_i$ in $\mathcal{O}(1)$ operations. Then, as shown in [13, Proposition 3.7], the following algorithm produces an admissible refinement P^a of the input partition P with $\#P^a \lesssim \#P$, taking not more than $\mathcal{O}(\#P)$ operations.

MakeAdmissible$[P] \to P^a$

$n := \max_{\triangle \in P} \mathrm{gen}(\triangle)$, $P_0^a := P_0$
`for` $i = 0, \ldots, n$ `do`
  define P_{i+1}^a *as the union of* P_{i+1} *and, when* $i \leq n-2$*, the collection of children of those* $\triangle \in P_i^a$ *that have a neighbour in* P_i *with grandchildren in* P_{i+2}
`od`
$P^a := P_n^a$

One easily verifies the following result (cf. [13, Prop. 3.4]), being the reason to consider admissible partitions.

Proposition 3.1 *Let* P^a *be an admissible partition. Then for any* $\triangle \in P^a$ *with* $\mathrm{gen}(\triangle) > 0$*, thus in particular for any* $\triangle \in P^a$ *that has a hanging vertex, all vertices of its parent are non-hanging vertices of* P^a*.*

Remark 3.2 The result of Proposition 3.1 will allow us to estimate interpolation errors locally. Generally, this will not be possible for partitions for which it is only required that the generations of triangles at both sides of each edge differ by at most one.

For a given partition P, let $\mathcal{S}_P \subset H_0^1(\Omega)$ denote the space of continuous, *piecewise linear* functions subordinate to P which vanish at $\partial\Omega$, i.e., $S_P = H_0^1(\Omega) \cap C(\Omega) \cap \prod_{\triangle \in P} P_1(\triangle)$. As is well-known, any $w_P \in \mathcal{S}_P$ is uniquely defined by its values on V_P. The solution $u_P \in \mathcal{S}_P$ of

$$a(u_P, w_P) = f(w_P), \quad (w_P \in \mathcal{S}_P), \tag{3.1}$$

is called the *Galerkin approximation* of the solution u of (2.1). Defining $L_P : \mathcal{S}_P \to (\mathcal{S}_P)' \supset H^{-1}(\Omega)$ by $(L_P u_P)(w_P) = a(u_P, w_P)$, the solution of (3.1) is given by $L_P^{-1} f$.

For approximating the right-hand side, we will make use of the spaces $\mathcal{S}_P^{0,-1} := \prod_{\triangle \in P} P_0(\triangle)$ and $\mathcal{S}_P^{1,-1} := \prod_{\triangle \in P} P_1(\triangle)$.

4 A residual based a posteriori error estimator

Lemma 4.1 *For any triangle $\triangle$, it holds that*

$$\|u\|_{L_2(\partial\triangle)} \lesssim \operatorname{diam}(\triangle)^{-\frac{1}{2}} \|u\|_{L_2(\triangle)} + \|u\|_{L_2(\triangle)}^{\frac{1}{2}} |u|_{H^1(\triangle)}^{\frac{1}{2}}, \quad (u \in H^1(\triangle)),$$

only dependent on the smallest angle in $\triangle$.

Proof A simple homogeneity argument shows that it is sufficient to prove the lemma for $\triangle$ with $\operatorname{diam}(\triangle) = 1$. Then the required estimate is a special case of the corresponding estimate proven for general d-dimensional Lipschitz domains in [9, Proof of Theorem 1.5.10 (bottom line of page 41)]. $\square$

Remark 4.2 Apparently not being aware of the result from [9], in [15] Verfürth gave an elementary, but technical proof of Lemma 4.1 relying on specific properties of a triangle (more generally of a d-simplex).

For a partition P, $f \in L_2(\Omega)$, $u \in \mathcal{S}_P$, we set the element and edge residuals by

$$\zeta_\triangle(f, u) := \min\{1, \operatorname{diam}(\triangle)\sqrt{1+\kappa^2}\}^2 \|f - \tfrac{\kappa^2}{1+\kappa^2} u_{P^a}\|_{L_2(\triangle)}^2, \quad (\triangle \in P),$$

$$\eta_e(u) := \tfrac{\min\{1, \operatorname{diam}(e)\sqrt{1+\kappa^2}\}}{(1+\kappa^2)^{3/2}} \|[\nabla u]_e \cdot \mathbf{n}_e\|_{L_2(e)}^2, \quad (e \in E_P),$$

where $\mathbf{n}_e$ is a unit vector orthogonal to e, and $[\nabla u]_e$ denotes the jump of ∇u in the direction of $\mathbf{n}_e$. We set the error estimator

$$\mathcal{E}(P, f, u) := [\sum_{\triangle \in P} \zeta_\triangle(f, u) + \sum_{e \in E_P} \eta_e(u)]^{\frac{1}{2}}.$$

Theorem 4.3 *For any $f \in L_2(\Omega)$ and any admissible partition P^a, with $u := L^{-1} f$ and $u_{P^a} := L_{P^a}^{-1} f$, we have*

$$\|u - u_{P^a}\|_E \leq C_1 \mathcal{E}(P^a, f, u_{P^a}),$$

for some absolute constant $C_1 > 0$.

Proof For any $w \in H_0^1(\Omega)$, $w_{P^a} \in \mathcal{S}_{P^a}$, by the Galerkin orthogonality integration by parts shows that

$$\begin{aligned}
|a(u-u_{P^a},w)| &= |a(u-u_{P^a},w-w_{P^a})| \\
&= \Big|\int_\Omega f(w-w_{P^a}) - a(u_{P^a},w-w_{P^a})\Big| \\
&= \Big|\sum_{\triangle\in P^a}\Big\{\int_\triangle \big(f+\tfrac{\operatorname{div}(\nabla u_{P^a})-\kappa^2 u_{P^a}}{1+\kappa^2}\big)(w-w_{P^a}) \qquad (4.1)\\
&\qquad\qquad -\int_{\partial\triangle}\tfrac{\partial_{\mathbf{n}} u_{P^a}}{1+\kappa^2}(w-w_{P^a})\Big\}\Big| \\
&\le \sum_{\triangle\in P^a}\|f-\tfrac{\kappa^2}{1+\kappa^2}u_{P^a}\|_{L_2(\triangle)}\|w-w_{P^a}\|_{L_2(\triangle)} \qquad (4.2)\\
&\qquad\qquad + \sum_{e\in E_{P^a}}\|\tfrac{[\nabla u_{P^a}]_e\cdot\mathbf{n}_e}{1+\kappa^2}\|_{L_2(e)}\|w-w_{P^a}\|_{L_2(e)},
\end{aligned}$$

where we used that in our case, since u_{P^a} is piecewise linear, $\operatorname{div}(\nabla u_{P^a})$ vanishes on each $\triangle$.

We select w_{P^a} as a quasi-interpolant of w. For any $v \in V_{P^a}$, we choose a $\triangle_v \in P^a$ with $v \in \triangle_v$. As shown in [12, p. 17-18], there exists a $\varphi(\triangle_v, v) \in L_\infty(\triangle_v)$ such that $\int_{\triangle_v} \varphi(\triangle_v, v)p = p(v)$ for any $p \in P_1(\triangle_v)$, and $\|\varphi(\triangle_v, v)\|_{L_\infty(\triangle_v)} \lesssim \operatorname{meas}(\triangle_v)^{-1}$, independently of $\triangle_v$. We now define $w_{P^a} \in \mathcal{S}_{P^a}$ by $w_{P^a}(v) = \int_{\triangle_v} \varphi(\triangle_v, v)w$, so that $|w_{P^a}(v)| \lesssim \operatorname{vol}(\triangle_v)^{-\frac{1}{2}}\|w\|_{L_2(\triangle_v)}$.

Each $\triangle \in P^a$ has between 0 and 3 hanging vertices, and in view of Proposition 3.1, $w_{P^a}|_\triangle$ is determined by the values of w_{P^a} in at most 4 vertices $v \in \bar{V}_{P^a}$, each of them being either a vertex of $\triangle$ or a vertex of its parent. We infer that there exists a uniformly bounded collection $P^a_\triangle \subset P^a$ such that $\triangle \in P^a_\triangle$, $\Omega_\triangle := \cup_{\tilde\triangle\in P^a_\triangle}\tilde\triangle$ is simply connected, $\operatorname{diam}(\Omega_\triangle) \lesssim \operatorname{diam}(\triangle)$,

$$\|w - w_{P^a}\|_{L_2(\triangle)} \lesssim \|w\|_{L_2(\Omega_\triangle)}, \qquad (4.3)$$

and, if one the $v \in \bar{V}_{P^a}$ for which $w_{P^a}(v)$ determine $w_{P^a}|_\triangle$ is on $\partial\Omega$, then $P^a_\triangle$ contains a $\tilde\triangle$ having an edge on $\partial\Omega$.

From a homogeneity argument, and either the fact that our interpolator reproduces any polynomial of first order degree, and, in particular, any constant, together with the Bramble-Hilbert lemma, or, in case one the $\tilde\triangle \in \Omega_\triangle$ has an edge on $\partial\Omega$, the Poincaré-Friedrichs inequality, we infer that

$$\begin{aligned}\|w - w_{P^a}\|_{L_2(\triangle)} &\lesssim \operatorname{diam}(\triangle)|w|_{H^1(\Omega_\triangle)} \\ &\leq \operatorname{diam}(\triangle)\sqrt{1+\kappa^2}\,|w|_{E,\Omega_\triangle} \qquad (4.4)\end{aligned}$$

and

$$\|w - w_{P^a}\|_{H^1(\triangle)} \lesssim |w|_{H^1(\Omega_\triangle)}. \qquad (4.5)$$

Noting that for $\kappa \geq 1$, $\|\cdot\|_{L_2(\Omega_\triangle)} \leq \sqrt{2}\,\|\cdot\|_{E,\Omega_\triangle}$, from (4.3) and (4.4) we obtain that

$$\|w - w_{P^a}\|_{L_2(\triangle)} \lesssim \min\{1, \operatorname{diam}(\triangle)\sqrt{1+\kappa^2}\}\|w\|_{E,\Omega_\triangle}. \qquad (4.6)$$

For each $e \in E_P$, there exists a $\triangle_e \in P^a$ with $e \subset \partial\triangle_e$ and $\operatorname{diam}(\triangle_e) \eqsim \operatorname{diam}(e)$. Lemma 4.1, (4.6) and (4.5) show that

$$\begin{aligned}&\|w - w_{P^a}\|_{L_2(e)} \\ &\quad\lesssim \operatorname{diam}(e)^{-\frac{1}{2}}\|w - w_{P^a}\|_{L_2(\triangle_e)} + \|w - w_{P^a}\|^{\frac{1}{2}}_{L_2(\triangle_e)}|w|^{\frac{1}{2}}_{H^1(\triangle_e)} \\ &\quad\lesssim \Big[\operatorname{diam}(e)^{-\frac{1}{2}}\min\{1, \operatorname{diam}(e)\sqrt{1+\kappa^2}\} + \\ &\qquad\qquad \min\{1, \operatorname{diam}(e)\sqrt{1+\kappa^2}\}^{\frac{1}{2}}\sqrt{1+\kappa^2}\Big]\|w\|_{E,\triangle_e} \\ &\quad\eqsim \min\{1, \operatorname{diam}(e)\sqrt{1+\kappa^2}\}^{\frac{1}{2}}(1+\kappa^2)^{\frac{1}{4}}\|w\|_{E,\triangle_e}. \qquad (4.7)\end{aligned}$$

By applying the Cauchy-Schwarz inequality to the sums from (4.2), and by substituting (4.6) and (4.7), we conclude that

$$\|u - u_{P^a}\|_E = \sup_{0 \neq w \in H^1_0(\Omega)} \frac{|a(u - u_{P^a}, w)|}{\|w\|_E} \lesssim \mathcal{E}(P^a, f, u_{P^a}).$$

□

Remark 4.4 The above proof is an easy generalization of the corresponding result from [15] for conforming partitions. Moreover, as shown in [15], the estimator easily generalizes to more dimensions and to finite elements of higher order, where then $\zeta_\triangle(f, u)$ should read as $\min\{1, \operatorname{diam}(\triangle)\sqrt{1+\kappa^2}\}^2\|f + \frac{1}{1+\kappa^2}\operatorname{div}(\nabla u) - \frac{\kappa^2}{1+\kappa^2}u_{P^a}\|^2_{L_2(\triangle)}$.

Remark 4.5 If, instead of the trace estimate of Lemma 4.1, one uses the more standard estimate

$$\|u\|_{L_2(\partial\triangle)} \lesssim \operatorname{diam}(\triangle)^{-\frac{1}{2}}\|u\|_{L_2(\triangle)} + \operatorname{diam}(\triangle)^{\frac{1}{2}}|u|_{H^1(\triangle)},$$

then one would find an estimator in which the terms $\eta_e(u)$ are replaced by $\frac{\operatorname{diam}(e)}{1+\kappa^2}\|[\nabla u]_e \cdot \mathbf{n}_e\|^2_{L_2(e)}$, which can be much larger.

5 The uniform saturation property

For a right-hand side f that is sufficiently close to being piecewise constant with respect to an admissible partition P^a, and a suitable adaptive refinement $\tilde{P}$ of $\tilde{P}^a$, we will show that a positive multiple of $\mathcal{E}(P^a, f, u_{P^a})$, and, in view of Theorem 4.3, thus of $\|u - u_{P^a}\|_E$, is a *lower bound* for $\|u_{\tilde{P}} - u_{P^a}\|_E$, where u is the solution of (2.1), and u_{P^a}, $u_{\tilde{P}}$ are its Galerkin approximations on P^a and $\tilde{P}$, respectively. This property is known as the saturation property. Since the aforementioned multiple will be independent of the parameter κ, we may speak about a uniform saturation property.

The approach that we will follow generalizes results from Morin, Nochetto and Siebert in [11] for the Poisson equation using conforming partitions, which extend on earlier work of Dörfler in [8]. Different from [11], in this section, we simply assume that the right-hand side is in $\mathcal{S}^{0,-1}_{P^a}$, after which, in the next section, we separately analyze the consequences when f does not satisfy this assumption and therefore has to be approximated. We start with a lemma that will allow us to bound an edge residual with $\mathrm{diam}(e)\sqrt{1+\kappa^2} \gtrsim 1$ in terms of the neighbouring element residuals.

Lemma 5.1 *For any two triangles $\triangle_1, \triangle_2$ that share a complete edge e of both $\triangle_1$ and $\triangle_2$, any $u \in C(\triangle_1 \cup \triangle_2) \cap \prod_{i=1}^2 P_1(\triangle_i)$, and any $f \in \prod_{i=1}^2 P_0(\triangle_i)$, we have*

$$\|[\nabla u]_e \cdot \mathbf{n}_e\|_{L_2(e)} \lesssim \mathrm{diam}(e)^{-\frac{3}{2}} \|f - u\|_{L_2(\triangle_1 \cup \triangle_2)},$$

only dependent on the smallest angle in $\triangle_1, \triangle_2$.

Proof Define $w \in C(\triangle_1 \cup \triangle_2) \cap \prod_{i=1}^2 P_1(\triangle_i)$ by $w(v) = \mathrm{diam}(e)$ for the vertex v of $\triangle_1$ not on e, and $w \equiv 0$ on $\triangle_2$. Since any continuous, piecewise linear u can be written as a linear combination of w and a $p \in P_1(\triangle_1 \cup \triangle_2)$, the latter having a zero jump of its normal derivative over e, we have to prove that

$$\|[\nabla w]_e \cdot \mathbf{n}_e\|_{L_2(e)} \lesssim \mathrm{diam}(e)^{-\frac{3}{2}} \inf_{p \in P_1(\triangle_1 \cup \triangle_2), f \in \prod_{i=1}^2 P_0(\triangle_i)} \|f - (w+p)\|_{L_2(\triangle_1 \cup \triangle_2)}.$$

Since $\|[\nabla w]_e \cdot \mathbf{n}_e\|_{L_2(e)} \eqsim \mathrm{diam}(e)^{\frac{1}{2}} \eqsim \mathrm{diam}(e)^{-\frac{3}{2}} \|w\|_{L_2(\triangle_1 \cup \triangle_2)}$, with $V := P_1(\triangle_1 \cup \triangle_2) + \prod_{i=1}^2 P_0(\triangle_i)$, we have to show that

$$\|w\|_{L_2(\triangle_1 \cup \triangle_2)} \lesssim \inf_{v \in V} \|w - v\|_{L_2(\triangle_1 \cup \triangle_2)}.$$

By homogeneity argument, it is sufficient to prove this inequality for $\operatorname{diam}(e) = 1$. For any $\triangle_1, \triangle_2$, it holds that $\inf_{v\in V} \|w - v\|_{L_2(\triangle_1\cup\triangle_2)}$ is a continuous, and since $w \notin V$, positive function of the angles in both triangles. Now a compactness argument completes the proof. $\square$

Corollary 5.2 *For admissible partitions P^a, $u_{P^a} \in \mathcal{S}_{P^a}$, $f_{P^a} \in \mathcal{S}^{0,-1}_{P^a}$, $\triangle_1, \triangle_2 \in P^a$ with $e = \triangle_1 \cap \triangle_2 \in E_{P^a}$, and $\operatorname{diam}(e)\sqrt{1+\kappa^2} \gtrsim 1$, we have*

$$\eta_e(u_{P^a}) \lesssim \zeta_{\triangle_1}(f_{P^a}, u_{P^a}) + \zeta_{\triangle_2}(f_{P^a}, u_{P^a}).$$

Proof By, for $i = 1, 2$, selecting $\tilde{\triangle}_i$ to be either $\triangle_i$ or one of its 4 children, e is an edge of both $\tilde{\triangle}_1$ and $\tilde{\triangle}_2$. Using that $\kappa \gtrsim 1$, an application of Lemma 5.1 shows that

$$\begin{aligned}
\eta_e(u_{P^a}) &= (1+\kappa^2)^{-\frac{3}{2}} \|[\nabla u_{P^a}]_e \cdot \mathbf{n}_e\|^2_{L_2(e)} \\
&\lesssim \operatorname{diam}(e)^{-3}(1+\kappa^2)^{-\frac{3}{2}} \|\tfrac{1+\kappa^2}{\kappa^2} f_{P^a} - u_{P^a}\|^2_{L_2(\tilde{\triangle}_1\cup\tilde{\triangle}_2)} \\
&= \left[\operatorname{diam}(e)\sqrt{1+\kappa^2}\right]^{-3} (\tfrac{1+\kappa^2}{\kappa^2})^2 \|f_{P^a} - \tfrac{\kappa^2}{1+\kappa^2} u_{P^a}\|^2_{L_2(\tilde{\triangle}_1\cup\tilde{\triangle}_2)} \\
&\lesssim \zeta_{\triangle_1}(f_{P^a}, u_{P^a}) + \zeta_{\triangle_2}(f_{P^a}, u_{P^a}).
\end{aligned}$$

$\square$

After this preparation to handle $\eta_e(u_{P^a})$ with $\operatorname{diam}(e)\sqrt{1+\kappa^2} \gtrsim 1$, we derive a local lower bound on the difference between the Galerkin approximations on a partition and a suitable local refinement of this partition.

Theorem 5.3 *Let P^a be an admissible partition and $\tilde{P}$ be a refinement of P^a. Let $f_{P^a} \in \mathcal{S}^{0,-1}_{P^a}$, and let $u_{P^a} = L^{-1}_{P^a} f_{P^a}$, $u_{\tilde{P}} = L^{-1}_{\tilde{P}} f_{P^a}$ be the corresponding Galerkin solutions.*

(a) *Let $\triangle_1, \triangle_2 \in P^a$ with $e := \triangle_1 \cap \triangle_2 \in E_{P^a}$. Assume that in $\tilde{P}$, both $\triangle_1, \triangle_2$ are replaced by at least* 16 *subtriangles by means of two recursive red-refinements. Then*

$$\|u_{\tilde{P}} - u_{P^a}\|^2_{E,\triangle_1\cup\triangle_2} \gtrsim \eta_e(u_{P^a}) + \sum_{i=1}^{2} \zeta_{\triangle_i}(f_{P^a}, u_{P^a}).$$

(b) *Let $\triangle \in P^a$ be replaced in $\tilde{P}$ by at least* 16 *subtriangles by means of two recursive red-refinements. Then*

$$\|u_{\tilde{P}} - u_{P^a}\|^2_{E,\triangle} \gtrsim \zeta_{\triangle}(f_{P^a}, u_{P^a}).$$

Proof (a) With $W := \{\omega \in \mathcal{S}_{\tilde{P}} : \operatorname{supp}\omega \subset \triangle_1 \cup \triangle_2\}$, we have that $\|u_{\tilde{P}} - u_{P^a}\|_{E,\triangle_1\cup\triangle_2} \geq \sup_{0\neq\phi\in W} \frac{a(u_{\tilde{P}}-u_{P^a},\phi)}{\|\phi\|_{E,\triangle_1\cup\triangle_2}}$. For $\phi \in W$, integration by parts shows that

$$\begin{aligned}a(u_{\tilde{P}} - u_{P^a}, \phi) &= \int_\Omega f_{P^a}\phi - a(u_{P^a}, \phi)\\ &= \sum_{i=1}^{2} \int_{\triangle_i} (f_{P^a} - \tfrac{\kappa^2}{1+\kappa^2} u_{P^a})\phi - \int_e \tfrac{[\nabla u_{P^a}]_e \cdot \mathbf{n}_e}{1+\kappa^2}\phi.\end{aligned}$$

In view of Corollary 5.2, it is sufficient to prove that

$$\begin{aligned}&\sup_{0\neq\phi\in W} \frac{\sum_{i=1}^{2}\int_{\triangle_i}(f_{P^a} - \frac{\kappa^2}{1+\kappa^2}u_{P^a})\phi - \int_e \frac{[\nabla u_{P^a}]_e\cdot\mathbf{n}_e}{1+\kappa^2}\phi}{\|\phi\|_{E,\triangle_1\cup\triangle_2}} \gtrsim \qquad (5.1)\\ &\min\{1, \operatorname{diam}(e)\sqrt{1+\kappa^2}\}\Big[\operatorname{diam}(e)^{-1}\|\tfrac{[\nabla u_{P^a}]_e\cdot\mathbf{n}_e}{1+\kappa^2}\|^2_{L_2(e)} + \\ &\qquad\qquad \sum_{i=1}^{2}\|f_{P^a} - \tfrac{\kappa^2}{1+\kappa^2}u_{P^a}\|^2_{L_2(\triangle_i)}\Big]^{\frac{1}{2}}.\end{aligned}$$

Since $\|\phi\|_{E,\triangle_1\cup\triangle_2} \eqsim \min\{1, \operatorname{diam}(e)\sqrt{1+\kappa^2}\}^{-1}\|\phi\|_{L_2(\triangle_1\cup\triangle_2)}$ $(\phi \in W)$, $\frac{[\nabla u_{P^a}]_e\cdot\mathbf{n}_e}{1+\kappa^2} \in P_0(e)$, and $(f_{P^a} - \frac{\kappa^2}{1+\kappa^2}u_{P^a})|_{\triangle_i} \in P_1(\triangle_i)$, it is sufficient to prove that for any $p_1 \in P_1(\triangle_1)$, $p_2 \in P_1(\triangle_2)$, $p_e \in P_0(e)$,

$$\begin{aligned}&\sup_{0\neq\phi\in W} \frac{\sum_{i=1}^{2}\int_{\triangle_i} p_i\phi - p_e\int_e\phi}{\|\phi\|_{L_2(\triangle_1\cup\triangle_2)}} \gtrsim \qquad (5.2)\\ &\qquad\qquad [\operatorname{diam}(e)^{-1}\|p_e\|^2_{L_2(e)} + \sum_{i=1}^{2}\|p_i\|^2_{L_2(\triangle_i)}]^{\frac{1}{2}},\end{aligned}$$

which by a homogeneity argument will follow from this statement for $\operatorname{diam}(e) = 1$, and thus $\operatorname{vol}(\triangle_1) \eqsim 1 \eqsim \operatorname{vol}(\triangle_2)$.

The space W contains the span of seven or nine standard nodal basis functions, three corresponding to vertices interior to $\triangle_1$, three corresponding to vertices interior to $\triangle_2$, and one or three corresponding to vertices interior to e (cf. Figure 5.1). By searching ϕ as a linear combination of these basis functions, where degrees of freedom associated to possible non-hanging vertices on the edge e other than the midpoint can be discarded, and by writing p_1, p_2, p_e with respect to the canonical basis for $P_1(\triangle_1)$, $P_1(\triangle_2)$ or $P_0(e)$, respectively, (5.2) for $\operatorname{diam}(e) = 1$ follows from

$$\sup_{0\neq\mathbf{c}\in\mathbb{R}^7} \frac{\langle\mathbf{Bd},\mathbf{c}\rangle}{\|\mathbf{c}\|} \gtrsim \|\mathbf{d}\|, \quad (\mathbf{d} \in \mathbb{R}^7),$$

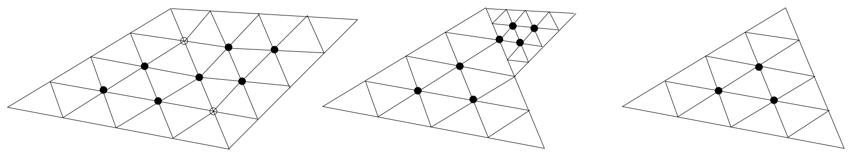

Fig. 5.1. Additional degrees of freedom required in $\tilde{P}$ in case (a) (left or middle picture) and (c) (right picture), respectively.

where with $\mathbf{C} = \frac{1}{32}\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} \mathrm{vol}(\triangle_1)\mathbf{C} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{vol}(\triangle_2)\mathbf{C} & \mathbf{0} \\ \mathbf{D}_1 & \mathbf{D}_2 & \frac{1}{2} \end{bmatrix}$ with $\mathbf{D}_1, \mathbf{D}_2$ being some uniformly bounded 3×1 matrices, or equivalently, from $\|\mathbf{B}^{-1}\| \lesssim 1$, which clearly holds true.

The proof of Part (b) poses no additional difficulties. $\square$

By summing over the local lower bounds from Theorem 5.3, we obtain the following global lower bound.

Corollary 5.4 *Let P^a be an admissible partition, $f_{P^a} \in \mathcal{S}^{0,-1}_{P^a}$, $F \subset P^a$ and $G \subset E_{P^a}$. Let $\tilde{P}$ be a refinement of P^a such that each $\triangle \in P^a$, that either is in F or has an edge which contains an $e \in G$, has been refined by means of at least two recursive red-refinements. Then for $u_{P^a} = L^{-1}_{P^a} f_{P^a}$ and $u_{\tilde{P}} = L^{-1}_{\tilde{P}} f_{P^a}$, it holds that*

$$\|u_{\tilde{P}} - u_{P^a}\|_E \geq c_2\Big[\sum_{\triangle\in F} \zeta_\triangle(f_{P^a}, u_{P^a}) + \sum_{e\in G} \eta_e(u_{P^a})\Big]^{\frac{1}{2}}, \tag{5.3}$$

for some absolute constant $c_2 > 0$.

Remark 5.5 In particular, if F and G are selected such that the right-hand side of (5.3) is bounded from below by some positive multiple of $\mathcal{E}(P^a, f_{P^a}, u_{P^a})$, then $\|u_{\tilde{P}} - u_{P^a}\|_E \gtrsim \mathcal{E}(P^a, f_{P^a}, u_{P^a}) \gtrsim \|u - u_{P^a}\|_E$ by Theorem 4.3, which is known as the *saturation property*, and which here thus holds uniformly in κ. Yet, recall that in Corollary 5.4, we have assumed that $f \in \mathcal{S}^{0,-1}_{P^a}$. In the next section we will prove a similar result for general f, making use of the present result for a sufficiently accurate piecewise constant approximation.

Remark 5.6 Obviously, Corollary 5.4 is also valid when $\mathcal{S}_{\tilde{P}}$ is replaced by the larger space $H^1_0(\Omega)$, showing that for $f_{P^a} \in \mathcal{S}^{0,-1}_{P^a}$ and with $u = L^{-1} f_{P^a}$, $u_{P^a} = L^{-1}_{P^a} f_{P^a}$,

$$\|u - u_{P^a}\|_E \geq c_2 \mathcal{E}(P^a, f_{P^a}, u_{P^a}). \tag{5.4}$$

For conforming partitions, a result similar to (5.4) was already proved in [15], even for right-hand sides $f \in \mathcal{S}^{1,-1}_{P^a}$. To prove an estimate like (5.2) in that case, it was necessary to make use of a 'bubble function' ϕ along the edge e with increasingly thin support when $\operatorname{diam}(e)\sqrt{1+\kappa^2} \to \infty$. The restriction to piecewise constant right-hand sides allows us to prove the uniform saturation property without being forced to apply anisotropic refinements in order to include such bubbles in the space W. Such refinements give rise to efficient approximations in case the solution exhibits strong layers, as when a non-vanishing right-hand side "meets" a homogeneous Dirichlet boundary condition. In other cases, for example as when the reaction-diffusion equation stems from a implicit time discretization of a "harmless" parabolic problem, one would like to avoid such refinements.

Remark 5.7 When one tries to generalize Theorem 5.3 to three space dimensions, one encounters the problem that after two recursive red-refinements of a tetrahedron only one interior vertex is created, whereas the other 34 are on its boundary. Indeed, recall from its proof, three interior degrees of freedom are needed to handle the linear function $f_{P^a} - \kappa^2 u_{P^a}$. Yet, for $\kappa = 0$, only the constant f_{P^a} has to be dealt with, and for this Poisson problem the resulting adaptive finite element method can be shown to be convergent. To be able to prove robustness for the reaction-diffusion equation, one has to consider three recursive red-refinements for each tetrahedron that is marked for refinement.

6 A uniformly convergent adaptive finite element method

By using the upper and lower bounds on the error in the Galerkin approximation proven in Theorem 4.3 and Corollary 5.4, respectively, we will construct a uniformly convergent adaptive finite element method. For both setting up the Galerkin systems and in the evaluation of the error estimator, we replace the given right-hand side by some piecewise constant approximation, with which we model the application of quadrature. Furthermore, we will allow that the Galerkin systems are solved only inexactly, as with the application of iterative solvers. We start with investigating the stability of the error estimator with respect to perturbations in the discrete solution.

Lemma 6.1 *There exists an absolute constant $C_3 > 0$ such that for any partition P, $f \in L_2(\Omega)$, $u_P, \tilde{u}_P \in \mathcal{S}_P$,*

$$|\mathcal{E}(P, f, u_P) - \mathcal{E}(P, f, \tilde{u}_P)| \le C_3 \|u_P - \tilde{u}_P\|_E.$$

Proof Using two applications of the triangle inequality in the form $\big|\|\cdot\|-\|\cdot\|\big|^2 \leq \|\cdot-\cdot\|^2$, first for vectors and then for functions, we have

$$
\begin{aligned}
&|\mathcal{E}(P,f,u_P)-\mathcal{E}(P,f,\tilde{u}_P)|^2\\
&=\Big|[\sum_{\triangle\in P}\zeta_\triangle(f,u_P)+\sum_{e\in E_P}\eta_e(u_P)]^{\frac12}-[\sum_{\triangle\in P}\zeta_\triangle(f,\tilde{u}_P)+\sum_{e\in E_P}\eta_e(\tilde{u}_P)]^{\frac12}\Big|^2\\
&\leq \sum_{\triangle\in P}(\zeta_\triangle(f,u_P)^{\frac12}-\zeta_\triangle(f,\tilde{u}_P)^{\frac12})^2+\sum_{e\in E_P}(\eta_e(u_P)^{\frac12}-\eta_e(\tilde{u}_P)^{\frac12})^2\\
&\leq \sum_{\triangle\in P}\|\tfrac{\kappa^2}{1+\kappa^2}(u_P-\tilde{u}_P)\|^2_{L_2(\triangle)}+\sum_{e\in E_P}\tfrac{\operatorname{diam}(e)}{1+\kappa^2}\|[\nabla(u_p-\tilde{u}_P)]_e\cdot\mathbf{n}_e\|^2_{L_2(e)}.
\end{aligned}
$$

The proof is completed by the observation that for any triangle $\triangle$, and $p\in P_1(\triangle)$,

$$\operatorname{diam}(e)\|\partial_{\mathbf{n}}p\|^2_{L_2(\partial\triangle)}\lesssim |p|^2_{H^1(\triangle)},$$

only dependent on the smallest angle in $\triangle$, which by a homogeneity argument easily follows from this statement for $\operatorname{diam}(e)=1$. $\square$

For some *fixed parameter* $\theta\in(0,1)$, we will make use of the following routine to determine a suitable adaptive refinement:

REFINE$[P^a,f,w_{P^a}]\to\tilde{P}$
% P^a is an admissible partition, $f\in\mathcal{S}^{0,-1}_{P^a}$ and $w_{P^a}\in\mathcal{S}_{P^a}$.
Select, in $\mathcal{O}(\#P^a)$ operations, $F\subset P^a$, $G\subset E_{P^a}$ such that

$$\sum_{\triangle\in F}\zeta_\triangle(f,w_{P^a})+\sum_{e\in G}\eta_e(w_{P^a})\geq\theta^2\,\mathcal{E}(P^a,f,w_{P^a})^2. \tag{6.1}$$

Construct the partition $\tilde{P}$ from P^a by the replacement by means of two recursive red-refinements of all $\triangle\in F$, as well as of all $\triangle\in P^a$ that have an edge that contains an $e\in G$.

Obviously, one would prefer to find F and G in (6.1) with smallest $\#(F\cup G)$, which however requires the sorting of all element and edge residuals by their values taking the order of $\#P^a|\log(\#P^a)|$ operations. To avoid the log-factor, we apply the following procedure. With $N:=\#(P^a\cup E_{P^a})$, we may discard all element and edge residuals with values less than or equal to $(1-\theta^2)\mathcal{E}(P^a,f,w_{P^a})^2/N$. With M being the maximum of all element and edge residuals, and q the smallest integer with $2^{-q-1}M\leq(1-\theta^2)\mathcal{E}(P^a,f,w_{P^a})^2/N$, we store the others in $q+1$ binary bins $[M,\frac12 M),[\frac12 M,\frac14 M),\ldots,[2^{-q}M,2^{-q-1}M)$. We then extract edge or element residuals from the bins, starting with the first bin, and when it got empty moving to the second bin and so on, and with that build sets F and G until (6.1) is satisfied. One may verify that the cardinality of the set $F \cup G$ created in this way is at most two times larger than the minimal cardinality, where the number of operations and storage locations required is $\lesssim \#P^a + q$, with $q < \log_2(MN/[(1-\theta^2)\mathcal{E}(P^a, f, w_{P^a})^2]) \leq \log_2(N/(1-\theta^2)) \lesssim \log_2(\#P^a) < \#P^a$.

Apart from **REFINE**, we assume that we have two more routines at our disposal. The first routine will be used to solve the Galerkin systems approximately.

GALSOLVE$[P^a, f_{P^a}, u^{(0)}_{P^a}, \varepsilon] \to \bar{u}_{P^a}$
% *P^a is an admissible partition, $f_{P^a} \in (\mathcal{S}_{P^a})'$, and $u^{(0)}_{P^a} \in \mathcal{S}_{P^a}$, the*
% *latter being an initial approximation for an iterative solver.*
% *With $u_{P^a} := L^{-1}_{P^a} f_{P^a}$, the output $\bar{u}_{P^a} \in \mathcal{S}_{P^a}$ satisfies*

$$\|u_{P^a} - \bar{u}_{P^a}\|_E \leq \varepsilon.$$

% *The call should require* $\lesssim \max\{1, \log(\varepsilon^{-1}\|u_{P^a} - u^{(0)}_{P^a}\|_E)\}\#P^a$
% *arithmetic operations.*

A possible realization of **GALSOLVE** is obtained by the application of Conjugate Gradients to the matrix-vector representation of $L_{P^a} u_{P^a} = f_{P^a}$ with respect to a basis for $\mathcal{S}_{P^a}$ that is well-conditioned, uniformly in $\#P^a$ and κ. In [7], wavelet bases were constructed for the spaces of continuous piecewise linears with respect to uniform (red-) refinements of P_0. These bases are uniformly stable with respect to both the L_2 and H^1 topologies, and so the normalized wavelets are stable with respect to $\|\cdot\|_E$, also uniformly in κ. In [13], it is shown that for any admissible partition P^a, a subset of the wavelet basis for the space corresponding to $\max_{\triangle \in P^a} \mathrm{gen}(\triangle)$ uniform refinements is a basis for $\mathcal{S}_{P^a}$, which is thus stable uniformly in $\#P^a$ and κ.

As an alternative for this wavelet approach, one can apply Conjugate Gradients using a modified BPX-preconditioner as proposed in [6].

A second routine, called **RHS**, will be needed to find a piecewise constant approximation for the right-hand side f that is sufficiently accurate. Since this might not be possible with respect to the current partition, a call of **RHS** may result in a further refinement. What is more, since in view of a possible next call of **REFINE**, it is required that **RHS** outputs an admissible partition, a refinement might be necessary even when it is not needed for approximating the right-hand side. Indeed, note that generally the output of **REFINE** and thus the input for **RHS** is not admissible.

RHS$[P, f, \varepsilon] \to [\tilde{P}^a, f_{\tilde{P}^a}]$
% P is a partition, $f \in H^{-1}(\Omega)$ and $\varepsilon > 0$. The output consists of
% an admissible refinement $\tilde{P}^a$ of P, and an $f_{\tilde{P}^a} \in \mathcal{S}^{0,-1}_{\tilde{P}^a}$ with
% $\|f - f_{\tilde{P}^a}\|_{E'} \leq \varepsilon$. The call should require $\mathcal{O}(\#P^a)$ operations.

Let us discuss some possible realizations of **RHS** for $f \in L_2(\Omega)$. For any partition P, let $Q^{0,-1}_P$ denote the $L_2(\Omega)$-orthogonal projector onto $\mathcal{S}^{0,-1}_P$, i.e., $(Q^{0,-1}_P f)|_\triangle = |\triangle|^{-1} \int_\triangle f$. Then with $\nu_\triangle(f) := \min\{1, \operatorname{diam}(\triangle)\sqrt{1+\kappa^2}\}^2 \|f - Q^{0,-1}_P f\|^2_{L_2(\triangle)}$, we infer that

$$
\begin{aligned}
|\int_\Omega (f - Q^{0,-1}_P f) w| &= |\int_\Omega (f - Q^{0,-1}_P f)(w - Q^{0,-1}_P w)| \\
&\lesssim [\sum_{\triangle \in P} \nu_\triangle(f)]^{\frac{1}{2}} \|w\|_E \qquad (w \in H^1_0(\Omega)),
\end{aligned}
$$

cf. (4.6), or $\|f - Q^{0,-1}_P f\|_{E'} \lesssim [\sum_{\triangle \in P} \nu_\triangle(f)]^{\frac{1}{2}}$.

Assuming that, for some $s \in [0,1]$, $f \in H^s(\Omega)$, or, more generally, $f \in \prod_{\triangle_0 \in P_0} H^s(\triangle_0)$, we have $\|f - Q^{0,-1}_P f\|_{L_2(\triangle)} \lesssim \operatorname{diam}(\triangle)^s \|f\|_{H^s(\triangle)}$ and so

$$
\begin{aligned}
&\|f - Q^{0,-1}_{\tilde{P}^a} f\|_{E'} \lesssim \qquad\qquad (6.2)\\
&\quad \max_{\triangle \in \tilde{P}^a} \Big[\min\{1, \operatorname{diam}(\triangle)\sqrt{1+\kappa^2}\} \operatorname{diam}(\triangle)^s\Big] \Big[\sum_{\triangle_0 \in P_0} \|f\|^2_{H^s(\triangle_0)}\Big]^{\frac{1}{2}}.
\end{aligned}
$$

So one may apply uniform refinements until the right-hand side of (6.2) is less than or equal to ε. When $s > 0$, this happens when $\max_{\triangle \in \tilde{P}^a} \operatorname{diam}(\triangle)$ is small enough, uniformly in κ.

In case quantitatively the smoothness of f varies largely over the domain, it will be advisable to construct **RHS** using adaptive refinements. Such a routine can be of greedy type, meaning that, starting with the input partition P, repeatedly a red-refinement is applied to that triangle $\triangle$ for which currently $\nu_\triangle(f)$ is the largest one, until $\sum_{\triangle \in P} \nu_\triangle(f)$ is less than or equal to a prescribed tolerance.

Alternatively, since $\nu_\triangle(f)$ is subadditive, i.e., $\sum_{i=1}^4 \nu_{\triangle_i}(f) \leq \nu_\triangle(f)$ with $\triangle_1, \ldots, \triangle_4$ being the 4 children of $\triangle$, one may also apply the so-called *Thresholding Second Algorithm* from [5]. In contrast to the greedy algorithm, this one outputs a refinement $\tilde{P}$ of the input partition P, with $\sum_{\triangle \in \tilde{P}} \nu_\triangle(f)$ less than or equal to some prescribed tolerance $\tilde{\varepsilon}$, which is always quasi-optimal, in the sense that some absolute constants $c_1, C_1 > 0$, $\#\tilde{P} \leq \#C_1\hat{P}$ for any refinement $\hat{P}$

of P with $\sum_{\triangle \in \hat{P}} \nu_\triangle(f) \leq c_1 \tilde{\varepsilon}$. Generally, both these adaptive algorithms have to be completed by a call of **MakeAdmissible** in order to return a partition that is admissible.

Both adaptive algorithms need the values $\nu_\triangle(f)$ that usually cannot be evaluated exactly. However, assuming some piecewise smoothness of f, one can compute equivalent quantities in $\mathcal{O}(1)$ operations. Furthermore, these algorithms require the ordering by their modulus of the error functionals $\nu_\triangle(f)$, or of modified error functionals, resulting in a log-factor in the operation count. This factor can be avoided by applying an inexact ordering based on binary binning as explained with the routine **REFINE**.

Remark 6.2 The fact that we replace the right-hand side f by a piecewise constant approximation in the evaluation of the error estimator allows us to consider $f \in H^{-1}(\Omega)$ with $f \notin L_2(\Omega)$, which thus the error estimator cannot be applied to. What is needed is that we are able to approximate f by piecewise constants within any prescribed tolerance in the $\|\cdot\|_{E'}$-norm. In [13, Example 7.3] an example is given of an $f \notin L_2(\Omega)$ which can be approximated by piecewise constants with a rate of convergence in the $\|\cdot\|_{E'}$-norm that does not limit the overall rate of convergence of the resulting adaptive algorithm for the approximation of the solution u. This example was formulated for $\kappa = 0$ but extends to any fixed κ. Considering $\kappa \to \infty$ however, the $\|\cdot\|_{E'}$-norm tends to the $\|\cdot\|_{L_2(\Omega)}$-norm, showing that a uniform convergence rate of piecewise constant functions towards a κ-independent functional $f \notin L_2(\Omega)$ cannot be expected.

Basically, the fact that, for piecewise constant right-hand sides, our error estimator $\mathcal{E}$ both provides an upper bound on the error in the Galerkin solution (cf. Theorem 4.3) and a lower bound on the difference of this Galerkin solution and the Galerkin solution on the adaptively refined partition (cf. Corollary 5.4) implies convergence of the adaptive method. In the next theorem a precise statement is given that incorporates additional terms due to the approximations of the right-hand side by piecewise constants needed to apply the aforementioned results, and a term as a consequence of the possibly inexact solution of the discrete system. A proof of this theorem can be found in [13, Theorem 6.7].

Theorem 6.3 *Let $f \in H^{-1}(\Omega)$, P^a be an admissible partition, $f_{P^a} \in \mathcal{S}^{0,-1}_{P^a}$, $\bar{u}_{P^a} \in \mathcal{S}_{P^a}$, $\tilde{P} = \mathbf{REFINE}[P^a, f_{P^a}, \bar{u}_{P^a}]$ or a refinement of it, and $f_{\tilde{P}} \in H^{-1}(\Omega)$. Then with the constants C_1, c_2 and C_3 as defined in Theorem 4.3, Corollary 5.4 and Lemma 6.1, respectively, and the* *parameter θ as has been used in the routine* **REFINE**, *for* $u = L^{-1}f$, $u_{P^a} = L_{P^a}^{-1} f_{P^a}$, $u_{\tilde{P}} = L_{\tilde{P}}^{-1} f_{\tilde{P}}$ *it holds that*

$$\|u - u_{\tilde{P}}\|_E \leq [1 - \tfrac{1}{2}(\tfrac{c_2\theta}{C_1})^2]^{\frac{1}{2}} \|u - u_{P^a}\|_E + 2c_2C_3\|u_{P^a} - \bar{u}_{P^a}\|_E \\ + 3\|f - f_{P^a}\|_{E'} + \|f - f_{\tilde{P}}\|_{E'}.$$

From Theorem 6.3 we deduce that a linearly convergent adaptive finite element method can be obtained by a repetition of the calls of successively **REFINE**, **RHS** and **GALSOLVE**. The question is what to choose for the tolerances in **RHS** and **GALSOLVE**. An option is, for some $\gamma < 1$, to select them in the ith iteration as being $\eqsim \gamma^i$. However, assuming for a moment that the estimate provided by this theorem is sharp, a choice of $\gamma < [1 - \frac{1}{2}(\frac{c_2\theta}{C_1})^2]^{\frac{1}{2}}$ would result in tolerances that asymptotically are unnecessarily small, making each iteration expensive, whereas for $\gamma > [1 - \frac{1}{2}(\frac{c_2\theta}{C_1})^2]^{\frac{1}{2}}$, the tolerances are responsible for a reduced convergence rate. Since, in addition, the constants C_1 and c_2 can only be estimated, it is not easy to make a sensible choice of the tolerances in this a priori fashion.

Therefore, as an alternative, we will construct an adaptive finite element routine **SOLVE**, in which the tolerances are chosen as some fixed, sufficiently small multiple of an a posteriori estimate of the error in the previous iterand. We will use our estimator $\mathcal{E}$, extended with some terms to cover the error in the right-hand side and that as a consequence of the inexact solution of the discrete system, see (6.3) in the following Proposition 6.4. A problem is that a monotonic decrease of these a posteriori error estimates cannot be guaranteed. In (6.4) and (6.6) in Proposition 6.4 however, we show that the estimator is equivalent to quantities that do decrease linearly. A proof of this proposition can be found in [13, Proposition 8.1]. The estimate (6.3) is a consequence of Theorem 4.3 and Lemma 6.1. The then non-trivial part of (6.4) is a consequence of (5.4) and Lemma 6.1, and (6.6) follows from Theorem 6.3 and (6.4).

Proposition 6.4 *Let* $C_4 := 1 + C_1C_3$, *where* $C_1, C_3 > 0$ *are the constants from Theorem 4.3 and Lemma 6.1. For* $f \in H^{-1}(\Omega)$, *any admissible partition* P^a, $\bar{u}_{P^a} \in \mathcal{S}_{P^a}$, $f_{P^a} \in L_2(\Omega)$, *with* $u = L^{-1}f$, $u_{P^a} = L_{P^a}^{-1} f_{P^a}$, *we have*

$$\|u - \bar{u}_{P^a}\|_E \leq C_1\mathcal{E}(P^a, f_{P^a}, \bar{u}_{P^a}) + \|f - f_{P^a}\|_{E'} + C_4\|u_{P^a} - \bar{u}_{P^a}\|_E. \tag{6.3}$$

With $f_{P^a} \in \mathcal{S}_{P^a}^{0,-1}$, *and for* ζ_{P^a} *being any upper bound on the error* $\|f - f_{P^a}\|_{E'} + C_4\|u_{P^a} - \bar{u}_{P^a}\|_E$, *we have that*

$$C_1\mathcal{E}(P^a, f_{P^a}, \bar{u}_{P^a}) + \zeta_{P^a} \eqsim \|u - \bar{u}_{P^a}\|_E + \zeta_{P^a}. \tag{6.4}$$

There exist $C_5, C_6 > 0$ *such that for* $\tilde{P} = \mathbf{REFINE}[P^a, f_{P^a}, \bar{u}_{P^a}]$ *or a refinement of it,* $\bar{u}_{\tilde{P}} \in \mathcal{S}_{\tilde{P}}$*,* $f_{\tilde{P}} \in H^{-1}(\Omega)$*,* $u_{\tilde{P}} := L_{\tilde{P}}^{-1} f_{\tilde{P}}$*, and* $\mu \in ([1 - \frac{1}{2}(\frac{c_2\theta}{C_1})^2]^{\frac{1}{2}}, 1)$*, if*

$$\zeta_{\tilde{P}} \leq C_5(\mu - [1 - \tfrac{1}{2}(\tfrac{c_2\theta}{C_1})^2]^{\frac{1}{2}})(1 + C_4)[C_1\mathcal{E}(P^a, f_{P^a}, \bar{u}_{P^a}) + \zeta_{P^a}], \quad (6.5)$$

where thus $\zeta_{\tilde{P}}$ *denotes an upper bound on* $\|f - f_{\tilde{P}}\|_{E'} + C_4\|u_{\tilde{P}} - \bar{u}_{\tilde{P}}\|_E$*, then*

$$\|u - \bar{u}_{\tilde{P}}\|_E + C_6\zeta_{\tilde{P}} \leq \mu[\|u - \bar{u}_{P^a}\|_E + C_6\zeta_{P^a}]. \quad (6.6)$$

We are ready to formulate our robust adaptive finite element routine for solving the reaction-diffusion equation.

$\mathbf{SOLVE}[f, \varepsilon, P^a, \bar{u}_{P^a}, e] \to [P^a, \bar{u}_{P^a}]$:
% *Let* $f \in H^{-1}(\Omega)$ *for which a valid routine* **RHS** *is available,* $\varepsilon > 0$*,*
% P^a *an admissible input partition,* $\bar{u}_{P^a} \in \mathcal{S}_{P^a}$ *and, with*
% $u := L^{-1}f$*,* $e \geq \max\{\varepsilon, \|u - \bar{u}_{P^a}\|_E\}$*.*
% *For a* $\mu \in ([1 - \frac{1}{2}(\frac{c_2\theta}{C_1})^2]^{\frac{1}{2}}, 1)$*, select a* $\delta \leq C_5(\mu - [1 - \frac{1}{2}(\frac{c_2\theta}{C_1})^2]^{\frac{1}{2}})$*.*

$[P^a, f_{P^a}] := \mathbf{RHS}[P^a, f, \delta e]$
$\bar{u}_{P^a} := \mathbf{GALSOLVE}[P^a, f_{P^a}, \bar{u}_{P^a}, \delta e]$
`while` $e := C_1\mathcal{E}(P^a, f_{P^a}, \bar{u}_{P^a}) + (1 + C_4)\delta e > \varepsilon$ `do`
  $P := \mathbf{REFINE}[P^a, f_{P^a}, \bar{u}_{P^a}]$
  $[P^a, f_{P^a}] := \mathbf{RHS}[P, f, \delta e]$
  $\bar{u}_{P^a} := \mathbf{GALSOLVE}[P^a, f_{P^a}, \bar{u}_{P^a}, \delta e]$
`od`

Proposition 6.5 *For some absolute constant* $C > 0$*,* $[P^a, \bar{u}_{P^a}] := \mathbf{SOLVE}[f, \varepsilon, P^a, \bar{u}_{P^a}, \varepsilon_0]$ *terminates, with* $\|u - \bar{u}_{P^a}\|_E \leq \varepsilon$*, after at most* M *iterations of the while-loop, with* M *being the smallest integer with* $C\mu^M \leq \varepsilon/\varepsilon_0$*.*

The number of operations required by a call of **SOLVE** *can be bounded by a fixed multiple of the sum of the cardinalities of all partitions that are created inside the algorithm.*

Proof After $\mathbf{GALSOLVE}[P^a, f_{P^a}, \bar{u}_{P^a}, \delta e]$, $\zeta_{P^a} := (1 + C_4)\delta e$ is an upper bound on $\|f - f_{P^a}\|_{E'} + C_4\|L_{P^a}^{-1} f_{P^a} - \bar{u}_{P^a}\|_E$. Considering two successive calls of **GALSOLVE**, this quantity ζ_{P^a} satisfies the recursion (6.5), and so with C_6 as in Proposition 6.4, we conclude that by each application of the triple **REFINE**, **RHS**, **GALSOLVE**, $\|u - \bar{u}_{P^a}\|_E + C_6\zeta_{P^a}$ decreases at least with a factor μ. Just before the first call of **REFINE**, we can bound $\|u - \bar{u}_{P^a}\|_E + C_6\zeta_{P^a} \leq \|u - L_{P^a}^{-1}u\|_E + (1 + C_6(1 + C_4))\delta\varepsilon_0 \leq (1 + (1 + C_6(1 + C_4))\delta)\varepsilon_0$. The proof of the first statement is completed by the observations that, by

(6.4), there exists an absolute constant $\tilde{C} > 0$ such that after each evaluation of $\mathcal{E}$, for the new e it holds that $e \leq \tilde{C}(\|u - \bar{u}_{P^a}\|_E + C_6\zeta_{P^a})$, whereas by (6.3), $\|u - \bar{u}_{P^a}\|_E \leq e$.

Since by assumption the number of operations required by any call of **REFINE** or **RHS** can be bounded by a fixed multiple of the cardinality of their output partition, for proving the statement about the cost of a call of **SOLVE** we only have to consider the cost of any call of **GALSOLVE** inside this algorithm. Before any application of **GALSOLVE**$[P^a, f_{P^a}, \bar{u}_{P^a}, \delta e]$, it holds that $\|u - \bar{u}_{P^a}\|_E \leq e$ and $\|f - f_{P^a}\|_{E'} \leq \delta e$. So for $u_{P^a} := L_{P^a}^{-1} f_{P^a}$ we have

$$\begin{aligned}
\|u - u_{P^a}\|_E &\leq \|u - L^{-1} f_{P^a}\|_E + \|L^{-1} f_{P^a} - u_{P^a}\|_E \\
&\leq \|u - L^{-1} f_{P^a}\|_E + \|L^{-1} f_{P^a} - \bar{u}_{P^a}\|_E \\
&\leq 2\|u - L^{-1} f_{P^a}\|_E + \|u - \bar{u}_{P^a}\|_E \\
&= 2\|f - f_{P^a}\|_{E'} + \|u - \bar{u}_{P^a}\|_E \leq (2\delta + 1)e,
\end{aligned}$$

where for the second inequality we have used that $\bar{u}_{P^a} \in \mathcal{S}_{P^a}$ and that u_{P^a} is the best approximation with respect to $\|\cdot\|_E$ of $L^{-1} f_{P^a}$ from $\mathcal{S}_{P^a}$. Since thus $\|u_{P^a} - \bar{u}_{P^a}\|_E \leq 2(\delta + 1)e$, because of our assumption on this routine the call **GALSOLVE**$[P^a, f_{P^a}, \bar{u}_{P^a}, \delta e]$ requires $\lesssim \log(\delta^{-1}) \# P^a \eqsim \# P^a$ operations. $\square$

Remark 6.6 As we have seen, (the upper bound for) the error decreases geometrically by the iteration inside **SOLVE**. Generally this cannot be expected without having at the same time a geometrically increase of the cardinality of the partitions, in which case thus by Proposition 6.5 the work required by a call of **SOLVE** is of the order of the cardinality of the final partition.

Although we demonstrated convergence, uniformly in κ, of the adaptive finite element method **SOLVE**, we have not shown that the method is at least as efficient, and in case of singular solutions, more efficient than the standard, non-adaptive finite element method based on uniform refinements. For $s > 0$, let us define the approximation classes

$$\mathcal{A}^s = \{u \in H_0^1(\Omega) : |u|_{\mathcal{A}^s} := \sup_{n \geq 0} n^s \inf_{\# P \leq n} \inf_{u_P \in \mathcal{S}_P} \|u - u_P\|_E < \infty\},$$

$$\bar{\mathcal{A}}^s = \{f \in H^{-1}(\Omega) : |f|_{\bar{\mathcal{A}}^s} := \sup_{n \geq 0} n^s \inf_{\# P \leq n} \inf_{f_P \in \mathcal{S}_P^{0,-1}} \|f - f_P\|_{E'} < \infty\},$$

where P is any partition of the type we consider. So for $u \in \mathcal{A}^s$, and similarly for $f \in \bar{\mathcal{A}}^s$, for any $\varepsilon > 0$ there exists a partition P and a $u_P \in \mathcal{S}_P$ with $\|u - u_P\|_E \leq \varepsilon$ where $\# P \lesssim \varepsilon^{-1/s} |u|_{\mathcal{A}^s}^{1/s}$. In [13], following earlier work in [3], for the Poisson problem and a routine **SOLVE** that was extended with a wavelet-based *coarsening routine*, we could prove *(quasi-) optimality* of the adaptive method in the following sense: Whenever for some $s > 0$, $u \in \mathcal{A}^s$ and $f \in \bar{\mathcal{A}}^s$, and assuming that, for this s and f, **RHS** is (quasi-) optimal in the sense that $\mathbf{RHS}[P, f, \varepsilon]$ returns a partition P^a with $\#P^a \lesssim \varepsilon^{-1/s}|f|_{\bar{\mathcal{A}}^s}^{1/s}$, the partition P^a returned by $\mathbf{SOLVE}[f, \varepsilon, P_0, \bar{u}_{P_0}, \varepsilon_0]$, where $\varepsilon_0 \lesssim \|f\|_{E'}$, satisfies

$$\#P^a \lesssim \varepsilon^{-1/s}(|u|_{\mathcal{A}^s}^{\frac{1}{s}} + |f|_{\bar{\mathcal{A}}^s}^{\frac{1}{s}}), \tag{6.7}$$

taking $\mathcal{O}(\varepsilon^{-1/s}(|u|_{\mathcal{A}^s}^{\frac{1}{s}} + |f|_{\bar{\mathcal{A}}^s}^{\frac{1}{s}}))$ operations.

Because of the robustness results we proved so far, the result (6.7) extends to the reaction-diffusion equation, uniformly in κ, in case we add a coarsening routine to the present **SOLVE**, where the wavelet basis applied in [13] should be replaced by the one from [7], which, properly scaled, is stable with respect to $\|\,\|_E$, uniformly in κ. The idea of a coarsening routine, that has to be applied after each fixed number of iterations of the triple **REFINE**, **RHS**, **GALSOLVE**, is to undo refinements that turn out hardly to contribute to the approximation, and in this way to ensure that during the iteration, the current upper bound e for the error $\|u - \bar{u}_{P^a}\|_E$ satisfies

$$\#P^a \lesssim e^{-1/s}. \tag{6.8}$$

Since, in any case for the Poisson problem, it is generally observed, see, e.g., [11], that adaptive finite element methods without a coarsening already generate a sequence of partitions for which (6.8) is valid for the best possible value of s that one generally may expect ($s = \frac{1}{2}$ for the Poisson problem in two dimensions with piecewise linears), we have chosen here not to include a coarsening routine in **SOLVE**.

Concerning membership of the classes $\mathcal{A}^s$ or $\bar{\mathcal{A}}^s$, for any fixed κ it is well-known that for $s \leq \frac{1}{2}$, $H_0^1(\Omega) \cap H^{1+2s}(\Omega) \subset \mathcal{A}^s$, and for $s \leq 1$, $H^{-1+2s}(\Omega) \subset \bar{\mathcal{A}}^s$, where for $u \in H_0^1(\Omega) \cap H^{1+2s}(\Omega)$ or $f \in H^{-1+2s}(\Omega)$, it is sufficient to consider uniform refinements to realize these rates. Obviously, the classes $\mathcal{A}^s$ and $\bar{\mathcal{A}}^s$ contain many more functions or functionals than only those from the aforementioned Sobolev spaces, which is the reason to consider adaptive methods anyway. Although $\mathcal{A}^s$ and $\bar{\mathcal{A}}^s$ are non-trivial for any $s > 0$, as they contain all $u \in \mathcal{S}_P$ or $f \in \mathcal{S}_P^{0,-1}$ for any partition P, only for $s \leq \frac{1}{2}$ or $s \leq 1$ membership of $\mathcal{A}^s$ or $\bar{\mathcal{A}}^s$ can be guaranteed by imposing suitable smoothness conditions. For partitions generated by the so-called newest vertex bisection, a characterization of $\mathcal{A}^s$ in terms of Besov spaces can be found in [4]. We expect the same results to be valid for the type of partitions that we consider. Furthermore, we expect that $u \in \mathcal{A}^s$ implies that $f = Lu \in \bar{\mathcal{A}}^s$.

For fixed κ, membership of $\mathcal{A}^s$ or $\bar{\mathcal{A}}^s$ characterizes the rate of convergence of the best approximations from $\mathcal{S}_P$ or $\mathcal{S}_P^{0,-1}$ for the best partitions P when $\#P \to \infty$. The situation changes when we consider both $\#P$ and κ as variables. In the following, let us consider the situation that, although at least one of f or $u = L^{-1}f$ depends on κ, both u and f are smooth, uniformly in κ, as is the case when f vanishes in a neighbourhood of $\partial\Omega$. Then for a given upper bound on $\#P$, up to an absolute multiplicative constant the best approximations are found in the spaces with respect to uniform refinements of P_0. For P being such a partition, and $h := \max_{\triangle \in P} \operatorname{diam}(\triangle)$, one easily verifies that

$$\inf_{u_P \in \mathcal{S}_P} \|u - u_P\|_E \lesssim \sqrt{\tfrac{\kappa^2}{1+\kappa^2} h^4 + \tfrac{1}{1+\kappa^2} h^2} \eqsim \tfrac{h}{\sqrt{1+\kappa^2}} \max\{h\sqrt{1+\kappa^2}, 1\},$$

whereas (6.2) shows that

$$\inf_{f_P \in \mathcal{S}_P^{0,-1}} \|f - f_P\|_{E'} \lesssim \min\{1, h\sqrt{1+\kappa^2}\} h,$$

where generally both these upper bounds are sharp. With the exact right-hand side f, the Galerkin solution would be the best approximation for u from $\mathcal{S}_P$ with respect to $\|\cdot\|_E$. Replacing f by $f_P \in \mathcal{S}_P^{0,-1}$ causes an additional error in the approximate solution with $\|\cdot\|_E$-norm equal to $\|f - f_P\|_{E'}$. Assuming that f_P is the best or quasi-best approximation for f from $\mathcal{S}_P^{0,-1}$ with respect to $\|\cdot\|_{E'}$, by comparing both bounds we infer that, generally, for $h\kappa^2 \gg 1$ the additional error due to this approximation of the right-hand side *dominates* the approximation error (h vs. h^2), whereas for $h\kappa^2 \ll 1$ it is neglectable ($h^2\sqrt{1+\kappa^2}$ vs. $\frac{h}{\sqrt{1+\kappa^2}}$). Recall that for $\kappa \to \infty$, L tends to the identity operator, and that the $\|\cdot\|_E$-norm of any fixed function in $H_0^1(\Omega)$ tends to the $\|\cdot\|_{L_2(\Omega)}$-norm, and the $\|\cdot\|_{E'}$ norm of any fixed function in $L_2(\Omega)$ tends to the $\|\cdot\|_{L_2(\Omega)}$-norm, giving another explanation for the degradation of the overall convergence for $h\kappa^2 \ll 1$ due to the piecewise constant approximation for the right-hand side.

In an attempt to remove this disadvantage of our replacements of the right-hand side by piecewise constant approximations, that we applied in order to guarantee uniform convergence of the adaptive algorithm, in the next section we will consider piecewise linear approximations. In the above setting of having smooth u and f, uniformly in κ, and uniform partitions, we have

$$\inf_{f_P \in \mathcal{S}_P^{1,-1}} \|f - f_P\|_{E'} \lesssim \min\{1, h\sqrt{1+\kappa^2}\} h^2,$$

so that the overall error will never be dominated by the error of the (quasi-) best piecewise linear approximation for the right-hand side. Moreover, in case the reaction-diffusion equation stems from an implicit time discretization of a parabolic or second order hyperbolic problem, the right-hand side is dominated by the discrete solution on the previous time step. Since this solution is continuous piecewise linear, in this setting it is not attractive to be forced to replace the right-hand side by a piecewise constant approximation.

7 Piecewise linear approximations for the right-hand side

In this section, we study the convergence of the routine **SOLVE**, in which a call $\mathbf{RHS}[P, f, \varepsilon]$ is now allowed to output, for some admissible refinement P^a of P, an approximation $f_{P^a} \in \mathcal{S}^{1,-1}_{P^a}$ with $\|f - f_{P^a}\|_{E'} \leq \varepsilon$. The only place in Sections 5 and 6 where it was used that $f_{P^a} \in \mathcal{S}^{0,-1}_{P}$ instead of $f_{P^a} \in \mathcal{S}^{1,-1}_{P}$ was in Lemma 5.1, and as a consequence in Corollary 5.2 and in (5.1). Lemma 5.1 is not valid for $f \in \prod_{i=1}^{2} P_1(\triangle)$, and without this lemma, instead of Theorem 5.3(a), one may verify that for $f \in \mathcal{S}^{1,-1}_{P^a}$ and with the other conditions as in that theorem, it holds that

$$\|u_{\tilde{P}} - u_{P^a}\|^2_{E,\triangle_1\cup\triangle_2} \gtrsim \min\Big\{1, \Big[\mathrm{diam}(e)\sqrt{1+\kappa^2}\Big]^{-1}\Big\}\eta_e(u_{P^a}) + \sum_{i=1}^{2}\zeta_{\triangle_i}(f_{P^a}, u_{P^a}). \tag{7.1}$$

So for $\mathrm{diam}(e)\kappa$ large, instead that the edge residual can be bounded in terms of the neighbouring element residuals (cf. Corollary 5.2), in this modified version (7.1) of Theorem 5.3(a), applicable to piecewise linear right-hand sides, it is possible that the edge residual dominates over the neighbouring element residuals. Taking into account (7.1), the remainder of Sections 5 and 6 can be copied, in the sense that the constant c_2 introduced in Corollary 5.4 should everywhere be replaced by

$$c_2(P^a, \kappa) := \tilde{c}_2 \min\Big\{1, \Big[\max_{\triangle\in P^a} \mathrm{diam}(\triangle)\sqrt{1+\kappa^2}\Big]^{-1}\Big\}^{\frac{1}{2}},$$

where $\tilde{c}_2 > 0$ is some absolute constant.

Remark 7.1 There is one place where c_2 does not have to be replaced by $c_2(P^a, \kappa)$. As shown in [15], also for piecewise linear right-hand sides, (5.4) is valid uniformly in κ, and so in combination with

Theorem 4.3, for P^a being an admissible partition, $f_{P^a} \in \mathcal{S}^{1,-1}_{P^a}$, $u = L^{-1} f_{P^a}$, $u_{P^a} = L^{-1}_{P^a} f_{P^a}$, we have

$$\|u - u_{P^a}\|_E \eqsim \mathcal{E}(P^a, f_{P^a}, u_{P^a}).$$

The replacement of c_2 by $c_2(P^a, \kappa)$ leads to the following conclusion. If **SOLVE** is applied with an input partition P^a and a $\delta \leq C_5(\mu - [1 - \frac{1}{2}(\frac{c_2(P^a,\kappa)\theta}{C_1})^2]^{\frac{1}{2}})$ for some $\mu \in ([1 - \frac{1}{2}(\frac{c_2(P^a,\kappa)\theta}{C_1})^2]^{\frac{1}{2}}, 1)$, then it converges with a rate $\leq \mu$. So, for a sufficiently small fixed $\delta > 0$, a uniform rate less than 1 is guaranteed for all κ and input partitions P^a for which $c_2(P^a, \kappa) \gtrsim 1$, i.e., for which

$$\kappa \max_{\triangle \in P^a} \operatorname{diam}(\triangle) \lesssim 1. \tag{7.2}$$

For example, thinking of an implicit time discretization of a parabolic problem, (7.2) reads as the mild condition $\max_{\triangle \in P^a} \operatorname{diam}(\triangle) \lesssim (\delta t)^{\frac{1}{2}}$.

If, however, **SOLVE** is called with a fixed initial partition, e.g., with P_0, then its rate of convergence ρ can only be shown to satisfy $1 - \rho \gtrsim (1 + \kappa^2)^{-\frac{1}{2}}$, where moreover the parameter δ should satisfy $\delta \lesssim (1 + \kappa^2)^{-\frac{1}{2}}$, so that the work required by each call of **GALSOLVE**$[P^a, f_{P^a}, \bar{u}_{P^a}, \delta e]$ inside **SOLVE** can only be bounded by $\mathcal{O}(\log(1+\kappa^2)\#P^a)$. Although this conclusion is a result of a worst case scenario, a rate ρ with with $1-\rho \eqsim (1+\kappa^2)^{-\frac{1}{2}}$ may indeed occur, even when the discrete systems are solved exactly, as is demonstrated by the following example.

Example 7.2 Let P_0 be the partition of $[0,1]^2$ into 4 isosceles right triangles, and let P_1 be the partition as a result of two uniform recursive red-refinements, see Figure 7.1. Let $f \in \mathcal{S}_{P_0} \subset \mathcal{S}^{1,-1}_{P_0}$ be defined

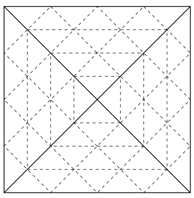

Fig. 7.1. P_0 and P_1

by $f(\frac{1}{2}, \frac{1}{2}) = 1$, and let $u := L^{-1} f$, $u_{P_0} := L^{-1}_{P_0} f$, and $u_{P_1} := L^{-1}_{P_1} f$. From Remark 7.1, we have that

$$r := \frac{\|u_{P_1} - u_{P_0}\|_E^2}{\|u - u_{P_1}\|_E^2} \eqsim \frac{\|u_{P_1} - u_{P_0}\|_E^2}{\mathcal{E}(P_1, f, u_{P_1})^2} = \frac{1267\sqrt{2}}{303}\kappa^{-1} + \mathcal{O}(\kappa^{-2}),$$

($\kappa \to \infty$), the latter as a result of a MAPLE computation. In particular, we found $\|u_{P_1} - u_{P_0}\|_E^2 \eqsim \kappa^{-4}$, $\sum_{\triangle \in P_1} \zeta_\triangle(f, u_{P^1}) \eqsim \kappa^{-4}$, and $\eta_e(u_{P_1}) \eqsim \kappa^{-3}$ for $e \in E_{P_1}$ that is part of an edge from E_{P_0}, and $\eta_e(u_{P_1}) \eqsim \kappa^{-7}$ otherwise. Using that $u_{P_1} - u_{P_0} \perp_E u - u_{P_1}$, we conclude that

$$1 - \frac{\|u - u_{P_1}\|_E}{\|u - u_{P_0}\|_E} = 1 - (1+r)^{-\frac{1}{2}} = \tfrac{1}{2}r + \mathcal{O}(r^2) \eqsim \kappa^{-1} \quad (\kappa \to \infty).$$

8 Numerical experiments

With $\Omega = (0,1)^2$ and an initial partition P_0 as indicated in Figure 8.1, we tested **SOLVE** both with (quasi-) best piecewise constant

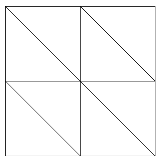

Fig. 8.1. P_0

and piecewise linear approximations for the right-hand side. In our experiments, we used $C_1 = 0.8$, $C_4 = 1$, $\delta = \frac{1}{9}$ and $\theta^2 = \frac{1}{2}$. Our routine **GALSOLVE** was based on the transformation to wavelet basis.

With

$$v_\kappa(z) := 1 - \frac{e^{-\frac{1}{2}\sqrt{2}\kappa z} + e^{-\frac{1}{2}\sqrt{2}\kappa(1-z)}}{1 + e^{-\frac{1}{2}\sqrt{2}\kappa}},$$

first we considered the reaction-diffusion problem (2.1) with right-hand side defined by

$$f_\kappa(x,y) = \tfrac{1}{2}\tfrac{\kappa^2}{1+\kappa^2}(v_\kappa(x) + v_\kappa(y)) + \tfrac{2\pi^2+\kappa^2}{1+\kappa^2}\sin(\pi x)\sin(\pi y),$$

so that the solution is given by

$$u_\kappa(x,y) = v_\kappa(x)v_\kappa(y) + \sin(\pi x)\sin(\pi y),$$

see Figure 8.2. As noted in Remark 2.1, $\|\cdot\|_{E'} \lesssim \|\cdot\|_{L_2(\Omega)}$ on $L_2(\Omega)$, and one easily verifies that

$$\|f_\kappa - [1 + \sin(\pi x)\sin(\pi y)]\|_{L_2(\Omega)} \lesssim \kappa^{-\frac{1}{2}}.$$

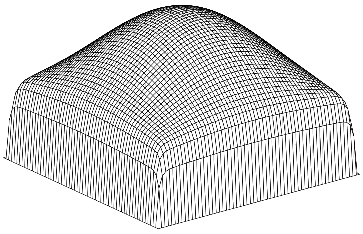

Fig. 8.2. Solution u_κ for $\kappa = 10^4$

So for $\kappa \to \infty$, f_κ tends to a smooth, κ-independent function that does not vanish at the boundary, and so the solution u_κ exhibits boundary layers that are typical for this equation. We included the term $\frac{2\pi^2+\kappa^2}{1+\kappa^2}\sin(\pi x)\sin(\pi y)$ in the right-hand f_κ side to avoid that $\lim_{\kappa\to\infty}\|f_\kappa - 1\|_{E'} = 0$, and, for any $\Sigma \Subset \Omega$, $\lim_{\kappa\to\infty}\|u_\kappa - 1\|_{E,\Sigma} = 0$, which case is special in the sense the constant 1 is included in alle finite element spaces. The numerical results obtained by **SOLVE** for $\kappa^2 = 0$, $\kappa^2 = 10^3$, and $\kappa^2 = 10^4$, and for piecewise constant and piecewise linear approximations of the right-hand side given in Figures 8.3 and 8.4, respectively, show that in all cases the energy norm of the error decays as $N^{-\frac{1}{2}}$, with N being the number of degrees of freedom, which for any fixed κ is the asymptotically optimal rate one generally can expect. Furthermore, the results show the same behaviour of the error as function of the CPU time, meaning that the work is proportional to the number of degrees of freedom. The asymptotically optimal slope $-\frac{1}{2}$ in log-log scale is indicated in all pictures by a dashed-dotted line. Although with piecewise constant approximations of the right-hand side sometimes refinements were made by a call of **RHS**, where we simply applied uniform refinements, with this solution u_κ these refinements appear hardly to influence the process since the results for piecewise constant and piecewise linear approximations are nearly the same. Although condition (7.2) was not valid, also with piecewise linear approximations of the right-hand side, in these experiments a uniform convergence rate can be observed. An illustration of a finite element partition generated by **SOLVE** is given in Figure 8.5. Although both the exact solution and the initial partition are symmetric with respect to both diagonals of $[0,1]^2$, this partition is not fully symmetric due to some randomness mainly caused by the binary binning in the routine **REFINE**.

To demonstrate that the piecewise linear approximations for the right-hand side can have advantages, finally we replaced the right-hand side f_κ by $\frac{2\pi^2+\kappa^2}{1+\kappa^2}\sin(\pi x)\sin(\pi y)$, so that the solution is given

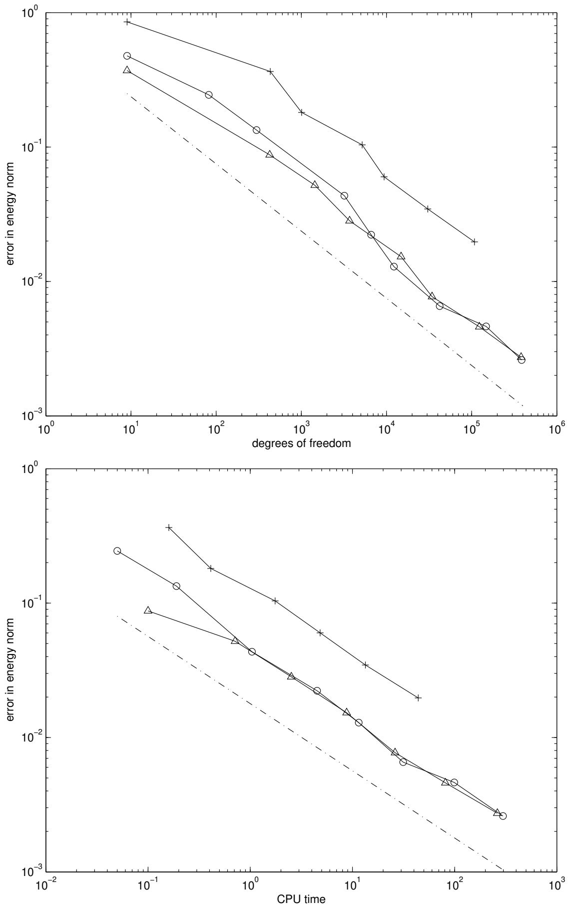

Fig. 8.3. Error vs. number of degrees of freedom or CPU time for $\kappa^2 = 0$ ($\star$), $\kappa^2 = 10^3$ ($\circ$) and $\kappa^2 = 10^4$ ($\triangle$), with piecewise *constant* approximations of the right-hand side

by $\sin(\pi x)\sin(\pi y)$. Because of this smooth solution and right-hand side we simply applied uniform refinements. The results for $\kappa^2 = 10^6$, given in Figure 8.6, for the (quasi-) best piecewise constant and piecewise linear approximations for the right-hand show the initially faster convergence with the piecewise linears. Note however that eventually, for the number of degrees of freedom tending to infinity, this advantage will vanish, since in the limit for both the best piecewise linear and piecewise constant approximations the error in the right-hand side is neglectable.

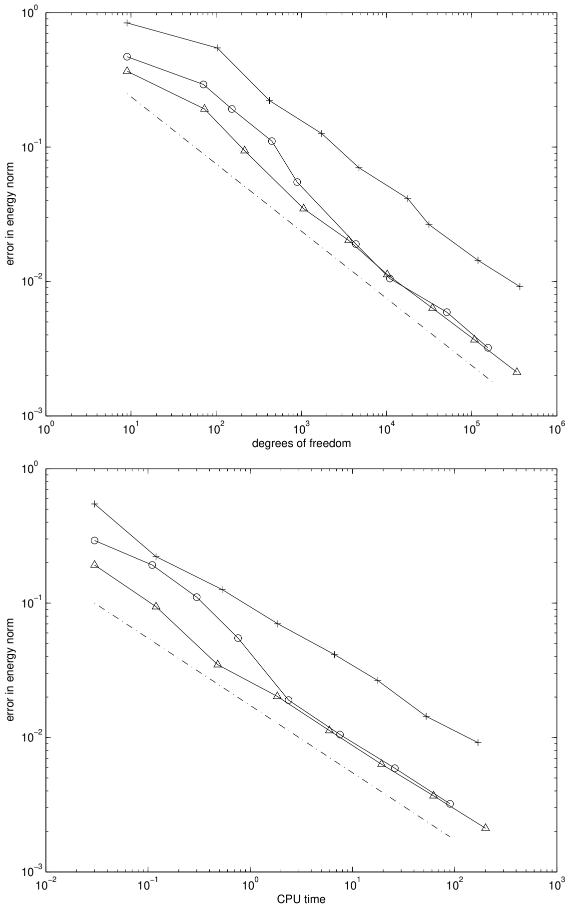

Fig. 8.4. Error vs. number of degrees of freedom or CPU time for $\kappa^2 = 0$ ($\star$), $\kappa^2 = 10^3$ ($\circ$) and $\kappa^2 = 10^4$ ($\triangle$), with piecewise *linear* approximations of the right-hand side

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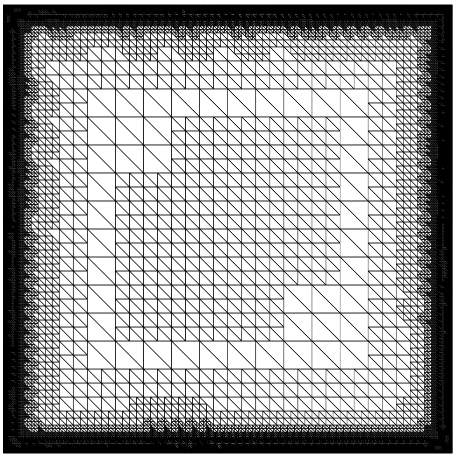

Fig. 8.5. Partition generated by **SOLVE** corresponding to $\kappa^2 = 10^4$, piecewise linear approximations of the right-hand side, and with 51135 degrees of freedom

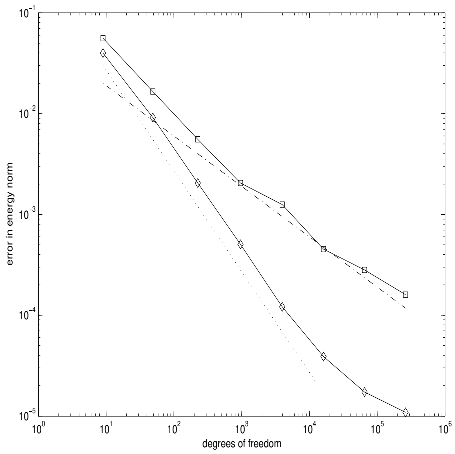

Fig. 8.6. Error vs. number of degrees of freedom for $\kappa^2 = 10^6$, right-hand side $\frac{2\pi^2+\kappa^2}{1+\kappa^2}\sin(\pi x)\sin(\pi y)$, and thus solution $\sin(\pi x)\sin(\pi y)$, and piecewise constant ($\Box$) or piecewise linear ($\diamond$) approximations for the right-hand side. The dashed-dotted and dotted lines indicate the slopes $-\frac{1}{2}$ and -1, respectively

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