Numerische Mathematik manuscript No.
(will be inserted by the editor)

Higher Order Mortar Finite Element Methods in 3D with Dual Lagrange Multiplier Bases*

B. P. Lamichhane[1], R. P. Stevenson[2], B. I. Wohlmuth[1]

[1] Institute of Applied Analysis and Numerical Simulation, University of Stuttgart, Germany, e-mail: {lamichhane,wohlmuth}@mathematik.uni-stuttgart.de
[2] Department of Mathematics, Utrecht University, The Netherlands, e-mail: stevenson@math.uu.nl

The date of receipt and acceptance will be inserted by the editor

Summary Mortar methods with dual Lagrange multiplier bases provide a flexible, efficient and optimal way to couple different discretization schemes or nonmatching triangulations. Here, we generalize the concept of dual Lagrange multiplier bases by relaxing the condition that the trace space of the approximation space at the slave side with zero boundary condition on the interface and the Lagrange multiplier space have the same dimension. We provide a new theoretical framework within this relaxed setting, which opens a new and simpler way to construct dual Lagrange multiplier bases for higher order finite element spaces. As examples, we consider quadratic and cubic tetrahedral elements and quadratic serendipity hexahedral elements. Numerical results illustrate the performance of our approach.

Key words. Mortar finite elements, Lagrange multipliers, biorthogonal bases, nonmatching triangulations.
AMS subject classification. 35N55, 65N30.

1 Introduction

Over the recent past years, mortar methods with dual Lagrange multiplier bases have become an active area of research, see, e.g., [28,

* This work was supported in part by the Deutsche Forschungsgemeinschaft, SFB 404, C12, the Netherlands Organization for Scientific Research and by the European Community's Human Potential Programme under contract HPRN-CT-2002-00286.

31, 18, 29, 19, 20]. Originally introduced to couple spectral and finite element approximations [5–7], mortar methods have a wide range of applications. Being a powerful tool to couple different discretization schemes or nonmatching triangulations, these nonconforming techniques provide a more flexible approach than standard conforming approaches. Examples where such situations occur are when, in a parallel environment, mesh generators are used independently on different subdomains, or when different physical models or, within one model, strongly different parameter values naturally lead to the use of different discretization schemes on different subdomains. The central idea of mortar techniques is to replace the strong, pointwise continuity across the interface by a weak one, which requires that the jump of the solution across the interface is orthogonal to a suitable Lagrange multiplier space. Under this relaxed condition, optimal a priori estimates of the discretization error have been demonstrated, cf. [4, 3, 7, 29, 18], and various iterative methods have been introduced to solve the arising linear system, see [9, 8, 31, 15, 27].

Generally, the mortar approach has the disadvantage that even when the boundary value problem is elliptic, the arising linear system is of saddle point type, usually for which iterative methods are known to be less efficient than for symmetric positive definite systems. However, when working with dual Lagrange multiplier bases, the degrees of freedom associated to the multiplier can be locally eliminated leading to a sparse, positive definite system, on which, for example, efficient multigrid methods can be applied, see [31, 29]. For the lowest order finite elements in 3D, dual Lagrange multiplier bases were constructed in [18, 29]. However, the construction of such bases for higher order elements is more complicated, see [19, 20, 23]. In the present paper, we develop a general framework facilitating the construction of Lagrange multiplier bases, in particular for higher order elements. In a first step, we remove the condition that the trace space of the approximation space at the slave side with zero boundary condition on the interface and the Lagrange multiplier space have the same dimension, with which the introduction of additional degrees of freedom at the multiplier side only to satisfy this condition on the dimensions is avoided. As a result, we can work with a subspace of the trace space for which it is easier to construct a dual Lagrange multiplier basis. In a second step, we describe a general procedure for the construction of multiplier basis functions near the interface boundaries, where one has to deal with the complication that homogeneous boundary conditions are incorporated in the trace space of the approximation space at the slave side, but not in the Lagrange multiplier space. Although the theory presented in this paper applies to general space dimensions, our concrete realizations of dual Lagrange multiplier bases will take place in the three dimensional case.

This paper is organized as follows: In the rest of this section, we describe the problem setting, and fix some notations related to the mortar method. In Section 2, we prove some optimal a priori estimates without assuming that the trace space with zero boundary condition on the interface from the slave side has the same dimension as the multiplier space. As in [18,12], we allow locally refined finite element partitions and geometrically nonconforming subdivisions into subdomains. Furthermore, as in [20,12], we do not require that the partitions match on the boundaries of the interfaces. For the case of having quasi-uniform meshes on each subdomain, we will prove an error estimate showing that the possibly disadvantageous effect of having at the slave side a finer mesh than at the master side is much milder than that is indicated by existing error estimates, e.g., in [29,20]. A general framework for the construction of dual multiplier bases, in particular for the modifications needed along the interface boundaries, is given in Section 3. Examples of dual multiplier bases are given for quadratic and cubic tetrahedral elements, and for quadratic serendipity hexahedral elements. Finally, in Section 4, we present some numerical results concerning the discretization errors in the L^2-norm, the piecewise H^1-norm, and in a weighted L^2-norm for the Lagrange multiplier.

We consider the following elliptic second order boundary value problem in variational form: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega a\nabla u \cdot \nabla v + cuvdx = f(v), \quad v \in H_0^1(\Omega), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded polytope, $a, c \in L^\infty(\Omega)$, $a > a_0 > 0$ and $c \geq 0$. The domain Ω is decomposed into K non-overlapping polytopes Ω_k, i.e.,

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k \quad \text{with} \quad \Omega_k \cap \Omega_\ell = \emptyset \quad \text{for} \quad k \neq \ell.$$

Defining, for each $1 \leq k \leq K$,

$$H_*^1(\Omega_k) := \{v \in H^1(\Omega_k), v_{|\partial\Omega\cap\partial\Omega_k} = 0\},$$

we assume that $f \in \prod_{k=1}^{K}(H_*^1(\Omega_k))'$. We assume that a and c are *piecewise smooth with respect to the subdivision into subdomains*, but allow that they have jumps over the interfaces. However, in this paper, no attempt is made to obtain results that hold uniformly in the sizes of such jumps, cf. [17].

For any $\{k, \ell\}$ with $1 \leq k \neq \ell \leq K$, for which the closures of Ω_k and Ω_ℓ have a $(d-1)$-dimensional intersection, we set $\overline{\gamma}_i = \overline{\Omega}_k \cap \overline{\Omega}_\ell$, where $1 \leq i \leq N$ is a number uniquely associated to this $\{k, \ell\}$, and thus with N being the total number of sets $\{k, \ell\}$ having this property. For each $1 \leq i \leq N$, one of the subdomains Ω_k and Ω_ℓ on both sides of the interface is referred to be at the slave or non-mortar side, and its index is denoted as $s(i)$, where the other is referred to be at the master or mortar side, with its index denoted as $m(i)$.

For each δ from some index set $\mathcal{I}$, and each $1 \leq k \leq K$, we assume a finite element space $X_k^\delta \subset H_*^1(\Omega_k)$ with respect to a partition of Ω_k into "elements", e.g., d-simplices or d-rectangles. We assume that for any $\delta \in \mathcal{I}$,

$$\begin{cases} \gamma_i \text{ is the union of complete } (d-1)\text{-dimensional faces of} \\ \text{elements at the slave side.} \end{cases} \tag{A.1}$$

Note that (A.1) does allow the partition into subdomains to be geometrically nonconforming; γ_i needs not to be a full face of both $\Omega_{m(i)}$ and $\Omega_{s(i)}$, or even not to be a full face of either $\Omega_{m(i)}$ or $\Omega_{s(i)}$, see Figure 1.1. In case the decomposition into subdomains is geometrically

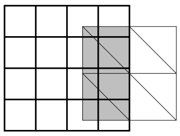

Fig. 1.1: An interface that is not a full face of either $\Omega_{m(i)}$ or $\Omega_{s(i)}$; it satisfies (A.1) when the square partitioned into triangles is a face of $\Omega_{s(i)}$, but not when this is a face of $\Omega_{m(i)}$.

conforming, (A.1) is automatically satisfied.

For $1 \leq i \leq N$ and $\delta \in \mathcal{I}$, we define the trace space

$$W_{0,i}^\delta = \{v^\delta|_{\gamma_i} : v^\delta \in X_{s(i)}^\delta,\ v^\delta|_{\partial\Omega_{s(i)} \setminus \gamma_i} = 0\}.$$

As a consequence of $X_{s(i)}^\delta \subset H_*^1(\Omega_{s(i)})$ being a finite element space, we can assume that $W_{0,i}^\delta \subset H_0^1(\gamma_i)$. We will refer to the partition of γ_i into the faces of elements as referred to in (A.1) as the finite element partition underlying $W_{0,i}^\delta$. For enforcing a weak coupling of the finite element functions over the interfaces, for each $1 \leq i \leq N$ and $\delta \in \mathcal{I}$

we introduce a Lagrange multiplier space $M_i^\delta \subset L^2(\gamma_i)$. Thinking of M_i^δ as being a finite element space, the partition underlying M_i^δ will be equal to that of $W_{0,i}^\delta$.

Setting the product spaces $X^\delta := \prod_{k=1}^K X_k^\delta$, $M^\delta := \prod_{i=1}^N M_i^\delta$, and defining the bilinear forms $a(v,w) = \sum_{k=1}^K \int_{\Omega_k} a\nabla v \cdot \nabla w + cvwdx$ and $b(v,\mu) = \sum_{i=1}^N \int_{\gamma_i} [v]_{\gamma_i} \mu d\sigma$, where $[v]_{\gamma_i}$ denotes the jump of v on γ_i from the master to the slave side, the mortar finite element discretization now consists in finding $(u^\delta, \lambda^\delta) \in X^\delta \times M^\delta$ such that

$$\begin{aligned} a(u^\delta, v^\delta) + b(v^\delta, \lambda^\delta) &= f(v^\delta), \quad && v^\delta \in X^\delta, \\ b(u^\delta, \mu^\delta) &= 0, && \mu^\delta \in M^\delta, \end{aligned} \tag{1.2}$$

or, equivalently, considering the first component only, in finding

$$u^\delta \in V^\delta := \{v^\delta \in X^\delta : b(v^\delta, \mu^\delta) = 0,\ \mu^\delta \in M^\delta\}$$

such that

$$a(u^\delta, v^\delta) = f(v^\delta), \quad v^\delta \in V^\delta. \tag{1.3}$$

Assuming that for any $\delta \in \mathcal{I}$ and $1 \le i \le N$, the constant function

$$1 \in M_i^\delta, \tag{A.2}$$

and that

$$\beta_i^\delta := \inf_{\mu_i^\delta \in M_i^\delta \setminus \{0\}} \sup_{w_i^\delta \in W_{0,i}^\delta \setminus \{0\}} \frac{\langle \mu_i^\delta, w_i^\delta \rangle_{L^2(\gamma_i)}}{\|\mu_i^\delta\|_{L^2(\gamma_i)} \|w_i^\delta\|_{L^2(\gamma_i)}} > 0, \tag{1.4}$$

the problem (1.2) and thus (1.3) are known to have a unique solution, cf. [7,10].

In this paper, in order to avoid the repeated use of generic but unspecified constants, by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on, where we, in particular, think of δ as well as the number of subdomains K. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2 A priori estimates

We start with establishing suitable mortar projectors without requiring that $\dim M_i^\delta = \dim W_{0,i}^\delta$, as is usually done in the literature.

Lemma 2.1 *Let M, W be subspaces of some Hilbert space H with $L := \dim M = \dim W < \infty$.*

(a). It holds

$$\beta := \inf_{\mu \in M\setminus\{0\}} \sup_{w \in W\setminus\{0\}} \frac{\langle \mu, w\rangle}{\|\mu\|\|w\|} > 0, \tag{2.1}$$

if and only if there exists a projector Π *onto* W *for which*

$$\langle \Pi v, \mu\rangle = \langle v, \mu\rangle \quad v \in H,\ \mu \in M.$$

Moreover, $\|\Pi\| = \beta^{-1}$.

(b). Let $\{m_1, \ldots, m_L\}$ *and* $\{w_1, \ldots, w_L\}$ *be bases for* M *and* W*, respectively. Then* $\beta > 0$ *if and only if the matrix* $\mathbf{R} := (\langle w_\ell, m_{\ell'}\rangle)_{1\le \ell,\ell' \le L}$ *is invertible. In particular, with* λ_M, Λ_M *being the largest or smallest constant such that*

$$\lambda_M \|\mathbf{c}\| \le \|\sum_{\ell=1}^{L} c_\ell m_\ell\| \le \Lambda_M \|\mathbf{c}\|, \quad \mathbf{c} = (c_\ell)_{1\le \ell \le L} \in \mathbb{R}^L,$$

and with λ_W, Λ_W *defined analogously for the basis for* W*, we have*

$$(\Lambda_M \Lambda_W)^{-1} \le \beta \|\mathbf{R}^{-1}\| \le (\lambda_M \lambda_W)^{-1}.$$

Furthermore, $\Pi v = \sum_\ell \langle v, m_\ell\rangle \sum_{\ell'} (\mathbf{R}^{-1})_{\ell,\ell'} w_{\ell'}$.

Proof For the non-trivial part of (a), see e.g. [13, Theorem 2.1(a)], and a straightforward computation shows (b).

Proposition 2.2 *Thanks to* (1.4)*, there exist a bounded projector* $\Pi_i^\delta : L^2(\gamma_i) \to L^2(\gamma_i)$ *onto* $W_{0,i}^\delta$ *with*

$$\langle \Pi_i^\delta v, \mu_i^\delta\rangle_{L^2(\gamma_i)} = \langle v, \mu_i^\delta\rangle_{L^2(\gamma_i)} \quad v \in L^2(\gamma_i),\ \mu_i^\delta \in M_i^\delta.$$

Such a projector will be referred to as being a mortar projector*. It can be selected such that* $\|\Pi_i^\delta\|_{L^2(\gamma_i)\to L^2(\gamma_i)} \le \sqrt{2}\,(\beta_i^\delta)^{-1}$.

Proof Let $Q_i^\delta : L^2(\gamma_i) \to W_{0,i}^\delta$ denote the $L^2(\gamma_i)$-orthogonal projector onto $W_{0,i}^\delta$, and let $\breve{Q}_i^\delta$ denote its restriction to M_i^δ. For any $\mu_i^\delta \in M_i^\delta$, we have

$$\begin{aligned}
\|\breve{Q}_i^\delta \mu_i^\delta\|_{L^2(\gamma_i)} &= \sup_{w_i^\delta \in W_{0,i}^\delta\setminus\{0\}} \frac{\langle \breve{Q}_i^\delta \mu_i^\delta, w_i^\delta\rangle_{L^2(\gamma_i)}}{\|w_i^\delta\|_{L^2(\gamma_i)}} \\
&= \sup_{w_i^\delta \in W_{0,i}^\delta\setminus\{0\}} \frac{\langle \mu_i^\delta, w_i^\delta\rangle_{L^2(\gamma_i)}}{\|w_i^\delta\|_{L^2(\gamma_i)}} \ge \beta_i^\delta \|\mu_i^\delta\|_{L^2(\gamma_i)}.
\end{aligned}$$

Let $U_i^\delta \subset W_{0,i}^\delta$ be defined by the orthogonal decomposition $W_{0,i}^\delta = \Im \breve{Q}_i^\delta \oplus^{\perp_{L^2(\gamma_i)}} U_i^\delta$, where $\Im \breve{Q}$ denotes the range of $\breve{Q}$. For any $\mu_i^\delta \in M_i^\delta$, $u_i^\delta \in U_i^\delta$, we have

$$\langle \mu_i^\delta + u_i^\delta, \breve{Q}_i^\delta \mu_i^\delta + u_i^\delta \rangle_{L^2(\gamma_i)} = \langle \breve{Q}_i^\delta \mu_i^\delta + u_i^\delta, \breve{Q}_i^\delta \mu_i^\delta + u_i^\delta \rangle_{L^2(\gamma_i)}, \tag{2.2}$$

and so from

$$\begin{aligned} \|\breve{Q}_i^\delta \mu_i^\delta + u_i^\delta\|_{L^2(\gamma_i)} &= \sqrt{\|\breve{Q}_i^\delta \mu_i^\delta\|_{L^2(\gamma_i)}^2 + \|u_i^\delta\|_{L^2(\gamma_i)}^2} \\ &\geq \beta_i^\delta \sqrt{\|\mu_i^\delta\|_{L^2(\gamma_i)}^2 + \|u_i^\delta\|_{L^2(\gamma_i)}^2} \\ &\geq \beta_i^\delta \tfrac{1}{2}\sqrt{2}\, \|\mu_i^\delta + u_i^\delta\|_{L^2(\gamma_i)}, \end{aligned}$$

we infer that [“H”, “W”, “M”] $= [L^2(\gamma_i), W_{0,i}, \tilde{M}_i^\delta := M_i^\delta + U_i^\delta]$ satisfies (2.1) with “β” $\geq \beta_i^\delta \frac{1}{2}\sqrt{2}$. Since (2.2) also shows that for any $w_i^\delta \in W_{0,i}^\delta \backslash \{0\}$ there exists a $\tilde{\mu}_i^\delta \in \tilde{M}_i^\delta$ with $\langle \tilde{\mu}_i^\delta, w_i^\delta \rangle_{L^2(\gamma_i)} \neq 0$, we conclude that $\dim W_{0,i}^\delta = \dim \tilde{M}_i^\delta$, and so the proof follows from an application of Lemma 2.1(a).

We assume that for $1 \leq i \leq N$, $\delta \in \mathcal{I}$ there exist $0 < \underline{h}_i^\delta \in L^\infty(\gamma_i)$ such that

$$\|w_i^\delta\|_{H^1(\gamma_i)} \lesssim \|(\underline{h}_i^\delta)^{-1} w_i^\delta\|_{L^2(\gamma_i)}, \quad w_i^\delta \in W_{0,i}^\delta, \tag{A.3}$$

and

$$\inf_{w_i^\delta \in W_{0,i}^\delta} \|(\underline{h}_i^\delta)^{-1}(w - w_i^\delta)\|_{L^2(\gamma_i)} + \|w_i^\delta\|_{H^1(\gamma_i)} \lesssim \|w\|_{H^1(\gamma_i)}, \; w \in H_0^1(\gamma_i). \tag{A.4}$$

If the finite element partition underlying $W_{0,i}^\delta$ is *shape regular* and *quasi-uniform*, both uniformly in δ, then with $\underline{h}_i^\delta$ being the constant function defined as the maximum of the diameters of the elements, (A.3) is the well-known inverse inequality, and (A.4) is guaranteed by the minimal requirement that $W_{0,i}^\delta$ contains all continuous functions that vanish at $\partial\gamma_i$ and that are piecewise linear with respect to the partition. Yet, for partitions that, instead of quasi-uniform, are only *locally quasi-uniform*, meaning that any two elements that have non-empty intersection have uniformly comparable diameters, (A.3) and (A.4) are also valid, now with $\underline{h}_i^\delta$ being defined as the piecewise constant function equal to the diameter of the underlying element. See, e.g., [11,24].

As expected, we will need that the β_i^δ, defined in (1.4), satisfy

$$\beta_i^\delta \gtrsim 1, \tag{A.5}$$

i.e., uniformly in $\delta \in \mathcal{I}$ and i. Then from Lemma 2.1 and Proposition 2.2 we know that there exist mortar projectors Π_i^δ for which $\|\Pi_i^\delta\|_{L^2(\gamma_i)\to L^2(\gamma_i)} \lesssim 1$. For the case that not all $\underline{h}_i^\delta$ are constant functions, we make the additional assumption that these projectors can be selected so that

$$\|(\underline{h}_i^\delta)^{-1}\Pi_i^\delta w\|_{L^2(\gamma_i)} \lesssim \|(\underline{h}_i^\delta)^{-1} w\|_{L^2(\gamma_i)}, \quad w \in L^2(\gamma_i). \tag{A.6}$$

Again assuming that the partition underlying $W_{i,0}^\delta$ is shape regular and locally quasi-uniform, (A.6) is valid when the Π_i^δ are *local* in the sense that $\Pi_i^\delta v$ vanishes on an element T whenever v vanishes on all elements that have distance to T less than some absolute multiple of $\mathrm{diam}(T)$.

Remark 2.3 Assume a basis $\{w_{i,1}^\delta, \ldots, w_{i,L}^\delta\}$ for $W_{i,0}^\delta$, and a collection $\{\tilde{m}_{i,1}^\delta, \ldots, \tilde{m}_{i,L}^\delta\}$ containing M_i^δ in its span, both which are *uniformly $L^2(\gamma_i)$-stable*, with which we mean that $\|\sum_{\ell=1}^L c_\ell w_{i,\ell}\|_{L^2(\gamma_i)}^2 \eqsim \sum_{\ell=1}^L |c_\ell|^2 \|w_{i,\ell}\|_{L^2(\gamma_i)}^2$, and analogously for the other collection. Suppose further that both collections consist of *local* functions, with which we mean that the support of any of these functions consists of a connected union of a (uniformly) bounded number of underlying elements. Finally, suppose that $\mathbf{R}_i^\delta := \left[\frac{\langle w_{i,\ell}^\delta, \tilde{m}_{i,\ell'}^\delta\rangle_{L^2(\gamma_i)}}{\|w_{i,\ell}^\delta\|_{L^2(\gamma_i)}\|\tilde{m}_{i,\ell'}^\delta\|_{L^2(\gamma_i)}}\right]_{1\le \ell,\ell'\le L}$ is (uniformly) bounded invertible, and that its inverse is local in the sense that $((\mathbf{R}_i^\delta)^{-1})_{\ell,\ell'} = 0$ whenever $\mathrm{dist}(\mathrm{supp}\, w_{i,\ell}^\delta, \mathrm{supp}\, \tilde{m}_{i,\ell'}^\delta)$ is larger than some absolute multiple of the maximum of $\mathrm{diam}(\mathrm{supp}\, w_{i,\ell}^\delta)$ and $\mathrm{diam}(\mathrm{supp}\, \tilde{m}_{i,\ell'}^\delta)$. Then from Lemma 2.1(b) we know that such local, and uniformly $L^2(\gamma_i)$-bounded mortar projectors exist.

Lemma 2.4 *$\Pi_i^\delta : L^2(\gamma_i) \to L^2(\gamma_i)$ and $\Pi_i^\delta : H_0^1(\gamma_i) \to H_0^1(\gamma_i)$ are bounded, uniformly in δ and i.*

Proof We already have seen that the first statement is a consequence of (A.5). Since Π_i^δ is a projector onto $W_{0,i}^\delta$, by (A.3), (A.6) and (A.4) for $w \in H_0^1(\gamma_i)$ we have

$$\begin{aligned}
\|\Pi_i^\delta w\|_{H^1(\gamma_i)} &\lesssim \inf_{w_i^\delta \in W_{0,i}^\delta} \|\Pi_i^\delta(w - w_i^\delta)\|_{H^1(\gamma_i)} + \|w_i^\delta\|_{H^1(\gamma_i)} \\
&\lesssim \inf_{w_i^\delta \in W_{0,i}^\delta} \|(\underline{h}_i^\delta)^{-1}\Pi_i^\delta(w - w_i^\delta)\|_{L^2(\gamma_i)} + \|w_i^\delta\|_{H^1(\gamma_i)} \\
&\lesssim \inf_{w_i^\delta \in W_{0,i}^\delta} \|(\underline{h}_i^\delta)^{-1}(w - w_i^\delta)\|_{L^2(\gamma_i)} + \|w_i^\delta\|_{H^1(\gamma_i)} \\
&\lesssim \|w\|_{H^1(\gamma_i)},
\end{aligned}$$

which shows the second statement.

Finally, for completeness, we mention the following technical assumption:

$$\begin{cases}\text{There exist } J_k^\delta : H^1(\Omega_k) \to X_k^\delta \text{ with } \|J_k^\delta\|_{H^1(\Omega_k)\to H^1(\Omega_k)} \lesssim 1,\\ \text{such that whenever } v|_{\partial\Omega_k} = v_k^\delta|_{\partial\Omega_k} \text{ for some } v_k^\delta \in X_k^\delta,\\ \text{then } (J_k^\delta v)|_{\partial\Omega_k} = v_k^\delta|_{\partial\Omega_k}.\end{cases} \tag{A.7}$$

For any given X_k^δ, the existence of such J_k^δ can be demonstrated as in [24].

Under the assumptions we made, and using Lemma 2.4, the error estimates proven in [20] easily generalize to the present more general setting. See [3,18,12] for preceding or related work. We will need that $a\frac{\partial u}{\partial \mathbf{n}_{\gamma_i}} \in (H^{\frac{1}{2}}(\gamma_i))'$. A sufficient condition is the mild regularity assumption $u \in H^{1+\varepsilon}(\Omega)$ for some arbitrary $\varepsilon > 0$, since then the trace theorem shows that $a\frac{\partial u}{\partial \mathbf{n}_{\gamma_i}} \in H^{-\frac{1}{2}+\varepsilon}(\gamma_i) \subset (H^{\frac{1}{2}}(\gamma_i))'$.

Theorem 2.5 *With u and $(u^\delta, \lambda^\delta)$ being the solutions of* (1.1) *and* (1.2)*, respectively, assume that $a\frac{\partial u}{\partial \mathbf{n}_{\gamma_i}} \in (H^{\frac{1}{2}}(\gamma_i))'$, $1 \le i \le N$. Then*

$$\sum_{k=1}^K \|u-u^\delta\|^2_{H^1(\Omega_k)} \lesssim \tag{2.3}$$
$$\inf_{v^\delta\in V^\delta} \sum_{k=1}^K \|u - v^\delta\|^2_{H^1(\Omega_k)} + \inf_{\mu^\delta\in M^\delta} \sum_{i=1}^N \|a\frac{\partial u}{\partial \mathbf{n}_{\gamma_i}} - \mu^\delta\|^2_{(H^{\frac{1}{2}}(\gamma_i))'},$$

where

$$\inf_{v^\delta\in V^\delta} \sum_{k=1}^K \|u - v^\delta\|^2_{H^1(\Omega_k)} \lesssim \inf_{x^\delta\in X^\delta} \Big\{ \sum_{k=1}^K \|u - x^\delta\|^2_{H^1(\Omega_k)} + \tag{2.4}$$
$$\sum_{i=1}^N \|(\underline{h}_i^\delta)^{-\frac{1}{2}}(u - x^\delta)|_{\Omega_{m(i)}}\|^2_{L^2(\gamma_i)} + \|(\underline{h}_i^\delta)^{-\frac{1}{2}}(u - x^\delta)|_{\Omega_{s(i)}}\|^2_{L^2(\gamma_i)} \Big\}.$$

With $\mathbf{n}_{\gamma_i}$ being the outward normal on γ_i from the master side, for the Lagrange multiplier we have

$$\sum_{i=1}^N \|a\frac{\partial u}{\partial \mathbf{n}_{\gamma_i}} - \lambda^\delta\|^2_{(H^{\frac{1}{2}}_{00}(\gamma_i))'} \tag{2.5}$$
$$\lesssim \sum_{k=1}^K \|u - u^\delta\|^2_{H^1(\Omega_k)} + \inf_{\mu^\delta\in M^\delta} \sum_{i=1}^N \|a\frac{\partial u}{\partial \mathbf{n}_{\gamma_i}} - \mu^\delta\|^2_{(H^{\frac{1}{2}}_{00}(\gamma_i))'}.$$

Proof Using (A.2), (2.3) follows from Strang's second lemma as in [20, proof of Theorem 2.3]. Because of (A.7), any $w_i^\delta \in W_{0,i}^\delta$ has an extension to a $x_{s(i)}^\delta \in X_{s(i)}^\delta$ with $\|x_{s(i)}^\delta\|_{H^1(\Omega_{s(i)})} \lesssim \|w_i^\delta\|_{H_{00}^{\frac{1}{2}}(\gamma_i)}$, see, e.g., [25, Lemma 5.1]. Now, following [20, proof Lemma 2.2], one derives that

$$\inf_{v^\delta \in V^\delta} \sum_{k=1}^{K} \|u - v^\delta\|_{H^1(\Omega_k)}^2 \lesssim \tag{2.6}$$
$$\inf_{x^\delta \in X^\delta} \sum_{k=1}^{K} \|u - x^\delta\|_{H^1(\Omega_k)}^2 + \sum_{i=1}^{N} \|\Pi_i^\delta [u - x^\delta]_{\gamma_i}\|_{H_{00}^{\frac{1}{2}}(\gamma_i)}^2 .$$

From (A.3), and (A.6), $\|\Pi_i^\delta\|_{L^2(\gamma_i)\to L^2(\gamma_i)} \lesssim 1$, using interpolation (cf. [26, ex. in §1.15.3]) we infer that $\|w_i^\delta\|_{H_{00}^{\frac{1}{2}}(\gamma_i)} \lesssim \|(\underline{h}_i^\delta)^{-\frac{1}{2}} w_i^\delta\|_{L^2(\gamma_i)}$ for any $w_i^\delta \in W_{0,i}^\delta$, and $\|(\underline{h}_i^\delta)^{-1}\Pi_i^\delta w\|_{L^2(\gamma_i)} \lesssim \|(\underline{h}_i^\delta)^{-1} w\|_{L^2(\gamma_i)}$ for any $w \in L^2(\gamma_i)$, respectively, so that (2.4) is valid. The bound (2.5) for the Lagrange multiplier follows from [3, Lemma 2.7], where the condition

$$\inf_{\mu^\delta \in M^\delta \setminus \{0\}} \sup_{\tilde{w} \in \tilde{X}^\delta \setminus \{0\}} \frac{b(\tilde{w}, \mu^\delta)^2 / \sum_{1\le i\le N} \|\mu^\delta\|_{(H_{00}^{\frac{1}{2}}(\gamma_i))'}^2}{\sum_{1\le k\le K} \|\tilde{w}^\delta\|_{H^1(\Omega_k)}^2 + \sum_{1\le i\le N} \|[\tilde{w}^\delta]_{\gamma_i}\|_{H_{00}^{\frac{1}{2}}(\gamma_i)}^2} \gtrsim 1$$

can be shown as in [20, Lemma 2.4].

Corollary 2.6 *For* $1 \le k \le N$ *and* $\delta \in \mathcal{I}$*, let* $1 < t_k \notin \mathbb{N} + \frac{1}{2}$ *and* $0 < h_k^\delta \in L^\infty(\Omega_k)$ *be such that* $\underline{h}_i^\delta := h_{s(i)}^\delta|_{\gamma_i} \in L^\infty(\gamma_i)$ *and* (A.3), (A.4) *and* (A.6) *are valid, and*

$$\inf_{x_k^\delta \in X_k^\delta} \|x_k - x_k^\delta\|_{H^1(\Omega_k)} + \|(h_k^\delta)^{-\frac{1}{2}}(x_k - x_k^\delta)|_{\partial\Omega_k}\|_{L^2(\partial\Omega_k)} \tag{2.7}$$
$$\lesssim \|h_k^\delta\|_{L^\infty(\Omega_k)}^{t_k - 1} \|x_k\|_{H^{t_k}(\Omega_k)}, \quad x_k \in H_*^1(\Omega_k) \cap H^{t_k}(\Omega_k),$$

$$\inf_{\mu_i^\delta \in M_i^\delta} \|\mu_i - \mu_i^\delta\|_{L^2(\gamma_i)} \lesssim \|\underline{h}_i^\delta\|_{L^\infty(\gamma_i)}^{t_{s(i)}-1} \|\mu_i\|_{H^{t_{s(i)}-1}(\gamma_i)}, \quad \mu_i \in H^{t_{s(i)}-1}(\gamma_i). \tag{2.8}$$

Assume that the solution u of (1.1) *is in* $\prod_{k=1}^K H^{t_k}(\Omega_k)$. *Then the solution* $(u^\delta, \lambda^\delta)$ *of* (1.2) *satisfies*

$$\sum_{k=1}^K \|u-u^\delta\|^2_{H^1(\Omega_k)} + \sum_{i=1}^N \|a\frac{\partial u}{\partial \mathbf{n}_{\gamma_i}} - \lambda^\delta\|^2_{(H_{00}^{\frac{1}{2}}(\gamma_i))'} \lesssim \tag{2.9}$$

$$\Big[1+\max_{1\le i\le N}\|h^\delta_{m(i)}/h^\delta_{s(i)}\|_{L^\infty(\gamma_i)}\Big]\sum_{k=1}^K \|h_k^\delta\|^{2t_k-2}_{L^\infty(\Omega_k)}\|u\|^2_{H^{t_k}(\Omega_k)}.$$

Proof The proof follows easily from the estimates (2.3) and (2.4), $\|\cdot\|_{(H_{00}^{\frac{1}{2}}(\gamma_i))'} \le \|\cdot\|_{(H^{\frac{1}{2}}(\gamma_i))'}$, an application of the trace theorem, and $\inf_{\mu_i^\delta\in M_i^\delta}\|\mu_i-\mu_i^\delta\|_{(H^{\frac{1}{2}}(\gamma_i))'} \lesssim \|h^\delta_{s(i)}\|^{t_{s(i)}-1}_{L^\infty(\Omega_{s(i)})}\|\mu_i\|_{H^{t_{s(i)}-\frac{3}{2}}(\gamma_i)}$ for any $\mu_i \in H^{t_{s(i)}-\frac{3}{2}}(\gamma_i)$, which is a consequence of (2.8).

Assuming X_k^δ contains all continuous, piecewise polynomials of degree $\lceil t_k-1\rceil$ that vanish on $\partial\Omega_k\cap\partial\Omega$ with respect to a shape regular and locally quasi-uniform finite element partition, with h_k^δ being the piecewise constant function equal to the diameter of the underlying element (2.7) is easily verified using a nodal, or, when $\lfloor t_k\rfloor \le \frac{d}{2}$, a modified Clément interpolator as introduced in [24]. Similarly, (2.8) is satisfied when M_i^δ contains all piecewise, or continuous piecewise polynomials of degree $\lceil t_{s(i)}-2\rceil$ with respect to the underlying partition inherited from the slave side.

Corollary 2.6 requires that $u \in \prod_{k=1}^K H^{t_k}(\Omega_k)$, which, in case $\min_k t_k > 2$, even for a convex polygon Ω generally can only be guaranteed when the right-hand side f vanishes at the corners of Ω. For less smooth u, using suitably locally refined partitions better results may be expected than those reflected by Corollary 2.6, that shows no benefits of local refinements. From the theory of nonlinear approximation, see, e.g. [14], in combination with the bounds from Theorem 2.5, we infer that we may hope for the same asymptotic behaviour of the error as function of the number of degrees of freedom under the relaxed assumption that u is the product space $\prod_{k=1}^K W_p^{t_k}(\Omega_k)$, for certain $p<2$, or, more precisely, that it is in such a product space with the Sobolev spaces $W_p^{t_k}(\Omega_k)$ replaced by certain Besov spaces $B^{t_k}_{p,q}(\Omega_k)$. A treatment of this topic is beyond the scope of this paper.

If the finite element partitions from slave and master side match on the interface boundaries $\partial\gamma_i$ (and there underly the same finite element space), and $t_k = t$ for all $1\le k\le K$, then starting from (2.6), and by using Lemma 2.4, (2.9) can be proven without the penalty factor $1+\max_{1\le i\le N}\|h^\delta_{m(i)}/h^\delta_{s(i)}\|_{L^\infty(\gamma_i)}$. This has been shown in [18] for $t=2$ and $d\le 3$ by selecting x^δ as the nodal interpolant of u, but it extends to arbitrary d and $t>1$ by taking x^δ as a quasi-interpolant of u that on $\partial\gamma_i$ only depends on $u|_{\partial\gamma_i}\in L^1(\partial\gamma_i)$ (cf. [24]), and by using that for each fixed $\varepsilon>0$, $H^{\frac{1}{2}}_{00}(\gamma_i)$ and the interpolation space $[H_0^{\frac{1}{2}+\varepsilon}(\gamma_i),H^{\frac{1}{2}-\varepsilon}(\gamma_i)]_{1/2}$ agree as sets and have equivalent norms. This *"matching condition"* on $\partial\gamma_i$ is readily met in the two-dimensional case, where the interface boundaries consist of isolated points, but it is an inconvenient restriction in higher dimensions.

In the remainder of this section, assuming *quasi-uniform meshes* on each subdomain without imposing the matching condition, we will demonstrate that the penalty factor $1+\max_{1\le i\le N}\|h^\delta_{m(i)}/h^\delta_{s(i)}\|_{L^\infty(\gamma_i)}$ is much too pessimistic. In Theorem 2.8, we will see that only the log of the quotient of the mesh sizes at master and slave sides, assuming that this quotient is larger than 1, may enter the upper bound of the error. Although, when having quasi-uniform meshes on each subdomain, for each interface the slave and master sides can be selected such that the mesh size at the master side is less than or equal to that at the slave side, it shows that a different choice of master and slave sides cannot do much harm. Moreover, it suggests that also for locally refined meshes, where uniform boundedness of $\|h^\delta_{m(i)}/h^\delta_{s(i)}\|_{L^\infty(\gamma_i)}$ cannot be guaranteed, the penalty factor from Corollary 2.6 is likely too pessimistic, in any case when locally the mesh sizes along the interfaces do not vary too strongly. A generalization of our result to locally refined meshes might be possible by using sufficiently smooth mesh indicator functions h^δ_k as used in [12]. However, the estimation of the unavoidable fractional Sobolev norms of functions that contain as factors powers of the non-constant functions h^δ_k will require quite some technicalities.

It is well-known that the interpolation space $[H^1_0(\gamma_i),L^2(\gamma_i)]_\theta$ is equal to $H_0^{1-\theta}(\gamma_i)$ for $\theta\in[0,\frac{1}{2})$, and that it is equal to $H^{1-\theta}(\gamma_i)$ for $\theta\in(\frac{1}{2},1]$, meaning that corresponding spaces are equal as sets and have equivalent norms. The space $[H^1_0(\gamma_i),L^2(\gamma_i)]_{1/2}$ is denoted as $H^{1/2}_{00}(\gamma_i)$, and it is strictly contained $H^{1/2}_0(\gamma_i)$ with a strictly finer topology. In the following we will consider these interpolation spaces for $\theta\downarrow\frac{1}{2}$. We cannot expect that the aforementioned norm equivalences hold uniformly in $\theta\in(\frac{1}{2},1]$ which motivates our following lemma.

Lemma 2.7 *For $1 \leq i \leq N$ and $\theta \in [0,1]\backslash\{\frac{1}{2}\}$, we have*

$$\sup_{v\in[H_0^1(\gamma_i),L^2(\gamma_i)]_\theta\backslash\{0\}} \frac{\|v\|_{[H_0^1(\gamma_i),L^2(\gamma_i)]_\theta}}{\|v\|_{H^{1-\theta}(\gamma_i)}} \lesssim \left|\tfrac{1}{2}-\theta\right|^{-1}.$$

Proof We may think of γ_i as being a polytope in $\mathbb{R}^{d-1}$. For any function v on γ_i, let $\tilde{v}$ denotes its extension on $\mathbb{R}^{d-1}$ with zero. As follows from [21, Lemma 11.3], we have $\|v\|_{[H_0^1(\gamma_i),L^2(\gamma_i)]_\theta} \lesssim \|\tilde{v}\|_{H^{1-\theta}(\mathbb{R}^{d-1})}$ uniformly in $u \in [H_0^1(\gamma_i),L^2(\gamma_i)]_\theta$ and $\theta \in [0,1]$. From [16, Lemma 1.3.2.6 and Theorem 1.4.4.4] it follows that for any $\theta \in [0,1]\backslash\{\frac{1}{2}\}$, there exists a constant $C_\theta > 0$ such that $\|\tilde{v}\|_{H^{1-\theta}(\mathbb{R}^{d-1})} \leq C_\theta \|v\|_{H^{1-\theta}(\gamma_i)}$ for all $v \in [H_0^1(\gamma_i),L^2(\gamma_i)]_\theta$. Inspection of the proof of this Theorem 1.4.4.4 reveals that C_θ is a result of the application of Hardy's inequality, and that it can be bounded by some absolute multiple of $\left|\frac{1}{2}-\theta\right|^{-1}$.

Theorem 2.8 *For $1 \leq k \leq N$ and $\delta \in \mathcal{I}$, let $1 < t_k \notin \mathbb{N}+\frac{1}{2}$ and $h_k > 0$ be a* constant *such that, with $\underline{h}_i^\delta := h_{s(i)}^\delta|_{\gamma_i}$, (A.3), (A.4) and (A.6) are valid, and*

$$\begin{aligned}\inf_{x_k^\delta\in X_k^\delta} &\|x_k - x_k^\delta\|_{H^1(\Omega_k)} \qquad (2.10)\\ &\lesssim (h_k^\delta)^{t_k-1}\|x_k\|_{H^{t_k}(\Omega_k)}, \quad x_k \in H_*^1(\Omega_k)\cap H^{t_k}(\Omega_k),\end{aligned}$$

$$\inf_{\mu_i^\delta\in M_i^\delta} \|\mu_i - \mu_i^\delta\|_{L^2(\gamma_i)} \lesssim (\underline{h}_i^\delta)^{t_{s(i)}-1}\|\mu_i\|_{H^{t_{s(i)}-1}(\gamma_i)}, \quad \mu_i \in H^{t_{s(i)}-1}(\gamma_i).$$

Assume that the solution u of (1.1) is in $\prod_{k=1}^K H^{t_k}(\Omega_k)$. Then the solution $(u^\delta,\lambda^\delta)$ of (1.2) satisfies

$$\begin{aligned}\sum_{k=1}^K &\|u-u^\delta\|^2_{H^1(\Omega_k)} + \sum_{i=1}^N \|a\frac{\partial u}{\partial \mathbf{n}_{\gamma_i}} - \lambda^\delta\|^2_{(H_{00}^{\frac{1}{2}}(\gamma_i))'}\\ &\lesssim \left[1+\log\left\lceil\max_{1\leq i\leq N} h_{m(i)}^\delta/h_{s(i)}^\delta\right\rceil\right]^2 \sum_{k=1}^K (h_k^\delta)^{2t_k-2}\|u\|^2_{H^{t_k}(\Omega_k)}.\end{aligned}$$

Proof Let us consider (2.6). We have $\|w_i^\delta\|_{H_{00}^{\frac{1}{2}}(\gamma_i)} \lesssim (\underline{h}_i^\delta)^{-\frac{1}{2}}\|w_i^\delta\|_{L^2(\gamma_i)}$ for any $w_i^\delta \in W_{0,i}^\delta$. So by interpolation and the reiteration theorem, we have $\|w_i^\delta\|_{H_{00}^{\frac{1}{2}}(\gamma_i)} \lesssim (\underline{h}_i^\delta)^{-\varepsilon}\|w_i^\delta\|_{[H_0^1(\gamma_i),L^2(\gamma_i)]_{\frac{1}{2}+\varepsilon}}$ for any $w_i^\delta \in W_{0,i}^\delta$ and $\varepsilon \in [0,\frac{1}{2}]$. Furthermore, by Lemma 2.4 and interpolation, we have that $\Pi_i^\delta : [H_0^1(\gamma_i), L^2(\gamma_i)]_\theta \to [H_0^1(\gamma_i), L^2(\gamma_i)]_\theta$ is bounded uniformly in $\delta \in \mathcal{I}$ and $\theta \in [0,1]$. So by Lemma 2.7, we find that

$$\begin{aligned}
&\|\Pi_i^\delta [u - x^\delta]_{\gamma_i}\|_{H_{00}^{\frac{1}{2}}(\gamma_i)} \\
&\quad \lesssim (h_i^\delta)^{-\varepsilon} \varepsilon^{-1} \big\{ \|(u - x^\delta)|_{\Omega_{m(i)}}\|_{H^{\frac{1}{2}-\varepsilon}(\gamma_i)} + \|(u - x^\delta)|_{\Omega_{s(i)}}\|_{H^{\frac{1}{2}-\varepsilon}(\gamma_i)} \big\} \\
&\quad \lesssim (h_i^\delta)^{-\varepsilon} \varepsilon^{-1} \big\{ \|u - x^\delta\|_{H^{1-\varepsilon}(\Omega_{m(i)})} + \|u - x^\delta\|_{H^{1-\varepsilon}(\Omega_{s(i)})} \big\},
\end{aligned}$$

where, for the last inequality, we have used that, for any $\eta \in [0, \frac{1}{2})$, the trace operator is bounded from $H^{1-\varepsilon}(\Omega_k) \to H^{\frac{1}{2}-\varepsilon}(\partial\Omega_k)$ uniformly in $\varepsilon \in [0, \eta]$. For $\varepsilon \leq \min_k t_k - 1$, a duality argument using $H^1(\Omega_k)$ as pivot space shows that $\inf_{x_k^\delta \in X_k^\delta} \|u - x_k^\delta\|_{H^{1-\varepsilon}(\Omega_k)} \lesssim (h_k^\delta)^{t_k - 1 + \varepsilon} \|u\|_{H^{t_k}(\Omega_k)}$. With $\zeta := \lceil \max_{1 \leq i \leq N} h_{m(i)}^\delta / h_{s(i)}^\delta \rceil$, from (2.3), (2.5), (2.6), and, to bound the error in the Lagrange multiplier, the trace theorem, we conclude the statement of the theorem with the factor $[1 + \log(\zeta)]^2$ replaced by $\varepsilon^{-2}[1 + \zeta^\varepsilon]^2$. By, when ζ is sufficiently large, taking $\varepsilon = \log(2)/\log(\zeta)$, the proof is completed.

3 Generalized dual Lagrange multipliers

3.1 General framework

In this subsection, after recalling the definition and the advantages of having dual Lagrange multiplier bases, we sketch a general procedure for their construction. Hereafter, in the following subsections, we will give examples of spaces $X^\delta = \prod_{k=1}^K X_k^\delta$ and $M^\delta = \prod_{i=1}^N M_i^\delta$ that satisfy all conditions we have imposed, and that can be equipped with such bases.

Let X_k^δ and M_i^δ be equipped with local, uniformly $L^2(\Omega_k)$- or $L^2(\gamma_i)$-stable bases $\{x_{k,j}^\delta : j \in D_k^\delta\}$ and $\{m_{i,\ell}^\delta : \ell \in I_i^\delta\}$, respectively. With

$$S_{0,i}^\delta := \{ j \in D_{s(i)}^\delta : x_{s(i),j}^\delta|_{\partial\Omega_{s(i)} \setminus \gamma_i} = 0,\ x_{s(i),j}^\delta|_{\gamma_i} \not\equiv 0 \},$$

we have $W_{0,i}^\delta = \operatorname{span}\{x_{s(i),j}^\delta|_{\gamma_i} : j \in S_{0,i}^\delta\}$.

We will select the bases for X_k^δ and M_i^δ in such a way that

$$I_i^\delta \subset S_{0,i}^\delta \quad \text{and} \tag{3.1}$$

$$\langle x_{s(i),\ell}^\delta|_{\gamma_i}, m_{i,\ell'}^\delta \rangle_{L^2(\gamma_i)} \begin{cases} \eqsim \|x_{s(i),\ell}^\delta|_{\gamma_i}\|_{L^2(\gamma_i)} \|m_{i,\ell'}^\delta\|_{L^2(\gamma_i)} & \text{if } \ell = \ell' \in I_i^\delta, \\ = 0 & \text{if } \ell \neq \ell' \in I_i^\delta, \end{cases}$$

i.e., suitably scaled, $\{m^\delta_{i,\ell} : \ell \in I^\delta_i\}$ and $\{x^\delta_{s(i),\ell}|_{\gamma_i} : \ell \in I^\delta_i\}$ are biorthogonal bases, which are also called dual to each other, explaining why we speak about dual Lagrange multipliers. Dual Lagrange multipliers were introduced earlier, with $I^\delta_i = S^\delta_{0,i}$ and so $\dim M^\delta_i = \dim W^\delta_{0,i}$, see [28,23,18,29]. Here we generalize the concept of dual Lagrange multipliers by allowing that $\dim M^\delta_i < \dim W^\delta_{0,i}$. Under the natural assumption that $\{x^\delta_{s(i),\ell}|_{\gamma_i} : \ell \in I^\delta_i\}$ is a uniformly $L^2(\gamma_i)$-stable system, note that (3.1) *implies that* $\beta^\delta_i \gtrsim 1$. Indeed, Lemma 2.1(b) shows then that this uniform "inf-sup condition" already holds with $W^\delta_{0,i}$ replaced by $\mathrm{span}\{x^\delta_{s(i),\ell}|_{\gamma_i} : \ell \in I^\delta_i\}$.

What is more, defining for $1 \le k \le K$, the set of "free" indices in $\bar{\Omega}_k$ by

$$\breve{D}^\delta_k = \{j \in D^\delta_k : j \notin I^\delta_p \text{ for any } 1 \le p \le N \text{ with } s(p) = k\},$$

the property (3.1) of having dual Lagrange multipliers implies that

$$\bigcup_{k=1}^{K} \left\{ x^\delta_{k,j} - \sum_{i=1}^{N} \sum_{\ell \in I^\delta_i} \frac{b(x^\delta_{k,j}, m^\delta_{i,\ell})}{b(x^\delta_{s(i),\ell}, m^\delta_{i,\ell})} x^\delta_{s(i),\ell} : j \in \breve{D}^\delta_k \right\}$$

is a basis for V^δ. The availability of such bases opens the way to apply efficient multi-level solvers to the elliptic system (1.3). It is generally observed that such schemes are more efficient than iterative solvers for the saddle point formulation (1.2).

Because of the incorporation of homogeneous Dirichlet boundary conditions at $\partial\gamma_i$ in $W^\delta_{0,i}$, but not in M^δ_i, the *construction* of dual Lagrange multipliers $\{m^\delta_{i,\ell} : \ell \in I^\delta_i\}$ requires a special treatment near $\partial\gamma_i$. In our examples, initially we ignore the interface boundary problems, and construct $\{x^\delta_{k,j} : j \in D^\delta_k\}$ and local, uniformly $L^2(\gamma_i)$-stable systems $\{\bar{m}^\delta_{i,\ell} : \ell \in \bar{I}^\delta_i\}$, generally with $\bar{I}^\delta_i \supsetneq I^\delta_i := \bar{I}^\delta_i \cap S^\delta_{0,i}$, for which

$$\langle x^\delta_{s(i),\ell}|_{\gamma_i}, \bar{m}^\delta_{i,\ell'} \rangle_{L^2(\gamma_i)} \begin{cases} \eqsim \|x^\delta_{s(i),\ell}|_{\gamma_i}\|_{L^2(\gamma_i)} \|\bar{m}^\delta_{i,\ell'}\|_{L^2(\gamma_i)} & \text{if } I^\delta_i \ni \ell = \ell', \\ = 0 & \text{if } I^\delta_i \ni \ell \neq \ell' \in \bar{I}^\delta_i, \end{cases} \tag{3.2}$$

cf. Figure 3.1. Typically, $\{x^\delta_{k,j} : j \in D^\delta_k\}$ and $\{\bar{m}^\delta_{i,\ell} : \ell \in \bar{I}^\delta_i\}$ will be assembled from local functions defined on the individual elements which in turn are push forwards of one pair of suitable sets of δ-independent functions defined on a reference element. For some $m \ge 2$, it will hold that X^δ_k contains all continuous piecewise polynomials of degree $m-1$ that vanish on $\partial\Omega_k \cap \partial\Omega$ with respect to a *shape regular*, and *locally quasi-uniform* partition. As a consequence, $W^\delta_{0,i}$ will contain in any

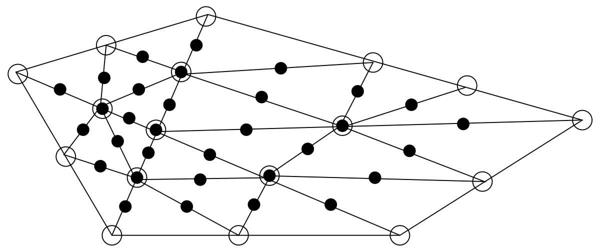

Fig. 3.1: Illustration of the sets $\bar{I}_i^\delta = \{\bigcirc\}$ and $S_{0,i}^\delta = \{\bullet\}$ on the interface γ_i for the example that will be discussed in §3.2.1.

case all continuous functions that vanish at $\partial\gamma_i$ and that are piecewise linear with respect to its underlying partition, meaning that (A.4) is satisfied. At the multiplier side, it will be ensured that there exists a collection of functionals $\{\psi_{i,\ell}^\delta : \ell \in \bar{I}_i^\delta\} \subset C(\gamma_i)'$, such that

$$v \mapsto \sum_{\ell \in \bar{I}_i^\delta} \psi_{i,\ell}^\delta(v) \bar{m}_{i,\ell}^\delta \quad \text{is a } \textit{quasi-interpolator} \text{ of order } m-2 \tag{3.3}$$

meaning that it preserves $P_{m-2}(\gamma_i)$, where, in addition, the $\psi_{i,\ell}^\delta$ are local with respect to the partition underlying $W_{0,i}^\delta$, $\psi_{i,\ell}^\delta \in L^\infty(\gamma_i)$, with $\psi_{i,\ell}^\delta(v) := \langle \psi_{i,\ell}^\delta, v \rangle_{L^2(\gamma_i)}$, and $\|\psi_{i,\ell}^\delta\|_{L^2(\gamma_i)} \eqsim \|\bar{m}_{i,\ell}^\delta\|_{L^2(\gamma_i)}^{-1}$. As a consequence, such a quasi-interpolator defines a uniformly $L^2(\gamma_i)$-bounded mapping onto $\text{span}\{\bar{m}_{i,\ell}^\delta : \ell \in \bar{I}_i^\delta\}$.

In view of (3.1), the problem to be solved is that generally $\bar{I}_i^\delta \supsetneq I_i^\delta$. Obviously, we cannot simply remove those $\bar{m}_{i,n}^\delta$ from $\{\bar{m}_{i,\ell}^\delta : \ell \in \bar{I}_i^\delta\}$ with $n \in \bar{I}_i^\delta \backslash I_i^\delta$, since generally the span of the remaining set will not have appropriate approximation properties. Therefore, together with the removal of such an $\bar{m}_{i,n}^\delta$, we will add suitable multiples of it to some of the remaining $\bar{m}_{i,\ell}^\delta$. For some fixed $q \geq \dim P_{m-2}(\gamma_i)$, for each $n \in \bar{I}_i^\delta \backslash I_i^\delta$, we select $\ell_1^n, \ldots, \ell_q^n \in I_i^\delta$ with $\text{dist}(\text{supp}\, \bar{m}_{i,\ell_r^n}^\delta, \text{supp}\, \bar{m}_{i,n}^\delta) \lesssim \text{diam}(\text{supp}\, \bar{m}_{i,n}^\delta)$, such that the problem

$$\underset{(\sigma_r^n)_{1 \leq r \leq q}}{\text{argmin}} \left\{ \sum_{r=1}^q \|\sigma_r^n \psi_{i,\ell_r^n}^\delta\|_{L^2(\gamma_i)}^2 : \sum_{r=1}^q \sigma_r^n \psi_{i,\ell_r^n}^\delta(p) = \psi_{i,n}^\delta(p),\ p \in P_{m-2}(\gamma_i) \right\} \tag{3.4}$$

has a solution with $|\sigma_r^n| \lesssim \|\bar{m}_{i,\ell_r^n}^\delta\|_{L^2(\gamma_i)} / \|\bar{m}_{i,n}^\delta\|_{L^2(\gamma_i)}$, $1 \leq r \leq q$. We remove n from $\bar{I}_i^\delta$ and thus $\bar{m}_{i,n}^\delta$ from the collection $\{\bar{m}_{i,\ell}^\delta : \ell \in \bar{I}_i^\delta\}$, and redefine

$$\bar{m}_{i,\ell_r^n}^\delta \leftarrow \bar{m}_{i,\ell_r^n}^\delta + \sigma_r^n \bar{m}_{i,n}^\delta, \quad 1 \leq r \leq q. \tag{3.5}$$

Note that the reduced and modified collection $\{\bar{m}^\delta_{i,\ell} : \ell \in \bar{I}^\delta_i\}$ still satisfies (3.2). Moreover, since before the removal of $\bar{m}^\delta_{i,n}$ and the redefinition of $\bar{m}^\delta_{i,\ell^n_r}$, we had $p = \sum_{\ell\in\bar{I}^\delta_i} \psi^\delta_{i,\ell}(p) m^\delta_{i,\ell}$ for any $p \in P_{m-2}(\gamma_i)$, from (3.4) we conclude that this property is retained. After the removal of all $n \in \bar{I}^\delta_i \backslash I^\delta_i$ and the corresponding updates of the $\bar{m}^\delta_{i,\ell^n_r}$, the collection $\{\bar{m}^\delta_{i,\ell} : \ell \in I^\delta_i\}$ will be written as $\{m^\delta_{i,\ell} : \ell \in I^\delta_i\}$, and we conclude that we have obtained (3.1).

To give useful error bounds, we have to verify whether the resulting space $M^\delta_i := \operatorname{span}\{m^\delta_{i,\ell} : \ell \in I^\delta_i\}$ constructed in this way does have appropriate approximation properties. For any element T from the underlying partition, by construction there exists a ball $B(T)$ with $\operatorname{diam}(B(T)) \lesssim \operatorname{diam}(T)$ such that $\operatorname{supp} \psi^\delta_{i,\ell} \subset B(T)$ for any of the uniformly bounded number of $\ell \in I^\delta_i$ with $\operatorname{supp} m^\delta_{i,\ell} \cap T \neq \emptyset$. By an application of the Bramble-Hilbert lemma, for any $s \in [0, m-1]$, $u \in H^s(\gamma_i)$, we have

$$
\begin{aligned}
&\|u - \sum_{\ell\in I^\delta_i} \langle \psi^\delta_{i,\ell}, u\rangle_{L^2(\gamma_i)} m^\delta_{i,\ell}\|_{L^2(T)} \\
&= \inf_{p\in P_{m-2}(\gamma_i)} \|u - p - \sum_{\ell\in I^\delta_i} \langle \psi^\delta_{i,\ell}, u-p\rangle_{L^2(\gamma_i)} m^\delta_{i,\ell}\|_{L^2(T)} \\
&\lesssim \inf_{p\in P_{m-2}(\gamma_i)} \|u-p\|_{L^2(B(T)\cap\gamma_i)} \lesssim \operatorname{diam}(T)^s \|u\|_{H^s(B(T)\cap\gamma_i)}
\end{aligned}
$$

which yields (2.8) for $t_{s(i)} \leq m$.

In our examples, the initial set $\bar{I}^\delta_i$ will naturally identify with a set of points on $\overline{\gamma_i}$, such that each element from the underlying partition contains a subset of $q = \dim P_{m-2}(\gamma_i)$ of these points on which the local interpolation problem to determine $p \in P_{m-2}(\gamma_i)$ is (uniformly) well-posed. Furthermore, in our first two examples, each $\bar{m}^\delta_{i,\ell}$ from the initial set $\{\bar{m}^\delta_{i,\ell} : \ell \in \bar{I}^\delta_i\}$ will be continuous at ℓ, with $\bar{m}^\delta_{i,\ell}(\ell) = 1$, $\|\bar{m}^\delta_{i,\ell}\|_{L^2(\gamma_i)} \eqsim (\operatorname{diam}(\operatorname{supp} \bar{m}^\delta_{i,\ell}))^{\frac{d-1}{2}}$, and

$$v \mapsto \sum_{\ell\in\bar{I}^\delta_i} v(\ell) \bar{m}^\delta_{i,\ell}$$

will be a quasi-interpolator of order $m-2$. In our third and final example, a slightly different situation will occur that we will discuss there. Obviously the functionals $v \mapsto v(\ell)$ are not in $L^\infty(\gamma_i)$, but, as outlined in [22, §2.1.1], we can always construct local $\psi^\delta_{i,\ell} \in L^\infty(\gamma_i)$ with $\|\psi^\delta_{i,\ell}\|_{L^2(\gamma_i)} \eqsim (\operatorname{diam}(\operatorname{supp} \bar{m}^\delta_{i,\ell}))^{-\frac{d-1}{2}}$ and $\psi^\delta_{i,\ell}(p) = p(\ell)$ for all $p \in P_{m-2}(\gamma_i)$, meaning that $v \mapsto \sum_{\ell \in \bar{I}_i^\delta} \psi_{i,\ell}^\delta(v)\bar{m}_{i,\ell}^\delta$ is a quasi-interpolator of the required type.

The set I_i^δ will be equal to $\bar{I}_i^\delta \cap \gamma_i$. Given $n \in \bar{I}_i^\delta \backslash I_i^\delta$, the selection of a suitable set $\ell_1^n, \ldots, \ell_q^n \in I_i^\delta$ now can boil down to the selection of an element T, having empty intersection with $\partial\gamma_i$ and with $\operatorname{dist}(n, T) \lesssim \operatorname{diam}(\operatorname{supp} \bar{m}_{i,n}^\delta)$, and inside this element, to the selection of a $P_{m-2}(\gamma_i)$-unisolvant set $\ell_1^n, \ldots, \ell_q^n$ of nodal points as above, with which the problem (3.4) has a unique solution. Equipping $P_{m-2}(\gamma_i)$ with the basis $\{p_{\ell_1^n}, \ldots, p_{\ell_q^n}\}$ defined by $p_{\ell_r^n}(\ell_{r'}^n) = \begin{cases} 1 \text{ if } r = r', \\ 0 \text{ if } r \neq r', \end{cases}$, this solution is given by

$$\sigma_k^n = \frac{\psi_{i,n}^\delta(p_{\ell_r^n})}{\psi_{i,\ell_r^n}^\delta(p_{\ell_r^n})} = p_{\ell_r^n}(n).$$

Note that the σ_r^n are uniformly bounded as required, and that they only depend on the polynomial degree $m-2$ and the location of the points n and $\ell_1^n, \ldots, \ell_q^n$.

Thinking of rectangular elements, if initially the nodal interpolator preserves $Q_{m-2}(\gamma_i)$, then, in view of [1], it seems recommendable to impose (3.4) for all $p \in Q_{m-2}(\gamma_i)$ so that this property is retained.

Finally we make some comments about how to work efficiently with the resulting dual Lagrange multipliers. Concerning the solution of (1.3), the basis $\cup_{i=1}^N \{m_{i,\ell}^\delta : \ell \in I_i^\delta\}$ for M^δ only enters via the matrices

$$\mathbf{M}_{i,M}^\delta := \left[b(x_{m(i),j}^\delta, m_{i,\ell}^\delta)\right]_{\ell \in I_i^\delta, j \in \check{D}_{m(i)}^\delta}, \mathbf{M}_{i,S}^\delta := \left[b(x_{s(i),j}^\delta, m_{i,\ell}^\delta)\right]_{\ell \in I_i^\delta, j \in \check{D}_{s(i)}^\delta},$$

and

$$\mathbf{S}_i^\delta := \operatorname{diag}\left[b(x_{s(i),\ell}^\delta, m_{i,\ell}^\delta)\right]_{\ell \in I_i^\delta}.$$

In view of this, instead of directly working with the basis functions $m_{i,\ell}^\delta$ of which those with supports near $\partial\gamma_i$ generally, because of the updates (3.5), have larger supports and have "irregular" definitions, it will be more efficient to apply the procedure outlined below. Let $\{\bar{m}_{i,\ell}^\delta : \ell \in \bar{I}_i^\delta\}$ be the initial collection satisfying (3.2), i.e., before the removal of any of the functions with indices in $\bar{I}_i^\delta \backslash I_i^\delta$ and the corresponding updates of q of the remaining functions. Let $\bar{\mathbf{M}}_{i,M}^\delta$, $\bar{\mathbf{M}}_{i,S}^\delta$ and $\bar{\mathbf{S}}_i^\delta$ be the matrices defined as $\mathbf{M}_{i,M}^\delta$, $\mathbf{M}_{i,S}^\delta$ and $\mathbf{S}_i^\delta$ with $\{m_{i,\ell}^\delta : \ell \in I_i^\delta\}$ replaced by $\{\bar{m}_{i,\ell}^\delta : \ell \in \bar{I}_i^\delta\}$, meaning that these matrices can be assembled via "regular" finite element computations.

The matrix $\mathbf{S}_i^\delta$ is now simply obtained from $\bar{\mathbf{S}}_i^\delta$ by deleting all rows and columns with indices from $\bar{I}_i^\delta \backslash I_i^\delta$. With $\mathbf{T}_i^\delta$ being the $\# I_i^\delta \times \# \bar{I}_i^\delta$

matrix containing in its ℓ'th row the coordinates of $m^\delta_{i,\ell'}$ in terms of $\{\bar{m}^\delta_{i,\ell} : \ell \in \bar{I}^\delta_i\}$, we have

$$\mathbf{M}^\delta_{i,M} = \mathbf{T}^\delta_i \bar{\mathbf{M}}^\delta_{i,M} \quad \text{and} \quad \mathbf{M}^\delta_{i,S} = \mathbf{T}^\delta_i \bar{\mathbf{M}}^\delta_{i,S}.$$

Fixing some numbering of $\bar{I}^\delta_i$, where indices from I^δ_i precede those from $\bar{I}^\delta_i \backslash I^\delta_i$, and with $\bar{\mathbf{e}}_\ell$ or $\mathbf{e}_\ell$ being the standard (column) basis vector of $\mathbb{R}^{\#\bar{I}^\delta_i}$ or $\mathbb{R}^{\# I^\delta_i}$, respectively, i.e., $\bar{\mathbf{e}}_\ell$ or $\mathbf{e}_\ell$ is 1 on the position corresponding to ℓ and is zero elsewhere, in view of (3.5) one infers that

$$\mathbf{T}^\delta_i = \sum_{\ell \in I^\delta_i} \mathbf{e}_\ell \bar{\mathbf{e}}^T_\ell + \sum_{n \in \bar{I}^\delta_i \backslash I^\delta_i} (\sum_{k=1}^{q} \sigma^n_r \mathbf{e}_{\ell^n_r}) \bar{\mathbf{e}}^T_n,$$

indicating an efficient way to apply $\mathbf{T}^\delta_i$.

Having solved u^δ from (1.3), one retrieves λ^δ represented in terms of $\cup_{i=1}^N \{m^\delta_{i,\ell} : \ell \in I^\delta_i\}$ by testing the equation $a(u^\delta, v^\delta) + b(v^\delta, \lambda^\delta) = f(v^\delta)$ for any v^δ of the form $x_{s(i),\ell}$ for $1 \leq i \leq N$, $\ell \in I^\delta_i$. By splitting the vector into N parts corresponding to the different interfaces, and by applying $(\mathbf{T}^\delta_i)^T$ to the part corresponding to γ_i, one obtains a representation of λ^δ in terms of $\cup_{i=1}^N \{\bar{m}^\delta_{i,\ell} : \ell \in \bar{I}^\delta_i\}$, which is more convenient for further processing.

3.2 Examples

In this subsection we will give three examples of spaces $X^\delta = \prod_{k=1}^K X^\delta_k$ and $M^\delta = \prod_{i=1}^N M^\delta_i$ that satisfy all conditions (A.2)-(A.7) we have imposed, and, moreover, that will be equipped with dual Lagrange multiplier bases. In our examples $\overline{\Omega} = \cup_{k=1}^K \overline{\Omega_k} \subset \mathbb{R}^3$ will be a polyhedron, and thus so are the subdomains Ω_k. In case of piecewise tensor product polynomial approximation on the slave domain, dual Lagrange multiplier bases can simply be made by taking tensor products of such bases on one-dimensional interfaces. Our examples concern non-tensor product approximations, where the construction is less straightforward. Examples of dual Lagrange multiplier spaces for Mortar finite elements can also be found in [28, 23, 30, 19], and in particular for three dimensional domains, in [8, 31, 18] dealing with linear tetrahedral elements, and in [20] dealing with quadratic serendipity hexahedral elements. Here, we consider quadratic and cubic tetrahedral elements on possibly locally refined partitions. In [20], we constructed dual Lagrange multipliers for a modified quadratic serendipity hexahedral finite element space that along the interface was enriched with bubble functions. Here, by taking an non-standard multiplier space, it will turn out that the addition of these bubble functions can be avoided. The existence of dual Lagrange multipliers depends as much on the choice of the basis for $X^\delta_{s(i)}$ as it does on the basis for M^δ_i. Apart that we allow that $\dim M^\delta_i < \dim W^\delta_{0,s(i)}$, another key point of our constructions will be that we do not simply resort on the nodal basis for $X^\delta_{s(i)}$.

3.2.1 Quadratic tetrahedral elements For an $1 \leq i \leq N$, let $(T^\delta_{s(i)})_{\delta\in\mathcal{I}}$ be a family of shape regular, locally quasi-uniform and conforming partitions of $\Omega_{s(i)}$ into tetrahedra T. Let $D^\delta_{s(i),v}$ and $D^\delta_{s(i),m}$ denote the set of vertices and midpoints of edges, respectively, which are not on $\partial\Omega$ of $T \in T^\delta_{s(i)}$, and let $D^\delta_{s(i)} = D^\delta_{s(i),v} \cup D^\delta_{s(i),m}$. We define $X^\delta_{s(i)} = C(\Omega_{s(i)}) \cap H^1_*(\Omega_{s(i)}) \cap \prod_{T\in T^\delta_{s(i)}} P_2(T)$, i.e., $m = 3$, and equip it with the nodal basis $\{x^\delta_{s(i),j} : j \in D^\delta_{s(i)}\}$. Note that with these definitions, and because of (A.1), the set $S^\delta_{0,i}$ is given by $D^\delta_{s(i)} \cap \gamma_i$.

For $\delta \in \mathcal{I}$, let $\triangle^\delta_i$ be the subdivision of γ_i into triangles $\triangle$ obtained by intersecting all $T \in T^\delta_{s(i)}$ with γ_i, and let $\bar{I}^\delta_i$ denote the set of all vertices of $\triangle \in \triangle^\delta_i$. We define $\{\bar{m}_{i,\ell} : \ell \in \bar{I}^\delta_i\}$ as the nodal basis for $C(\gamma_i) \cap \prod_{\triangle\in\triangle^\delta_i} P_1(\triangle)$. Now we apply a basis transformation at the primal side. For any $j \in I^\delta_i = \bar{I}^\delta_i \cap S^\delta_{0,i}$, which is equal to $D^\delta_{s(i),v} \cap \gamma_i$, we redefine

$$x^\delta_{s(i),j} \leftarrow x^\delta_{s(i),j} + \frac{1}{12} \sum_{\{\tilde{j}\in D^\delta_{s(i),m} : j,\tilde{j}\in\triangle \text{ for some } \triangle\in\triangle^\delta_i\}} x^\delta_{s(i),\tilde{j}}. \tag{3.6}$$

Then, as we will verify below, we have

$$\langle x^\delta_{s(i),\ell}|_{\gamma_i}, \bar{m}^\delta_{i,\ell'}\rangle_{L^2(\gamma_i)} \begin{cases} \eqsim \|x^\delta_{s(i),\ell}|_{\gamma_i}\|_{L^2(\gamma_i)} \|\bar{m}^\delta_{i,\ell'}\|_{L^2(\gamma_i)} & \text{if } I^\delta_i \ni \ell = \ell', \\ = 0 & \text{if } I^\delta_i \ni \ell \neq \ell' \in \bar{I}^\delta_i, \end{cases} \tag{3.7}$$

i.e., (3.2) is valid. Obviously $\{x^\delta_{s(i),j} : j \in D^\delta_{s(i)}\}$, $\{\bar{m}^\delta_{i,\ell} : \ell \in \bar{I}^\delta_i\}$ and $\{x^\delta_{s(i),j}|_{\gamma_i} : j \in S^\delta_{0,s(i)}\}$ are uniformly $L^2(\Omega_{s(i)})$- or $L^2(\gamma_i)$-stable collections of local functions.

The mapping $v \mapsto \sum_{\ell\in\bar{I}^\delta_i} v(\ell)\bar{m}^\delta_{i,\ell}$ is a quasi-interpolator of order $m - 2 = 1$. The general procedure outlined in §3.1 to remove the degrees of freedom associated to $n \in \bar{I}^\delta_i \backslash I^\delta_i$, while retaining the local approximation properties reads in this case as follows: For any vertex n of a $\triangle \in \triangle^\delta_i$ that is on $\partial\gamma_i$, select a $\tilde{\triangle} \in \triangle^\delta_i$ that has empty intersection with $\partial\gamma_i$ and $\operatorname{dist}(\triangle, \tilde{\triangle}) \lesssim \operatorname{diam}(\triangle)$. Remove n from $\bar{I}^\delta_i$ and thus $\bar{m}^\delta_{i,\ell}$ from $\{\bar{m}^\delta_{i,\ell} : \ell \in \bar{I}^\delta_i\}$, and with (μ_1, μ_2, μ_3) being the barycentric coordinates of n with respect to the vertices ℓ_1, ℓ_2, ℓ_3 of

$\tilde{\triangle}$, update $\bar{m}^{\delta}_{\ell_r} \leftarrow \bar{m}^{\delta}_{\ell_r} + \mu_r \bar{m}^{\delta}_n$, $1 \leq r \leq 3$. After the removal of all $n \in \bar{I}^{\delta}_i \backslash I^{\delta}_i$, we end up with a set $\{m^{\delta}_{i,\ell} : \ell \in I^{\delta}_i\}$ of (generalized) dual Lagrange multipliers, which spans a space M^{δ}_i that has the required local approximation properties.

To show (3.7), let ϕ^l_r, $r = 1, \cdots, 3$ and ϕ^q_r, $r = 1, \cdots, 6$ be the linear and quadratic finite element nodal basis functions on a reference triangle $\hat{\triangle}$ with $\mathrm{vol}(\hat{\triangle}) = 1$, respectively, using a numbering of the points as in see Figure 3.2. A direct computation gives that

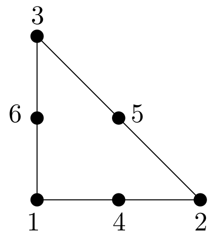

Fig. 3.2: Numbering of the vertices and midpoints of edges on the reference triangle.

$$\left[\langle \phi^q_r, \phi^l_{r'} \rangle_{L^2(\hat{\triangle})}\right]_{1 \leq r \leq 6, 1 \leq r' \leq 3} = \frac{1}{60} \begin{bmatrix} 2 & -1 & -1 & 8 & 4 & 8 \\ -1 & 2 & -1 & 8 & 8 & 4 \\ -1 & -1 & 2 & 4 & 8 & 8 \end{bmatrix}^T .$$

Hence, if we define

$$\tilde{\phi}^q_1 := \phi^q_1 + \frac{1}{12}(\phi^q_4 + \phi^q_6),$$

and similarly $\tilde{\phi}^q_2$ and $\tilde{\phi}^q_3$ using symmetry in the barycentric coordinates, we obtain

$$\left[\langle \tilde{\phi}^q_r, \phi^l_{r'} \rangle_{L^2(\hat{\triangle})}\right]_{1 \leq r, r' \leq 3} = \frac{1}{18} I,$$

which, in view of the definition (3.6), by using affine bijections between the reference triangle $\hat{\triangle}$ and all $\triangle \in \triangle^{\delta}_i$ easily leads to (3.7). The basis transformation (3.6) was made as a result of the above construction of the basis transformation on the reference triangle. Yet, note that not all local basis transformations give rise to corresponding global basis transformations. Essential properties of the modified basis functions on the reference triangle are symmetry in the barycentric coordinates, and the fact that a basis function associated to a point, here a vertex or a midpoint of an edge, vanishes on all edges that do not contain that point.

Finally, for the case that the family of triangulations $(\triangle_i^\delta)_{\delta\in\mathcal{I}}$ is not quasi-uniform, we will verify the existence of mortar projectors Π_i^δ that are local, so that (A.6) is valid. Following the approach from Remark 2.3, we extend $\{m_{i,\ell}^\delta : \ell \in I_i^\delta\}$ to a suitable collection $\{m_{i,\ell}^\delta : \ell \in S_{0,i}^\delta\}$. It is easy to construct functions $m_r \in P_2(\hat{\triangle})$ for $r = 4, 5, 6$ on the reference triangle $\hat{\triangle}$ that are symmetric in the barycentric coordinates, and, with $\tilde{\phi}_r^q = \phi_r^q$ for $r = 4, 5, 6$, satisfy $\langle \tilde{\phi}_r^q, m_{r'} \rangle_{L^2(\hat{\triangle})} = \begin{cases} 0 \text{ if } r' \neq r \in \{1, \ldots, 6\}, \\ 1 \text{ if } r' = r. \end{cases}$ By lifting m_4, m_5, m_6 to all $\triangle \in \triangle_i^\delta$ by using affine bijections, and by connecting those pairs that are associated to the same edge, and finally by leaving out those functions associated to edges on $\partial\gamma_i$, we obtain a collection of auxiliary functions $\{m_{i,\ell}^\delta : \ell \in S_{0,i}^\delta \backslash I_i^\delta\}$ on γ_i. The union $\{m_{i,\ell}^\delta : \ell \in S_{0,i}^\delta\}$ is a uniformly $L^2(\gamma_i)$-stable set of local functions. With respect to the partition of $S_{0,i}^\delta$ into I_i^δ and $S_{0,i}^\delta \backslash I_i^\delta$, the matrix $\mathbf{R}_i^\delta := \left[\frac{\langle x_{i,\ell}^\delta, m_{i,\ell'}^\delta|_{\gamma_i} \rangle_{L^2(\gamma_i)}}{\|x_{i,\ell}^\delta|_{\gamma_i}\|_{L^2(\gamma_i)} \|m_{i,\ell'}^\delta\|_{L^2(\gamma_i)}} \right]_{\ell,\ell' \in S_{0,i}^\delta}$ is of the form $\begin{bmatrix} \mathbf{D}_{i,1}^\delta & 0 \\ \mathbf{A}_i^\delta & \mathbf{D}_{i,2}^\delta \end{bmatrix}$, with $\mathbf{D}_{i,1}^\delta$, $\mathbf{D}_{i,2}^\delta$ being diagonal matrices that are uniformly bounded invertible, and $\mathbf{A}_i^\delta$ being a uniformly sparse and bounded matrix. We infer that $\mathbf{R}_i^\delta$ is uniformly bounded invertible with a local inverse, showing that a uniformly $L^2(\gamma_i)$-bounded local mortar projector exists.

As an alternative for our definition $M_i^\delta = \text{span}\{m_i^\delta : i \in I_i^\delta\}$, one may use such auxiliary functions to define $M_i^\delta = \text{span}\{m_i^\delta : i \in S_{0,i}^\delta\}$, with which one gets into the situation of having quasi-dual Lagrange multipliers as in [20], and $\dim W_{0,i}^\delta = \dim M_i^\delta$. This approach, however, leads to a more complicated coupling between master and slave sides. In the cases where we have tested both approaches, the discretization errors were almost equal.

3.2.2 Quadratic serendipity hexahedral elements For an $1 \leq i \leq N$, let $(P_{s(i)}^\delta)_{\delta\in\mathcal{I}}$ be a family of shape regular, and conforming partitions of $\Omega_{s(i)}$ into parallelepipeds P. So here, in order not to be forced to handle cases with hanging nodes, we restrict ourselves to quasi-uniform partitions. Let $D_{s(i),v}^\delta$ and $D_{s(i),m}^\delta$ denote the set of vertices and midpoints of edges, respectively, which are not on $\partial\Omega$ of $P \in P_{s(i)}^\delta$, and let $D_{s(i)}^\delta = D_{s(i),v}^\delta \cup D_{s(i),m}^\delta$. We define $X_{s(i)}^\delta$ as the quadratic serendipity finite element space with respect to $P^\delta_{s(i)}$, i.e., $m = 3$, with zero boundary conditions on $\partial\Omega_{s(i)} \cap \partial\Omega$, and equip it with the nodal basis $\{x^\delta_{s(i),j} : j \in D^\delta_{s(i)}\}$. Note that the set $S^\delta_{0,i}$ is given by $D^\delta_{s(i)} \cap \gamma_i$.

For $\delta \in \mathcal{I}$, let $\square^\delta_i$ be the subdivision of γ_i into parallelograms $\square$ obtained by intersecting all $P \in P^\delta_{s(i)}$ with γ_i, and let $\bar{I}^\delta_i$ denote the set of all vertices of $\square \in \square^\delta_i$. We apply the following basis transformation at the primal side. For any $j \in I^\delta_i = \bar{I}^\delta_i \cap S^\delta_{0,i}$, which is equal to $D^\delta_{s(i),v} \cap \gamma_i$, we redefine

$$x^\delta_{s(i),j} \leftarrow x^\delta_{s(i),j} + \frac{1}{5} \sum_{\{\tilde{\jmath} \in D^\delta_{s(i),m} : j, \tilde{\jmath} \in \square \text{ for some } \square \in \square^\delta_i\}} x^\delta_{s(i),\tilde{\jmath}}. \tag{3.8}$$

We will construct an $L^2(\gamma_i)$-stable collection $\{\bar{m}_{i,\ell} : \ell \in \bar{I}^\delta_i\}$ of local functions such that

$$\langle x^\delta_{s(i),\ell}|_{\gamma_i}, \bar{m}^\delta_{i,\ell'} \rangle_{L^2(\gamma_i)} \begin{cases} \eqsim \|x^\delta_{s(i),\ell}|_{\gamma_i}\|_{L^2(\gamma_i)} \|\bar{m}^\delta_{i,\ell'}\|_{L^2(\gamma_i)} & \text{if } I^\delta_i \ni \ell = \ell', \\ = 0 & \text{if } I^\delta_i \ni \ell \neq \ell' \in \bar{I}^\delta_i, \end{cases} \tag{3.9}$$

i.e., such that (3.2) is valid. Obviously $\{x^\delta_{s(i),j} : j \in D^\delta_{s(i)}\}$ and $\{x^\delta_{s(i),j}|_{\gamma_i} : j \in S^\delta_{0,s(i)}\}$ are uniformly $L^2(\Omega_{s(i)})$- or $L^2(\gamma_i)$-stable collections of local functions.

To show (3.9) and to construct $\{\bar{m}_{i,\ell} : \ell \in \bar{I}^\delta_i\}$, using the numbering of the points as in Figure 3.3, let ϕ^b_r for $r = 1, \cdots, 4$, and ϕ^s_i for $i = 1, \cdots, 8$ denote the bilinear and serendipity finite element nodal basis functions on the reference square $\hat{\square} = (0,1)^2$, respectively. A

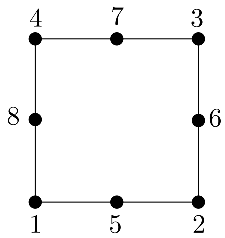

Fig. 3.3: Numbering of the vertices and midpoints of edges on the reference square.

direct computation gives that

$$\left[\langle \phi^s_r, \phi^b_1 \rangle_{L^2(\hat{\square})}\right]_{1 \le r \le 8} = \frac{1}{36} \begin{bmatrix} 0 & -1 & -1 & -1 & 4 & 2 & 2 & 4 \end{bmatrix}^T,$$

where the other entries follow using symmetry. Because of the missing interior degree of freedom, for the serendipity element we are not able to construct a biorthogonal system by only making a transformation at the 'primal' side. Therefore, for some $\alpha \neq 1$, we set

$$\tilde{\phi}_1^b := \frac{\phi_1^b + \alpha(\phi_3^b - \phi_2^b - \phi_4^b)}{1-\alpha},$$

and define $\tilde{\phi}_2^b$, $\tilde{\phi}_3^b$ and $\tilde{\phi}_4^b$ analogously. The corresponding collection of global functions $\{\bar{m}_{i,\ell}^\delta : \ell \in \bar{I}_i^\delta\}$ is now formally defined by $\bar{m}_{i,\ell}^\delta|_\Box = \tilde{\phi}_1^b \circ F_\Box^{-1}$ for any $\Box \in \Box_i^\delta$ that contains ℓ as a vertex, and where $F_\Box$ is an affine bijection between $\hat{\Box}$ and $\Box$ mapping point 1 onto ℓ, and $\bar{m}_{i,\ell}$ is zero elsewhere. Note that the $\bar{m}_{i,\ell}^\delta$ are not continuous, and that $\{\bar{m}_{i,\ell}^\delta : \ell \in \bar{I}_i^\delta\}$ does *not* span the space of all continuous piecewise bilinears with respect to $\Box_i^\delta$. Yet, although $v \mapsto \sum_{i\in \bar{I}_i^\delta} v(\ell)\bar{m}_{i,\ell}^\delta$ does not preserve $Q_1(\gamma_i)$, since $\tilde{\phi}_1^b + \tilde{\phi}_2^b = \phi_1^b + \phi_2^b$, $\tilde{\phi}_2^b + \tilde{\phi}_3^b = \phi_2^b + \phi_3^b$, $\tilde{\phi}_3^b + \tilde{\phi}_4^b = \phi_3^b + \phi_4^b$, and $\tilde{\phi}_1^b + \tilde{\phi}_2^b + \tilde{\phi}_3^b + \tilde{\phi}_4^b = \phi_1^b + \phi_2^b + \phi_3^b + \phi_4^b$ it does preserve $P_1(\gamma_i)$, i.e., it is a quasi-interpolator of order $m-2=1$.

Defining, for some θ,

$$\tilde{\phi}_1^s := \phi_1^s + \theta(\phi_5^s + \phi_8^s),$$

and analogously $\tilde{\phi}_2^s$, $\tilde{\phi}_3^s$ and $\tilde{\phi}_4^s$, a direct computation shows that $\left[\langle \tilde{\phi}_r^s, \tilde{\phi}_{r'}^b \rangle_{L^2(\hat{\Box})}\right]_{1\le r,r'\le 4}$ is diagonal if and only if $\alpha = \frac{1}{6}$ and $\theta = \frac{1}{5}$, in which case this matrix is equal to $\frac{1}{20}I$. With $\alpha = \frac{1}{6}$ substituted into the definition of $\{\bar{m}_{i,\ell}^\delta : \ell \in \bar{I}_i^\delta\}$, we conclude (3.9).

The procedure outlined in §3.1 to remove the degrees of freedom associated to $n \in \bar{I}_i^\delta \backslash I_i^\delta$ reads here as follows: For any vertex n of a $\Box \in \Box_i^\delta$ that is on $\partial\gamma_i$, select a $\tilde{\Box} \in \Box_i^\delta$ that has empty intersection with $\partial\gamma_i$ and $\operatorname{dist}(\Box, \tilde{\Box}) \lesssim \operatorname{diam}(\Box)$. Remove n from $\bar{I}_i^\delta$ and thus $\bar{m}_{i,\ell}^\delta$ from $\{\bar{m}_{i,\ell}^\delta : \ell \in \bar{I}_i^\delta\}$, and with (μ_1, μ_2, μ_3) being the barycentric coordinates of n with respect to three vertices ℓ_1, ℓ_2, ℓ_3 of $\tilde{\Box}$, update $\bar{m}_{\ell_r}^\delta \leftarrow \bar{m}_{\ell_r}^\delta + \mu_r \bar{m}_n^\delta$, $1 \le r \le 3$. After the removal of all $n \in \bar{I}_i^\delta \backslash I_i^\delta$, we end up with a set $\{m_{i,\ell}^\delta : \ell \in I_i^\delta\}$ of (generalized) dual Lagrange multipliers, which spans a space M_i^δ that has the required local approximation properties.

3.2.3 Cubic tetrahedral elements For an $1 \le i \le N$, let $(T_{s(i)}^\delta)_{\delta\in\mathcal{I}}$ and $(\triangle_i^\delta)_{\delta\in\mathcal{I}}$ be families as in §3.2.1 of partitions of $\Omega_{s(i)}$ into tetrahedra, and of γ_i into triangles, respectively. We define $X_{s(i)}^\delta = C(\Omega_{s(i)}) \cap$

$H^1_*(\Omega_{s(i)}) \cap \prod_{T\in T^\delta_{s(i)}} P_3(T)$, and $\bar{M}^\delta_i = C(\gamma_i) \cap \prod_{\triangle\in\triangle^\delta_i} P_2(\triangle)$, i.e., $m = 4$.

To define a basis $\{\bar{m}^\delta_{i,\ell} : \ell \in \bar{I}^\delta_i\}$ for $\bar{M}^\delta_i$, and a modification of the nodal basis $\{x^\delta_{s(i),j} : j \in D^\delta_{s(i)}\}$ for $X^\delta_{s(i)}$, which only affects $\{x^\delta_{s(i),j}|_{\gamma_i} : j \in S^\delta_{0,i}\}$, it is sufficient to consider a reference triangle $\hat{\triangle}$ with $\mathrm{vol}(\hat{\triangle}) = 1$. Using a numbering of the nodal points as indicated in Figure 3.4, let $\{\phi^c_1, \dots, \phi^c_{10}\}$ and $\{\phi^q_1, \dots, \phi^q_6\}$ be the nodal bases

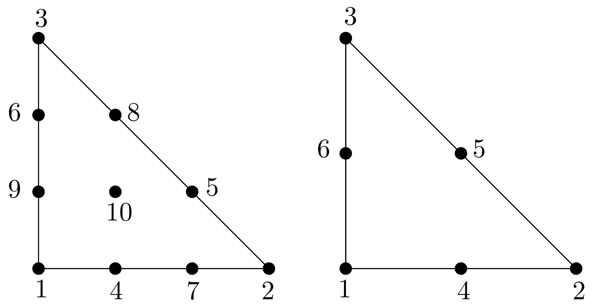

Fig. 3.4: Numbering of nodal points at primal and multiplier side for the cubic tetrahedral case.

for $P_3(\hat{\triangle})$ and $P_2(\hat{\triangle})$, respectively. We define transformed bases by

$$\begin{aligned}
\tilde{\phi}^c_1 &= 60\phi^c_1 + \frac{20}{9}(\phi^c_4 + \phi^c_9 - \phi^c_6 - \phi^c_7 - \phi^c_{10}),\\
\tilde{\phi}^c_4 &= 10(\phi^c_4 + \phi^c_7) - \frac{5}{6}\phi^c_{10}, \quad \tilde{\phi}^c_{10} = \phi^c_{10}, \quad \tilde{\phi}^c_7 = \phi^c_4 - \phi^c_7,
\end{aligned}$$

and

$$\tilde{\phi}^q_1 = \phi^q_1 - \frac{1}{8}(\phi^q_4 + \phi^q_6), \quad \tilde{\phi}^q_4 = \phi^q_4, \tag{3.10}$$

and the others by permuting barycentric coordinates. All these basis functions at primal and multiplier side can be naturally associated to either a vertex, an edge (at the primal side two at each edge) or the center of $\hat{\triangle}$, where they vanish on the opposite edge, on the other two edges, or on all edges, respectively, and finally, where they are either symmetric or anti-symmetric in permutations of the barycentric coordinates. A direct computation shows that

$$[\langle \tilde{\phi}^c_r, \tilde{\phi}^q_{r'}\rangle_{L^2(\hat{\triangle})}]_{1\le r,r'\le 6} = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{I} & 0 \\ \hline 0 & 0 & -\frac{1}{8} & \\ -\frac{1}{8} & 0 & 0 & I \\ 0 & -\frac{1}{8} & 0 & \end{array}\right].$$

Although this matrix is thus not diagonal, the fact that, with respect to a splitting of the degrees of freedom into those associated to vertices and those associated to edges, it is block lower triangular with diagonal blocks equal to identity matrices which will allow for the construction of local dual Lagrange multiplier bases.

By lifting $\{\tilde{\phi}^c_1, \ldots, \tilde{\phi}^c_{10}\}$ and $\{\tilde{\phi}^q_1, \ldots, \tilde{\phi}^q_6\}$ to all $\triangle \in \triangle^\delta_i$ using affine bijections, and by gluing them continuously over the interfaces, and, at the primal side, by leaving out those functions associated to vertices or edges on $\partial\gamma_i$, we end up with the modified collection $\{x^\delta_{s(i),j}|_{\gamma_i} : j \in S^\delta_{0,i}\}$, and a basis $\{\breve{m}^\delta_{i,\ell} : \ell \in \bar{I}^\delta_i\}$ for $\bar{M}^\delta_i$. Here as a natural index set $\bar{I}^\delta_i$, we use the union of the sets $\bar{I}^\delta_{i,v}$ and $\bar{I}^\delta_{i,m}$ of all vertices and midpoints of edges of $\triangle \in \triangle^\delta_i$, respectively. Defining $I^\delta_{i,v} = \bar{I}^\delta_{i,v} \cap \gamma_i$, $I^\delta_{i,m} = \bar{I}^\delta_{i,m} \cap \gamma_i$ and $I^\delta_i = I^\delta_{i,v} \cup I^\delta_{i,m}$, we write $S^\delta_{0,i}$ as $I^\delta_i \cup (S^\delta_{0,i} \backslash I^\delta_i)$, where $I^\delta_{i,v}$ and $I^\delta_{i,m}$ are the index sets for those $x^\delta_{s(i),j}|_{\gamma_i}$ resulting from the functions $\tilde{\phi}^c_1, \tilde{\phi}^c_2, \tilde{\phi}^c_3$ and $\tilde{\phi}^c_4, \tilde{\phi}^c_5, \tilde{\phi}^c_6$, respectively, and where $S^\delta_{0,i} \backslash I^\delta_i$ is some index set for those $x^\delta_{s(i),j}|_{\gamma_i}$ resulting from the functions $\tilde{\phi}^c_7, \tilde{\phi}^c_8, \tilde{\phi}^c_9, \tilde{\phi}^c_{10}$.

For $\ell \in I^\delta_i$, we have

$$\langle x^\delta_{s(i),\ell}|_{\gamma_i}, \breve{m}^\delta_{i,\ell}\rangle_{L^2(\gamma_i)} = d^\delta_\ell := \sum_{\{\triangle \in \triangle^\delta_i : \triangle \ni \ell\}} \mathrm{vol}(\triangle),$$

and for any $\ell' \in \bar{I}^\delta_{i,v}$, and

$$\ell \in \gamma_i \cap \bar{I}^\delta_{i,\mathrm{opp}}(\ell') := \{\ell \in \bar{I}^\delta_{i,m} : \exists \triangle = \triangle_{\ell,\ell'} \in \triangle^\delta_i \text{ with } \ell', \ell \in \triangle \\ \text{and } \ell \text{ is on the edge opposite to } \ell'\},$$

see Figure 3.5, we have

$$\langle x^\delta_{s(i),\ell}|_{\gamma_i}, \breve{m}^\delta_{i,\ell'}\rangle_{L^2(\gamma_i)} = -\frac{1}{8}\mathrm{vol}(\triangle_{\ell,\ell'}),$$

whereas $\langle x^\delta_{s(i),\ell}|_{\gamma_i}, \breve{m}^\delta_{i,\ell'}\rangle_{L^2(\gamma_i)} = 0$ for all other $\ell \in I^\delta_i$, $\ell' \in \bar{I}^\delta_i$.

Defining the transformed basis $\{\bar{m}^\delta_{i,\ell} : \ell \in \bar{I}^\delta_i\}$ by

$$\bar{m}^\delta_{i,\ell'} := \begin{cases} \breve{m}^\delta_{i,\ell'} + \sum_{\ell \in \bar{I}^\delta_{i,\mathrm{opp}}(\ell')} \frac{\frac{1}{8}\mathrm{vol}(\triangle_{\ell,\ell'})}{d^\delta_\ell} \breve{m}^\delta_{i,\ell} & \text{if } \ell' \in \bar{I}^\delta_{i,v}, \\ \breve{m}^\delta_{i,\ell'} & \text{if } \ell' \in \bar{I}^\delta_{i,m}, \end{cases} \tag{3.11}$$

we find

$$\langle x^\delta_{s(i),\ell}|_{\gamma_i}, \bar m^\delta_{i,\ell'}\rangle_{L^2(\gamma_i)} = \begin{cases} d^\delta_\ell & \text{if } I^\delta_i \ni \ell = \ell', \\ 0 & \text{if } I^\delta_i \ni \ell \neq \ell' \in \bar I^\delta_i, \end{cases}$$

so that (3.2) is valid by $d^\delta_\ell \eqsim \|x^\delta_{s(i),\ell}|_{\gamma_i}\|_{L^2(\gamma_i)} \|\bar m^\delta_{i,\ell}\|_{L^2(\gamma_i)}$.

Concerning implementation, since for $\ell \in \bar I^\delta_{i,v}$, $\bar m^\delta_{i,\ell}$ has a much larger support than $\breve m^\delta_{i,\ell}$, instead of computing the matrices $\bar{\mathbf{M}}^\delta_{i,M}$ and $\bar{\mathbf{M}}^\delta_{i,S}$ mentioned in §3.1, it is much more efficient to write them as the composition of the basis transformation defined by (3.11), that can be applied at low cost, and the matrices $\breve{\mathbf{M}}^\delta_{i,M}$ and $\breve{\mathbf{M}}^\delta_{i,S}$ one gets by replacing $\{\bar m^\delta_{i,\ell} : \ell \in \bar I^\delta_i\}$ by $\{\breve m^\delta_{i,\ell} : \ell \in \bar I^\delta_i\}$.

Finally, we have to remove $\bar m^\delta_{i,\ell}$ for $\ell \in \partial\gamma_i$. Let $\{\bar n_{i,\ell} : \ell \in \bar I^\delta_i\}$ be the nodal basis for $\bar M^\delta_i$. As discussed in §3.1, there exists a collection of functions $\{\breve\psi_{i,\ell} : \ell \in \bar I^\delta_i\}$, which is local with respect to $\triangle^\delta_i$, and with $\|\breve\psi_{i,\ell}\|_{L^2(\gamma_i)} \lesssim \|\bar n_{i,\ell}\|^{-1}_{L^2(\gamma_i)}$, such that $\breve\psi_{i,\ell}(p) = p(\ell)$ for $p \in P_2(\gamma_i)$. With, for $\ell \in \bar I^\delta_{i,m}$, $\bar I^\delta_{i,\mathrm{adj}}(\ell) := \{\ell' \in \bar I^\delta_{i,v} : \ell, \ell' \text{ are on one edge of a } \triangle \in \triangle^\delta_i\}$, see Figure 3.5, we define

$$\psi_{i,\ell} = \begin{cases} \breve\psi_{i,\ell} & \text{if } \ell \in \bar I^\delta_{i,v}, \\ \breve\psi_{i,\ell} + \frac{1}{8} \sum\limits_{\ell' \in \bar I^\delta_{i,\mathrm{adj}}(\ell)} \breve\psi_{i,\ell'} - \sum\limits_{\{\ell' \in \bar I^\delta_{i,v} : \ell \in \bar I^\delta_{i,\mathrm{opp}}(\ell')\}} \frac{\frac{1}{8}\mathrm{vol}(\triangle_{\ell,\ell'})}{d^\delta_\ell} \breve\psi_{i,\ell'} & \text{if } \ell \in \bar I^\delta_{i,m}. \end{cases}$$

From (3.10) and (3.11), we infer that

$$v \mapsto \sum_{\ell \in \bar I^\delta_i} \psi_{i,\ell}(v) \bar m_{i,\ell} = \sum_{\ell \in \bar I^\delta_i} \breve\psi_{i,\ell}(v) \bar n_{i,\ell},$$

which is thus a uniformly $L^2(\gamma_i)$-bounded quasi-interpolator of order $m - 2 = 2$.

Now for any $n \in \bar I^\delta_i \backslash I^\delta_i$, vertex or midpoint of a $\triangle \in \triangle^\delta_i$, we select a $\tilde\triangle \in \triangle^\delta_i$ with $\mathrm{dist}(\triangle, \tilde\triangle) \lesssim \mathrm{diam}(\triangle)$, such that

$$\bar I^\delta_i(\tilde\triangle) := (\bar I^\delta_i \cap \tilde\triangle) \cup \cup_{\ell \in \bar I^\delta_{i,m} \cap \tilde\triangle} \{\ell' \in \bar I^\delta_{i,v} : \ell \in \bar I^\delta_{i,\mathrm{opp}}(\ell')\} \subset \gamma_i,$$

see Figure 3.5. Using that, because of the locally quasi-uniform partition, all $\psi^\delta_{i,\ell}$ for $\ell \in I^\delta_i(\tilde\triangle)$ have comparable $L^2(\gamma_i)$-norms, we solve $(\sigma^n_\ell)_{\ell \in \bar I^\delta_i(\tilde\triangle)}$ from

$$\underset{(\sigma^n_\ell)_{\ell \in \bar I^\delta_i(\tilde\triangle)}}{\mathrm{argmin}} \left\{ \sum_{\ell \in \bar I^\delta_i(\tilde\triangle)} |\sigma^n_\ell|^2 : \sum_{\ell \in \bar I^\delta_i(\tilde\triangle)} \sigma^n_\ell \psi^\delta_{i,\ell}(p) = \psi^\delta_{i,n}(p), \quad p \in P_2(\gamma_i) \right\}.$$

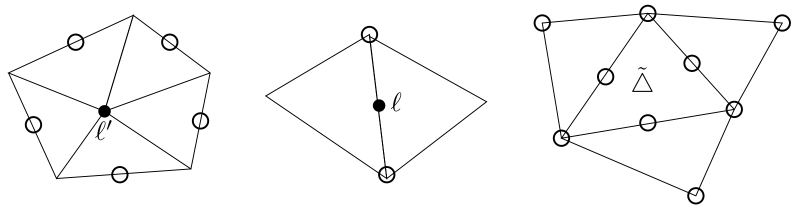

Fig. 3.5: $\bar{I}^{\delta}_{i,\mathrm{opp}}(\ell')$, $\bar{I}^{\delta}_{i,\mathrm{adj}}(\ell)$ and $\bar{I}^{\delta}_{i}(\tilde{\triangle})$

Since $\mathrm{span}\{\psi^{\delta}_{i,\ell} : \ell \in \bar{I}^{\delta}_{i}(\tilde{\triangle})\} = \mathrm{span}\{\breve{\psi}^{\delta}_{i,\ell} : \ell \in \bar{I}^{\delta}_{i}(\tilde{\triangle})\} \supset \mathrm{span}\{\breve{\psi}_{i,\ell} : \ell \in \bar{I}^{\delta}_{i} \cap \tilde{\triangle}\}$, and the transformation from $\{\psi^{\delta}_{i,\ell} : \ell \in \bar{I}^{\delta}_{i}(\tilde{\triangle})\}$ to $\{\breve{\psi}^{\delta}_{i,\ell} : \ell \in \bar{I}^{\delta}_{i}(\tilde{\triangle})\}$ is uniformly bounded invertible, we infer this problem has a unique solution with $|\sigma^{n}_{\ell}| \lesssim 1$. We remove n from $\bar{I}^{\delta}_{i}$ and thus $\bar{m}^{\delta}_{i,n}$ from the collection $\{\bar{m}^{\delta}_{i,\ell} : \ell \in \bar{I}^{\delta}_{i}\}$, and redefine

$$\bar{m}^{\delta}_{i,\ell} \leftarrow \bar{m}^{\delta}_{i,\ell} + \sigma^{n}_{\ell}\bar{m}^{\delta}_{i,n}, \quad \ell \in \bar{I}^{\delta}_{i}(\tilde{\triangle}).$$

After removal of all $n \in \bar{I}^{\delta}_{i} \backslash I^{\delta}_{i}$, the obtained collection $\{m^{\delta}_{i,\ell} : \ell \in I^{\delta}_{i}\}$ of (generalized) dual Lagrange multipliers spans a space M^{δ}_{i} that has the required local approximation properties.

Finally, for the case that the family of triangulations $(\triangle^{\delta}_{i})_{\delta \in \mathcal{I}}$ is not quasi-uniform, with the aid of Remark 2.3, the existence of uniformly $L^2(\gamma_i)$-bounded and local mortar projectors Π^{δ}_{i} can be verified in a similar way as in §3.2.1.

4 Numerical Results

In this section, we present some numerical results illustrating the performance of the quadratic tetrahedral and quadratic serendipity hexahedral mortar finite elements from §3.2.1 and §3.2.2. For the primal variable, we give the discretization errors in the $\|\cdot\|_1 := [\sum_{k=1}^{K} \|\cdot\|^2_{H^1(\Omega_k)}]^{\frac{1}{2}}$-norm, as also in the $\|\cdot\|_{L^2(\Omega)}$-norm. Using exclusively quasi-uniform partitions and having piecewise smooth solutions in our numerical examples, with h denoting the largest diameter of any underlying element, Corollary 2.6 shows that the $\|\cdot\|_1$-norm of the error is of order h^2. Using duality arguments, one can prove, see e.g. [9], that the error in $L^2(\Omega)$-norm is of order h^3. Since the $\|\cdot\|_{(H^{\frac{1}{2}}_{00}(\gamma_i))'}$-norms are not so easy to compute, we measure the discretization errors in the flux across the interfaces in a mesh-dependent

Lagrange multiplier norm, that for tetrahedral elements is defined by

$$\|\mu\|_\delta^2 := \sum_{i=1}^{N} \sum_{\triangle \in \triangle_i^\delta} \mathrm{diam}(\triangle) \|\mu|_\triangle\|^2_{L^2(\triangle)},$$

and that has an analogous definition for the hexahedral elements with $\triangle_i^\delta$, $\triangle$ replaced by $\square_i^\delta$, $\square$. Theoretically, the asymptotic rates of errors in the weighted Lagrange multiplier norm are of order h^2 for quadratic mortar finite elements, see [20]. However, under the assumption that the errors in the primal variable are equally distributed, which is the case here because of the smooth solution, the asymptotic rates can be shown to be of order $h^{5/2}$, see [30].

In our examples, we combine tetrahedral and hexahedral elements on the different subdomains. We point out that different Lagrange multipliers should be used depending on the hexahedral or tetrahedral partition on the slave side of the interface. Starting with some partition into tetrahedra or hexahedra on each subdomain, we create sequences of partitions by uniform dyadic refinements. We use $I = \mathbb{N}_0$ as index set for δ, where then δ refers to the level of refinement, that is, the number of uniform dyadic refinement steps that has been applied.

For solving the arising linear systems, we have used the multigrid method applied to a reformulation of (1.2) as a positive definite system on the product space X^δ as introduced in [31]. This multigrid method has a level independent convergence rate and is of optimal computational complexity, see [31, 29]. Our implementation is based on the finite element toolbox ug, [2].

In all our examples, we observe convergence rates for $\|u - u^\delta\|_{L^2(\Omega)}$, $\|u - u^\delta\|_1$, and $\|\lambda - \lambda^\delta\|_\delta$ that approach the values 8, 4, and $2^{\frac{5}{2}} \approx 5.66$, respectively, that when having smooth solutions are expected for dyadic refinements and quadratic/linear approximation for the primal/Lagrange multiplier variables.

In our first example, we consider $-\Delta u = f$ in Ω, where Ω is composed of five cubes $\Omega_1 := (0,1)^3$, $\Omega_2 := (1,2) \times (0,1)^2$, $\Omega_3 := (0,1)^2 \times (1,2)$, $\Omega_4 := (-1,0) \times (0,1)^2$ and $\Omega_5 := (0,1)^2 \times (-1,0)$. Here, subdomain Ω_1, which has an initial partition into 27 hexahedra, is the slave subdomain and the others are master subdomains. In each of the master subdomains, the initial partition consists of 6 tetrahedra. The right hand side f and the Dirichlet boundary conditions are chosen such that the exact solution is given by

$$u(x, y, z) = e^{-0.25(x^2+y^2+z^2)}(\cos(5x + z) + 3\sin(4y + z)).$$

In Figure 4.1, the decomposition of the domain, the initial finite element partitions, and the isolines of the solution at the interface $z = 1$ are shown. The discretization errors are given in Table 4.1.

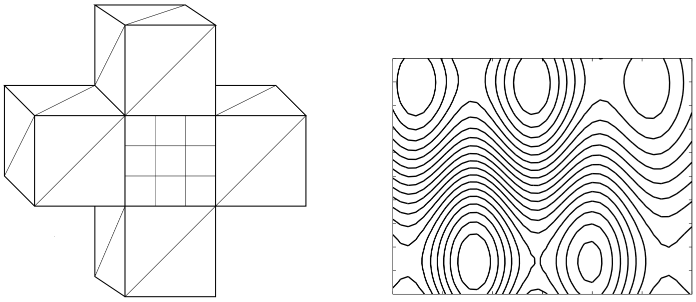

Fig. 4.1: Decomposition of the domain and initial partitions (left), isolines of the solution at the plane $z = 1$ (right) for Example 1.

Table 4.1. Discretization errors for Example 1.

δ	# elem.	$\|u-u^\delta\|_{L^2(\Omega)}$		$\|u-u^\delta\|_1$		$\|\lambda-\lambda^\delta\|_\delta$	
0	51	3.24e-1		5.75e-1		3.20e+0	
1	408	5.02e-2	6.5	1.70e-1	3.4	8.44e-1	3.8
2	3264	6.79e-3	7.4	4.70e-2	3.6	1.60e-1	5.3
3	26112	8.52e-4	8.0	1.19e-2	3.9	2.94e-2	5.4

In our second example, we consider a domain and problem from [20]. Here, $\Omega := (0,2)\times(0,1)\times(0,2)$ is decomposed into four subdomains $\Omega_1 := (0,1)^3$, $\Omega_2 := (0,1)^2\times(1,2)$, $\Omega_3 := (1,2)\times(0,1)^2$ and $\Omega_4 := (1,2)\times(0,1)\times(1,2)$, with Ω_2 and Ω_3 being slave subdomains, and Ω_1 and Ω_4 being master subdomains. On both Ω_2 and Ω_3 the initial partition consists of 27 hexahedra, and on both Ω_1 and Ω_4 it consists of 6 hexahedra. The problem for this example is given by a reaction-diffusion equation

$$-\mathrm{div}(a\nabla u) + u = f \quad \text{in} \quad \Omega,$$

where $a = 1$ in Ω_1 and Ω_4, and $a = 10$ in Ω_2 and Ω_3. The right hand side f and the Dirichlet boundary conditions are chosen such that the exact solution is given by

$$u(x,y,z) = (x-1)y(z-1)e^{-(x-1)^2-y^2-(z-1)^2}\cos(2\,x+2\,y+2\,z)/a.$$

Note that u has a jump in the normal derivative across the interface, whereas the flux is continuous. In Figure 4.2, the decomposition of the domain together with the initial finite element partitions, the isolines of the solution on the plane $y = \frac{1}{2}$, and the flux of the exact solution at the interface $x = 1$ are shown. The discretization errors

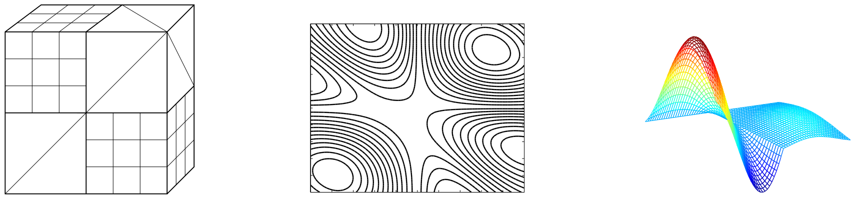

Fig. 4.2: Decomposition of the domain and initial partitions (left), isolines of the solution at the plane $y = \frac{1}{2}$ (middle) and flux of the exact solution at the interface $x = 1$ (right) for Example 2.

are given in Tables 4.2.

Table 4.2. Discretization errors for Example 2.

δ	# elem.	$\|u - u^\delta\|_{L^2(\Omega)}$		$\|u - u^\delta\|_1$		$\|\lambda - \lambda^\delta\|_\delta$	
0	66	7.94e-2		1.57e-1		5.19e-2	
1	528	1.69e-2	4.7	4.99e-2	3.1	4.11e-3	13
2	4224	2.35e-3	7.2	1.48e-2	3.4	6.26e-4	6.5
3	33792	2.73e-4	8.6	3.87e-3	3.8	1.04e-4	6.0

Our third example is given by $-\Delta u = f$ in Ω, where the non-convex Ω is composed of eight cubes $\Omega_1 := (0,1)^3$, $\Omega_2 := (1,2) \times (0,1)^2$, $\Omega_3 := (2,3) \times (0,1)^2$, $\Omega_4 := (0,1)^2 \times (1,3)$, $\Omega_5 := (2,3) \times (0,1) \times (1,3)$, $\Omega_6 := (0,1)^2 \times (3,4)$, $\Omega_7 := (1,2) \times (0,1) \times (3,4)$ and $\Omega_8 := (2,3) \times (0,1) \times (3,4)$. Here, subdomains Ω_1, Ω_3, Ω_6 and Ω_8 are taken as slave subdomains and the others are master subdomains. The initial partitions consist of tetrahedra for Ω_1 and Ω_3, and of hexahedra for the other subdomains, and they are illustrated in Figure 4.3. The right hand side f and the Dirichlet boundary conditions are chosen such that the exact solution is given by

$$u(x, y, z) = e^{-\frac{1}{4}(x+y+z)}(\sin(5xy) + \cos(5yz))(x + y^2 + z^2).$$

The isolines of this solution at the interface $z = 1$ can also be found in Figure 4.3. Finally, the discretization errors are given in Table 4.3.

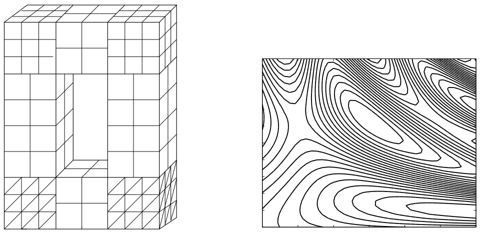

Fig. 4.3: Decomposition of the domain and initial partitions (left), isolines of the solution at the plane $z = 1$ (right) for Example 3.

Table 4.3. Discretization errors for Example 3.

δ	# elem.	$\|u-u^\delta\|_{L^2(\Omega)}$		$\|u-u^\delta\|_1$		$\|\lambda-\lambda^\delta\|_\delta$	
0	426	6.94e-1		9.13e-1		1.07e+1	
1	3408	1.43e-1	4.9	4.18e-1	2.2	5.79e+0	1.8
2	27264	1.98e-2	7.2	1.10e-1	3.8	1.34e+0	4.3
3	218112	2.63e-3	7.5	2.85e-2	3.9	1.71e-1	7.8

In our last example, we consider an example in linear elasticity. We point out that the theory can easily be extended to this case with standard arguments. For this example, we take the domain and the problem from [18]. Here, the computational domain Ω, which is a beam, is decomposed into three subdomains Ω_1, Ω_2 and Ω_3 with $\Omega_1 := (0,50) \times (0,10) \times (0,2)$, $\Omega_2 := (0,50) \times (3,7) \times (2,11)$ and $\Omega_3 := (0,50) \times (0,10) \times (11,13)$. The decomposition of the domain, is given in the left picture of Figure 4 with an initial partition. Here, Ω_2 is the slave subdomain, so that, although the decomposition is geometrically nonconforming, condition (A.1) is satisfied. Our linear elasticity problem is

$$-\nabla \cdot \sigma(\mathbf{u}) = 0 \quad \text{in} \quad \Omega$$

with $\mathbf{u} = 0$ on Γ_D and $\sigma(\mathbf{u})\mathbf{n} = \mathbf{f}$ on Γ_N, where $\sigma(\mathbf{u})$ is a second order tensor defined as

$$\sigma(\mathbf{u}) = 2\mu\varepsilon(\mathbf{u}) + \lambda \text{trace}\, \varepsilon(\mathbf{u}) I, \quad \varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + [\nabla\mathbf{u}]^T\right)$$

with $\mu = 8.2$, $\lambda = 10$ (kg/cm^3), and I is the identity matrix of size 3×3. Here, Γ_D is the part of the boundary of Ω with $x = 0$ and $x = 50$ so that the left and the right sides of each subdomain are fixed, and $\Gamma_N := \partial\Omega \backslash \Gamma_D$. The function $\mathbf{f} = (f_1, f_2, f_3)$ on Γ_N is given as $f_1 = f_2 = 0$, and

$$f_3 = \begin{cases} -0.35 & \text{if } 22 \le x \le 28 \text{ and } z = 13 \\ 0 & \text{otherwise} \end{cases}$$

so that a constant vertical force is applied from the top boundary ($z = 13$) of Ω. We have computed the solution with quadratic finite elements. The resulting deformation of the beam is given in the right picture of Figure 4.

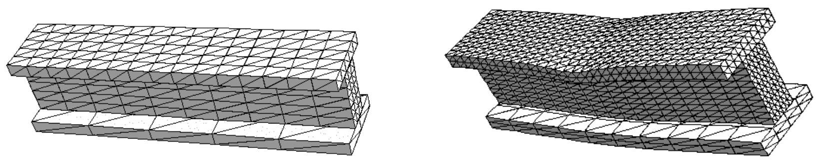

Fig. 4.4: Decomposition of the domain with initial partition (left) and the distorted mesh after refining once (right)

References

1. D. N. Arnold, D. Boffi, and R. S. Falk. Approximation by quadrilateral finite elements. *Math. Comp.*, 71(239):909–922, 2002.
2. P. Bastian, K. Birken, K. Johannsen, S. Lang, N. Neuß, H. Rentz–Reichert, and C. Wieners. UG – a flexible software toolbox for solving partial differential equations. *Computing and Visualization in Science*, 1:27–40, 1997.
3. F. Ben Belgacem. The mortar finite element method with Lagrange multipliers. *Numer. Math.*, 84:173–197, 1999.
4. F. Ben Belgacem and Y. Maday. The mortar element method for three dimensional finite elements. M^2AN, 31:289–302, 1997.
5. C. Bernardi, N. Debit, and Y. Maday. Coupling finite element and spectral methods: First results. *Math. Comp.*, 54:21–39, 1990.
6. C. Bernardi, Y. Maday, and A.T. Patera. Domain decomposition by the mortar element method. In H. Kaper et al., editor, *Asymptotic and numerical methods for partial differential equations with critical parameters*, pages 269–286. Reidel, Dordrecht, 1993.
7. C. Bernardi, Y. Maday, and A.T. Patera. A new nonconforming approach to domain decomposition: the mortar element method. In H. Brezzi et al., editor, *Nonlinear partial differential equations and their applications*, pages 13–51. Paris, 1994.
8. D. Braess and W. Dahmen. Stability estimates of the mortar finite element method for 3–dimensional problems. *East–West J. Numer. Math.*, 6:249–264, 1998.
9. D. Braess, W. Dahmen, and C. Wieners. A multigrid algorithm for the mortar finite element method. *SIAM J. Numer. Anal.*, 37:48–69, 1999.
10. F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*. Springer–Verlag, New York, 1991.
11. P. Clément. Approximation by finite element functions using local regularization. *RAIRO Anal. Numér.*, 9:77–84, 1975.

12. W. Dahmen, B. Faermann, I.G. Graham, W. Hackbusch, and S.A. Sauter. Inverse inequalities on non-quasiuniform meshes and application to the mortar element method. IGPM report 201, RWTH Aachen, April 2001.
13. W. Dahmen and R.P. Stevenson. Element-by-element construction of wavelets satisfying stability and moment conditions. *SIAM J. Numer. Anal.*, 37(1):319–352, 1999.
14. R. DeVore. Nonlinear approximation. *Acta Numer.*, 7:51–150, 1998.
15. J. Gopalakrishnan and J. Pasciak. Multigrid for the mortar finite element method. *SIAM J. Numer. Anal.*, 37:1029–1052, 2000.
16. P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.
17. J. Huang and J. Zou. A mortar element method for elliptic problems with discontinuous coefficients. *IMA J. Numer. Anal.*, 22(4):549–576, 2002.
18. C. Kim, R.D. Lazarov, J.E. Pasciak, and P.S. Vassilevski. Multiplier spaces for the mortar finite element method in three dimensions. *SIAM J. Numer. Anal.*, 39:519–538, 2001.
19. B.P. Lamichhane and B.I. Wohlmuth. Higher order dual Lagrange multiplier spaces for mortar finite element discretizations. *CALCOLO*, 39:219–237, 2002.
20. B.P. Lamichhane and B.I. Wohlmuth. A quasi-dual Lagrange multiplier space for serendipity mortar finite elements in 3D. M^2AN, 38:73–92, 2004.
21. J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I.* Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.
22. P. Oswald. *Multilevel finite element approximation: Theory and applications.* B.G. Teubner, Stuttgart, 1994.
23. P. Oswald and B.I. Wohlmuth. On polynomial reproduction of dual FE bases. In N. Debit, M. Garbey, R. Hoppe, J. Pèriaux, D. Keyes, and Y. Kuznetsov, editors, *Thirteenth International Conference on Domain Decomposition Methods*, pages 85–96, 2001.
24. L. R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.
25. P. Seshaiyer and M. Suri. Uniform *hp* convergence results for the mortar finite element method. *Math. Comp.*, 69(230):521–546, 2000.
26. H. Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1978.
27. C. Wieners and B.I. Wohlmuth. Duality estimates and multigrid analysis for saddle point problems arising from mortar discretizations. *SISC*, 24:2163–2184, 2003.
28. B.I. Wohlmuth. A mortar finite element method using dual spaces for the Lagrange multiplier. *SIAM J. Numer. Anal.*, 38:989–1012, 2000.
29. B.I. Wohlmuth. *Discretization Methods and Iterative Solvers Based on Domain Decomposition*, volume 17 of *LNCS*. Springer Heidelberg, 2001.
30. B.I. Wohlmuth. A comparison of dual Lagrange multiplier spaces for mortar finite element discretizations. M^2AN, 36:995–1012, 2002.
31. B.I. Wohlmuth and R.H. Krause. Multigrid methods based on the unconstrained product space arising from mortar finite element discretizations. *SIAM J. Numer. Anal.*, 39:192–213, 2001.