COMPOSITE WAVELET BASES WITH EXTENDED STABILITY AND CANCELLATION PROPERTIES*

ROB STEVENSON†

Abstract. The efficient solution of operator equations using wavelets requires that they generate a Riesz basis for the underlying Sobolev space, and that they have cancellation properties of a sufficiently high order. Suitable biorthogonal wavelets were constructed on reference domains as the n-cube, which bases using a domain decomposition approach have been used as building blocks to construct biorthogonal composite wavelets on general domains or manifolds, where, in order to end up with local wavelets, biorthogonality was realized with respect to a modified L_2-scalar product. The use of this modified scalar product restricts the application of the wavelets to problems of orders strictly larger than -1, and wavelets with supports that extend to more than one patches generally have no cancellation properties. In this paper, we construct local composite wavelets that are sufficiently close to being biorthogonal with respect to the standard L_2-scalar product, so that they generate Riesz bases for the Sobolev spaces H^s for *full range of s* that is allowed by the continuous gluing of functions over the patch interfaces, the properties of the primal and dual approximation spaces on the reference domain, and, in the manifold case, by the regularity of the manifold. Moreover, all wavelets have *cancellation properties of the full order* induced by the approximation properties of the dual spaces on the reference domain. We illustrate our findings by a concrete realization of wavelets on a perturbed sphere.

Key words. Wavelets, Riesz bases, cancellation properties, domain decomposition, boundary integral equations.

AMS subject classifications. 46B15, 46E35, 65N55, 65R20, 65T60.

1. Introduction. The use of wavelet bases for solving operator equations, as *partial differential equations* or *(boundary) integral equations*, has a number of advantages, cf. [Dah97, Coh03]. Assuming that the operator defines a boundedly invertible mapping between a (Sobolev) space H and its dual, and that, properly scaled, the wavelets are stable in H-metric, i.e., the infinite collection of wavelets generates a *Riesz basis* for H, then the stiffness matrix in wavelet coordinates resulting from a Ritz-Galerkin discretization is well-conditioned uniformly in its size, guaranteeing a uniform rate of convergence of an iterative method for inverting this matrix. In case of a differential operator, this stiffness matrix is not truly sparse, but has the well-known "finger structure". However, for multiplying with this matrix one may switch to single-scale basis, with respect to which the stiffness matrix is sparse. For integral operators, however, the stiffness matrix with respect to both single-scale and wavelet basis is densely populated. Here the second important property of wavelets can be exploited of having vanishing moments or, more generally, *cancellation properties*, meaning that the integral of a wavelet against a smooth function vanishes with a certain order of the length scale of the wavelet. When, depending on the order of the operator and the order of approximation, this order of the cancellation properties is sufficiently large, then the stiffness matrix with respect to the wavelet basis can be a priori compressed to a sparse one without reducing the order of convergence, with which also for solving these integral equations a method of linear complexity is obtained ([Sch98, DHS02]).

Instead of projecting the operator equation onto a fixed finite dimensional space,

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†Department of Mathematics, Utrecht University, P.O. Box 80.010, NL-3508 TA Utrecht, The Netherlands (stevenson@math.uu.nl).

or a sequence of such spaces, and afterwards solving the resulting matrix-vector problem with an iterative method, the availability of a *Riesz basis* for H opens an attractive alternative for approximating the solution by adaptive wavelet methods ([CDD01, CDD02]). By writing this unknown solution in terms of this basis and testing the equation for all basis functions, one obtains an infinite dimensional matrix vector problem, that is *equivalent* to the operator equation, and that is well-posed in ℓ_2-metric, meaning that it can be solved using an iterative method. In each iteration of such a method, the application of the infinite stiffness matrix to the current approximation vector has to be approximated. Here the concept of *adaptivity* enters; the accuracy with which a column is approximated grows with the modulus of the corresponding entry of the vector. The resulting method, extended with a so-called coarsening routine to remove small entries, can be proven to be optimal in the following sense. Whenever, for a certain range of s, the solution is in a class of functions for which the error of the best N-term approximations from the wavelet basis decays like N^{-s}, then the sequence of approximations produced by this adaptive method has the same rate of convergence, whereas the computational cost is equivalent to their support sizes. A necessary condition for this statement is that the stiffness matrix is sufficiently close to a sparse matrix, which depends on the smoothness of the wavelets and, again, on the *order of the cancellation properties* ([Ste04]).

Aiming at the aforementioned applications, this paper deals with the construction on general n-dimensional domains or manifolds of wavelets that, properly scaled, generate Riesz bases for a range of Sobolev spaces, and satisfy cancellation properties of any required order. To be able to choose this order independently from the order of approximation, we will consider biorthogonal wavelets. Their construction starts with two nested sequences of approximation spaces that both satisfy Jackson and Bernstein estimates ("multiresolution analyses"), where then the primal and dual wavelets are sought as bases of the biorthogonal complements of successive approximation spaces at primal and dual side, respectively. In case the primal and dual approximation spaces can be equipped with bases of local, biorthogonal scaling functions, then local primal wavelets are found by applying the biorthogonal projector onto a local basis of *some* complement space of two successive primal approximation spaces, where then also the corresponding dual wavelets are local. Actually, for only constructing local primal wavelets, a reduced set of assumptions already suffices, which for simplicity we will ignore in this introduction. Note that in algorithms for solving operator equations, the dual wavelets usually do not play any role.

Biorthogonal scaling functions have been constructed on the real line ([CDF92]), and as adaptations of these, on the interval ([DKU99]). By taking tensor products, one obtains biorthogonal scaling functions on the n-dimensional unit cube. To construct biorthogonal scaling functions and wavelets on general domains and manifolds, a domain decomposition approach has been developed in [DS99a] (see [CTU99, CM00] for related approaches). The domain or manifold of interest is written as a disjoint union of smooth parametric images of the unit cube. The biorthogonal scaling functions on the cube are lifted to the patches, and, assuming that the decomposition satisfies some matching condition, they are continuously connected over the interfaces. With respect to a *modified L_2-scalar product*, defined by ignoring the Jacobian determinants in the definition of the canonical L_2-scalar product, the resulting collections of scaling functions are biorthogonal, and, as on the unit cube, wavelets, called *composite wavelets*, can be constructed using the biorthogonal projector. There are however two principal limitations related to the realization of biorthogonality with respect to the modified L_2-scalar product. First of all, wavelets with supports that extend to more than one patches, generally have *no cancellation properties* with respect to the canonical L_2-scalar product, so that results concerning matrix compression do not apply to entries involving such wavelets. Secondly, with respect to the interpretation of a wavelet as a functional using the duality pairing in terms of the canonical L_2-scalar product, generally the resulting wavelets *cannot* generate a *Riesz basis for H^s for $s \leq -\frac{1}{2}$*, so that for operators of order $2s \leq -1$, like the single-layer potential operator, neither optimal preconditioning results are valid, nor the adaptive wavelet method can be applied.

These limitations were already recognized in [DS99a], and in [DS99b], Dahmen and Schneider developed an elegant approach to construct wavelets on general domains or manifolds that, properly scaled, generate Riesz bases for H^s for in principal any s, and that have cancellation properties of any required order. Unfortunately, so far with this approach it seems not easy to construct wavelets that have competitive quantitative properties.

In this paper, we reconsider the approach from [DS99a], except that, in view of the aforementioned limitations, we make use of the canonical L_2-scalar product. Although, generally, the lifted and connected scaling functions are not biorthogonal with respect to this scalar product, we can derive a general formula for the corresponding biorthogonal wavelets. However, since this formula involves the inverse of the matrix consisting of all L_2-scalar products between primal and dual scaling functions, which matrix is thus generally not diagonal, these wavelet have global supports. On the other hand, this matrix is nearly diagonal so that its inverse can be well approximated by sparse matrices which gives rise to local, approximate wavelets. We derive general conditions under which, properly scaled, such approximate wavelets generate a Riesz basis for H^s for the *full range of s* that is allowed by the continuous gluing of the scaling functions over the interfaces, the properties of the primal and dual approximation spaces on the cube, and, in the manifold case, by the regularity of the manifold. We give three possibilities for the construction of approximate wavelets that are local, generate Riesz bases for H^s for the aforementioned full range of s, and have *cancellation properties of the full order* induced by the approximation properties of the dual spaces on the unit cube. Firstly, we show that the approximation of the inverse of the matrix of L_2-scalar products of primal and dual scaling functions by a suitable, fixed number of Jacobi iterations yields such approximate wavelets. Secondly, since these wavelets have relatively large supports, we show that away from the patch interfaces they can be replaced by the wavelets one gets by ignoring the Jacobian determinants, which are the wavelets yielded by the approach from [DS99a]. Thirdly, we show that also along the patch interfaces suitable approximate wavelets with smaller supports can be constructed, which however will involve solving some local systems. Although several proofs will be quite involved, we emphasize that the implementation of the approximate wavelets is relatively straightforward.

In [DS99c, Ste03], we constructed wavelet bases for Lagrange finite element spaces based on a subdivision of polygonal domains into n-simplices. In this paper, we include the option that these finite element wavelets, or more precisely the underlying scaling functions are used as building blocks for wavelets on general (non-polygonal) domains or manifolds, where then the unit n-cube as reference domain should be replaced by some reference n-simplex.

This paper is organized as follows. In the remainder of this section we fix a few notations. In Section 2, we define the type of domains and manifolds and their parametrizations that we will consider, as well as the Sobolev spaces, that may involve zero order Dirichlet boundary conditions, which we are going to equip with Riesz bases. In Section 3, we collect all assumptions on the multiresolution analyses on the reference domain. The induced, continuous multiresolution analyses on the target domain or manifold are defined in Section 4. Although we consider a more general framework, the main construction principles from Sections 2-4 can already be found in [DS99a]. Biorthogonal space decompositions and the, generally, globally supported biorthogonal wavelets are constructed in Section 5. Sections 6 and 7, which form the main part of this paper, are devoted to the construction of local, approximate wavelets. Finally, in Section 8 we show examples of approximate wavelets on a perturbed sphere, and give some numerically computed condition numbers.

In order to avoid the repeated use of generic but unspecified constants, in this paper by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

Let H be a separable Hilbert space with with scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. For a countable collection Σ of functions in H, which we formally view as a (column) vector, and for $\mathbf{c} = (c_\sigma)_{\sigma \in \Sigma}$ a vector of scalars, with $\mathbf{c}^T \Sigma$ we will mean the expansion $\sum_{\sigma \in \Sigma} c_\sigma \sigma$. The span of Σ will be denoted as $\mathcal{S}(\Sigma)$. For $x \in H$, with $\langle \Sigma, x \rangle$ and $\langle x, \Sigma \rangle$ we will mean the column- and row-vectors with coefficients $\langle \sigma, x \rangle$ and $\langle x, \sigma \rangle$, $\sigma \in \Sigma$. When $\tilde{\Sigma}$ is another countable collection in H, with $\langle \Sigma, \tilde{\Sigma} \rangle$ we denote the matrix $(\langle \sigma, \tilde{\sigma} \rangle)_{\sigma \in \Sigma, \tilde{\sigma} \in \tilde{\Sigma}}$. For $V \subset H$ being a dense, continuously embedded Banach space, as usual we will use $\langle \cdot, \cdot \rangle$ sometimes also to denote the duality pairing $\langle \cdot, \cdot \rangle_{V \times V'}$, which, with the aforementioned meaning, can also be applied to collections from V and/or V'.

On the spaces of (possibly infinite) scalar vectors or matrices, we will exclusively use the ℓ_2-scalar product, ℓ_2-norm or the resulting operator norm, that we therefore simply denote by $\langle \cdot, \cdot \rangle$ or $\|\cdot\|$, respectively. A collection Σ is called a *Riesz system* when $\|\mathbf{c}^T \Sigma\| \eqsim \|\mathbf{c}\|$, i.e., when $\langle \Sigma, \Sigma \rangle$ is boundedly invertible, and Σ is called a *Riesz basis* when it is in addition a basis for H. When Σ depends on a parameter, we will speak about *uniform Riesz systems* (or bases) when the above equivalence holds uniformly over the values this parameter may attain. We set $\|\Sigma\| = \|\langle \Sigma, \Sigma \rangle\|^{\frac{1}{2}}$, and collect a few properties related to this definition:

PROPOSITION 1.1.
(i) $\sup_{\mathbf{c} \neq 0} \frac{\|\mathbf{c}^T \Sigma\|}{\|\mathbf{c}\|} = \|\Sigma\|$,
(ii) $\|\langle \Sigma, \tilde{\Sigma} \rangle\| \leq \|\Sigma\| \|\tilde{\Sigma}\|$,
(iii) $\|\Sigma + \tilde{\Sigma}\| \leq \|\Sigma\| + \|\tilde{\Sigma}\|$,
(iv) *For a matrix* $\mathbf{A}$, $\|\mathbf{A}\Sigma\| \leq \|\mathbf{A}\| \|\Sigma\|$.

Proof. For (i), use $\|\mathbf{c}^T \Sigma\|^2 = \langle \langle \Sigma, \Sigma \rangle \mathbf{c}, \mathbf{c} \rangle$. Part (ii) follows from $|\langle \langle \Sigma, \tilde{\Sigma} \rangle \mathbf{c}, \tilde{\mathbf{c}} \rangle| = |\langle \mathbf{c}^T \Sigma, \tilde{\mathbf{c}}^T \tilde{\Sigma} \rangle| \leq \|\mathbf{c}\| \|\tilde{\mathbf{c}}\| \|\Sigma\| \|\tilde{\Sigma}\|$ because of (i). Part (iii) follows easily from (ii). For (iv), use $\langle \mathbf{A}\Sigma, \mathbf{A}\Sigma \rangle = \mathbf{A} \langle \Sigma, \Sigma \rangle \mathbf{A}^*$. ☐

2. Domains and function spaces. For some $n' \geq n \geq 1$, let Γ be a n-dimensional bounded manifold in $\mathbb{R}^{n'}$, with or without a boundary. For $\square$ denoting the interior of either the n-cube $[0,1]^n$ or, despite of its notation, of some reference n-simplex, we assume that Γ is given as

$$\overline{\Gamma} = \cup_{q=1}^M \overline{\Gamma_q}, \text{ with } \Gamma_q \cap \Gamma_{q'} = \emptyset \text{ when } q \neq q', \text{ and } \Gamma_q = \kappa_q(\square),$$

where $\kappa_q : \mathbb{R}^n \to \mathbb{R}^{n'}$ be some smooth, regular parametrizations. With Π we will denote the collection of all affine mappings from $\square$ onto $\square$. So in case $\square$ is the interior of an n-simplex, this collection consists of the permutations of the $n+1$ barycentric coordinates, and otherwise it consists of the compositions of any permutation of the n Cartesian coordinates and reflections of type $x \mapsto (x_1, \ldots, x_{i-1}, 1-x_i, x_{i+1}, \ldots, x_n)$ $(1 \le i \le n)$. We assume that the splitting of Γ into the patches Γ_q is conforming in the sense for any $q \neq q'$, $\overline{\Gamma_q} \cap \overline{\Gamma'_q}$ is either empty, or

$$\kappa_q^{-1}(\overline{\Gamma_q} \cap \overline{\Gamma'_q}) \text{ is a face of } \square,$$

and, in addition, that the parametrizations can be chosen such that the following matching condition is satisfied: There exists a $\pi \in \Pi$ with

$$\kappa_{q'} \circ \pi \circ \kappa_q^{-1} = I \quad \text{on } \overline{\Gamma_q} \cap \overline{\Gamma'_q}. \tag{$\mathcal{M}$}$$

Here and in the remainder of this paper, with a "face" of $\square$, we mean a (complete, closed) face of any dimension $0 \le k \le n-1$, i.e., for $n = 3$, it is either a vertex, an edge or a facet. Note that our setting allows Γ to be a bounded domain in $\mathbb{R}^n$, as well as an open or closed bounded manifold in $\mathbb{R}^{n'}$ for some $n' > n$.

We include the possibility that homogeneous, zero order Dirichlet boundary conditions are prescribed on some part $\partial\Gamma_D \subset \overline{\Gamma}\backslash\Gamma$, for which, for all $1 \le q \le M$,

$$\kappa_q^{-1}(\partial\Gamma_D \cap \overline{\Gamma}_q) \text{ is a, possibly empty, union of faces of } \square\,. \tag{2.1}$$

For some $s_\Gamma > 0$, we assume that, globally,

$$\Gamma \in C^{s_\Gamma} \text{ when } s_\Gamma \notin \mathbb{N}, \text{ or } \Gamma \in C^{s_\Gamma - 1,1} \text{ when } s_\Gamma \in \mathbb{N},$$

which means that for $0 \le s < s_\Gamma \notin \mathbb{N}$, or $0 \le s \le s_\Gamma \in \mathbb{N}$, the Sobolev spaces

$$\mathcal{H}^s(\Gamma) := \begin{cases} H^s_{0,\partial\Gamma_D}(\Gamma) & \text{when } s \le 1, \\ H^s(\Gamma) \cap H^1_{0,\partial\Gamma_D}(\Gamma) & \text{when } s > 1 \end{cases}$$

can be defined in the usual way using a partition of unity relative to some atlas. For $s > 0$ in the above range, $\mathcal{H}^{-s}(\Gamma)$ will be understood as being the dual of $\mathcal{H}^s(\Gamma)$.

With μ being the induced Lebesgue measure on Γ, the inner product on $L_2(\Gamma)$ is given by

$$\langle u, v\rangle_{L_2(\Gamma)} = \int_\Gamma u\overline{v}d\mu = \sum_{q=1}^M \langle u \circ \kappa_q, v \circ \kappa_q\rangle_{L_2(\square), |\partial\kappa_q|}, \tag{2.2}$$

where, for $w \in L^\infty(\square)$ with $w > 0$ a.e., $\langle \hat{u}, \hat{v}\rangle_{L_2(\square),w} := \int_\square \hat{u}(z)\overline{\hat{v}(z)}w(z)dz$, and $|\partial\kappa_q| : z \mapsto |\partial\kappa_q(z)|$ are the Jacobian determinants. We will also make use of a modified inner product

$$\langle\!\langle u, v\rangle\!\rangle_0 := \sum_{q=1}^M \langle u \circ \kappa_q, v \circ \kappa_q\rangle_{L_2(\square)}, \tag{2.3}$$

which is equivalent to $\langle\, ,\, \rangle_{L_2(\Gamma)}$ in the sense that $|\!|\!| \cdot |\!|\!|_0 := \langle\!\langle \cdot, \cdot\rangle\!\rangle_0^{\frac{1}{2}} \eqsim \| \cdot \|_{L_2(\Gamma)}$. More generally, for any $s \ge 0$, we define

$$\langle\!\langle u, v\rangle\!\rangle_s = \sum_{q=1}^M \langle u \circ \kappa_q, v \circ \kappa_q\rangle_{H^s(\square)},$$

and let $\mathcal{H}_s(\Gamma)$ denote the closure with respect to $|\!|\!|\cdot|\!|\!|_s := \langle\!\langle\cdot,\cdot\rangle\!\rangle_s^{\frac{1}{2}}$ of the set all *globally continuous*, and with respect to the subdivision $\overline{\Gamma} = \cup_{q=1}^M \overline{\Gamma_q}$, *piecewise* C^∞ functions on Γ that are *zero on* $\partial\Gamma_D$, and define $\mathcal{H}_{-s}(\Gamma) = (\mathcal{H}_s(\Gamma))'$. For $0 \leq s < s_\Gamma \notin \mathbb{N}$ or $0 \leq s \leq s_\Gamma \in \mathbb{N}$, it holds that $\|\cdot\|_{H^s(\Gamma)} \eqsim |\!|\!|\cdot|\!|\!|_s$ on $H^s(\Gamma)$. Furthermore, if $s < \frac{3}{2}$, then the functions in the aforementioned set generate a dense subset in $\mathcal{H}^s(\Gamma)$. Using in addition duality, we infer that

$$\mathcal{H}^s(\Gamma) \asymp \mathcal{H}_s(\Gamma) \quad (|s| < \tfrac{3}{2} \text{ with } |s| < s_\Gamma \notin \mathbb{N} \text{ or } |s| \leq s_\Gamma \in \mathbb{N}), \tag{2.4}$$

meaning that both spaces agree as sets and have equivalent norms. The spaces $\mathcal{H}_s(\Gamma)$ will only serve as auxiliary spaces to be able to prove that the wavelets we are going to construct generate, properly scaled, a Riesz basis for $\mathcal{H}^s$ for the full range of s, in case this range is limited by the regularity of Γ to a *closed* range $[-s_\Gamma, s_\Gamma]$.

3. Multiresolution analyses on the reference domain. On the reference domain, we will need two nested sequences of approximation spaces that satisfy Jackson and Bernstein estimates. We will assume that these spaces are equipped with single-scale bases, that after lifting to the patches can be continuously connected over the interfaces. Furthermore, we will assume that the rate of approximation given by the Jackson estimate is realized by some concrete projector, with which it will be shown that the induced multiresolution analysis on Γ satisfies the corresponding Jackson estimate. We will make some assumptions (($\mathcal{J}1$) and ($\mathcal{J}2$)) connecting primal and dual multiresolution analyses to ensure the existence and uniform boundedness of the biorthogonal projector (cf. Proposition 5.2), and finally we will assume the existence of a suitable "initial stable completion".

For $j \in \mathbb{N}_0$, let $I_j^\square \subset \overline{\square}$ be some index set with

$$\pi(I_j^\square) = I_j^\square \quad (\pi \in \Pi), \quad \sup_{y\in\square} \#(I_j^\square \cap B(y; 2^{-j})) \lesssim 1.$$

For completeness, for $A \subset \mathbb{R}^n$ and $\delta \geq 0$, we set $B(A;\delta) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) \leq \delta\}$ with $B(\emptyset;\delta) = \emptyset$. For $j \in \mathbb{N}_0$, we assume a collection $\Phi_j^\square = (\phi_{j,x}^\square)_{x\in I_j^\square} \subset C(\overline{\square})$, usually referred to as being *scaling functions*, such that

($\mathcal{L}$) $\exists$ constant $\varepsilon > 0$, $\operatorname{supp} \phi_{j,x}^\square \subset B(x; \varepsilon 2^{-j})$.

($\mathcal{V}$) $\phi_{j,x}^\square$ vanishes on any face of $\square$ that does not contain x.

($\mathcal{S}$) $\phi_{j,x}^\square = \phi_{j,\pi(x)}^\square \circ \pi, \quad (\pi \in \Pi)$.

($\mathcal{R}$) $\Phi_j^\square$ is a uniform $L_2(\square)$-Riesz system.

($\mathcal{J}$) There exists a collection of functionals $\Lambda_j^\square = (\lambda_{j,x}^\square)_{x\in I_j^\square} \subset C(\overline{\square})'$ such that

  (i) $\exists$ constant $\vartheta > 0, \operatorname{supp} \lambda_{j,x}^\square \subset B(x; \vartheta 2^{-j})$.

  (ii) If $x \in \partial\square$, then $\operatorname{supp} \lambda_{j,x}^\square$ is contained in the lowest dimensional face of $\square$ that contains x.

  (iii) $\langle u, \lambda_{j,\pi(x)}^\square\rangle_{L_2(\square)} = \langle u\circ\pi, \lambda_{j,x}^\square\rangle_{L_2(\square)}, \quad (\pi \in \Pi)$.

  (iv) $|\langle u, \lambda_{j,x}^\square\rangle_{L_2(\square)}| \lesssim 2^{-jn/2}\|u\|_{L_\infty(\operatorname{supp}\lambda_{j,x}^\square)}$.

  (v) $\langle \Phi_j^\square, \Lambda_j^\square\rangle_{L_2(\square)} = I$.

  (vi) For some $\frac{n}{2} < d \in \mathbb{N}$, $P_{d-1}(\square) \subset \mathcal{S}(\Phi_j^\square)$.

($\mathcal{N}$) $\mathcal{S}(\Phi_j^\square) \subset \mathcal{S}(\Phi_{j+1}^\square)$.

($\mathcal{B}$) For some $\gamma > 0$, and any $s \in [0, \gamma)$, it holds that

$$\|u_j\|_{H^s(\square)} \lesssim 2^{sj}\|u_j\|_{L_2(\square)}, \quad (u_j \in \mathcal{S}(\Phi_j^\square)).$$

Note that, in particular, ($\mathcal{J}$)(ii) implies that for x being a vertex of $\square$, $\langle u, \lambda_{j,x}^\square\rangle_{L_2(\square)}$ is a multiple of $u(x)$. Examples of such collections will be given at the end of this section.

PROPOSITION 3.1. *For the projector $P_j^\square : u \mapsto \langle u, \Lambda_j^\square\rangle_{L_2(\square)}\Phi_j^\square$ onto $\mathcal{S}(\Phi_j^\square)$, we have*

$$\|u - P_j^\square u\|_{L_2(A)} \lesssim 2^{-dj}|u|_{H^d(B(A;(\vartheta+3\varepsilon)2^{-j})\cap\square)}, \quad (A \subset \square, u \in H^d(\square)).$$

Proof. Using ($\mathcal{J}$)(iv) and $d > \frac{n}{2}$, by applying the Sobolev embedding theorem on a Lipschitz domain with volume of order one as well as $\|\cdot\|_{H^d} \eqsim [\|\cdot\|_{L_2}^2 + |\cdot|_{H^d}^2]^{\frac{1}{2}}$ on such a domain, together with a homogeneity argument, we infer that

$$|\langle u, \lambda_{j,y}^\square\rangle_{L_2(\square)}| \lesssim \|u\|_{L_2(B(y;\vartheta 2^{-j})\cap\square)} + 2^{-dj}|u|_{H^d(B(y;\vartheta 2^{-j})\cap\square)}. \tag{3.1}$$

Since $P_j^\square$ reproduces $P_{d-1}(\square)$, for each $x \in I_j^\square$ and $p \in P_{d-1}(\square)$ we have

$$\begin{aligned}
&\|u - P_j^\square u\|_{L_2(B(x;\varepsilon 2^{-j})\cap\square)} = \|u - p - P_j^\square(u - p)\|_{L_2(B(x;\varepsilon 2^{-j})\cap\square)} \\
&\leq \|u - p\|_{L_2(B(x;\varepsilon 2^{-j})\cap\square)} + \|\sum_{y \in I_j^\square \cap B(x;2\varepsilon 2^{-j})} \langle u - p, \lambda_{j,y}^\square\rangle_{L_2(\square)}\phi_{j,y}\|_{L_2(B(x;\varepsilon 2^{-j})\cap\square)} \\
&\lesssim \|u - p\|_{L_2(B(x;\varepsilon 2^{-j})\cap\square)} + \sum_{y \in I_j^\square \cap B(x;2\varepsilon 2^{-j})} \|u - p\|_{L_2(B(y;\vartheta 2^{-j})\cap\square)} + 2^{-dj}|u|_{H^d(B(y;\vartheta 2^{-j})\cap\square)} \\
&\lesssim \|u - p\|_{L_2(B(x;(\vartheta+2\varepsilon)2^{-j})\cap\square)} + 2^{-dj}|u|_{H^d(B(x;(\vartheta+2\varepsilon)2^{-j})\cap\square)},
\end{aligned}$$

where we have used that $\sup_{z\in\square} \#(I_j^\square \cap B(z; 2^{-j})) \lesssim 1$. By taking the infimum over p, using the Bramble-Hilbert lemma we find that $\|u - P_j^\square u\|_{L_2(B(x;\varepsilon 2^{-j})\cap\square)} \lesssim 2^{-dj}|u|_{H^d(B(x;(\vartheta+2\varepsilon)2^{-j})\cap\square)}$. Since $P_0(\square) \subset \mathcal{S}(\Phi_j^\square)$, $\{B(x; \varepsilon 2^{-j}); x \in I_j^\square\}$ is a covering of $\square$, so that $\|u - P_j^\square u\|_{L_2(A)}^2 \leq \sum_{\{x\in I_j^\square : B(x;\varepsilon 2^{-j})\cap A\neq\emptyset\}} \|u - P_j^\square u\|_{B(x;\varepsilon 2^{-j})\cap\square)}^2$. By again using that $\sup_{z\in\square} \#(I_j^\square \cap B(z; 2^{-j})) \lesssim 1$, we conclude the statement of the proposition. ∎

Remark 3.2. If, when $p \in (2, \infty]$ we assume that $\|\phi_{j,x}\|_{L_p(\square)} \lesssim 2^{(\frac{n}{2}-\frac{n}{p})j}$, and, when $p \in [1, 2)$ that $d > \frac{n}{p}$, then an easy adaptation of the above proof shows that

$$\|u - P_j^\square u\|_{L_p(A)} \lesssim 2^{-dj}\|u\|_{W_p^d(B(A;(\vartheta+3\varepsilon)2^{-j})\cap\square)}, \quad (A \subset \square, u \in W_p^d(\square)).$$

Apart from the above collection $\Phi_j^\square$ of *primal* scaling functions, for $j \in \mathbb{N}_0$ we assume the existence of a collection $\tilde{\Phi}_j^\square = (\tilde{\phi}_{j,x}^\square)_{x\in I_j^\square} \subset C(\overline{\square})$ of *dual* scaling functions that also satisfies all of ($\mathcal{L}$)-($\mathcal{B}$) with the same index set $I_j^\square$, but with generally different parameters and functionals in ($\mathcal{B}$) and ($\mathcal{J}$) that we will denote as $\tilde{\gamma} > 0$, $\tilde{d} > \frac{n}{2}$, $\tilde{\Lambda}_j^\square$, and $\tilde{\varepsilon}, \tilde{\vartheta} > 0$. The resulting projector $\tilde{P}_j^\square : u \mapsto \langle u, \tilde{\Lambda}_j^\square\rangle_{L_2(\square)}\tilde{\Phi}_j^\square$ satisfies the analogue of Proposition 3.1 with $(d, \vartheta, \varepsilon)$ replaced by $(\tilde{d}, \tilde{\vartheta}, \tilde{\varepsilon})$.

Since $\Phi_j^\square$ and $\tilde{\Phi}_j^\square$ are uniform $L_2(\square)$-Riesz systems, the matrix $\langle \Phi_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square)}$ defines a uniformly bounded linear operator on $\ell_2(I_j^\square)$. A relation between $\mathcal{S}(\Phi_j^\square)$ and $\mathcal{S}(\tilde{\Phi}_j^\square)$ is established by assuming that that its real part satisfies

$$(\mathcal{I}1) \qquad \Re\langle \Phi_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square)} \gtrsim I,$$

which implies that the $L_2(\square)$-angle between any $u_j \in \mathcal{S}(\Phi_j^\square)$ and $\mathcal{S}(\tilde{\Phi}_j^\square)$, and any $\tilde{u}_j \in \mathcal{S}(\tilde{\Phi}_j^\square)$ and $\mathcal{S}(\Phi_j^\square)$, is less than $\frac{\pi}{2}$, uniformly in j.

Finally, for $j \in \mathbb{N}_0$ we assume the existence of collections $\Theta_j^\square = (\theta_{j,x}^\square)_{x \in I_j^\square}$ with

$$(\mathcal{I}2) \qquad \langle \Theta_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square)} = \operatorname{diag}\langle \Theta_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square)} \gtrsim I,$$

and, for some index sets $J_j^\square \subset \overline{\square}$ with $J_j^\square \cap I_j^\square = \emptyset$, $\pi(J_j^\square) = J_j^\square$ $(\pi \in \Pi)$ and $\sup_{y\in\square} \#(J_j^\square \cap B(y; 2^{-j})) \lesssim 1$, collections $\Xi_j^\square = (\xi_{j,x}^\square)_{x\in J_j^\square}$ such that the union $\Upsilon_{j+1}^\square := \left[(\Theta_j^\square)^T \quad (\Xi_j^\square)^T\right]^T$ satisfies $(\mathcal{L})$-$(\mathcal{R})$, and $\mathcal{S}(\Upsilon_{j+1}^\square) = \mathcal{S}(\Phi_{j+1}^\square)$.

"Classical" wavelet constructions start with assuming biorthogonal scaling functions, i.e., $\langle \Phi_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square)} = I$, in which case $(\mathcal{I}1)$ and $(\mathcal{I}2)$ are satisfied with $\Theta_j^\square = \Phi_j^\square$. When there is no need for locally supported dual wavelets, biorthogonality of the scaling functions can be relaxed to the conditions given here, with generally $\Theta_j^\square$ different from $\Phi_j^\square$, and in particular not contained in $\mathcal{S}(\Phi_j)$. For the case that $\Theta_j^\square = \Phi_j^\square$, in the literature the set $\Xi_j^\square$ is sometimes called an initial "stable" completion of $\Phi_j^\square$, that is, a completion of $\Phi_j^\square$ to a uniform $L_2(\square)$-Riesz basis for $\mathcal{S}(\Phi_{j+1}^\square)$. The wavelets to be constructed are then thought of being the target stable completion.

Remark 3.3. The condition $(\mathcal{I}2)$ can be further relaxed, which turned out to be useful in [Ste03]. Instead of assuming that $\langle \Theta_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square)}$ is diagonal, more generally it will also be sufficient when, for some fixed p, $I_j^\square$ is the union of disjoint sets $I_{j,1}^\square, \ldots, I_{j,p}^\square$, with $\pi(I_{j,i}^\square) = I_{j,i}^\square$ $(\pi \in \Pi,\ 1 \leq i \leq p)$, such that, with respect to this partitioning, $\langle \Theta_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square)}$ is a block triangular matrix, with diagonal blocks that are uniformly boundedly invertible diagonal matrices.

The assumption $(\mathcal{I}2)$, or the relaxed assumption from Remark 3.3, implies that *both* the $L_2(\square)$-angle between any $u_j \in \mathcal{S}(\Theta_j^\square)$ and $\mathcal{S}(\tilde{\Phi}_j^\square)$, and any $\tilde{u}_j \in \mathcal{S}(\tilde{\Phi}_j^\square)$ and $\mathcal{S}(\Theta_j^\square)$, is less than $\frac{\pi}{2}$, uniformly in j, *and* that the matrix $\langle \Theta_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square)}^{-1}$ is *uniformly local*, with which we mean that only for $x, y \in I_j^\square$ with $|x-y| \lesssim 2^{-j}$ the corresponding entry might be nonzero. The latter property will be needed to be able to construct wavelets that are local.

Because of $\mathcal{S}(\Phi_j^\square) \subset \mathcal{S}(\Phi_{j+1}^\square)$, $\mathcal{S}(\tilde{\Phi}_j^\square) \subset \mathcal{S}(\tilde{\Phi}_{j+1}^\square)$ and $\mathcal{S}(\Upsilon_{j+1}^\square) = \mathcal{S}(\Phi_{j+1}^\square)$, it holds that $\Phi_j^\square = \langle \Phi_j^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)} \Phi_{j+1}^\square$, $\tilde{\Phi}_j^\square = \langle \tilde{\Phi}_j^\square, \tilde{\Lambda}_{j+1}^\square \rangle_{L_2(\square)} \tilde{\Phi}_{j+1}^\square$, and $\Upsilon_{j+1}^\square = \langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)} \Phi_{j+1}^\square$, where $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)}$ is uniformly bounded invertible.

Lemma 3.4. *For the matrix* $\mathbf{R}_j^\square$ *being either* $\langle \Phi_j^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)}$, $\langle \tilde{\Phi}_j^\square, \tilde{\Lambda}_{j+1}^\square \rangle_{L_2(\square)}$, $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)}$ *or* $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)}^{-1}$, *it holds that*

(a) $(\mathbf{R}_j^\square)_{\pi(x),\pi(y)} = (\mathbf{R}_j^\square)_{x,y}$ *(*$\pi \in \Pi$*).*

(b) $(\mathbf{R}_j^\square)_{x,y} = 0$ *when* y *is on a face of* $\square$ *that does not contain* x.

Proof. The assumption $(\mathcal{S})$ for $\Phi_j^\square$, $\tilde{\Phi}_j^\square$ or $\Upsilon_{j+1}^\square$, together with the assumption $(\mathcal{J})$(iii) for $\Lambda_{j+1}^\square$ or $\tilde{\Lambda}_{j+1}^\square$ implies (a) for $\langle \Phi_j^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)}$, $\langle \tilde{\Phi}_j^\square, \tilde{\Lambda}_{j+1}^\square \rangle_{L_2(\square)}$ or $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)}$, and thus also for $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)}^{-1}$.

The assumption $(\mathcal{V})$ for $\Phi_j^\square$, $\tilde{\Phi}_j^\square$ or $\Upsilon_{j+1}^\square$, together with the assumption $(\mathcal{J})$(ii) for $\Lambda_{j+1}^\square$ or $\tilde{\Lambda}_{j+1}^\square$ implies (b) for $\langle \Phi_j^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)}$, $\langle \tilde{\Phi}_j^\square, \tilde{\Lambda}_{j+1}^\square \rangle_{L_2(\square)}$ or $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)}$. Now let e be a face of $\square$. With respect to the partitioning of the index sets for $\Upsilon_{j+1}^\square$ and $\Lambda_{j+1}^\square$ into indices on e and indices not on e, $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)}$ is a 2×2 upper block triangular matrix, and thus so is its inverse, which shows (b) for $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)}^{-1}$. ∎

Obviously, by assumption $(\mathcal{L})$ for $\Phi_j^\square$ and $\Upsilon_{j+1}^\square$, together with the assumption $(\mathcal{J})$(i) for $\Lambda_{j+1}^\square$, the matrices $\langle \Phi_j^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)}$ and $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)}$ are uniformly local, and so uniformly sparse. As a consequence of this, for the wavelets we are going to construct, the basis transformation from wavelet- to single scale basis will be of optimal computational complexity. Analogously, assumption $(\mathcal{L})$ for $\tilde{\Phi}_j^\square$ together with $(\mathcal{J})$(i) for $\tilde{\Lambda}_{j+1}^\square$ shows that $\langle \tilde{\Phi}_j^\square, \tilde{\Lambda}_{j+1}^\square \rangle_{L_2(\square)}$ is uniformly local.

For some applications, it is also essential to have a basis transformation from single scale to wavelet basis that is of optimal computational complexity. In that case, one has to assume *both* that

$$\Theta_j^\square = \Phi_j^\square,$$

with which $(\mathcal{I}1)$ can be dropped since it is either implied by $(\mathcal{I}2)$ or it can be replaced by the condition from Remark 3.3 (cf. Propositions 5.2 and 5.4(a)), *and* also that

$$\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)}^{-1} \text{ is } \textit{uniformly local.} \tag{3.2}$$

Note that, for $\Theta_j^\square = \Phi_j^\square$, $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square \rangle_{L_2(\square)}^{-1}$ is the basis transformation from $\Phi_{j+1}^\square$ to the two-level basis $\Phi_j^\square \cup \Xi_j^\square$.

All conditions imposed in this section are satisfied by the collections $\Phi_j^\square$, $\tilde{\Phi}_j^\square$, $\Theta_j^\square$, $\Xi_j^\square$ underlying the *finite element wavelets* introduced in [DS99c]. With this construction, $\mathcal{S}(\Phi_j^\square)$, $\mathcal{S}(\tilde{\Phi}_j^\square)$ are standard Lagrange finite element spaces, so that $\gamma = \tilde{\gamma} = \frac{3}{2}$, of orders d and $\tilde{d}$, respectively, with respect to a j times repeated uniform dyadic refinement of an initial simplicial partition of a polygon. For the present setting, we take this polygon to be a reference *n-simplex*. Thinking of $\tilde{d} \geq d$, these orders are chosen such that there is an $m \in \mathbb{N}$ with $2^m(d-1) = \tilde{d}-1$, so that with the initial partition at the dual side being the reference simplex itself, and at the primal side it is being created by applying m dyadic recursive refinements to this simplex, we have $\dim \mathcal{S}(\Phi_j^\square) = \dim \mathcal{S}(\tilde{\Phi}_j^\square)$. So each "element" at the dual side is equal to a macro-element at the primal side consisting of 2^m "elements". The collections $\tilde{\Phi}_j^\square$ at the dual side, and $\Phi_j^\square$, $\Theta_j^\square$, $\Xi_j^\square$ at the primal side are now assembled in the standard finite element fashion from local collections, of a small, fixed dimension, living on the individual elements or macro-elements, respectively, each of these collections being a copy, or more precisely, a push-forward using an affine bijection of such a collection created once and for all on a reference (macro-) element. The functionals from $\tilde{\Lambda}_j^\square$ and $\Lambda_j^\square$ are assembled in the same manner from local collections, and are either simply scaled function evaluations in the "nodal points" $I_j^\square$, or local linear combinations of these. Actually, in the present paper, we will repeat the idea of assembling functions and functionals from collections defined on (macro-) elements, which in turn are push-forwards of collections defined on a reference (macro-) element, where the role of the (macro-) elements will now be played by the patches Γ_q, and that of the reference (macro-) element by $\kappa_q^{-1}(\Gamma_q)$. A difference is that the number of patches is fixed, and that, as a consequence, the dimension of the local collections grows with the level. The major difficulty we have to deal with is that generally κ_q is not affine, so that the Jacobian determinants are not constants.

In [Ste03], we reconsidered the finite element wavelets, and constructed collections, also satisfying all conditions we impose here, with $\Theta_j^\square = \Phi_j^\square$, so that also the resulting dual wavelets are locally supported. In this case, the dual spaces, although they consist of continuous piecewise polynomials, are not standard finite element spaces.

Other examples of collections $\Phi_j^\square$, $\tilde{\Phi}_j^\square$, $\Theta_j^\square$, $\Xi_j^\square$ satisfying our assumptions, with $\Theta_j^\square = \Phi_j^\square$, and with $\square$ now being the *n-cube*, are given in [DS99a], which underly the construction of *biorthogonal spline wavelets*. These collections are slight modifications of those developed in [DKU99], and, for $n > 1$, they are simply generated using tensor products from univariate collections $\Phi_j^{[0,1]}$, $\tilde{\Phi}_j^{[0,1]}$, $\Xi_j^{[0,1]}$ defined on $[0,1]$. For given $\tilde{d} \geq d \geq 2$ with $d + \tilde{d}$ even, $\mathcal{S}(\Phi_j^{[0,1]})$ is the spline space of order d with respect a dyadic subdivision of $[0,1]$ into subintervals of length 2^{-j}, thus with $\gamma = d - \frac{1}{2}$, and $\tilde{\Phi}_j^{[0,1]}$ is such that $P_{\tilde{d}-1}[0,1] \subset \mathcal{S}(\tilde{\Phi}_j^{[0,1]})$ and $\langle \Phi_j^{[0,1]}, \tilde{\Phi}_j^{[0,1]} \rangle_{L_2([0,1])} = I$, where $\tilde{\gamma}$ grows linearly with $\tilde{d}$. For x not near the endpoints 0 or 1, $\phi_{j,x}^{[0,1]} = 2^{j/2}\phi(2^j \cdot -x)$ and $\tilde{\phi}_{j,x}^{[0,1]} = 2^{j/2}\tilde{\phi}(2^j \cdot -x)$, where $(\phi, \tilde{\phi})$ is a biorthogonal pair constructed in [CDF92]. Also the functionals from $\Lambda_j^\square$ and $\tilde{\Lambda}_j^\square$ are constructed from the collections of univariate functionals $\Lambda_j^{[0,1]}$ and $\tilde{\Lambda}_j^{[0,1]}$ using tensor products, where, except when $x = 0$ or 1, $\lambda_{j,x}^{[0,1]} = \tilde{\phi}_{j,x}^{[0,1]}$, $\tilde{\lambda}_{j,x}^{[0,1]} = \phi_{j,x}^{[0,1]}$, and where they are simply scaled function evaluations in 0 or 1, respectively, otherwise.

4. Induced, continuous multiresolution analyses on Γ. By lifting the collections of functions on $\square$ to the patches of Γ, and by connecting those that do not vanish at the interfaces continuously with ones from other patches, we will construct nested sequences of primal and dual spaces that satisfy Jackson estimates and Bernstein inequalities.

We define the index sets $I_j \subset \overline{\Gamma} \backslash \partial\Gamma_D$, and analogously J_j, by

$$I_j = (\cup_{q=1}^M \kappa_q(I_j^\square)) \cap (\overline{\Gamma} \backslash \partial\Gamma_D),$$

By $(\mathcal{M})$ and $\pi(I_j^\square) = I_j^\square$ $(\pi \in \Pi)$, for any $1 \leq q, q' \leq M$ with $\overline{\Gamma_q} \cap \overline{\Gamma_{q'}} \neq \emptyset$, the sets $\kappa_q(I_j^\square)$ and $\kappa_{q'}(I_j^\square)$ restricted to this interface coincide. For $x \in \overline{\Gamma}$, we set $k(x) = \#\{q : x \in \overline{\Gamma_q}\}$.

For $j \in \mathbb{N}_0$, we define the collection $\Phi_j = (\phi_{j,x})_{x \in I_j} \subset C(\overline{\Gamma})$ by

$$\phi_{j,x}(y) = k(x)^{-\frac{1}{2}} \begin{cases} \phi^\square_{j,\kappa_q^{-1}(x)}(\kappa_q^{-1}(y)) & \text{when } x, y \in \overline{\Gamma_q} \text{ for some } 1 \leq q \leq M, \\ 0 & \text{elsewhere.} \end{cases} \tag{4.1}$$

Note that by $(\mathcal{S})$, $(\mathcal{V})$ and (2.1), $\phi_{j,x}$ is well-defined, indeed continuous and vanishes on $\partial\Gamma_D$. By assumption $(\mathcal{L})$, the collection Φ_j is *uniformly local*, with which we mean that $x \in \operatorname{supp} \phi_{j,x}$, and that $d_\Gamma(x,y) \lesssim 2^{-j}$ for any $y \in \operatorname{supp} \phi_{j,x}$, where $d_\Gamma(x,y)$ denotes the geodesic distance of x and y over Γ, i.e., the length of the shortest curve on Γ connecting x and y.

With $I_{j,q} : \ell_2(I_j^{\square}) \to \ell_2(I_j)$ defined by

$$(I_{j,q}\mathbf{c}_j^{\square})_x = k(x)^{-\frac{1}{2}} \begin{cases} \mathbf{c}^{\square}_{j,\kappa_q^{-1}(x)} & x \in \overline{\Gamma_q}, \\ 0 & \text{otherwise}, \end{cases} \tag{4.2}$$

we have $\sum_{q=1}^{M} I_{j,q} I_{j,q}^T = I$. By construction of Φ_j from $\Phi_j^{\square}$, we have

$$\langle \Phi_j, \Phi_j \rangle_{L_2(\Gamma)} = \sum_{q=1}^{M} I_{j,q} \langle \Phi_j^{\square}, \Phi_j^{\square} \rangle_{L_2(\square),|\partial\kappa_q|} I_{j,q}^T, \tag{4.3}$$

so that, because of $\langle \Phi_j^{\square}, \Phi_j^{\square} \rangle_{L_2(\square),|\partial\kappa_q|} \eqsim \langle \Phi_j^{\square}, \Phi_j^{\square} \rangle_{L_2(\square)} \eqsim I$ by $|\partial\kappa_q| \eqsim 1$ and $(\mathcal{R})$, Φ_j *is a uniform* $L_2(\Gamma)$*-Riesz system.*

PROPOSITION 4.1. *Setting* $\Lambda_j = (\lambda_{j,x})_{x \in I_j} \subset C(\overline{\Gamma})'$ *by*

$$\lambda_{j,x}(u) = k(x)^{\frac{1}{2}} \lambda^{\square}_{j,\kappa_q^{-1}(x)}(u \circ \kappa_q) \quad \textit{when } x \in \overline{\Gamma}_q,$$

we have $\langle \Phi_j, \Lambda_j \rangle_{L_2(\Gamma)} = I$. *The projector* $P_j : u \mapsto \langle u, \Lambda_j \rangle_{L_2(\Gamma)} \Phi_j$ *onto* $\mathcal{S}(\Phi_j)$ *satisfies*

$$\|(I - P_j)u\|_{L_2(\breve{\Gamma})} \lesssim 2^{-dj} \sum_{q=1}^{M} |u \circ \kappa_q|_{H^d(B(\kappa_q^{-1}(\breve{\Gamma} \cap \Gamma_q);(\vartheta+3\varepsilon)2^{-j}) \cap \square)} \quad (\breve{\Gamma} \subset \Gamma,\, u \in \mathcal{H}_d(\Gamma)).$$

Proof. Assumption $(\mathcal{J})$(ii) shows that, for $u \in C(\overline{\Gamma})$, $\lambda_{j,x}(u)$ is well-defined, also when x is on an interface between patches, and, because of (4.1), that $(P_j u) \circ \kappa_q = P_j^{\square}(u \circ \kappa_q)$ when u vanishes on $\partial\Gamma_D$. Condition $(\mathcal{J})$(v) shows that $\langle \Phi_j, \Lambda_j \rangle_{L_2(\Gamma)} = I$. By Proposition 3.1, we have

$$\|(I - P_j)u\|_{L_2(\breve{\Gamma})} \eqsim \sum_{q=1}^{M} \|((I - P_j)u) \circ \kappa_q\|_{L_2(\kappa_q^{-1}(\breve{\Gamma} \cap \Gamma_q))},$$

$$= \sum_{q=1}^{M} \|(I - P_j^{\square})(u \circ \kappa_q)\|_{L_2(\kappa_q^{-1}(\breve{\Gamma} \cap \Gamma_q))} \lesssim 2^{-dj} \sum_{q=1}^{M} |u \circ \kappa_q|_{H^d(B(\kappa_q^{-1}(\breve{\Gamma} \cap \Gamma_q);(\vartheta+3\varepsilon)2^{-j}) \cap \square)}.$$

□

By substituting $\breve{\Gamma} = \Gamma$ in Proposition 4.1, we have the following *Jackson estimate*:

$$\inf_{u_j \in \mathcal{S}(\Phi_j)} \|u - u_j\|_{L_2(\Gamma)} \lesssim 2^{-dj} \|u\|_d, \qquad (u \in \mathcal{H}_d(\Gamma)). \tag{4.4}$$

A direct consequence of $(\mathcal{B})$ is the following *Bernstein inequality*: For $s \in [0, \gamma)$,

$$\|u_j\|_s \lesssim 2^{sj} \|u_j\|_{L_2(\Gamma)}, \quad (u_j \in \mathcal{S}(\Phi_j)). \tag{4.5}$$

Thanks to properties of a Sobolev scale, (4.5) gives rise to the following extended version:

LEMMA 4.2. *For any* $t \le s < \gamma$ *with* $t \le 0$,

$$\|u_j\|_s \lesssim 2^{(s-t)j} \|u_j\|_t, \quad (u_j \in \mathcal{S}(\Phi_j)).$$

Proof. For $s=0$, the statement follows from $\|\cdot\|_0 \lesssim \|\cdot\|_t^{\frac{\gamma'}{\gamma'-t}}\|\cdot\|_{\gamma'}^{\frac{-t}{\gamma'-t}}$ on $\mathcal{H}_{\gamma'}(\Gamma)$ for some $\gamma' \in (0,\gamma')$, and $\|\cdot\|_{\gamma'} \lesssim 2^{\gamma' j}\|\cdot\|_0$ on $\mathcal{S}(\Phi_j)$. An additional application of $\|\cdot\|_s \lesssim 2^{sj}\|\cdot\|_0$ on $\mathcal{S}(\Phi_j)$ when $s>0$, or $\|\cdot\|_s \lesssim \|\cdot\|_0^{\frac{-t+s}{-t}}\|\cdot\|_t^{\frac{-s}{-t}}$ on $L_2(\Gamma)$ when $s<0$ shows it for any $s\in[t,\gamma)$. ∎

As $\Phi_j^{\square}$, via (4.1), gave rise to a uniformly local, uniform $L_2(\Gamma)$-Riesz system Φ_j, analogously the collections $\Upsilon_{j+1}^{\square} = \left[(\Theta_j^{\square})^T \quad (\Xi_j^{\square})^T\right]^T$ and $\tilde{\Phi}_j^{\square}$ yield *uniformly local, uniform $L_2(\Gamma)$-Riesz systems* $\Upsilon_{j+1} = \left[\Theta_j^T \quad \Xi_j^T\right]^T$ and $\tilde{\Phi}_j$, respectively.

We have the analogue of Proposition 4.1 at the dual side, with functionals and a projector denoted as $\tilde{\Lambda}_j = (\tilde{\lambda}_{j,x})_{x\in I_j}$ and $\tilde{P}_j$, respectively, and with Φ_j, d, ϑ, ε replaced by $\tilde{\Phi}_j$, $\tilde{d}$, $\tilde{\vartheta}$, $\tilde{\varepsilon}$. In particular, we have the *Jackson estimate*

$$\inf_{u_j\in\mathcal{S}(\tilde{\Phi}_j)} \|u-u_j\|_{L_2(\Gamma)} \lesssim 2^{-\tilde{d}j}\|u\|_{\tilde{d}}, \qquad (u\in\mathcal{H}_{\tilde{d}}(\Gamma)), \tag{4.6}$$

and furthermore also the *Bernstein inequality*: For any $s\in[0,\tilde{\gamma})$,

$$\|u_j\|_s \lesssim 2^{sj}\|u_j\|_{L_2(\Gamma)}, \quad (u_j\in\mathcal{S}(\tilde{\Phi}_j)), \tag{4.7}$$

which can be extended analogously to Lemma 4.2.

The inclusions $\mathcal{S}(\Phi_j^{\square}) \subset \mathcal{S}(\Phi_{j+1}^{\square})$, $\mathcal{S}(\tilde{\Phi}_j^{\square}) \subset \mathcal{S}(\tilde{\Phi}_{j+1}^{\square})$ and $\mathcal{S}(\Upsilon_{j+1}^{\square}) = \mathcal{S}(\Phi_{j+1}^{\square})$, and their representations with respect to the canonical bases give rise to the following counterparts on Γ:

Proposition 4.3. *It holds that $\mathcal{S}(\Phi_j) \subset \mathcal{S}(\Phi_{j+1})$, $\mathcal{S}(\tilde{\Phi}_j) \subset \mathcal{S}(\tilde{\Phi}_{j+1})$ and $\mathcal{S}(\Upsilon_{j+1}) = \mathcal{S}(\Phi_{j+1})$. In particular,*

$$\Phi_j = \sum_{q=1}^M I_{j,q}\langle\Phi_j^{\square},\Lambda_{j+1}^{\square}\rangle_{L_2(\square)} I_{j+1,q}^T \Phi_{j+1},\ \Upsilon_{j+1} = \sum_{q=1}^M I_{j+1,q}\langle\Upsilon_{j+1}^{\square},\Lambda_{j+1}^{\square}\rangle_{L_2(\square)} I_{j+1,q}^T \Phi_{j+1},$$

$$\tilde{\Phi}_j = \sum_{q=1}^M I_{j,q}\langle\tilde{\Phi}_j^{\square},\tilde{\Lambda}_{j+1}^{\square}\rangle_{L_2(\square)} I_{j+1,q}^T \tilde{\Phi}_{j+1},\ \Phi_{j+1} = \sum_{q=1}^M I_{j+1,q}\langle\Upsilon_{j+1}^{\square},\Lambda_{j+1}^{\square}\rangle_{L_2(\square)}^{-1} I_{j+1,q}^T \Upsilon_{j+1}.$$

Proof. Let $1\le q'\le M$ be arbitrary, but fixed. For any $I_{j+1}\ni y\notin\overline{\Gamma_{q'}}$, it holds that $\phi_{j+1,y}\circ\kappa_{q'} = 0$ on $\square$. Lemma 3.4(b) shows that if for $y\in I_{j+1}\cap\overline{\Gamma_{q'}}$, $x\in I_j$ and $1\le q\le M$, $(I_{j,q}\langle\Phi_j^{\square},\Lambda_{j+1}^{\square}\rangle_{L_2(\square)} I_{j+1,q}^T)_{x,y}\neq 0$, then $x\in\overline{\Gamma_{q'}}$, where, using Lemma 3.4(a), it follows that $(\sum_{q=1}^M I_{j,q}\langle\Phi_j^{\square},\Lambda_{j+1}^{\square}\rangle_{L_2(\square)} I_{j+1,q}^T)_{x,y} = (I_{j,q'}\langle\Phi_j^{\square},\Lambda_{j+1}^{\square}\rangle_{L_2(\square)} I_{j+1,q'}^T)_{x,y}k(y)$. Using that $\Phi_j\circ\kappa_q = I_{j,q}\Phi_j^{\square}$, we conclude that on $\square$,

$$\begin{aligned}
&\sum_{q=1}^M I_{j,q}\langle\Phi_j^{\square},\Lambda_{j+1}^{\square}\rangle_{L_2(\square)} I_{j+1,q}^T \Phi_{j+1}\circ\kappa_{q'}\\
&= I_{j,q'}\langle\Phi_j^{\square},\Lambda_{j+1}^{\square}\rangle_{L_2(\square)} I_{j+1,q'}^T \operatorname{diag}[k(y)]_{y\in I_{j+1}} I_{j+1,q'}\Phi_{j+1}^{\square}\\
&= I_{j,q'}\langle\Phi_j^{\square},\Lambda_{j+1}^{\square}\rangle_{L_2(\square)}\Phi_{j+1}^{\square} = I_{j,q'}\Phi_j^{\square} = \Phi_j\circ\kappa_{q'},
\end{aligned}$$

showing that $\Phi_j = \sum_{q=1}^M I_{j,q}\langle\Phi_j^{\square},\Lambda_{j+1}^{\square}\rangle_{L_2(\square)} I_{j+1,q}^T\Phi_{j+1}$, and thus in particular that $\mathcal{S}(\Phi_j)\subset\mathcal{S}(\Phi_{j+1})$.

The other statements are proven analogously. This includes the last one, since the only requirements are that $\Phi_{j+1}^{\square} = \langle\Upsilon_{j+1}^{\square},\Lambda_{j+1}^{\square}\rangle_{L_2(\square)}^{-1}\Upsilon_{j+1}^{\square}$, and that $\langle\Upsilon_{j+1}^{\square},\Lambda_{j+1}^{\square}\rangle_{L_2(\square)}^{-1}$ has the properties from Lemma 3.4. ∎

Remark 4.4. Knowing $\mathcal{S}(\Phi_j) \subset \mathcal{S}(\Phi_{j+1})$, we have $\Phi_j = \langle \Phi_j, \Lambda_{j+1}\rangle_{L_2(\Gamma)} \Phi_{j+1}$, and so we find that $\langle \Phi_j, \Lambda_{j+1}\rangle_{L_2(\Gamma)} = \sum_{q=1}^{M} I_{j,q} \langle \Phi_j^\square, \Lambda_{j+1}^\square\rangle_{L_2(\square)} I_{j+1,q}^T$, which can also be derived directly from the definitions of Φ_j and Λ_{j+1}. We find similar expressions for $\langle \tilde{\Phi}_j, \tilde{\Lambda}_{j+1}\rangle_{L_2(\Gamma)}$ and $\langle \Upsilon_{j+1}, \Lambda_{j+1}\rangle_{L_2(\Gamma)}$, as well as $\langle \Upsilon_{j+1}, \Lambda_{j+1}\rangle_{L_2(\Gamma)}^{-1} = \sum_{q=1}^{M} I_{j+1,q} \langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square\rangle_{L_2(\square)}^{-1} I_{j+1,q}^T$. The locality assumptions $(\mathfrak{L})$ and $(\mathfrak{J})$(i) for the involved functions and functionals, respectively, directly show that $\langle \Phi_j, \Lambda_{j+1}\rangle_{L_2(\Gamma)}$, $\langle \tilde{\Phi}_j, \tilde{\Lambda}_{j+1}\rangle_{L_2(\Gamma)}$, and $\langle \Upsilon_{j+1}, \Lambda_{j+1}\rangle_{L_2(\Gamma)}$ are *uniformly local*, with which we mean that they may have nonzero entries only for pairs (x, y) with $\mathrm{dist}_\Gamma(x, y) \lesssim 2^{-j}$. The above expression of $\langle \Upsilon_{j+1}, \Lambda_{j+1}\rangle_{L_2(\Gamma)}^{-1}$ in terms of $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square\rangle_{L_2(\square)}^{-1}$ shows that if the latter matrix is uniformly local, cf. (3.2), then so is $\langle \Upsilon_{j+1}, \Lambda_{j+1}\rangle_{L_2(\Gamma)}^{-1}$.

5. Biorthogonal space decompositions and wavelets. We have constructed primal and dual sequences of nested spaces that satisfy Jackson and Bernstein estimates. To conclude existence and stability, with respect to a range of Sobolev norms, of the corresponding biorthogonal space decompositions, the only thing left to show is the existence and uniform $L_2(\Gamma)$-boundedness of the biorthogonal projector.

Results similar to the next lemma are often used in the context of saddle point problems. A proof of (the non-trivial part of) this lemma can be found in, e.g., [DS99c, Theorem 2.1(a)].

LEMMA 5.1. *Let V, U be closed subspaces of a Hilbert space H. Then the following statements are equivalent:*

(a) $\gamma := \inf_{0\neq u\in U} \sup_{0\neq v\in V} \frac{|\langle u,v\rangle|}{\|u\|\|v\|} > 0$, *and for any $v \in V$, there exists a $u \in U$ with* $\langle u, v\rangle \neq 0$.

(b) *There exists a bounded projector $Q : H \to H$ with $\Im(Q) = V$ and $\Im(I - Q) = U^\perp$, which is therefore appropriately called a* biorthogonal projector.

In either case it holds that $\gamma = \|Q\|^{-1}$, and the adjoint Q^* satisfies $\Im(Q^*) = U$ and $\Im(I - Q^*) = V^\perp$.

When Σ and Δ are Riesz bases for U and V, respectively, then (a) *or* (b) *are equivalent to the existence of a bounded inverse of $\langle \Sigma, \Delta\rangle : \ell_2(\Delta) \to \ell_2(\Sigma)$. In that case it holds that*

$$\|\langle \Sigma, \Sigma\rangle^{-1}\|^{-\frac{1}{2}} \|\langle \Delta, \Delta\rangle^{-1}\|^{-\frac{1}{2}} \leq \frac{\|Q\|}{\|\langle \Sigma, \Delta\rangle^{-1}\|} \leq \|\Sigma\|\|\Delta\|.$$

To be able to transfer results valid on the reference parameter domain to the manifold, in particular those concerning $L_2(\square)$- or $L_2(\Gamma)$-angles between spaces, we will have to assume that the coarsest "mesh" is sufficiently fine in order to control the influence of the generally non-constant Jacobian determinants.

PROPOSITION 5.2. *For $j \geq j_0$ being large enough, there exists a uniformly bounded projector $Q_j : L_2(\Gamma) \to L_2(\Gamma)$ with $\Im(Q_j) = \mathcal{S}(\Phi_j)$ and $\Im(I - Q_j) = \mathcal{S}(\tilde{\Phi}_j)^{\perp_{L_2(\Gamma)}}$.*

Proof. Setting

$$\Delta_{j,q}^\square = \mathrm{diag}(|\partial\kappa_q(x)|)_{x\in I_j^\square},$$

by $(\mathfrak{L})$ for both $\Phi_j^\square, \tilde{\Phi}_j^\square$, the smoothness of $z \mapsto |\partial\kappa_q(z)|$, and the uniform boundedness of $\|\phi_{j,x}^\square\|_{L_2(\square)}$, $\|\tilde{\phi}_{j,x}^\square\|_{L_2(\square)}$, we have

$$\|\langle \Phi_j^\square, \tilde{\Phi}_j^\square\rangle_{L_2(\square), |\partial\kappa_q|} - (\Delta_{j,q}^\square)^{\frac{1}{2}} \langle \Phi_j^\square, \tilde{\Phi}_j^\square\rangle_{L_2(\square)} (\Delta_{j,q}^\square)^{\frac{1}{2}}\| \lesssim 2^{-j}. \tag{5.1}$$

By assumption (J1) and $|\partial\kappa_q| \gtrsim 1$, we have

$$\Re((\Delta_{j,q}^{\square})^{\frac{1}{2}}\langle\Phi_j^{\square},\tilde{\Phi}_j^{\square}\rangle_{L_2(\square)}(\Delta_{j,q}^{\square})^{\frac{1}{2}}) = (\Delta_{j,q}^{\square})^{\frac{1}{2}}\Re\langle\Phi_j^{\square},\tilde{\Phi}_j^{\square}\rangle_{L_2(\square)}(\Delta_{j,q}^{\square})^{\frac{1}{2}} \gtrsim I,$$

so that for $j \geq j_0$ large enough, $\Re\langle\Phi_j^{\square},\tilde{\Phi}_j^{\square}\rangle_{L_2(\square),\,|\partial\kappa_q|} \gtrsim I$. Similarly to (4.3), we find that

$$\Re\langle\Phi_j,\tilde{\Phi}_j\rangle_{L_2(\Gamma)} = \sum_{q=1}^{M} I_{j,q}\Re\langle\Phi_j^{\square},\tilde{\Phi}_j^{\square}\rangle_{L_2(\square),\,|\partial\kappa_q|}I_{j,q}^T \gtrsim \sum_{q=1}^{M} I_{j,q}I_{j,q}^T = I.$$

Since, for $j \geq j_0$, apparently $\langle\Phi_j,\tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ is uniformly bounded invertible, and Φ_j and $\tilde{\Phi}_j$ are uniform $L_2(\Gamma)$-Riesz systems, an application of Lemma 5.1 completes the the proof. $\square$

For $j \geq j_0$, the nesting $\mathcal{S}(\tilde{\Phi}_j) \subset \mathcal{S}(\tilde{\Phi}_{j+1})$ gives $Q_j^* = Q_{j+1}^*Q_j^*$ or $Q_j = Q_jQ_{j+1}$, from which it follows that

$$\Im(Q_{j+1}-Q_j) = \mathcal{S}(\Phi_{j+1}) \cap \mathcal{S}(\tilde{\Phi}_j)^{\perp_{L_2(\Gamma)}}.$$

Analogously, $\mathcal{S}(\Phi_j) \subset \mathcal{S}(\Phi_{j+1})$ implies that

$$\Im(Q_{j+1}^*-Q_j^*) = \mathcal{S}(\tilde{\Phi}_{j+1}) \cap \mathcal{S}(\Phi_j)^{\perp_{L_2(\Gamma)}}.$$

From the Jackson estimates and Bernstein inequalities at primal and dual side (4.4), (4.5), (4.6) and (4.7), and the existence and uniform $L_2(\Gamma)$-boundedness of the biorthogonal projectors Q_j from Proposition 5.2, we have

Theorem 5.3 (cf. e.g. [Dah96], [DS99c, Theorem 2.1]). *With $Q_{j_0-1} := 0$, it holds that*

$$\|\!|\sum_{j=j_0}^{\infty} w_j\|\!|_s^2 \lesssim \sum_{j=j_0}^{\infty} 4^{sj}\|w_j\|_{L_2(\Gamma)}^2, \qquad (w_j \in \Im(Q_j-Q_{j-1}),\, s \in (-\tilde{d},\gamma)), \tag{5.2}$$

and

$$\sum_{j=j_0}^{\infty} 4^{sj}\|(Q_j-Q_{j-1})u\|_{L_2(\Gamma)}^2 \lesssim \|\!|u\|\!|_s^2, \qquad (u \in \mathcal{H}_s(\Gamma),\, s \in (-\tilde{\gamma},d)). \tag{5.3}$$

For $s \in (-\min\{\tilde{\gamma},\tilde{d}\},\min\{\gamma,d\})$, $(w_j)_{j\geq j_0} \mapsto \sum_{j=j_0}^{\infty} w_j$ and $u \mapsto ((Q_j-Q_{j-1})u)_{j\geq j_0}$, which mappings are bounded in the sense of (5.2) and (5.3), are each others inverse.

Analogous results are valid with (Q_j) replaced by (Q_j^*) and with interchanged roles of (γ,d) and $(\tilde{\gamma},\tilde{d})$.

Next, we construct a uniform $L_2(\Gamma)$-Riesz basis for $\Im(Q_{j+1}-Q_j)$, which elements are called *wavelets*.

Proposition 5.4.

(a) *For $j \geq j_0$ being large enough, there exists a uniformly bounded projector $\bar{Q}_j : L_2(\Gamma) \to L_2(\Gamma)$ with $\Im(\bar{Q}_j) = \mathcal{S}(\Theta_j)$ and $\Im(I-\bar{Q}_j) = \mathcal{S}(\tilde{\Phi}_j)^{\perp_{L_2(\Gamma)}}$.*

(b) *This projector can be computed as $\bar{Q}_ju = \langle u,\tilde{\Phi}_j\rangle_{L_2(\Gamma)}\langle\Theta_j,\tilde{\Phi}_j\rangle_{L_2(\Gamma)}^{-1}\Theta_j$.*

(c) *The collection of wavelets*

$$\boxed{\Psi_j := \Xi_j - \langle\Xi_j,\tilde{\Phi}_j\rangle_{L_2(\Gamma)}\langle\Theta_j,\tilde{\Phi}_j\rangle_{L_2(\Gamma)}^{-1}\Theta_j,} \tag{5.4}$$

is a uniform $L_2(\Gamma)$-Riesz basis for $\mathcal{S}(\Phi_{j+1}) \cap \mathcal{S}(\tilde{\Phi}_j)^{\perp_{L_2(\Gamma)}}$.

So by taking j_0 to be the maximum of the values from (a) *and that of Proposition 5.2, for* $s \in (-\min\{\tilde{\gamma}, \tilde{d}\}, \min\{\gamma, d\})$,

$$\Phi_{j_0} \cup \cup_{j \geq j_0} 2^{-sj} \Psi_j \text{ is a Riesz basis for } \mathcal{H}_s(\Gamma),$$

and thus, in view of (2.4), *when in addition* $|s| < \frac{3}{2}$, $|s| < s_\Gamma \notin \mathbb{N}$ *or* $|s| \leq s_\Gamma \in \mathbb{N}$, *it is a Riesz basis for* $\mathcal{H}^s(\Gamma)$.

Proof. (a) Since Θ_j and $\tilde{\Phi}_j$ are uniform $L_2(\Gamma)$-Riesz bases, by Lemma 5.1 we have to show that, for $j \geq j_0$ being large enough, $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)}$ is uniformly boundedly invertible. In case ($\mathcal{J}$2) is valid, then obviously $\Re \langle \Theta_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square)} \gtrsim I$, and so the proof of Proposition 5.2 can be copied.

We now consider the case that ($\mathcal{J}$2) is relaxed to the assumption from Remark 3.3. From $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)} = \sum_{q=1}^M I_{j,q} \langle \Theta_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square), |\partial \kappa_q|} I_{j,q}^T$ similarly to (4.3), and

$$\| \langle \Theta_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square), |\partial \kappa_q|} - \langle \Theta_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square)} \Delta_{j,q}^\square \| \lesssim 2^{-j}$$

similarly to (5.1), we find

$$\| \langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)} - \sum_{q=1}^M I_{j,q} \langle \Theta_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square)} \Delta_{j,q}^\square I_{j,q}^T \| \lesssim 2^{-j}. \tag{5.5}$$

With respect to the partitioning of I_j into p disjoint sets as induced by the partitioning of $I_j^\square$, one may verify that $\sum_{q=1}^M I_{j,q} \langle \Theta_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square)} \Delta_{j,q}^\square I_{j,q}^T$ is a uniformly bounded block triangular matrix, with diagonal blocks that are uniformly boundedly invertible diagonal matrices, meaning in particular that the whole matrix is uniformly boundedly invertible, and so is $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)}$ when $j \geq j_0$ being large enough.

(b) Using that $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)}$ is uniformly bounded invertible, one easily verifies that $\bar{Q}_j$, as given in (b), indeed has the properties listed in (a).

(c) Let $u \in \mathcal{S}(\Phi_{j+1})$, then $u = \mathbf{c}_j^T \Theta_j + \mathbf{d}_j^T \Xi_j$ with $\|u\|_{L_2(\Gamma)} \eqsim (\|\mathbf{c}_j\|^2 + \|\mathbf{d}_j\|^2)^{\frac{1}{2}}$. If, in addition, $u \in \mathcal{S}(\tilde{\Phi}_j)^{\perp_{L_2(\Gamma)}}$, then $u = (I - \bar{Q}_j)u = (I - \bar{Q}_j)\mathbf{d}_j^T \Xi_j = \mathbf{d}_j^T \Psi_j$, and so with $\|u\|_{L_2(\Gamma)} \lesssim (1 + \|\bar{Q}_j\|_{L_2(\Gamma) \to L_2(\Gamma)}) \|\mathbf{d}_j^T \Xi_j\|_{L_2(\Gamma)} \lesssim \|\mathbf{d}_j\| \lesssim \|u\|_{L_2(\Gamma)}$. Noting that $\Psi_j \subset \mathcal{S}(\Phi_{j+1}) \cap \mathcal{S}(\tilde{\Phi}_j)^{\perp_{L_2(\Gamma)}}$, we conclude that it is a uniform $L_2(\Gamma)$-Riesz basis for this space. The last statements are now consequences of Theorem 5.3. $\square$

Note that Proposition 4.1 at the dual side implies that $\Psi_j = (\psi_{j,x})_{x \in J_j}$, yielded by (5.4), satisfies

$$\begin{aligned} |\langle \psi_{j,x}, u \rangle_{L_2(\Gamma)}| &= |\langle \psi_{j,x}, (I - \tilde{P}_j)u \rangle_{L_2(\Gamma)}| \\ &\lesssim 2^{-\tilde{d}j} \sum_{q=1}^M |u \circ \kappa_q|_{H^{\tilde{d}}(B(\kappa_q^{-1}(\operatorname{supp} \psi_{j,x} \cap \Gamma_q); (\tilde{\vartheta} + 3\tilde{\varepsilon})2^{-j}) \cap \square)} \quad (u \in \mathcal{H}_{\tilde{d}}(\Gamma)), \end{aligned} \tag{5.6}$$

which property of the collections Ψ_j, with $\tilde{\vartheta} + 3\tilde{\varepsilon}$ replaced by an arbitrary but fixed $\tilde{\eta} \geq 0$ and the local semi-norms $|\cdot|_{H^{\tilde{d}}(\cdots)}$ replaced by the local norms $\|\cdot\|_{H^{\tilde{d}}(\cdots)}$, will be referred to as the *uniform cancellation property of order* $\tilde{d}$.

6. Stability of approximate wavelet bases. Similarly to (4.3), the definition of the collections Θ_j and $\tilde{\Phi}_j$ via (4.1) shows that

$$\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)} = \sum_{q=1}^M I_{j,q} \, \langle \Theta_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square), |\partial \kappa_q|} I_{j,q}^T.$$

So if both $\langle\Theta_j^\square,\tilde{\Phi}_j^\square\rangle_{L_2(\square)}$ is diagonal ((J2)) and, for each q, $z\mapsto|\partial\kappa_q(z)|$ is a *constant function*, then $\langle\Theta_j,\tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ is diagonal, and thus so is its inverse, and the collection of wavelets Ψ_j given in (5.4) is *uniformly local*. If, instead of (J2), the relaxed assumption from Remark 3.3 is made, then again $\langle\Theta_j,\tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ has a uniformly local inverse, and so the same conclusion is valid. However, unfortunately, only a restricted class of manifolds can be described as the union of patches that are the images of $\square$ under parametrizations that have constant Jacobians. In case *not all Jacobians are constants*, then, generally, $\langle\Theta_j,\tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ is not diagonal and it has a densely populated inverse, so that (5.4) yields wavelets Ψ_j that have *global supports*.

A possibility to circumvent this problem, pursued in [DS99a], is to carry out the whole wavelet construction outlined so far using the modified scalar product $\langle\!\langle\, ,\, \rangle\!\rangle_0$ instead of $\langle\, ,\, \rangle_{L_2(\Gamma)}$. Indeed, if $\langle\Theta_j^\square,\tilde{\Phi}_j^\square\rangle_{L_2(\square)}$ is diagonal, then $\langle\!\langle\Theta_j,\tilde{\Phi}_j\rangle\!\rangle_0=\sum_{q=1}^M I_{j,q}\,\langle\Theta_j^\square,\tilde{\Phi}_j^\square\rangle_{L_2(\square)}I_{j,q}^T$ is diagonal, and so uniformly local wavelets are obtained. The same is true under the relaxed assumptions from Remark 3.3. What is more, by employing this scalar product, it is always possible to take the coarsest level $j_0=0$.

As was already recognized in [DS99a], this approach however has two limitations: Firstly, the obtained wavelets will be orthogonal to the constant function with respect to $\langle\!\langle\, ,\, \rangle\!\rangle_0$. As a consequence, if the function

$$J:\cup_{q=1}^M\Gamma_q\to\mathbb{R}:x\mapsto|\partial\kappa_{q'}(\kappa_{q'}^{-1}(x))|\quad\text{when } x\in\Gamma_q'$$

has discontinuities, or more precisely, cannot be extended to a continuous function on Γ, then wavelets with supports that are not contained in one patch will generally *not* have a zero mean value with respect to the canonical Lebesgue measure on Γ, meaning that they have *no cancellation property* with respect to $\langle\cdot,\cdot\rangle_{L_2(\Gamma)}$. The application of wavelets we focus on is that for the solution of differential- or integral equations in variational form using the duality pairing with respect to $\langle\cdot,\cdot\rangle_{L_2(\Gamma)}$ (taking a different scalar product here yields other disadvantages, cf. [DS99c, §1.2]). For obtaining nearly sparse representations of these operators in wavelet coordinates, and with that algorithms of optimal computational complexity, the wavelets should have a cancellation property of sufficiently high order (cf. [DHS02, Ste04] or the surveys [Dah97, Coh03]), with respect thus to $\langle\cdot,\cdot\rangle_{L_2(\Gamma)}$.

The second limitation has to do with the interpretation, for $s<0$, of the statement that $\Phi_{j_0}\cup\cup_{j\geq j_0}2^{-sj}\Psi_j$ is a Riesz basis for $\mathcal{H}_s(\Gamma)$, which is a consequence of Theorem 5.3. In case biorthogonality is realized with respect to $\langle\cdot,\cdot\rangle_{L_2(\Gamma)}$, then an expansion in terms of the basis $\Phi_{j_0}\cup\cup_{j\geq j_0}2^{-sj}\Psi_j$ should be interpreted as an element of $\mathcal{H}_s(\Gamma)$, i.e., as a functional, using the embedding $L_2(\Gamma)\to\mathcal{H}_s(\Gamma):u\mapsto(v\mapsto\langle v,u\rangle_{L_2(\Gamma)})$. Replacing $\langle\cdot,\cdot\rangle_{L_2(\Gamma)}$ by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_0$ means that also the embedding should be changed into $u\mapsto(v\mapsto\langle\!\langle v,u\rangle\!\rangle_0)$. One can show (cf. [NS03, §4]) that if J has discontinuities, then for $s\leq-\frac{1}{2}$ a set $\Phi_{j_0}\cup\cup_{j\geq j_0}2^{-sj}\Psi_j$ which is a Riesz basis for $\mathcal{H}_s(\Gamma)$ using the embedding $u\mapsto(v\mapsto\langle\!\langle v,u\rangle\!\rangle_0)$ *cannot* be a Riesz basis for $\mathcal{H}_s(\Gamma)$ using the embedding $u\mapsto(v\mapsto\langle v,u\rangle_{L_2(\Gamma)})$. For $s>-\frac{1}{2}$, the property of being a Riesz basis for $\mathcal{H}_s(\Gamma)$ is the same for both embeddings. Again, since in applications duality pairing with respect to $\langle\cdot,\cdot\rangle_{L_2(\Gamma)}$ is used, in this paper, the property of a collection of functions to be a Riesz basis for $\mathcal{H}_s(\Gamma)$ for $s<0$ will always be interpreted with respect to the canonical embedding $u\mapsto(v\mapsto\langle v,u\rangle_{L_2(\Gamma)})$.

In this paper, we propose another approach to solve the problem that generally $\langle\Theta_j,\tilde{\Phi}_j\rangle_{L_2(\Gamma)}^{-1}$ is densely populated, so that the wavelets yielded by (5.4) have global supports. As we have seen in (5.5), with

$$\bar{\mathbf{D}}_j := \sum_{q=1}^{M} I_{j,q}\langle\Theta_j^{\square}, \tilde{\Phi}_j^{\square}\rangle_{L_2(\square)}\Delta_{j,q}^{\square} I_{j,q}^T, \quad \|\langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)} - \bar{\mathbf{D}}_j\| \lesssim 2^{-j}, \tag{6.1}$$

where, because of the assumption (J2) or the relaxed assumption from Remark 3.3, $\bar{\mathbf{D}}_j$ is uniformly bounded invertible, and it is either diagonal, or, with respect to a suitable partitioning of I_j into p sets with p fixed, it is block triangular with diagonal blocks that are diagonal matrices, showing that its inverse is uniformly local. In other words, close to the collections of the wavelets Ψ_j, there are collections of uniformly local functions of which suitable ones might be applied instead. In the following main theorem of this paper we derive general criteria under which such *approximate wavelets* satisfy the *same* conditions as Ψ_j concerning *both* stability with respect to a range of Sobolev norms *and* the order of the cancellation property, where moreover, in contrast to Ψ_j, they are uniformly local.

THEOREM 6.1. *If, for $j \geq j_0$,*

(i) $\breve{\Psi}_j = (\breve{\psi}_{j,x})_{x\in J_j} \subset \mathcal{S}(\Phi_{j+1})$ *is uniformly local,*

(ii) $\breve{\Psi}_j$ *has the uniform cancellation property of order $\tilde{d}$ (w.r.t. $\langle\cdot,\cdot\rangle_{L_2(\Gamma)}$),*

(iii) *for some $\omega \in (0,1)$, $\|\Psi_j - \breve{\Psi}_j\|_{L_2(\Gamma)} \lesssim \omega^j$,*

then, possibly for a larger value of j_0, for $s \in (-\min\{\tilde{\gamma}, \tilde{d}\}, \min\{\gamma, d\})$,

$$\Phi_{j_0} \cup \cup_{j\geq j_0} 2^{-sj}\breve{\Psi}_j \text{ is a Riesz basis for } \mathcal{H}_s(\Gamma),$$

and thus, in view of (2.4)*, when in addition $|s| < \frac{3}{2}$, $|s| < s_\Gamma \notin \mathbb{N}$ or $|s| \leq s_\Gamma \in \mathbb{N}$, it is a Riesz basis for $\mathcal{H}^s(\Gamma)$.*

The rather lengthy proof of this theorem is postponed to Appendix A. Here we emphasize that the theorem is not valid without either Condition (iii) or Condition (ii) (unless $\omega \leq 2^{-\tilde{d}}$).

Remark 6.2. The approximate wavelets $\breve{\Psi}_j$ we are going to construct in the next section will be of type $\breve{\Psi}_j = \Xi_j - \mathbf{Z}_j\Theta_j$, where $\mathbf{Z}_j$ is a uniformly local $\#J_j \times \#I_j$ matrix. Since the basis transformation $\langle\Upsilon_{j+1}, \Lambda_{j+1}\rangle_{L_2(\Gamma)}^T$ from $\Upsilon_{j+1} = \begin{bmatrix}\Theta_j^T & \Xi_j^T\end{bmatrix}^T$ to Φ_{j+1} is uniformly local, so is the basis transformation from $\begin{bmatrix}\Theta_j^T & \breve{\Psi}_j^T\end{bmatrix}^T$ to Φ_{j+1}, and the transformation from multi-scale basis $\Phi_{j_0} \cup \cup_{k=j_0}^{j-1}\breve{\Psi}_k$ to single-scale basis Φ_j has linear complexity.

In the special case that $\Theta_j = \Phi_j$ and $\langle\Upsilon_{j+1}, \Lambda_{j+1}\rangle_{L_2(\Gamma)}^{-T}$ is uniformly local (cf. Remark 4.4), the basis transformation from Φ_{j+1} to $\begin{bmatrix}\Phi_j^T & \breve{\Psi}_j^T\end{bmatrix}^T$ will also be uniformly local, and so the inverse transformation from single-scale basis Φ_j to multi-scale basis $\Phi_{j_0} \cup \cup_{k=j_0}^{j-1}\breve{\Psi}_k$ has also linear complexity. However, since $\mathcal{S}(\breve{\Psi}_j)$ is not exactly $L_2(\Gamma)$-orthogonal to $\mathcal{S}(\Phi_j)$, the corresponding dual wavelets are not explicitly given.

Remark 6.3. Under the assumptions as in Remark 3.2, by using their uniform locality one may verify that, for some constant $\tilde{\eta} \geq 0$ and $p \in [1,\infty]$, the approximate wavelets $\breve{\Psi}_j$ that will be constructed in the next section satisfy

$$|\langle\breve{\psi}_{j,x}, u\rangle_{L_2(\Gamma)}| \lesssim 2^{-(\tilde{d}+\frac{n}{2}-\frac{p}{2})j} \sum_{q=1}^{M} \|u \circ \kappa_q\|_{W_p^{\tilde{d}}(B(\kappa_q^{-1}(\operatorname{supp}\psi_{j,x}\cap\Gamma_q);\tilde{\eta}2^{-j})\cap\square)},$$

for $u \in \mathcal{W}_{p,\tilde{d}}(\Gamma)$, which space is defined as $\mathcal{H}_{\tilde{d}}(\Gamma)$ with the $H^{\tilde{d}}(\square)$-norms replaced by $W_p^{\tilde{d}}(\square)$-norms. This generalization of the uniform cancellation property of order $\tilde{d}$ has some useful applications for proving results about matrix compression.

7. Construction of uniformly local approximate wavelet bases.

7.1. Approximating $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)}^{-1}$ by a fixed number of steps of an iterative method. Our first plan to construct suitable approximate wavelets is to use $\bar{\mathbf{D}}_j$, or more efficiently, the restriction of $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)}$ to the non-zero pattern of $\bar{\mathbf{D}}_j$ as a preconditioner to construct a uniformly local, but sufficiently accurate approximation of $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)}^{-1}$. Actually, since generally $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)}$, and also $\langle \Xi_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)}$ cannot be evaluated exactly, we apply this procedure starting with sufficiently accurate approximations. Thinking in particular of $\bar{\mathbf{D}}_j$ being diagonal, we will denote the resulting collection of approximate wavelets as Ψ_j^{Jc}, where "Jc" refers to Jacobi iteration.

Theorem 7.1. *Let $\mathbf{A}_j$, $\mathbf{B}_j$ be uniformly local matrices with*

$$\|\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)} - \mathbf{A}_j\| \lesssim 2^{-\tilde{d}j}, \quad \|\langle \Xi_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)} - \mathbf{B}_j\| \lesssim 2^{-\tilde{d}j},$$

and let $\mathbf{D}_j$ be the restriction of $\mathbf{A}_j$ to the non-zero pattern of $\bar{\mathbf{D}}_j$. Then, for $j \geq j_0$ large enough,

$$\Psi_j^{\mathrm{Jc}} := \Xi_j - \mathbf{B}_j \sum_{k=0}^{\tilde{d}-1} (I - \mathbf{D}_j^{-1}\mathbf{A}_j)^k \mathbf{D}_j^{-1} \Theta_j$$

is uniformly local, it has the uniform cancellation property of order $\tilde{d}$, and finally, for $s \in (-\min\{\tilde{\gamma}, \tilde{d}\}, \min\{\gamma, d\})$,

$$\Phi_{j_0} \cup \cup_{j \geq j_0} 2^{-sj} \Psi_j^{\mathrm{Jc}} \text{ is a Riesz basis for } \mathcal{H}_s(\Gamma),$$

and thus, in view of (2.4), *when in addition $|s| < \frac{3}{2}$, $|s| < s_\Gamma \notin \mathbb{N}$ or $|s| \leq s_\Gamma \in \mathbb{N}$, it is a Riesz basis for $\mathcal{H}^s(\Gamma)$.*

Proof. Because of the assumption ($\mathcal{J}$2) or the relaxed assumption from Remark 3.3, $\bar{\mathbf{D}}_j$ is uniformly boundedly invertible, and it is either diagonal, or, with respect to a suitable partitioning of I_j into p sets with p fixed, it is block triangular with diagonal blocks that are diagonal matrices, so that in any case its inverse is uniformly local. Furthermore, it holds that $\|\bar{\mathbf{D}}_j - \langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)}\| \lesssim 2^{-j}$, and so $\|\bar{\mathbf{D}}_j - \mathbf{A}_j\| \lesssim 2^{-j}$, and, by the uniform sparsity of $\bar{\mathbf{D}}_j$ and $\mathbf{D}_j$,

$$\|\bar{\mathbf{D}}_j - \mathbf{D}_j\| \lesssim \max_{x,y \in I_j} |(\bar{\mathbf{D}}_j - \mathbf{D}_j)_{x,y}| \leq \max_{x,y \in I_j} |(\bar{\mathbf{D}}_j - \mathbf{A}_j)_{x,y}| \leq \|\bar{\mathbf{D}}_j - \mathbf{A}_j\| \lesssim 2^{-j}.$$

We conclude that for $j \geq j_0$ being large enough, $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)}$, $\mathbf{A}_j$ and $\mathbf{D}_j$ are uniformly bounded invertible, $\mathbf{D}_j^{-1}$ is uniformly local, and $\|\mathbf{D}_j - \mathbf{A}_j\| \leq \|\mathbf{D}_j - \bar{\mathbf{D}}_j\| + \|\bar{\mathbf{D}}_j - \mathbf{A}_j\| \lesssim 2^{-j}$. We infer that

$$\|\mathbf{A}_j^{-1} - \sum_{k=0}^{\tilde{d}-1} (I - \mathbf{D}_j^{-1}\mathbf{A}_j)^k \mathbf{D}_j^{-1}\| = \|\mathbf{A}_j^{-1}((\mathbf{D}_j - \mathbf{A}_j)\mathbf{D}_j^{-1})^{\tilde{d}}\| \lesssim 2^{-\tilde{d}j},$$

so that $\|\langle \Xi_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)} \langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)}^{-1} - \mathbf{B}_j \sum_{k=0}^{\tilde{d}-1} (I - \mathbf{D}_j^{-1}\mathbf{A}_j)^k \mathbf{D}_j^{-1}\| \lesssim 2^{-\tilde{d}j}$ using the assumptions. By definition of Ψ_j in (5.4), we conclude that

$$\|\Psi_j - \Psi_j^{\mathrm{Jc}}\|_{L_2(\Gamma)} \lesssim 2^{-\tilde{d}j} \|\Theta_j\|_{L_2(\Gamma)} \lesssim 2^{-\tilde{d}j}. \tag{7.1}$$

Since furthermore $\mathbf{B}_j \sum_{k=0}^{\tilde{d}-1}(I - \mathbf{D}_j^{-1}\mathbf{A}_j)^k \mathbf{D}_j^{-1}$, and thus Ψ_j^{Jc} is uniformly local, in view of Theorem 6.1 the only thing left to show is that Ψ_j^{Jc} has the uniform cancellation property of order $\tilde{d}$. Although Ψ_j is known to have the uniform cancellation property of order $\tilde{d}$, we cannot conclude this immediately from (7.1) for Ψ_j^{Jc}, because generally the wavelets from Ψ_j have global supports (cf. the definition of the uniform cancellation property (5.6)).

Using $\|\Psi_j^{\mathrm{Jc}}\|_{L_2(\Gamma)} \leq \|\Psi_j^{\mathrm{Jc}} - \Psi_j\|_{L_2(\Gamma)} + \|\Psi_j\|_{L_2(\Gamma)} \lesssim 1$, from the analogue of Proposition 4.1 at the dual side, for $u \in \mathcal{H}_{\tilde{d}}(\Gamma)$ we have

$$(7.2)\quad |\langle \psi_{j,x}^{\mathrm{Jc}}, (I - \tilde{P}_j)u\rangle_{L_2(\Gamma)}| \lesssim 2^{-\tilde{d}j} \sum_{q=1}^{M} |u \circ \kappa_q|_{H^{\tilde{d}}(B(\kappa_q^{-1}(\operatorname{supp}\psi_{j,x}^{\mathrm{Jc}} \cap \Gamma_q);(\tilde{\vartheta}+3\tilde{\varepsilon})2^{-j})\cap\square)}.$$

Let $I_j(x) := \{y \in I_j : \operatorname{supp}\tilde{\phi}_{j,y} \cap \operatorname{supp}\psi_{j,x}^{\mathrm{Jc}} \neq \emptyset\}$. Noting that $\#I_j(x)$ is bounded, uniformly in j and $x \in J_j$, we have

$$\begin{aligned}
(7.3)\quad & |\langle \psi_{j,x}^{\mathrm{Jc}}, \tilde{P}_j u\rangle_{L_2(\Gamma)}| \\
&= |\langle \psi_{j,x}^{\mathrm{Jc}}, \sum_{y\in I_j(x)} \tilde{\lambda}_{j,y}(u)\tilde{\phi}_{j,y}\rangle_{L_2(\Gamma)}| = |\langle \psi_{j,x}^{\mathrm{Jc}} - \psi_{j,x}, \sum_{y\in I_j(x)} \tilde{\lambda}_{j,y}(u)\tilde{\phi}_{j,y}\rangle_{L_2(\Gamma)}| \\
&\lesssim 2^{-\tilde{d}j} \sum_{q=1}^{M} \| \sum_{z\in\kappa_q^{-1}(I_j(x))} \tilde{\lambda}_{j,z}^{\square}(u\circ\kappa_q)\tilde{\phi}_{j,z}^{\square}\|_{L_2(\square)} \lesssim 2^{-\tilde{d}j} \max_{1\leq q\leq M,\, z\in\kappa_q^{-1}(I_j(x))} |\tilde{\lambda}_{j,z}^{\square}(u\circ\kappa_q)|.
\end{aligned}$$

It is possible that for some $1 \leq q' \leq M$, $\kappa_{q'}^{-1}(I_j(x)) \neq \emptyset$, whereas $\operatorname{supp}\psi_{j,x}^{\mathrm{Jc}} \cap \Gamma_{q'} = \emptyset$. Yet, in that case, each $z' \in \kappa_{q'}^{-1}(I_j(x))$ is on a face of $\square$ and there exist $1 \leq q \leq M$, $z \in \kappa_q^{-1}(I_j(x))$ with $\kappa_{q'}(z') = \kappa_q(z)$, and thus $\tilde{\lambda}_{j,z'}^{\square}(u\circ\kappa_{q'}) = \tilde{\lambda}_{j,z}^{\square}(u\circ\kappa_q)$, and $\operatorname{supp}\psi_{j,x}^{\mathrm{Jc}} \cap \Gamma_q \neq \emptyset$ with in particular $\operatorname{dist}(z, \kappa_q^{-1}(\operatorname{supp}\psi_{j,x}^{\mathrm{Jc}})) \leq \tilde{\varepsilon}2^{-j}$, meaning that it is sufficient to consider this case. Since $|\tilde{\lambda}_{j,z}^{\square}(u\circ\kappa_q)| \lesssim \|u\circ\kappa_q\|_{L_2(B(z,\tilde{\vartheta}2^{-j})\cap\square)} + 2^{-\tilde{d}j}|u\circ\kappa_q|_{H^{\tilde{d}}(B(z,\tilde{\vartheta}2^{-j})\cap\square)}$, as follows from (3.1) at the dual side, we conclude that

$$\begin{aligned}
(7.4)\qquad |\langle \psi_{j,x}^{\mathrm{Jc}}, \tilde{P}_j u\rangle_{L_2(\Gamma)}| \lesssim & 2^{-\tilde{d}j} \sum_{q=1}^{M} [\|u\circ\kappa_q\|_{L_2(B(\kappa_q^{-1}(\operatorname{supp}\psi_{j,x}^{\mathrm{Jc}}\cap\Gamma_q);(\tilde{\vartheta}+\tilde{\varepsilon})2^{-j})\cap\square)} \\
& + 2^{-\tilde{d}j}|u\circ\kappa_q|_{H^{\tilde{d}}(B(\kappa_q^{-1}(\operatorname{supp}\psi_{j,x}^{\mathrm{Jc}}\cap\Gamma_q);(\tilde{\vartheta}+\tilde{\varepsilon})2^{-j})\cap\square)}].
\end{aligned}$$

The combination of (7.2) and (7.4) completes the proof. □

The construction of the approximate wavelets Ψ_j^{Jc} in Theorem 7.1 has some similarities to the construction of approximate "prewavelets" in [VW99], where the inverse of a mass matrix with respect to a standard finite element basis is approximated by a number of steps of an iterative method, as the Jacobi or symmetric Gauss-Seidel method. A difference is that in our case the matrix $\mathbf{D}_j$ converges to $\mathbf{A}_j$ for $j \to \infty$, so that, as a consequence, the approximation of $\mathbf{A}_j^{-1}$ by means of a sufficiently large, fixed number of steps of the linear iteration corresponding to the splitting $\mathbf{A}_j = \mathbf{D}_j - [\mathbf{D}_j - \mathbf{A}_j]$ yields approximate wavelets that generate Riesz bases for the full range of s.

Compared to the approximate wavelets one gets by simply replacing $\langle\cdot,\cdot\rangle_{L_2(\Gamma)}$ by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_0$ in (5.4), in particular for larger $\tilde{d}$ the approximate wavelets $\psi_{j,x}^{\mathrm{Jc}}$ have large supports. Although this has not so much an effect on the multi-scale to single scale transform, that can be implemented being much more efficient than suggested by the sizes of the supports, it is a disadvantage for example when it concerns the compression of the stiffness matrix of an integral operator with respect to these approximate wavelets. In the following two subsections, we construct approximate wavelets with smaller supports. As a preparation, the next proposition facilitates the verification of the third condition from Theorem 6.1 in case different constructions of approximate wavelets are used on different parts of Γ.

PROPOSITION 7.2. *Let $\omega \in (0,1)$ and let $\breve{\Psi}_j = (\breve{\psi}_{j,x})_{x\in J_j}$ be uniformly local. Then $\|\Psi_j - \breve{\Psi}_j\|_{L_2(\Gamma)} \lesssim \omega^j$ if and only if $\sup_{x\in J_j} \|\psi_{j,x} - \breve{\psi}_{j,x}\|_{L_2(\Gamma)} \lesssim \omega^j$.*

Proof. Let $\sup_{x\in J_j} \|\psi_{j,x} - \breve{\psi}_{j,x}\|_{L_2(\Gamma)} \lesssim \omega^j$. Selecting $m \in \mathbb{N}$ such that $2^{-m} \leq \omega$, from the proof of Theorem 7.1 we learn that there exists a uniformly local Ψ_j^{Jc} with $\|\Psi_j - \Psi_j^{\mathrm{Jc}}\|_{L_2(\Gamma)} \lesssim 2^{-mj} \leq \omega^j$, and so $\sup_{x\in J_j} \|\psi_{j,x} - \psi_{j,x}^{\mathrm{Jc}}\|_{L_2(\Gamma)} \lesssim \omega^j$ and thus $\sup_{x\in J_j} \|\psi_{j,x}^{\mathrm{Jc}} - \breve{\psi}_{j,x}\|_{L_2(\Gamma)} \lesssim \omega^j$. Since both Ψ_j^{Jc} and $\breve{\Psi}_j$ are uniformly local, this implies $\|\Psi_j^{\mathrm{Jc}} - \breve{\Psi}_j\|_{L_2(\Gamma)} \lesssim \omega_j$, and thus that $\|\Psi_j - \breve{\Psi}_j\|_{L_2(\Gamma)} \leq \|\Psi_j - \Psi_j^{\mathrm{Jc}}\|_{L_2(\Gamma)} + \|\Psi_j^{\mathrm{Jc}} - \breve{\Psi}_j\|_{L_2(\Gamma)} \lesssim \omega^j$. The proof of the other implication is trivial. $\square$

7.2. Ignoring the Jacobian determinants away from the interfaces. We will show that away from the patch interfaces, we may replace the wavelets from Ψ_j^{Jc} by the corresponding ones from $\Psi_j^{(0)} := \Xi_j - \langle\!\langle \Xi_j, \tilde{\Phi}_j\rangle\!\rangle_0 \langle\!\langle \Theta_j, \tilde{\Phi}_j\rangle\!\rangle_0^{-1}\Theta_j$, which collection one obtains by replacing all Jacobian determinants by the constant 1 in the wavelet formula (5.4). We set

$$I_j^{\square,\circ} = \{y \in I_j^{\square} : (\langle \Theta_j^{\square}, \tilde{\Phi}_j^{\square}\rangle_{L_2(\square)}^{-1})_{x,y} = 0 \text{ for all } x \in I_j^{\square} \cap \partial\square\}.$$

By assumption ($\mathcal{J}2$) or the relaxed assumption from Remark 3.3, $\langle \Theta_j^{\square}, \tilde{\Phi}_j^{\square}\rangle_{L_2(\square)}^{-1}$ is uniformly local, and has nonzero diagonal entries. This means that $I_j^{\square,\circ} \cap \partial\square = \emptyset$, and that for any $y \in I_j^{\square}$ with $2^j \operatorname{dist}(y, \partial\square)$ sufficiently large, independently of j, it holds that $y \in I_j^{\square,\circ}$. In particular, in case $\langle \Theta_j^{\square}, \tilde{\Phi}_j^{\square}\rangle_{L_2(\square)}$ is diagonal, then $I_j^{\square,\circ} = I_j^{\square} \backslash (I_j^{\square} \cap \partial\square)$. Furthermore, we set $I_j^{\circ} = \cup_{q=1}^{M} \kappa_q(I_j^{\square,\circ})$, and

$$J_j^{\circ} = \{x \in J_j : k(x) = 1 \text{ and } \langle \xi_{j,x}, \tilde{\phi}_{j,y}\rangle_{L_2(\Gamma)} = 0 \text{ for all } y \in I_j \backslash I_j^{\circ}\}.$$

which set is designed such that for $x \in J_j^{\circ}$, $\psi_{j,x}^{(0)}$ is fully supported inside one patch $\overline{\Gamma_q}$.

THEOREM 7.3. *The set $\breve{\Psi}_j = \{\breve{\psi}_{j,x} : x \in J_j\}$, defined by $\breve{\psi}_{j,x} = \psi_{j,x}^{(0)}$ when $x \in J_j^{\circ}$, and $\breve{\psi}_{j,x} = \psi_{j,x}^{\mathrm{Jc}}$ when $x \in J_j \backslash J_j^{\circ}$, is uniformly local, it has the uniform cancellation property of order $\tilde{d}$, and for any $s \in (-\min\{\tilde{\gamma}, \tilde{d}\}, \min\{\gamma, d\})$ and j_0 large enough,*

$$\Phi_{j_0} \cup \cup_{j\geq j_0} 2^{-sj}\breve{\Psi}_j \text{ is a Riesz basis for } \mathcal{H}_s(\Gamma),$$

and thus, in view of (2.4), when in addition $|s| < \frac{3}{2}$, $|s| < s_\Gamma \notin \mathbb{N}$ or $|s| \leq s_\Gamma \in \mathbb{N}$, it is a Riesz basis for $\mathcal{H}^s(\Gamma)$.

Proof. It holds that $\langle\!\langle \Theta_j, \tilde{\Phi}_j\rangle\!\rangle_0 = \sum_{q=1}^{M} I_{j,q}\langle \Theta_j^{\square}, \tilde{\Phi}_j^{\square}\rangle_{L_2(\square)} I_{j,q}^T$ and, with $J_{j,q} : \ell_2(J_j^{\square}) \to \ell_2(J_j)$ defined by $(J_{j,q}\mathbf{c}_j^{\square})_x = k(x)^{-\frac{1}{2}} \begin{cases} \mathbf{c}_{j,\kappa_q^{-1}(x)}^{\square} & x \in \overline{\Gamma_q}, \\ 0 & \text{otherwise.} \end{cases}$ (cf. (4.2)), that $\langle\!\langle \Xi_j, \tilde{\Phi}_j\rangle\!\rangle_0 = \sum_{q=1}^{M} J_{j,q}\langle \Xi_j^{\square}, \tilde{\Phi}_j^{\square}\rangle_{L_2(\square)} I_{j,q}^T$.

Similarly to the construction of $\bar{\mathbf{D}}_j$ as an approximation for $\langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ in (6.1), with

$$\bar{\mathbf{B}}_j := \sum_{q=1}^{M} J_{j,q}\langle\Xi_j^\square, \tilde{\Phi}_j^\square\rangle_{L_2(\square)}\Delta_{j,q}^\square I_{j,q}^T, \quad \|\langle\Xi_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)} - \bar{\mathbf{B}}_j\| \lesssim 2^{-j}$$

and so, for $j \geq j_0$ large enough, we have

$$\|\langle\Xi_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}\langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}^{-1} - \bar{\mathbf{B}}_j\bar{\mathbf{D}}_j^{-1}\| \lesssim 2^{-j}. \tag{7.5}$$

We conclude that the collection

$$\hat{\Psi}_j := \Xi_j - \bar{\mathbf{B}}_j\bar{\mathbf{D}}_j^{-1}\Theta_j \tag{7.6}$$

is both uniformly local and satisfies $\|\Psi_j - \hat{\Psi}_j\|_{L_2(\Gamma)} \lesssim 2^{-j}$.

For $y \in I_j^\circ$, $x \in I_j$, we have $(\langle\!\langle\Theta_j, \tilde{\Phi}_j\rangle\!\rangle_0 \sum_{q=1}^M I_{j,q}\langle\Theta_j^\square, \tilde{\Phi}_j^\square\rangle_{L_2(\square)}^{-1} I_{j,q}^T)_{x,y} = \delta_{x,y}$, or $(\langle\!\langle\Theta_j, \tilde{\Phi}_j\rangle\!\rangle_0^{-1})_{\bullet,y} = (\sum_{q=1}^M I_{j,q}\langle\Theta_j^\square, \tilde{\Phi}_j^\square\rangle_{L_2(\square)}^{-1} I_{j,q}^T)_{\bullet,y}$, and analogously $(\bar{\mathbf{D}}_j)_{\bullet,y}^{-1} = (\sum_{q=1}^M I_{j,q}(\Delta_{j,q}^\square)^{-1}\langle\Theta_j^\square, \tilde{\Phi}_j^\square\rangle_{L_2(\square)}^{-1} I_{j,q}^T)_{\bullet,y}$. As a consequence, for $x \in J_j^\circ$ we have $(\langle\!\langle\Xi_j, \tilde{\Phi}_j\rangle\!\rangle_0\langle\!\langle\Theta_j, \tilde{\Phi}_j\rangle\!\rangle_0^{-1})_{x,\bullet} = (\bar{\mathbf{B}}_j\bar{\mathbf{D}}_j^{-1})_{x,\bullet}$, or $\psi_{j,x}^{(0)} = \hat{\psi}_{j,x}$.

From the uniform locality of both Ψ_j^{Jc} and $\hat{\Psi}_j$, $\|\Psi_j - \Psi_j^{\mathrm{Jc}}\|_{L_2(\Gamma)} \lesssim 2^{-\tilde{d}j}$ and $\|\Psi_j - \hat{\Psi}_j\|_{L_2(\Gamma)} \lesssim 2^{-j}$, and thus, by Proposition 7.2, $\|\Psi_j - \breve{\Psi}_j\|_{L_2(\Gamma)} \lesssim 2^{-j}$, in view of Theorem 6.1 the only thing left to show is that $\breve{\Psi}_j$ has the uniform cancellation property of order $\tilde{d}$. Because of Theorem 7.1, we only have to consider $\breve{\psi}_{j,x} = \psi_{j,x}^{(0)}$ for $x \in J_j^\circ$. Let $x \in J_j^\circ$, say $x \in \Gamma_q$, and let u be a globally continuous, patchwise smooth function on Γ that is zero on $\partial\Gamma_D$. Let v be some arbitrary extension of the mapping $x \mapsto u(x)|\partial\kappa_q(\kappa_q^{-1}(x))|$ on Γ_q to a globally continuous, patchwise smooth function on Γ that is zero on $\partial\Gamma_D$. Since $\operatorname{supp}\psi_{j,x}^{(0)} \subset \overline{\Gamma_q}$, from Proposition 4.1 at the dual side we have

$$\begin{aligned}
|\langle\psi_{j,x}^{(0)}, u\rangle_{L_2(\Gamma)}| &= |\langle\!\langle\psi_{j,x}^{(0)}, v\rangle\!\rangle_0| = |\langle\!\langle\psi_{j,x}^{(0)}, ((I - \tilde{P}_j)v)|_{\operatorname{supp}\psi_{j,x}^{(0)}}\rangle\!\rangle_0| \\
&\lesssim 2^{-\tilde{d}j}|v \circ \kappa_q|_{H^{\tilde{d}}(B(\kappa_q^{-1}(\operatorname{supp}\psi_{j,x}^{(0)});(\tilde{\vartheta}+3\tilde{\varepsilon})2^{-j})\cap\square)} \\
&\lesssim 2^{-\tilde{d}j}\|u \circ \kappa_q\|_{H^{\tilde{d}}(B(\kappa_q^{-1}(\operatorname{supp}\psi_{j,x}^{(0)});(\tilde{\vartheta}+3\tilde{\varepsilon})2^{-j})\cap\square)},
\end{aligned}$$

which completes the proof. □

For $x \in J_j^\circ$, generally it holds that $\|\psi_{j,x} - \psi_{j,x}^{(0)}\|_{L_2(\Gamma)} \eqsim 2^{-j}$. So in contrast to Ψ_j^{Jc}, for the approximate wavelets $\breve{\Psi}_j$ from this subsection, generally $\|\Psi_j - \breve{\Psi}_j\|_{L_2(\Gamma)} \not\lesssim 2^{-\tilde{d}j}$ when $\tilde{d} > 1$. The same will hold true for the collections $\breve{\Psi}_j$ that will be constructed in the next subsection.

7.3. Approximate wavelets with small supports near the interfaces. As we saw in the previous subsection, we can replace $\psi_{j,x}^{\mathrm{Jc}}$ by $\psi_{j,x}^{(0)}$, that generally has a much smaller support, for all those $x \in J_j$ for which $\psi_{j,x}^{(0)}$ is fully supported inside one patch $\overline{\Gamma_q}$. In this subsection, we investigate whether we can replace also the remaining $\psi_{j,x}^{\mathrm{Jc}}$ by appropriate approximate wavelets with smaller supports. Recall the definition

$$\Psi_j^{\mathrm{Jc}} := \Xi_j - \mathbf{B}_j \sum_{k=0}^{\tilde{d}-1}(I - \mathbf{D}_j^{-1}\mathbf{A}_j)^k\mathbf{D}_j^{-1}\Theta_j.$$

Responsible for the large supports are the terms $\mathbf{B}_j(I-\mathbf{D}_j^{-1}\mathbf{A}_j)^k\mathbf{D}_j^{-1}$ for large k, which have coefficients of order 2^{-jk}. Starting with $k=\tilde{d}-1$, for some $x\in J_j\backslash J_j^\circ$ let us write $[\mathbf{B}_j(I-\mathbf{D}_j^{-1}\mathbf{A}_j)^k\mathbf{D}_j^{-1}\Theta_j]_x$ as $\sum_{y\in I_j}\sigma_{j,x,y}\theta_{j,y}$, which sum thus contributes to $\psi_{j,x}^{\mathrm{Jc}}$. Let us pick one term $\sigma_{j,x,y}\theta_{j,y}$ with $2^j\operatorname{dist}_\Gamma(x,y)$ relatively large. Select a $\breve{y}\in I_j$ close to x, and, with $c_{y,\breve{y}}:=\int_\Gamma\theta_{j,y}d\mu/\int_\Gamma\theta_{j,\breve{y}}d\mu$, replace $\sigma_{j,x,y}\theta_{j,y}$ by $\sigma_{j,x,y}c_{y,\breve{y}}\theta_{j,\breve{y}}$. Since, by $\langle\theta_{j,y}-c_{y,\breve{y}}\theta_{j,\breve{y}},1\rangle_{L_2(\Gamma)}=0$, we have $|\langle\theta_{j,y}-c_{y,\breve{y}}\theta_{j,\breve{y}},u\rangle_{L_2(\Gamma)}|\lesssim 2^{-j}$ depending on some local H^1-norms of the patchwise pull-backs of $u\in\mathcal{H}^1$, and $|\sigma_{j,x,y}|\lesssim 2^{-kj}$, we infer that the uniform cancellation property of order $\tilde{d}$ is retained, and that the modified $\psi_{j,x}^{\mathrm{Jc}}$ has $L_2(\Gamma)$-distance $\lesssim 2^{-kj}$ to the original one. By repeating this process for all terms $\sigma_{j,x,y}\theta_{j,y}$ with $2^j\operatorname{dist}_\Gamma(x,y)$ large, we can reduce the size of the support. What is more, afterwards we can handle the terms $\mathbf{B}_j(I-\mathbf{D}_j^{-1}\mathbf{A}_j)^k\mathbf{D}_j^{-1}$ for each $k=\tilde{d}-2,\ldots,1$ similarly, if we now replace terms $\sigma_{j,x,y}\theta_{j,y}$ by a sufficiently large linear combination of $\theta_{j,\breve{y}}$'s such that the difference is $L_2(\Gamma)$-orthogonal to all continuous functions that are patchwise push-forwards of polynomials of order $\tilde{d}-1-k$.

Although above lines show that also for $x\in J_j\backslash J_j^\circ$ we can construct appropriate approximate wavelets with smaller supports that those of the $\psi_{j,x}^{\mathrm{Jc}}$, the above sketched procedure for obtaining them is rather awkward. In the remainder of this subsection, we approach the problem from the opposite side. Throughout this subsection, let

- $\hat{\Psi}_j=\{\hat{\psi}_{j,x}:x\in J_j\}\subset\mathcal{S}(\Phi_{j+1})$ be uniformly local, with
- $\hat{\psi}_{j,x}=\psi_{j,x}^{(0)}$ when $x\in J_j^\circ$, and
- $\|\Psi_j-\hat{\Psi}_j\|_{L_2(\Gamma)}\lesssim 2^{-j}$,

where, however, when $\tilde{d}>1$, $\{\hat{\psi}_{j,x}:x\in J_j\backslash J_j^\circ\}$ has *not* the uniform cancellation property of order $\tilde{d}$. Examples of such collections $\{\hat{\psi}_{j,x}:x\in J_j\backslash J_j^\circ\}$ are the first order approximations $\Xi_j-\mathbf{B}_j\mathbf{D}_j^{-1}\Theta_j$ or $\Xi_j-\bar{\mathbf{B}}_j\bar{\mathbf{D}}_j^{-1}\Theta_j$, restricted to $J_j\backslash J_j^\circ$. The approach that will be followed is, for each $x\in J_j\backslash J_j^\circ$, to search for a *correction term* having a small, uniformly local support, and $L_2(\Gamma)$-norm $\lesssim 2^{-j}$, that when added to $\hat{\psi}_{j,x}$ yields a collection $\breve{\Psi}_j$ that has the uniform cancellation property of order $\tilde{d}$, and thus, by $\|\Psi_j-\breve{\Psi}_j\|_{L_2(\Gamma)}\leq\|\Psi_j-\hat{\Psi}_j\|_{L_2(\Gamma)}+\|\hat{\Psi}_j-\breve{\Psi}_j\|_{L_2(\Gamma)}\leq 2^{-j}$, fulfills all our requirements.

When $\square$ is an n-simplex, we will search the aforementioned correction term for $\hat{\psi}_{j,x}$ such that the resulting $\breve{\psi}_{j,x}$ is $L_2(\Gamma)$-orthogonal to all continuous functions that are patchwise push-forwards of polynomials of order $\tilde{d}-2$. The fact that $\hat{\psi}_{j,x}$ is close to $\psi_{j,x}$ will be the reason that it is sufficient to impose this condition only for order $\tilde{d}-2$ instead of order $\tilde{d}-1$ as one may have expected. When homogeneous Dirichlet boundary conditions are incorporated in the definition of the Sobolev spaces, it is sufficient to impose orthogonality only with respect to the continuous, patchwise push-forwards of polynomials of order $\tilde{d}-2$ that locally satisfy these boundary conditions. What is more, since these boundary conditions are also incorporated in the dual multiresolution analysis, in order to be able to find a correction with $L_2(\Gamma)$-norm $\lesssim 2^{-j}$, actually it will be necessary to impose orthogonality only with respect to this smaller class of functions. When $\square$ is the n-cube $[0,1]^n$, the space of continuous functions that are patchwise push-forwards of polynomials of order $\tilde{d}-2$ has locally not enough degrees of freedom, and instead of $P_{\tilde{d}-2}(\square)$ we have to work with the tensor product space $Q_{\tilde{d}-2}(\square)$ of the univariate polynomial spaces $P_{\tilde{d}-2}([0,1])$ in the n coordinate directions.

In the following, let $I\!\!P_{\tilde{d}-2}(\Gamma):=C(\Gamma)\cap\prod_{q=1}^M\kappa_q(I\!\!P_{\tilde{d}-2}(\square))$, where $I\!\!P_{\tilde{d}-2}(\square):=$

$P_{\tilde{d}-2}(\square)$ when $\square$ is the interior of an n-simplex, and $I\!P_{\tilde{d}-2}(\square) := Q_{\tilde{d}-2}(\square)$ when $\square = (0,1)^n$. In the latter case, in addition to the assumption that $P_{\tilde{d}-1}(\square) \subset \mathcal{S}(\tilde{\Phi}_j^{\square})$ (($\mathfrak{J}$)(ii) at the dual side), in this subsection we assume that

$$(\tilde{\mathfrak{J}}\mathrm{e}) \qquad Q_{\tilde{d}-2}(\square) \subset \mathcal{S}(\tilde{\Phi}_j^{\square}).$$

We start with showing that the approach that was sketched above, assuming that it is realizable, indeed gives rise to approximate wavelets that have the uniform cancellation property of order $\tilde{d}$. For $x \in \overline{\Gamma}$ and $\rho \geq 0$, we set $B_\Gamma(x;\rho) = \{y \in \overline{\Gamma} : d_\Gamma(x,y) \leq \rho\}$. Furthermore, we set $\bar{I}_j = \cup_{q=1}^M \kappa_q(I_j^{\square})$, i.e., without the exclusion of possible points on $\partial\Gamma_D$, and define $\tilde{\phi}_{j,y}$ and $\tilde{\lambda}_{j,y}$ similarly as in (4.1) and in Proposition 4.1, respectively, but now also for $y \in \bar{I}_j \backslash I_j$. Because of our assumption ($\tilde{\mathfrak{J}}$e) and the inclusion of possible points on $\partial\Gamma_D$, the projector $u \mapsto \sum_{y\in\bar{I}_j} \langle u, \tilde{\lambda}_{j,y}\rangle_{L_2(\Gamma)} \tilde{\phi}_{j,y}$ preserves any $v \in I\!P_{\tilde{d}-2}(\Gamma)$.

Theorem 7.4. *Let $\breve{\Psi}_j = \{\breve{\psi}_{j,x} : x \in J_j\} \subset \mathcal{S}(\Phi_{j+1})$ be a uniformly local collection, with $\|\Psi_j - \breve{\Psi}_j\|_{L_2(\Gamma)} \lesssim 2^{-j}$, $\breve{\psi}_{j,x} = \psi_{j,x}^{(0)}$ when $x \in J_j^\circ$, and such that, for some constant $\rho \geq 0$, for $x \in J_j \backslash J_j^\circ$,*

$$\breve{\psi}_{j,x} \perp_{L_2(\Gamma)} (I\!P_{\tilde{d}-2}(\Gamma) \cap H^1_{0,\partial\Gamma_D \cap B_\Gamma(x;\rho 2^{-j})}(\Gamma)).$$

Then $\breve{\Psi}_j$ has the uniform cancellation property of order $\tilde{d}$, and for j_0 large enough, and for $s \in (-\min\{\tilde{\gamma}, \tilde{d}\}, \min\{\gamma, d\})$,

$$\Phi_{j_0} \cup \cup_{j \geq j_0} 2^{-sj} \breve{\Psi}_j \text{ is a Riesz basis for } \mathcal{H}_s(\Gamma),$$

and thus, in view of (2.4)*, when in addition $|s| < \frac{3}{2}$, $|s| < s_\Gamma \notin \mathbb{N}$ or $|s| \leq s_\Gamma \in \mathbb{N}$, it is a Riesz basis for $\mathcal{H}^s(\Gamma)$.*

Proof. In view of Theorem 6.1, it is sufficient to show that $\{\breve{\psi}_{j,x} : x \in J_j \backslash J_j^\circ\}$ has the uniform cancellation property of order $\tilde{d}$.

For any $m \in \mathbb{N}$, there exists a regular subdivision of $\overline{\square}$ into m^n cubes or simplices $\overline{\square_{m,1}}, \overline{\square_{m,2}}, \ldots \overline{\square_{m,m^n}}$ with $\square_{m,i} \cap \square_{m,\hat{\imath}} = \emptyset$ when $i \neq \hat{\imath}$, which decomposition is invariant under any $\pi \in \Pi$.

Assuming $\tilde{d} > 2$, there exists a set $K \subset \overline{\square}$ consisting of $\dim I\!P_{\tilde{d}-2}(\square)$ interpolation points, invariant under all $\pi \in \Pi$, such that each $p \in I\!P_{\tilde{d}-2}(\square)$ is uniquely determined by its values in these points, and moreover that for any face of $\square$, the restriction of p to that face is uniquely determined by its values in the interpolation points on that face. For any $1 \leq i \leq m^n$, let $K_{m,i}$ be the image of K under an arbitrary affine bijection between $\overline{\square}$ and $\overline{\square_{m,i}}$. For any i and $\pi \in \Pi$, and with $\hat{\imath}$ such that $\pi(\square_{m,i}) = \square_{m,\hat{\imath}}$, we have $\pi(K_{m,i}) = K_{m,\hat{\imath}}$.

For some $m = m(j)$ with $|m - 2^j| \lesssim 1$, $x \in J_j \backslash J_j^\circ$, and $q \in F_{j,x} := \{1 \leq q' \leq M : \operatorname{supp} \breve{\psi}_{j,x} \cap \Gamma_{q'} \neq \emptyset\}$, we can select $1 \leq i(x,q) \leq m^n$, such that

$$(B(\kappa_q^{-1}(\operatorname{supp} \breve{\psi}_{j,x} \cap \Gamma_q); (\tilde{\vartheta} + 3\tilde{\varepsilon})2^{-j}) \cap \square) \;\cup\; \kappa_q^{-1}(B_\Gamma(x;\rho 2^{-j}) \cap \Gamma_q) \subset \square_{m,i(x,q)},$$

where $\cup_{q\in F_{j,x}} \kappa_q(\overline{\square_{m,i(x,q)}})$ is connected.

Assuming $\tilde{d} > 2$, given $u \in \mathcal{H}_{\tilde{d}}(\Gamma)$, we define $v \in \prod_{q\in F_{j,x}} \kappa_q(I\!P_{\tilde{d}-2}(\square))$ by $v \circ \kappa_q = u \circ \kappa_q$ on $K_{m,i(x,q)}$. By construction, the interpolation points coincide at the interfaces between patches Γ_q for $q \in F_{j,x}$, and so is v well-defined, $v \in C(\cup_{q\in F_{j,x}} \overline{\Gamma_q})$, and it can be extended to a function in $I\!P_{\tilde{d}-2}(\Gamma)$.

Concerning boundary conditions, if $z \in \overline{\square_{m,i(x,q)}} \cap \kappa_q^{-1}(\partial\Gamma_D)$, and e is the lowest dimensional face of $\square$, say of dimension k, containing z, then, by the properties of the subdivision, $\square_{m,i(x,q)}$ has a face of dimension k on e, and so, since $u \circ \kappa_q$ vanishes on e by (2.1), so does $v \circ \kappa_q$.

Since, apparently $v \in I\!\!P_{\tilde{d}-2}(\Gamma) \cap H^1_{0,\partial\Gamma_D \cap B_\Gamma(x;\rho 2^{-j})}(\Gamma)$, we have $\langle \breve{\psi}_{j,x}, v\rangle_{L_2(\Gamma)} = 0$. Furthermore, we have $v = \sum_{y \in \bar{I}_j} \langle v, \tilde{\lambda}_{j,y}\rangle_{L_2(\Gamma)} \tilde{\phi}_{j,y}$. From the locality assumption $(\mathcal{L})$ at the dual side, the right-hand side of the last expression restricted to supp$\breve{\psi}_{j,x}$ depends only on terms for $y \in \bar{I}_j$ in $\cup_{q \in F_{j,x}} \kappa_q(B(\kappa_q^{-1}(\operatorname{supp} \breve{\psi}_{j,x} \cap \overline{\Gamma_q}); \tilde{\varepsilon} 2^{-j}) \cap \overline{\square})$. For any of such y on $\partial\Gamma_D$, i.e., $y \in \bar{I}_j \backslash I_j$, say with $y \in \partial\Gamma_D \cap \overline{\Gamma_q}$, $v \circ \kappa_q$ vanishes on the lowest dimensional face of $\square$ that contains $\kappa_q^{-1}(y)$, and $(\mathcal{J})$(i) at the dual side shows that $\langle v, \tilde{\lambda}_{j,y}\rangle_{L_2(\Gamma)} = 0$. We conclude that $(I - \tilde{P}_j)v$ vanishes on supp$\breve{\psi}_{j,x}$.

Standard arguments using the Sobolev embedding theorem (recall that $\tilde{d} > \frac{n}{2}$), the Bramble-Hilbert lemma, and the fact that for any $q \in F_{j,x}$ the above interpolation reproduces any $u|_{\Gamma_q} \in \kappa_q(P_{\tilde{d}-2}(\square))$, show that for $q \in F_{j,x}$,

(7.7)
$$\|(u-v)\circ\kappa_q\|_{L_2(\square_{m,i(x,q)})} \lesssim 2^{-(\tilde{d}-1)j}|u\circ\kappa_q|_{H^{\tilde{d}-1}(\square_{m,i(x,q)})} + 2^{-\tilde{d}j}|u\circ\kappa_q|_{H^{\tilde{d}}(\square_{m,i(x,q)})}.$$

Writing

$$\langle \breve{\psi}_{j,x}, u\rangle_{L_2(\Gamma)} = \langle \breve{\psi}_{j,x}, u - v\rangle_{L_2(\Gamma)} = \langle \breve{\psi}_{j,x}, (I-\tilde{P}_j)u\rangle_{L_2(\Gamma)} + \langle \breve{\psi}_{j,x}, \tilde{P}_j(u-v)\rangle_{L_2(\Gamma)},$$

we have $|\langle \breve{\psi}_{j,x}, (I-\tilde{P}_j)u\rangle_{L_2(\Gamma)}| \lesssim 2^{-\tilde{d}j}\sum_{q\in F_{j,x}} |u\circ\kappa_q|_{H^{\tilde{d}}(\square_{m,i(x,q)})}$ by Proposition 4.1 at the dual side. Since $\|\breve{\psi}_{j,x} - \psi_{j,x}\|_{L_2(\Gamma)} \lesssim 2^{-j}$, similar to (7.3) and (7.4) we have

$$\begin{aligned}
&|\langle \breve{\psi}_{j,x}, \tilde{P}_j(u-v)\rangle_{L_2(\Gamma)}| \\
&\qquad \lesssim 2^{-j}\sum_{q\in F_{j,x}} [\|(u-v)\circ\kappa_q\|_{L_2(\square_{m,i(x,q)})} + 2^{-\tilde{d}j}|u\circ\kappa_q|_{H^{\tilde{d}}(\square_{m,i(x,q)})}] \\
&\qquad \lesssim \sum_{q\in F_{j,x}} [2^{-\tilde{d}j}|u\circ\kappa_q|_{H^{\tilde{d}-1}(\square_{m,i(x,q)})} + 2^{-(\tilde{d}+1)j}|u\circ\kappa_q|_{H^{\tilde{d}}(\square_{m,i(x,q)})}]
\end{aligned}$$

by (7.7), which completes the proof for the case $\tilde{d} > 2$.

For $\tilde{d} = 2$, we define v as the constant function equal to u in some point of $\cup_{q\in F_{j,x}} \kappa_q(\overline{\square_{m,i(x,q)}})$, where, in case this set has non-empty intersection with $\partial\Gamma_D$, this point should be on $\partial\Gamma_D$, so that both $\langle \breve{\psi}_{j,x}, v\rangle_{L_2(\Gamma)} = 0$ and $(I-\tilde{P}_j)v$ vanishes on supp $\breve{\psi}_{j,x}$.

From the Sobolev embedding theorem and the Bramble-Hilbert lemma, we infer that for any $q \in F_{j,x}$,

$$2^{-jn/2}\sup_{x,y\in\overline{\square_{m,i(x,q)}}} |u\circ\kappa_q(x) - u\circ\kappa_q(y)| \lesssim 2^{-j}|u\circ\kappa_q|_{H^1(\square_{m,i(x,q)})} + 2^{-\tilde{d}j}|u\circ\kappa_q|_{H^{\tilde{d}}(\square_{m,i(x,q)})},$$

so that from the definition of v and the continuity of u, we have

$$\|(u-v)\circ\kappa_q\|_{L_2(\square_{m,i(x,q)})} \lesssim \sum_{q'\in F_{j,x}} [2^{-j}|u\circ\kappa_{q'}|_{H^1(\square_{m,i(x,q')})} + 2^{-\tilde{d}j}|u\circ\kappa_{q'}|_{H^{\tilde{d}}(\square_{m,i(x,q')})}],$$

with which the proof can be completed as in the case $\tilde{d} > 2$. $\square$

Next, starting with the initial collection $\hat{\Psi}_j$, we discuss *the construction of* $\breve{\Psi}_j$ as in Theorem 7.4. Let $\rho > 0$ be a sufficiently large constant such that for any $x \in J_j$, $u \in C(\overline{\Gamma}) \cap H^1_{0,\partial\Gamma_D \cap B_\Gamma(x;\rho 2^{-j})}(\Gamma))$, and $y \in \bar{I}_j \backslash I_j$ with $\operatorname{supp} \tilde{\phi}_{j,y} \cap \operatorname{supp} \hat{\psi}_{j,x} \neq \emptyset$, it holds that $\langle u, \tilde{\lambda}_{j,y}\rangle_{L_2(\Gamma)} = 0$. In view of (2.1), and ($\mathcal{L}$), ($\mathcal{J}$)(i),(ii) at the dual side, it is sufficient is to take $\rho \geq \tilde{\varepsilon} + \tilde{\vartheta}$.

Let us first consider the *special case* $\tilde{d} = 2$ assuming that $|\int \xi_{j,x} d\mu| \gtrsim 2^{-jn/2}$ $(j \in J_j \backslash J_j^\circ)$. If $x \in J_j \backslash J_j^\circ$ is such that $\partial\Gamma_D \cap B_\Gamma(x;\rho 2^{-j}) \neq \emptyset$, then $I\!\!P_{\tilde{d}-2}(\Gamma) \cap H^1_{0,\partial\Gamma_D \cap B_\Gamma(x;\rho 2^{-j})}(\Gamma) = \{0\}$, and we can take $\breve{\psi}_{j,x} = \hat{\psi}_{j,x}$. Otherwise, we take $\breve{\psi}_{j,x} := \hat{\psi}_{j,x} - [\int_\Gamma \hat{\psi}_{j,x} d\mu / \int_\Gamma \xi_{j,x} d\mu]\xi_{j,x}$. Obviously $\breve{\psi}_{j,x} \perp_{L_2(\Gamma)} 1$, and so we only have to show that $|\int_\Gamma \hat{\psi}_{j,x} d\mu| \lesssim 2^{-j(1+n/2)}$, so that $\|\psi_{j,x} - \breve{\psi}_{j,x}\|_{L_2(\Gamma)} \lesssim 2^{-j}$. With $1_{j,x} := \sum_{\{y \in I_j : \operatorname{supp} \tilde{\phi}_{j,y} \cap \operatorname{supp} \hat{\psi}_{j,x} \neq \emptyset\}} \langle 1, \tilde{\lambda}_{j,y}\rangle_{L_2(\Gamma)} \tilde{\phi}_{j,y}$, from $1 = \sum_{y \in \bar{I}_j} \langle 1, \tilde{\lambda}_{j,y}\rangle_{L_2(\Gamma)} \tilde{\phi}_{j,y}$ and the definition of ρ, we have $(1 - 1_{j,x})|_{\operatorname{supp} \hat{\psi}_{j,x}} = 0$, and so $|\int_\Gamma \hat{\psi}_{j,x} d\mu| = |\langle \hat{\psi}_{j,x}, 1_{j,x}\rangle_{L_2(\Gamma)}| = |\langle \hat{\psi}_{j,x} - \psi_{j,x}, 1_{j,x}\rangle_{L_2(\Gamma)}| \lesssim 2^{-j} \|1_{j,x}\|_{L_2(\Gamma)} \lesssim 2^{-j(1+n/2)}$. Note that the support of $\breve{\psi}_{j,x}$ is equal to that of $\hat{\psi}_{j,x}$.

In this special case, we corrected $\hat{\psi}_{j,x}$ with a multiple of $\xi_{j,x}$. Equally well, we could have corrected it with a linear combination of one or more $\theta_{j,y}$. In the following theorem it will be shown that the latter approach always applies if a sufficiently large, but constant number of $\theta_{j,y}$ are taken into account. We will solve the resulting, generally, underdetermined problem in a least squares sense, which explains the notation $\psi^{\mathrm{ls}}_{j,x}$ that we will use.

THEOREM 7.5. *For a sufficiently large constant $\delta > 0$, and with $\Theta_j(x;\delta) := \{\theta_{j,y} : y \in I_j \cap B_\Gamma(x;\delta 2^{-j})\}$, for any $x \in J_j \backslash J_j^\circ$ the problem of determining*

$$\underset{\psi^{\mathrm{ls}}_{j,x} \in \hat{\psi}_{j,x} + \mathcal{S}(\Theta_j(x;\delta))}{\operatorname{argmin}} \{\|\psi^{\mathrm{ls}}_{j,x} - \hat{\psi}_{j,x}\|_{L_2(\Gamma)} : \psi^{\mathrm{ls}}_{j,x} \perp_{L_2(\Gamma)} (I\!\!P_{\tilde{d}-2}(\Gamma) \cap H^1_{0,\partial\Gamma_D \cap B_\Gamma(x;\rho 2^{-j})}(\Gamma))\}$$

has a unique solution, and for the resulting $\breve{\Psi}_j$ we have $\|\breve{\Psi}_j - \hat{\Psi}_j\|_{L_2(\Gamma)} \lesssim 2^{-j}$, so that Theorem 7.4 applies.

Proof. In the following, let $\eta > 0$ be a constant such that

$$\operatorname{supp} \tilde{\phi}_{j,y}, \operatorname{supp} \theta_{j,y} \subset B_\Gamma(y;\eta 2^{-j}) \quad (y \in I_j), \tag{7.8}$$

and let $\delta > 0$ be a constant that will be chosen later on, which, in any case, is sufficiently large such that

$$\operatorname{supp} \hat{\psi}_{j,x} \subset B_\Gamma(x;(\delta+\eta)2^{-j}) \quad (x \in J_j). \tag{7.9}$$

In the remainder of this proof, we will use the abbreviation

$$B = B_\Gamma(x;(\delta+\eta)2^{-j}).$$

For any $x \in \overline{\Gamma}$ and $p \in I\!\!P_{\tilde{d}-2}(\Gamma)$, it holds that

$$p - \sum_{y \in I_j \cap B_\Gamma(x;\delta 2^{-j})} \langle p, \tilde{\lambda}_{j,y}\rangle_{L_2(\Gamma)} \tilde{\phi}_{j,y} = \sum_{y \in \bar{I}_j \backslash (I_j \cap B_\Gamma(x;\delta 2^{-j}))} \langle p, \tilde{\lambda}_{j,y}\rangle_{L_2(\Gamma)} \tilde{\phi}_{j,y},$$

and so, by the properties of the $\tilde{\phi}_{j,y}$ and $\tilde{\lambda}_{j,y}$, for some constant $\zeta > 0$

$$\begin{aligned}
&\|p - \sum_{y\in I_j\cap B_\Gamma(x;\delta 2^{-j})} \langle p, \tilde{\lambda}_{j,y}\rangle_{L_2(\Gamma)}\tilde{\phi}_{j,y}\|_{L_2(B)}\\
&\lesssim [\sum_{\{y\in \bar{I}_j\setminus(I_j\cap B_\Gamma(x;\delta 2^{-j})):\operatorname{supp}\tilde{\phi}_{j,y}\cap B\neq\emptyset\}} |\langle p, \tilde{\lambda}_{j,y}\rangle_{L_2(\Gamma)}|^2]^{\frac{1}{2}}\\
&\lesssim 2^{-jn/2}\|p\|_{L_\infty(B_\Gamma(x;(\delta+\eta+\zeta)2^{-j}))} \times [\#\{y\in \bar{I}_j\setminus(I_j\cap B_\Gamma(x;\delta 2^{-j})) : \operatorname{supp}\tilde{\phi}_{j,y}\cap B\neq\emptyset\}]^{\frac{1}{2}}\\
&\lesssim \delta^{-n/2}\|p\|_{L_2(B)} \times \delta^{(n-1)/2},
\end{aligned}$$

where we used that p is a function from a fixed, finite dimensional space. With $\tilde{\Phi}_j(x,\delta) := \{\tilde{\phi}_{j,y} : y \in I_j \cap B_\Gamma(x, \delta 2^{-j})\}$, we conclude that

$$\inf_{\tilde{v}_j\in\mathcal{S}(\tilde{\Phi}_j(x,\delta))} \|p-\tilde{v}_j\|_{L_2(B)} \lesssim \delta^{-\frac{1}{2}}\|p\|_{L_2(B)} \quad (p\in I\!P_{\tilde{d}-2}(\Gamma)).$$

Recall that $\bar{\mathbf{D}}_j := \sum_{q=1}^M I_{j,q}\langle\Theta_j^\square, \tilde{\Phi}_j^\square\rangle_{L_2(\square)}\Delta_{j,q}^\square I_{j,q}^T$ is uniformly boundedly invertible, and it is either diagonal, or, with respect to a suitable partitioning of I_j into p sets with p fixed, it is block triangular with diagonal blocks that are diagonal matrices. So any principal submatrix of $\bar{\mathbf{D}}_j$ is uniformly boundedly invertible, also with respect to the selection of this submatrix. From $\|\langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)} - \bar{\mathbf{D}}_j\| \lesssim 2^{-j}$, we infer that for $j \geq j_0$ sufficiently large, any principal submatrix of $\langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ is uniformly boundedly invertible, again also with respect to the selection of this submatrix. By using in addition that by (7.8) for any $y \in I_j \cap B_\Gamma(x, \delta 2^{-j})$, $\tilde{\phi}_{j,y}$ and $\theta_{j,y}$ vanish outside B, we infer that $\tilde{\Phi}_j(x,\delta)$ and $\Theta_j(x,\delta)$ are uniform $L_2(B)$-Riesz systems, with $\langle\tilde{\Phi}_j(x,\delta), \Theta_j(x,\delta)\rangle_{L_2(B)}$ being uniformly bounded invertible, where here as well as in the following "uniform" not only refers to $j \geq j_0$, but also to x and δ, as long as the latter has not been fixed.

By Lemma 5.1 there exists a uniformly bounded projector $\hat{Q}_j(x,\delta) : L_2(B) \to L_2(B)$ with $\Im(\hat{Q}_j(x,\delta)) = \mathcal{S}(\tilde{\Phi}_j(x,\delta))$ and $\Im(I-\hat{Q}_j(x,\delta)) = \mathcal{S}(\Theta_j(x,\delta))^{\perp_{L_2(B)}}$. From

$$\|p - \hat{Q}_j(x,\delta)p\|_{L_2(B)} \leq (1 + \|\hat{Q}_j(x,\delta)\|_{L_2(B)\to L_2(B)}) \inf_{\tilde{v}_j\in\mathcal{S}(\tilde{\Phi}_j(x,\delta))} \|p-\tilde{v}_j\|_{L_2(B)}$$

we conclude that by fixing δ to be a sufficiently large constant, for $p \in I\!P_{\tilde{d}-2}$ we have

$$\|\hat{Q}_j(x,\delta)p\|_{L_2(B)} \geq \|p\|_{L_2(B)} - \|p - \hat{Q}_j(x,\delta)p\|_{L_2(B)} \gtrsim \|p\|_{L_2(B)},$$

and thus

$$\langle\hat{Q}_j(x,\delta)^*\hat{Q}_j(x,\delta)p, p\rangle_{L_2(B))} \gtrsim \|p\|_{L_2(B)}^2 \gtrsim \|\hat{Q}_j(x,\delta)^*\hat{Q}_j(x,\delta)p\|_{L_2(B)}\|p\|_{L_2(B)}.$$

Since this result is in particular valid for any $p \in I\!P_{\tilde{d}-2} \cap H^1_{0,\partial\Gamma_D\cap B_\Gamma(x;\rho 2^{-j})}(\Gamma)$, and furthermore $\Im(\hat{Q}_j(x,\delta)^*\hat{Q}_j(x,\delta)) \subset \mathcal{S}(\Theta_j(x,\delta))$, we have that

$$\inf_{0\neq p\in I\!P_{\tilde{d}-2}\cap H^1_{0,\partial\Gamma_D\cap B_\Gamma(x;\rho 2^{-j})}(\Gamma)} \sup_{0\neq v_j\in\mathcal{S}(\Theta_j(x,\delta))} \frac{|\langle v_j, p\rangle_{L_2(B)}|}{\|v_j\|_{L_2(B)}\|p\|_{L_2(B)}} \gtrsim 1. \tag{7.10}$$

It is well-known that this so-called inf-sup condition implies that for any $u \in L_2(B)$, the problem of finding $v_j \in \mathcal{S}(\Theta_j(x,\delta))$ with minimal $L_2(B)$-norm satisfying

$$\langle v_j, p\rangle_{L_2(B)} = \langle u, p\rangle_{L_2(B)} \quad (p \in I\!P_{\tilde{d}-2} \cap H^1_{0,\partial\Gamma_D\cap B_\Gamma(x;\rho 2^{-j})}(\Gamma)), \tag{7.11}$$

has a unique solution, that we will denote as $v_j = Q_j(x,\delta)u$, for which $\|v_j\|_{L_2(B)} \lesssim \|u\|_{L_2(B)}$. Obviously, $Q_j(x,\delta)$ is projector onto $\mathcal{S}(\Theta_j(x,\delta))$ with $\ker(Q_j(x,\delta)) = ((\mathbb{P}_{\tilde{d}-2} \cap H^1_{0,\partial\Gamma_D \cap B_\Gamma(x;\rho 2^{-j})}(\Gamma))|_B)^{\perp_{L_2(B)}}$. Since $\operatorname{supp} v_j \subset B$ for any $v_j \in \mathcal{S}(\Theta_j(x,\delta))$, for u being supported in B, the $L_2(B)$-scalar products in (7.11) read as $L_2(\Gamma)$-scalar products, and so from $\operatorname{supp} \hat{\psi}_{j,x} \subset B$ we conclude that the constrained minimization problem from the theorem has the unique solution $\psi^{\mathrm{ls}}_{j,x} = (I - Q_j(x,\delta))\hat{\psi}_{j,x}$.

What is left to show is that $\|\psi^{\mathrm{ls}}_{j,x} - \hat{\psi}_{j,x}\|_{L_2(\Gamma)} \lesssim 2^{-j}$, which will follow from the fact that $\hat{\psi}_{j,x}$ is already nearly orthogonal to $\mathbb{P}_{\tilde{d}-2}(\Gamma) \cap H^1_{0,\partial\Gamma_D \cap B_\Gamma(x;\rho 2^{-j})}(\Gamma)$. Indeed, as we have seen, by assumption ($\tilde{\mathfrak{J}}$e), for any $p \in \mathbb{P}_{\tilde{d}-2}(\Gamma)$, we have $p = \sum_{y \in \bar{I}_j} \langle p, \tilde{\lambda}_{j,y}\rangle_{L_2(\Gamma)} \tilde{\phi}_{j,y}$. So when in addition $p \in H^1_{0,\partial\Gamma_D \cap B_\Gamma(x;\rho 2^{-j})}(\Gamma)$, by definition of the constant ρ, for $p_{j,x} := \sum_{\{y \in I_j : \operatorname{supp} \tilde{\phi}_{j,y} \cap \operatorname{supp} \hat{\psi}_{j,x} \neq \emptyset\}} \langle p, \tilde{\lambda}_{j,y}\rangle_{L_2(\Gamma)} \tilde{\phi}_{j,y}$, we have $(p - p_{j,x})|_{\operatorname{supp} \hat{\psi}_{j,x}} = 0$. Furthermore, we have

$$\begin{aligned} \|p_{j,x}\|_{L_2(\Gamma)} &\lesssim [\sum_{\{y \in I_j : \operatorname{supp} \tilde{\phi}_{j,y} \cap \operatorname{supp} \hat{\psi}_{j,x} \neq \emptyset\}} |\langle p, \tilde{\lambda}_{j,y}\rangle_{L_2(\Gamma)}|^2]^{\frac{1}{2}} \\ &\lesssim 2^{-jn/2} \|p\|_{L_\infty(B_\Gamma(x;(\delta+\eta+\zeta)2^{-j}))} \lesssim \|p\|_{L_2(B)}, \end{aligned}$$

and so, since $p_{j,x} \in \mathcal{S}(\tilde{\Phi}_j)$, we obtain

$$\begin{aligned} |\langle \hat{\psi}_{j,x}, p\rangle_{L_2(B)}| &= |\langle \hat{\psi}_{j,x}, p_{j,x}\rangle_{L_2(B)}| = |\langle \hat{\psi}_{j,x}, p_{j,x}\rangle_{L_2(\Gamma)}| = |\langle \hat{\psi}_{j,x} - \psi_{j,x}, p_{j,x}\rangle_{L_2(\Gamma)}| \\ &\lesssim 2^{-j} \|p_{j,x}\|_{L_2(\Gamma)} \lesssim 2^{-j} \|p\|_{L_2(B)}. \end{aligned}$$

Since $\Im(Q_j(x,\delta)^*) = \ker(Q_j(x,\delta))^{\perp_{L_2(B)}} = (\mathbb{P}_{\tilde{d}-2}(\Gamma) \cap H^1_{0,\partial\Gamma_D \cap B_\Gamma(x;\rho 2^{-j})}(\Gamma))|_B$, we have

$$\begin{aligned} \|Q_j(x,\delta)\hat{\psi}_{j,x}\|_{L_2(B)} &= \sup_{0 \neq v \in L_2(B)} \frac{|\langle Q_j(x,\delta)\hat{\psi}_{j,x}, v\rangle_{L_2(B)}|}{\|v\|_{L_2(B)}} \\ &= \sup_{0 \neq v \in L_2(B)} \frac{|\langle \hat{\psi}_{j,x}, Q_j(x,\delta)^* v\rangle_{L_2(B)}|}{\|v\|_{L_2(B)}} \\ &\lesssim 2^{-j} \|\hat{\psi}_{j,x}\|_{L_2(B)} \|Q_j(x,\delta)^*\|_{L_2(B) \to L_2(B)} \lesssim 2^{-j} \|\hat{\psi}_{j,x}\|_{L_2(B)}, \end{aligned}$$

that is, $\|\psi^{\mathrm{ls}}_{j,x} - \hat{\psi}_{j,x}\|_{L_2(\Gamma)} \lesssim 2^{-j}$. $\square$

Generally we cannot evaluate integrals over Γ exactly, and so the computation of $\psi^{\mathrm{ls}}_{j,x}$, defined in Theorem 7.5, will not be exact. Let $B \subset \Gamma$ be as in the proof of this theorem, i.e., $\operatorname{supp} \hat{\psi}_{j,x}, \operatorname{supp} \theta_{j,y} \subset B$ for all $\theta_{j,y} \in \Theta_j(x;\delta)$ and $\operatorname{diam}(B) \lesssim 2^{-j}$. Let $\{p_1, \ldots, p_n\}$ be a uniformly $L_2(B)$-Riesz basis for $\mathbb{P}_{\tilde{d}-2}(\Gamma) \cap H^1_{0,\partial\Gamma_D \cap B_\Gamma(x;\rho 2^{-j})}(\Gamma)$ restricted to B. Using such a basis, and the canonical, uniformly $L_2(B)$-Riesz basis for $\mathcal{S}(\Theta_j(x;\delta))$, the inf-sup condition (7.10) shows that the constrained minimization problem defining $\psi^{\mathrm{ls}}_{j,x}$ corresponds to a uniformly well-conditioned matrix-vector problem, where the entries of the saddle-point type matrix are either of the form $\langle \theta_{j,y}, \theta_{j,y'}\rangle_{L_2(\Gamma)}$ or $\langle \theta_{j,y}, p_i\rangle_{L_2(\Gamma)}$, and the entries of the right-hand side are either zero or of the form $\langle \hat{\psi}_{j,x}, p_i\rangle_{L_2(\Gamma)}$. Computing all these entries with an absolute error $\lesssim 2^{-\tilde{d}j}$, we obtain an approximation for $\psi^{\mathrm{ls}}_{j,x}$ with an $L_2(\Gamma)$-error $\lesssim 2^{-\tilde{d}j}$, so that the uniform cancellation property of order $\tilde{d}$ is still guaranteed.

In case $\mathcal{S}(\Theta_j) \neq \mathcal{S}(\Phi_j)$, then it is not of any interest that $\breve{\Psi}_j - \Xi_j \in \mathcal{S}(\Theta_j)$ (cf. Remark 6.2). In that case, in Theorem 7.5 we may search $\psi^{\mathrm{ls}}_{j,x}$ in the larger space

$\hat{\psi}_{j,x} + \mathcal{S}(\Phi_{j+1}(x;\delta))$, with $\mathcal{S}(\Phi_{j+1}(x;\delta)) := \{\phi_{j+1,y} : x \in I_{j+1} \cap B_\Gamma(x;\delta 2^{-j})\}$, which opens the possibility that we may take a smaller δ, and so reduce the support of the resulting $\psi^{\mathrm{ls}}_{j,x}$.

8. Numerical example. We consider a 2-dimensional Lipschitz manifold $\Gamma = \cup_{i=1}^{4} \overline{\Gamma_q} \subset \mathbb{R}^3$ as illustrated in Figure 8.1, which, together with its parametrization that satisfies ($\mathcal{M}$), is defined as follows. Let P be a tetrahedron in $\mathbb{R}^3$, with vertices on the unit sphere, geometric centroid in $(0,0,0)$, and one of its four facets $F_1, \ldots, F_4$, say F_4, parallel to and below the $x_3 = 0$ plane. Let $\boldsymbol{T}$ $(= \square)$ be a reference 2-simplex

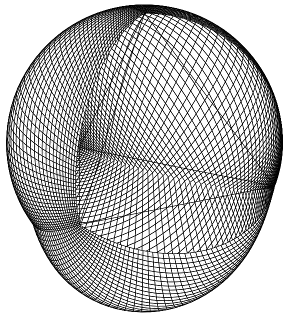

FIG. 8.1. *The manifold Γ, excluding one patch, and the tetrahedron P.*

in $\mathbb{R}^2$, with $\mathrm{vol}(\boldsymbol{T}) = 1$, and for $1 \leq q \leq 4$, let $B_q : \boldsymbol{T} \to F_q$ some affine bijection. The parametrizations $\kappa_q : \boldsymbol{T} \to \Gamma_q$ are defined by $\kappa_q(z) = B_q(z)/\|B_q(z)\|$ for $1 \leq q \leq 3$, and by $\kappa_4(z) = B_q(z)/\|B_q(z)\| - \frac{27}{4}(0,0,\lambda_1(z)\lambda_2(z)\lambda_3(z))$, where $(\lambda_1(z), \lambda_2(z), \lambda_3(z))$ are the barycentric coordinates of z with respect to $\boldsymbol{T}$. So without the perturbation by this cubic bubble, Γ would be the unit sphere. We added this perturbation term so that $J : x \mapsto |\partial\kappa_q(\kappa_q^{-1}(x))|$ when $x \in \Gamma_q$, cannot be extended to a continuous function on Γ. This means that constructions based on ignoring the Jacobian determinants will yield wavelets of which those that have supports that intersect an interface between Γ_4 and one of the three other patches have no cancellation properties. Note that $z \mapsto |\partial\kappa_q(z)|$ are not constant functions.

We consider two examples of collections $\Phi_j^\square$, $\tilde{\Phi}_j^\square$, $\Theta_j^\square$, $\Xi_j^\square$, $\Lambda_j^\square$, $\tilde{\Lambda}_j^\square$, both based on the construction of finite element wavelets from [DS99c]. With $\tau_0 = \{\boldsymbol{T}\}$, let $\tau_0, \tau_1, \ldots$ be the sequence of triangulations of $\boldsymbol{T}$ with τ_{j+1} generated from τ_j by a uniform dyadic refinement, let V_j be the set of vertices of all $T \in \tau_j$, and for $p \in \mathbb{N}_{>0}$, let $S_j^p = C(\overline{\boldsymbol{T}}) \cap \prod_{T \in \tau_j} P_p(T)$.

Let $\boldsymbol{\Phi} = (\boldsymbol{\phi}_\alpha)_{\alpha \in \boldsymbol{I}}$, $\tilde{\boldsymbol{\Phi}} = (\tilde{\boldsymbol{\phi}}_\alpha)_{\alpha \in \boldsymbol{I}}$, $\boldsymbol{\Theta} = (\boldsymbol{\theta}_\alpha)_{\alpha \in \boldsymbol{I}}$, $\boldsymbol{\Xi} = (\boldsymbol{\xi}_\alpha)_{\alpha \in \boldsymbol{J}}$, and $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}_\alpha)_{\alpha \in \boldsymbol{I}}$, $\tilde{\boldsymbol{\Lambda}} = (\tilde{\boldsymbol{\lambda}}_\alpha)_{\alpha \in \boldsymbol{I}}$ be collections in $C(\overline{\boldsymbol{T}})$ or $C(\overline{\boldsymbol{T}})'$, respectively, where both the index sets $\boldsymbol{I}$ and $\boldsymbol{J}$, and all collections of functions or functionals are symmetric in the barycentric coordinates. For $j \in \mathbb{N}$, and, for each $T \in \tau_j$, with $B_T : \boldsymbol{T} \to T$ being some arbitrary affine bijection, let $I_j^\square = \cup_{T \in \tau_j} B_T(\boldsymbol{I})$, $J_j^\square = \cup_{T \in \tau_j} B_T(\boldsymbol{J})$, and for $x \in \overline{\boldsymbol{T}}$, let $k_j^\square(x) := \#\{T \in \tau_j : x \in \overline{T}\}$. The 'local' index sets $\boldsymbol{I}$ and $\boldsymbol{J}$ will be chosen such that $J_j^\square = I_{j+1}^\square \backslash I_j^\square$. We define the collections $\Phi_j^\square := (\phi_{j,x}^\square)_{x \in I_j^\square}$, and analogously $\tilde{\Phi}_j^\square$, $\Theta_j^\square$ and $\Xi_j^\square = (\xi_{j,y}^\square)_{y \in J_j^\square}$, and the collections $\Lambda_j^\square = (\lambda_{j,x}^\square)_{x \in I_j^\square}$, and analogously $\tilde{\Lambda}_j^\square$, by

$$\phi_{j,x}^\square(z) = 2^j k_j^\square(x)^{-\frac{1}{2}} \begin{cases} \boldsymbol{\phi}_{B_T^{-1}(x)}(B_T^{-1}(z)) & \text{if } x, z \in T \in \tau_j, \\ 0 & \text{elsewhere,} \end{cases} \tag{8.1}$$

and

$$\lambda^{\square}_{j,x}(u) = 2^{-j} k^{\square}_j(x)^{\frac{1}{2}} \boldsymbol{\lambda}_{B_T^{-1}(x)}(u \circ B_T) \quad \text{when } x \in \overline{T} \text{ for some } T \in \tau_j, \tag{8.2}$$

respectively. The 'local' collections will be designed such that the 'global' collections given by (8.1) or (8.2) are well-defined, where moreover all functions are continuous. Furthermore, it will hold that $\langle \boldsymbol{\Phi}, \boldsymbol{\Lambda}\rangle_{L_2(\square)} = \langle \tilde{\boldsymbol{\Phi}}, \tilde{\boldsymbol{\Lambda}}\rangle_{L_2(\square)} = \langle \boldsymbol{\Theta}, \tilde{\boldsymbol{\Phi}}\rangle_{L_2(\square)} = I$, so that $\langle \Phi^{\square}_j, \Lambda^{\square}_j\rangle_{L_2(\square)} = \langle \tilde{\Phi}^{\square}_j, \tilde{\Lambda}^{\square}_j\rangle_{L_2(\square)} = \langle \Theta^{\square}_j, \tilde{\Phi}^{\square}_j\rangle_{L_2(\square)} = I$, and all other conditions imposed in §3 will be also satisfied with parameters $\gamma = \tilde{\gamma} = \frac{3}{2}$, $d = 2$, and either $\tilde{d} = 2$ or $\tilde{d} = 3$. However, in both our examples $\mathcal{S}(\boldsymbol{\Theta}) \neq \mathcal{S}(\boldsymbol{\Phi})$, so that the basis transformation from single-scale to multi-scale basis will not have a linear complexity.

In our example for $\tilde{d} = 2$, we have $\boldsymbol{I} = V_0$, $\boldsymbol{J} = V_1 \backslash V_0$, $\boldsymbol{\Phi} = \tilde{\boldsymbol{\Phi}}$ with $\mathcal{S}(\boldsymbol{\Phi}) = S^1_0$, and $\mathcal{S}(\boldsymbol{\Theta} \cup \boldsymbol{\Xi}) = S^1_1$. The collections were introduced in [DS99c, §4.5.1], and they are determined by the illustrations in Figure 8.2 by using symmetry in the barycentric coordinates. The collection $\boldsymbol{\Lambda} = \tilde{\boldsymbol{\Lambda}}$ is just the set of function evaluations on V_0. Note

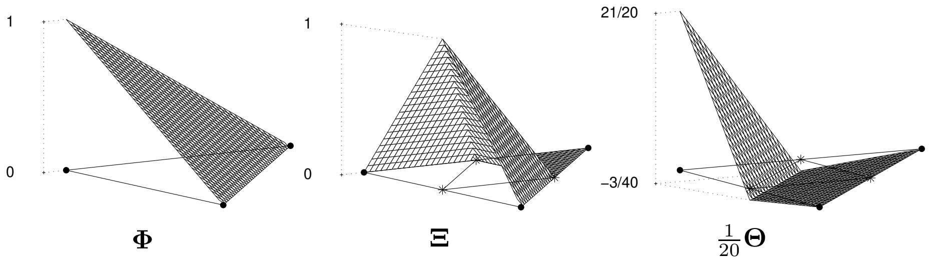

FIG. 8.2. *"Local building blocks" for $\tilde{d} = 2$.*

that $I^{\square}_j = V_j$, $\mathcal{S}(\Phi^{\square}_j) = \mathcal{S}(\tilde{\Phi}^{\square}_j) = S^1_j$. So, in particular, for the resulting collection Φ_j on Γ, we have $\mathcal{S}(\Phi_j) = C(\Gamma) \cap \prod_{q=1}^4 \kappa_q(S^1_j)$.

In our example for $\tilde{d} = 3$, we have $\boldsymbol{I} = V_1$, $\boldsymbol{J} = V_2 \backslash V_1$, $\mathcal{S}(\boldsymbol{\Phi}) = S^1_1$, $\mathcal{S}(\tilde{\boldsymbol{\Phi}}) = S^2_0$, and $\mathcal{S}(\boldsymbol{\Theta} \cup \boldsymbol{\Xi}) = S^1_2$. The collections we use are modifications of those introduced in [DS99c, §4.5.4], and yield better conditioned wavelet bases. They are illustrated in Figure 8.3, and their derivation and precise definition will be presented in [NS04], together with examples for other values of d and $\tilde{d}$. The collection $\boldsymbol{\Lambda}$ is the set of function evaluations on V_1. Writing $\tilde{\boldsymbol{\Phi}}$ as an invertible 6×6 matrix $\mathbf{B}$ applied to the nodal basis of S^2_0, $\tilde{\boldsymbol{\Lambda}}$ is obtained by applying $\mathbf{B}^{-T}$ to the collection of function evaluations on V_1. Note that $I^{\square}_j = V_{j+1}$, $\mathcal{S}(\Phi^{\square}_j) = S^1_{j+1}$ and $\mathcal{S}(\tilde{\Phi}^{\square}_j) = S^2_j$. So for the resulting collections Φ_j and $\tilde{\Phi}_j$ on Γ, we have $\mathcal{S}(\Phi_j) = C(\Gamma) \cap \prod_{q=1}^4 \kappa_q(S^1_{j+1})$ and $\mathcal{S}(\tilde{\Phi}_j) = C(\Gamma) \cap \prod_{q=1}^4 \kappa_q(S^2_j)$, respectively.

We have implemented the approximate wavelet constructions from §7.2 and §7.3, that away from the patch interfaces both yield the approximate wavelets from the collection $\Psi^{(0)}_j = \Xi_j - \langle\!\langle \Xi_j, \tilde{\Phi}_j\rangle\!\rangle_0 \langle\!\langle \Theta_j, \tilde{\Phi}_j\rangle\!\rangle_0^{-1} \Theta_j$ obtained by ignoring the Jacobian determinants. The pull-backs of these wavelets to the parameter domain are illustrated in Figure 8.4, which functions are thus continuous piecewise linear with respect to the indicated triangulation.

For obtaining approximate wavelets along the patch interfaces we need sufficiently accurate approximations $\mathbf{A}_j$ and $\mathbf{B}_j$ for $\langle \Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ and $\langle \Xi_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}$, respectively, which we define by the application of a quadrature rule that is exact on polynomials

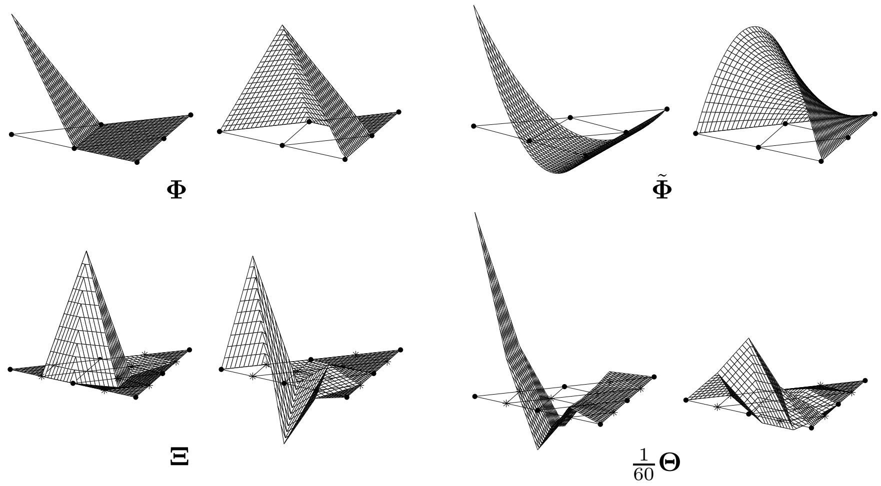

FIG. 8.3. *"Local building blocks" for $\tilde{d} = 3$.*

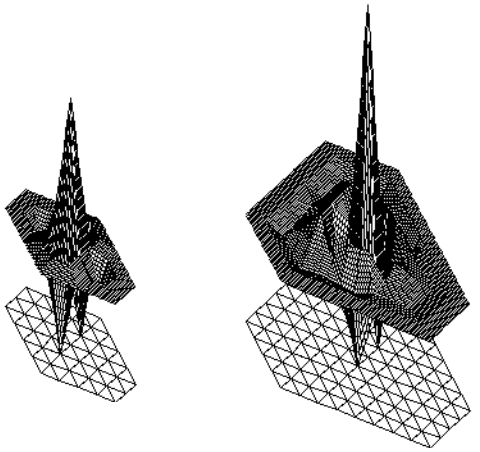

FIG. 8.4. *Wavelets $\psi_{j,x}^{(0)}$ away from the patch interfaces for $\tilde{d} = 2$ and $\tilde{d} = 3$ (one of the two different types), and their supports in terms of the underlying triangulation.*

of degree 5 on all triangles from the coarsest triangulation with respect to which the pull-backs of $\Theta_j, \tilde{\Phi}_j, \Xi_j$ are piecewise polynomial. Although we will write $\int_\Gamma \dots$ and $\langle \cdot, \cdot \rangle_{L_2(\Gamma)}$ in the following, on all places where such integrals appear actually we have approximated them by the application of this composite quadrature scheme.

With the approach from §7.2, wavelets along the patch interfaces are taken from the collection $\Psi_j^{\mathrm{Jc}} = \Xi_j - \mathbf{B}_j \sum_{k=0}^{\tilde{d}-1} (I - \mathbf{D}_j^{-1}\mathbf{A}_j)^k \mathbf{D}_j^{-1} \Theta_j$, where $\mathbf{D}_j := \operatorname{diag} \mathbf{A}_j$. Illustrations of the, naturally joined, patchwise pull-backs of these wavelets can be found in Figure 8.5. For $j = 0$, the Neumann series does not converge, and as a consequence $\sum_{k=0}^{\tilde{d}-1} (I - \mathbf{D}_0^{-1}\mathbf{A}_0)^k \mathbf{D}_0^{-1}$ provides a very poor approximation for $\mathbf{A}_0^{-1}$. We redefined $\breve{\Psi}_0 := \Xi_0 - \mathbf{B}_0 \mathbf{A}_0^{-1} \Theta_0$.

For the construction from §7.3 we have to specify a 'first order approximation' $\hat{\Psi}_j$, which we define as $\hat{\Psi}_j = \Xi_j - \mathbf{B}_j \mathbf{D}_j^{-1} \Theta_j$, as well as, for each $x \in J_j \backslash J_j^\circ$, a subspace

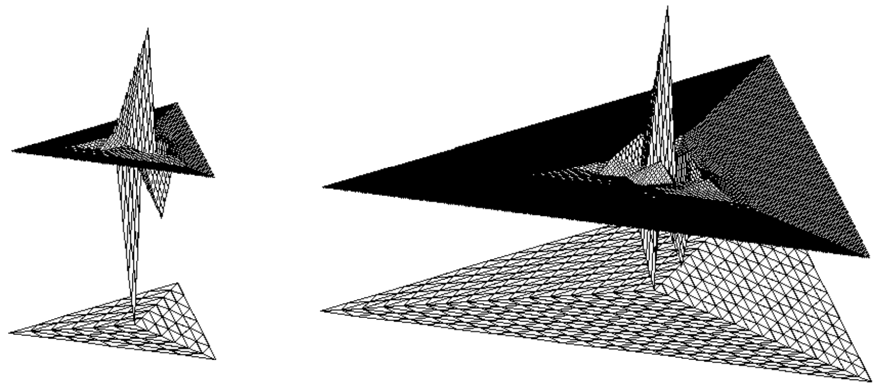

FIG. 8.5. *Wavelets $\psi^{\mathrm{Jc}}_{4,x}$ for $\tilde{d}=2$ and $\psi^{\mathrm{Jc}}_{3,x}$ for $\tilde{d}=3$ near the "north pole", and their supports in terms of the underlying triangulation.*

$A_{j,x} \subset \mathcal{S}(\Phi_{j+1})$ that defines $\psi^{\mathrm{ls}}_{j,x}$ via

$$\underset{\psi^{\mathrm{ls}}_{j,x} \in \hat{\psi}_{j,x}+A_{j,x}}{\operatorname{argmin}} \{\|\psi^{\mathrm{ls}}_{j,x} - \hat{\psi}_{j,x}\|_{L_2(\Gamma)} : \psi^{\mathrm{ls}}_{j,x} \perp_{L_2(\Gamma)} \mathbb{P}_{\tilde{d}-1}(\Gamma)\}.$$

For $\tilde{d}=2$, we take $A_{j,x} = \mathcal{S}(\{\xi_{j,x}\})$, so that $\psi^{\mathrm{ls}}_{j,x} = \hat{\psi}_{j,x} + \alpha\xi_{j,x}$ with α such that $\int_\Gamma \psi^{\mathrm{ls}}_{j,x} = 0$. For $\tilde{d}=3$, we take $A_{j,x} = \mathcal{S}(\{\phi_{j+1,y} : \operatorname{supp}\phi_{j+1,y} \subset \operatorname{supp}\hat{\psi}_{j,x}\})$, which space turns out to be sufficiently large so that the constrained minimization problem has a solution $\psi^{\mathrm{ls}}_{j,x}$, with $\|\psi^{\mathrm{ls}}_{j,x} - \hat{\psi}_{j,x}\|_{L_2(\Gamma)} \lesssim 2^{-j}$. The, naturally joined, patchwise pull-backs of the resulting $\psi^{\mathrm{ls}}_{j,x}$ are illustrated in Figure 8.6. By definition they have the *same supports* as the corresponding $\psi^{(0)}_{j,x}$ that one obtains by ignoring the Jacobian determinants also along the interfaces. As with the Jacobi iteration

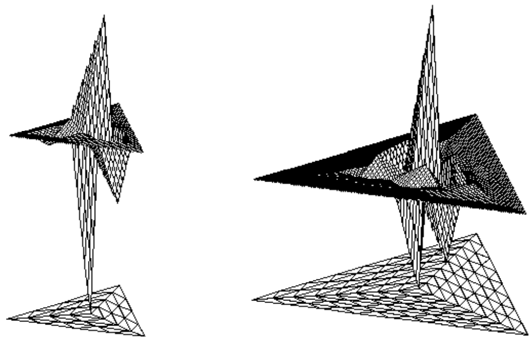

FIG. 8.6. *Wavelets $\psi^{\mathrm{ls}}_{4,x}$ for $\tilde{d}=2$ and $\psi^{\mathrm{ls}}_{3,x}$ for $\tilde{d}=3$ near the "north pole", and their supports in terms of the underlying triangulation.*

approach, for $j=0$, we redefined $\breve{\Psi}_0 := \Xi_0 - \mathbf{B}_0\mathbf{A}_0^{-1}\Theta_0$.

With

$$\kappa_{\Sigma,\|\cdot\|} := \sup_{0\neq \mathbf{c}=(c_\sigma)_{\sigma\in\Sigma}} \frac{\left\|\sum_{\sigma\in\Sigma} c_\sigma \frac{\sigma}{\|\sigma\|}\right\|^2}{\|\mathbf{c}\|^2} \Bigg/ \inf_{0\neq \mathbf{c}=(c_\sigma)_{\sigma\in\Sigma}} \frac{\left\|\sum_{\sigma\in\Sigma} c_\sigma \frac{\sigma}{\|\sigma\|}\right\|^2}{\|\mathbf{c}\|^2},$$

and, for the obtained collections $\breve{\Psi}_j$ of approximate wavelets using either the approach from §7.2 (Jacobi approximation along the interfaces) or that of §7.3 (least squares approximation along the interfaces), with

$$\breve{\Psi}^{(\ell)} := \begin{cases} \Phi_0 \cup \cup_{k=0}^{\ell-1} \breve{\Psi}_k & \text{when } \tilde{d}=2, \\ \Phi_0 \cup \cup_{k=0}^{\ell-2} \breve{\Psi}_k & \text{when } \tilde{d}=3, \end{cases}$$

for $\ell \leq 6$ we give $\kappa_{\breve{\Psi}^{(\ell)},\|\cdot\|_{L_2(\Gamma)}}$, $\kappa_{\breve{\Psi}^{(\ell)},|\!|\!|\cdot|\!|\!|_1}$, $\kappa_{\breve{\Psi}^{(\ell)},|\!|\!|\cdot|\!|\!|_{-1,6}}$, where on $C(\Gamma)\cap\prod_{q=1}^4 \kappa_q(S_m^1)$, $|\!|\!|u|\!|\!|_{-1,m} := \sup_{v\in C(\Gamma)\cap\prod_{q=1}^4 \kappa_q(S_m^1)} \frac{|\langle u,v\rangle_{L_2(\Gamma)}|}{|\!|\!|v|\!|\!|_1}$. The uniform boundedness in $|\!|\!|\cdot|\!|\!|_1$ of the $L_2(\Gamma)$-orthogonal projector onto $C(\Gamma)\cap\prod_{q=1}^4 \kappa_q(S_m^1)$, which is a consequence of Theorem 5.3, shows that $|\!|\!|\cdot|\!|\!|_{-1} \eqsim |\!|\!|\cdot|\!|\!|_{-1,m}$ on $C(\Gamma)\cap\prod_{q=1}^4 \kappa_q(S_\ell^1)$ uniformly in $\ell \leq m$. Recall that for both $\tilde{d}=2$ and $\tilde{d}=3$, $\breve{\Psi}^{(\ell)}$ is basis for $C(\Gamma)\cap\prod_{q=1}^4 \kappa_q(S_\ell^1)$. Since the functions from Φ_0, and for $\tilde{d}=2$, from $\breve{\Psi}_0$ and $\breve{\Psi}_1$, and for $\tilde{d}=3$, from $\breve{\Psi}_0$ have global supports anyway, for computing the condition numbers we replaced each of these collections by orthonormalized versions by multiplying them by the inverse of the square root of their mass matrix with respect to either $\langle\cdot,\cdot\rangle_{L_2(\Gamma)}$, $\langle\!\langle\cdot,\cdot\rangle\!\rangle_1$ or $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{-1,6}$, the latter being the scalar product corresponding to the norm $\|\cdot\|_{-1,6}$. The results are presented in Tables 8.1 and 8.2. Although it turns out that unfortunately,

TABLE 8.1
Condition numbers for $\tilde{d}=2$.

ℓ	Jacobi approximation			least squares approximation		
	$\kappa_{\Psi^{(\ell)},\|\cdot\|_{L_2(\Gamma)}}$	$\kappa_{\breve{\Psi}^{(\ell)},|\!|\!|\cdot|\!|\!|_1}$	$\kappa_{\breve{\Psi}^{(\ell)},|\!|\!|\cdot|\!|\!|_{-1,6}}$	$\kappa_{\breve{\Psi}^{(\ell)},\|\cdot\|_{L_2(\Gamma)}}$	$\kappa_{\breve{\Psi}^{(\ell)},|\!|\!|\cdot|\!|\!|_1}$	$\kappa_{\breve{\Psi}^{(\ell)},|\!|\!|\cdot|\!|\!|_{-1,6}}$
1	1.35e0	2.30e0	3.06e0	1.35e0	2.30e0	3.09e0
2	1.45e0	7.86e0	8.65e0	1.82e0	7.64e0	1.09e1
3	1.75e1	4.82e1	2.79e1	2.42e1	6.84e1	3.41e1
4	1.79e1	6.51e1	4.53e1	2.46e1	9.06e1	5.21e1
5	1.79e1	7.66e1	6.20e1	2.46e1	1.06e2	6.69e1
6	1.79e1	8.25e1	7.34e1	2.46e1	1.14e2	7.84e1

TABLE 8.2
Condition numbers for $\tilde{d}=3$.

ℓ	Jacobi approximation			least squares approximation		
	$\kappa_{\Psi^{(\ell)},\|\cdot\|_{L_2(\Gamma)}}$	$\kappa_{\breve{\Psi}^{(\ell)},|\!|\!|\cdot|\!|\!|_1}$	$\kappa_{\breve{\Psi}^{(\ell)},|\!|\!|\cdot|\!|\!|_{-1,6}}$	$\kappa_{\breve{\Psi}^{(\ell)},\|\cdot\|_{L_2(\Gamma)}}$	$\kappa_{\breve{\Psi}^{(\ell)},|\!|\!|\cdot|\!|\!|_1}$	$\kappa_{\breve{\Psi}^{(\ell)},|\!|\!|\cdot|\!|\!|_{-1,6}}$
2	2.80e0	1.26e1	4.18e1	2.80e0	1.26e1	4.22e0
3	1.36e1	5.47e1	3.36e1	1.85e1	8.51e1	3.82e1
4	1.81e1	7.88e1	5.76e1	2.34e1	1.19e2	6.86e1
5	1.97e1	8.83e1	8.00e1	2.44e1	1.32e2	9.65e1
6	2.03e1	9.20e1	9.64e1	2.47e1	1.36e2	1.34e2

in particular in the $\|\cdot\|_{-1,6}$-norm, the condition numbers haven't really stabilized, we stopped our computations at $\ell = 6$ since mainly due to the normalization of the wavelets, in particular with respect to the $\|\cdot\|_{-1,6}$-norm, on higher levels they become too time consuming.

Appendix A. Proof of Theorem 6.1. The proof will consist of steps **(I)**-**(VI)**.

Although basically **(V)** and **(VI)** can be found in [Ste98, Theorem 3.1], which in turn was based on [Xu92, Appendix], for convenience we include a complete proof.

(I) Since, as shown in Proposition 5.4, for $j \geq j_0$ large enough, Ψ_j is a uniform $L_2(\Gamma)$-Riesz system, and by Assumption (iii),

$$\begin{aligned}&\|\langle \breve{\Psi}_j, \breve{\Psi}_j\rangle_{L_2(\Gamma)} - \langle \Psi_j, \Psi_j\rangle_{L_2(\Gamma)}\| \\ &= \|\langle \breve{\Psi}_j - \Psi_j, \Psi_j\rangle_{L_2(\Gamma)} + \langle \Psi_j, \breve{\Psi}_j - \Psi_j\rangle_{L_2(\Gamma)} + \langle \breve{\Psi}_j - \Psi_j, \breve{\Psi}_j - \Psi_j\rangle_{L_2(\Gamma)}\| \lesssim \omega^j,\end{aligned}$$

we infer that, possibly for a larger value of j_0, for $j \geq j_0$, $\breve{\Psi}_j$ is a uniform $L_2(\Gamma)$-Riesz system.

(II) Next we investigate whether $\mathcal{S}(\Phi_j) + \mathcal{S}(\breve{\Psi}_j)$ is a uniformly $L_2(\Gamma)$-stable *two-level* decomposition of $\mathcal{S}(\Phi_{j+1})$:

Proposition A.1. *Let V, W be subspaces of a Hilbert space H. Then equivalent are*
(a) $H = V + W$ *and* $\alpha := \sup_{0\neq v\in V,\, 0\neq w\in W} \frac{|\langle v,w\rangle|}{\|v\|\|w\|} < 1$,

(b) *There exists a bounded projector $Q : H \to H$ with $\Im(Q) = V$ and $\Im(I - Q) = W$. Moreover, $\|Q\| = (1-\alpha^2)^{-\frac{1}{2}}$.*

Now let (a)*, or equivalently* (b)*, be satisfied, and let $\breve{W}$ be another subspace of H for which there exists a linear mapping $R : \breve{W} \to W$ with $\|I - R\| < \frac{1-\alpha}{1+\alpha}$. Then*

$$\breve{\alpha} := \sup_{0\neq v\in V,\, 0\neq \breve{w}\in \breve{W}} \frac{|\langle v, \breve{w}\rangle|}{\|v\|\|\breve{w}\|} \leq \alpha + (1+\alpha)\|I - R\| < 1.$$

With $\breve{Q} : H \to H$ being the bounded projector with $\Im(\breve{Q}) = V$ and $\Im(I - \breve{Q}) = \breve{W}$, it holds that

$$\|Q - \breve{Q}\| \leq \frac{\|I - R\|}{(1-\alpha^2)^{\frac{1}{2}}(1-\breve{\alpha}^2)^{\frac{1}{2}}}.$$

Proof. If (a) is valid, then $H = V \oplus W$, and so there exists a projector Q with $\Im(Q) = V$ and $\Im(I - Q) = W$. For any $u \in H$, we have

$$\begin{aligned}\|u\|^2 = \|Qu + (I-Q)u\|^2 &\geq \|Qu\|^2 - 2|\langle Qu, (I-Q)u\rangle| + \|(I-Q)u\|^2 \\ &\geq \|Qu\|^2 - 2\alpha\|Qu\|\|(I-Q)u\| + \|(I-Q)u\|^2 \geq (1-\alpha^2)\|Qu\|^2,\end{aligned}$$

or $\|Q\| \leq (1-\alpha^2)^{-\frac{1}{2}}$.

Now let (b) be valid. Suppose there exist nonzero $v \in V$, $w \in W$ such that $\mu := \frac{|\langle v,w\rangle|}{\|v\|\|w\|} > (1 - \|Q\|^{-2})^{\frac{1}{2}}$. Then there exist nonzero $v \in V$, $w \in W$ with $\langle v, w\rangle = -\mu\|v\|\|w\|$, moreover which can be chosen such that $\|w\| = -\mu\|v\|$. From

$$\|Q\|^{-2}\|v\|^2 \leq \|v + w\|^2 = \|v\|^2 + 2\langle v, w\rangle + \|w\|^2 = (1-\mu^2)\|v\|^2$$

we conclude a contradiction, so that $\sup_{0\neq v\in V,\, 0\neq w\in W} \frac{|\langle v,w\rangle|}{\|v\|\|w\|} \leq (1 - \|Q\|^{-2})^{\frac{1}{2}}$.

Now let (a) or, equivalently, (b) be valid, and let $\breve{W}$ be a subspace as in the proposition. For any $v \in V$, $\breve{w} \in \breve{W}$,

$$\begin{aligned}|\langle v, \breve{w}\rangle| = |\langle v, R\breve{w}\rangle + \langle v, (I-R)\breve{w}\rangle| &\leq \alpha\|v\|\|R\breve{w}\| + \|v\|\|(I-R)\breve{w}\| \\ &\leq \alpha\|v\|\|\breve{w}\| + (1+\alpha)\|v\|\|(I-R)\breve{w}\| \leq (\alpha + (1+\alpha)\|I-R\|)\|v\|\|\breve{w}\|,\end{aligned}$$

showing the statement about $\breve{\alpha}$. The last statement follows from $\|Q\| = (1-\alpha^2)^{-\frac{1}{2}}$, $\|I - \breve{Q}\| = (1-\breve{\alpha}^2)^{-\frac{1}{2}}$, and $Q - \breve{Q} = Q(I-R)(I-\breve{Q})$ by $QR = 0$ and $Q\breve{Q} = \breve{Q}$. □

Returning to the proof of Theorem 6.1, with, for $j \geq j_0$, $Q_j^{(j+1)} := Q_j|_{\mathcal{S}(\Phi_{j+1})}$, it holds that $\Im(Q_j^{(j+1)}) = \mathcal{S}(\Phi_j)$ and $\Im(I - Q_j^{(j+1)}) = \mathcal{S}(\Psi_j)$. Setting $R_j : \mathcal{S}(\breve{\Psi}_j) \to \mathcal{S}(\Psi_j) : \mathbf{c}_j^T \breve{\Psi}_j \mapsto \mathbf{c}_j^T \Psi_j$, by Assumption (iii) and **(I)** we have

$$\|(I - R_j)\mathbf{c}_j^T \breve{\Psi}_j\|_{L_2(\Gamma)} = \|\mathbf{c}_j^T(\breve{\Psi}_j - \Psi_j)\|_{L_2(\Gamma)} \leq \|\mathbf{c}\| \|\breve{\Psi}_j - \Psi_j\|_{L_2(\Gamma)} \lesssim \|\mathbf{c}_j^T \breve{\Psi}_j\|_{L_2(\Gamma)} \omega^j,$$

or $\|I - R_j\|_{L_2(\Gamma) \to L_2(\Gamma)} \lesssim \omega^j$. From Proposition A.1 we conclude that, possibly for a larger value of j_0, for $j \geq j_0$ there exists a uniformly $L_2(\Gamma)$-bounded projector $\breve{Q}_j^{(j+1)} : \mathcal{S}(\Phi_{j+1}) \to \mathcal{S}(\Phi_{j+1})$ with

$$\Im(\breve{Q}_j^{(j+1)}) = \mathcal{S}(\Phi_j), \quad \Im(I - \breve{Q}_j^{(j+1)}) = \mathcal{S}(\breve{\Psi}_j),$$

and

$$\|Q_j^{(j+1)} - \breve{Q}_j^{(j+1)}\|_{L_2(\Gamma) \to L_2(\Gamma)} \lesssim \omega^j. \tag{A.1}$$

(III) By Assumption (i), $\breve{\Psi}_j$ is uniformly local. Since furthermore, as shown in **(I)**, $\breve{\Psi}_j$ is a uniform $L_2(\Gamma)$-Riesz system, that, by Assumption (ii), has the uniform cancellation property of order $\tilde{d}$, for some $\tilde{\eta} \geq 0$ we have

$$|\langle \mathbf{c}_j^T \breve{\Psi}_j, u\rangle_{L_2(\Gamma)}| \leq \sum_{x \in J_j} |c_{j,x}| |\langle \breve{\psi}_{j,x}, u\rangle_{L_2(\Gamma)}| \quad (\mathbf{c}_j \in \ell_2(J_j),\, u \in \mathcal{H}_{\tilde{d}}(\Gamma))$$

$$\lesssim 2^{-j\tilde{d}} \|\mathbf{c}_j\| \,[\sum_{x \in J_j} \sum_{q=1}^{M} \|u \circ \kappa_q\|^2_{H^{\tilde{d}}(B(\kappa_q^{-1}(\operatorname{supp} \breve{\psi}_{j,x} \cap \Gamma_q); \tilde{\eta} 2^{-j}) \cap \square)}]^{\frac{1}{2}} \lesssim 2^{-j\tilde{d}} \|\mathbf{c}_j^T \breve{\Psi}_j\|_{L_2(\Gamma)} \|u\|_{\tilde{d}},$$

or

$$\|\cdot\|_{-\tilde{d}} \lesssim 2^{-\tilde{d}j} \|\cdot\|_{L_2(\Gamma)} \quad \text{on } \mathcal{S}(\breve{\Psi}_j). \tag{A.2}$$

By the uniform $L_2(\Gamma)$-boundedness of $\breve{Q}_j^{(j+1)}$ for $j \geq j_0$, an direct consequence of the last result is that $\|I - \breve{Q}_j^{(j+1)}\|_{0 \to -\tilde{d}} \lesssim 2^{-\tilde{d}j}$. By the Jackson estimate at the dual side (4.6), and the uniform $L_2(\Gamma)$-boundedness of Q_j for $j \geq j_0$, we have $\|I - Q_j^{(j+1)}\|_{0 \to -\tilde{d}} \leq \|I - Q_j\|_{0 \to -\tilde{d}} \lesssim 2^{-\tilde{d}j}$, and so $\|Q_j^{(j+1)} - \breve{Q}_j^{(j+1)}\|_{0 \to -\tilde{d}} \lesssim 2^{-\tilde{d}j}$. Now an application of the extended Bernstein inequality Lemma 4.2 shows that $\|Q_j^{(j+1)} - \breve{Q}_j^{(j+1)}\|_{-\tilde{d} \to -\tilde{d}} \lesssim 1$, and so by interpolation using (A.1) that

$$\|Q_j^{(j+1)} - \breve{Q}_j^{(j+1)}\|_{s \to s} \lesssim \omega^{(1+\frac{s}{\tilde{d}})j}, \quad (s \in [-\tilde{d}, 0]). \tag{A.3}$$

(IV) Knowing (A.3), we are ready to investigate the stability of the *multi-level* decomposition defined by the collections $\breve{\Psi}_j$. We are going to construct a projector $\breve{Q}_j$ defined on $\mathcal{H}_s(\Gamma)$ for some range of s, such that, for $j \geq j_0$, $\Im(\breve{Q}_j) = \mathcal{S}(\Phi_j)$ and $\Im(\breve{Q}_{j+1} - \breve{Q}_j) = \mathcal{S}(\breve{\Psi}_j)$.

By writing $Q_j = \sum_{k=j_0}^{j} Q_k - Q_{k-1}$, for any $s \in (-\min\{\tilde{\gamma}, \tilde{d}\}, \min\{\gamma, d\})$ and $u \in \mathcal{H}_s(\Gamma)$, Theorem 5.3 shows that $\|Q_j u\|_s^2 \lesssim \sum_{k=j_0}^{j} 4^{sk} \|(Q_k - Q_{k-1})u\|^2_{L_2(\Gamma)} \leq \sum_{k=j_0}^{\infty} 4^{sk} \|(Q_k - Q_{k-1})u\|^2_{L_2(\Gamma)} \lesssim \|u\|_s^2$ or $\|Q_j\|_{s \to s} \lesssim 1$.

For $\ell \geq j \geq j_0$, we define $\breve{Q}_j^{(\ell)} : \mathcal{S}(\Phi_\ell) \to \mathcal{S}(\Phi_j)$ by $\breve{Q}_\ell^{(\ell)} = I$ and, for $j < \ell$, by $\breve{Q}_j^{(\ell)} = \breve{Q}_j^{(j+1)} \breve{Q}_{j+1}^{(\ell)}$. For some arbitrary, but fixed $s \in (-\min\{\tilde{\gamma}, \tilde{d}\}, 0]$, we set

$$\rho_j^{(\ell)} := \max_{j_0 \leq k \leq j} \|Q_k \breve{Q}_j^{(\ell)}\|_{s \to s}, \quad \varepsilon_j := \max_{j_0 \leq k \leq j} \|Q_k(\breve{Q}_j^{(j+1)} - Q_j^{(j+1)})\|_{s \to s}.$$

From $Q_k \breve{Q}_j^{(\ell)} = Q_k(\breve{Q}_j^{(j+1)} - Q_j^{(j+1)})\breve{Q}_{j+1}^{(\ell)} + Q_k \breve{Q}_{j+1}^{(\ell)}$, we find $\rho_j^{(\ell)} \leq (\varepsilon_j + 1)\rho_{j+1}^{(\ell)}$, and so by $\rho_\ell^{(\ell)} = \max_{j_0 \leq k \leq j} \|Q_k\|_{s \to s} \lesssim 1$, and $\varepsilon_j \lesssim \omega^{(1+\frac{s}{d})j}$ by (A.3), we infer that

$$\|\breve{Q}_j^{(\ell)}\|_{s \to s} \leq \rho_j^{(\ell)} \lesssim \prod_{k=j}^{\ell-1} (\varepsilon_k + 1) \lesssim 1 + \sum_{k=j}^{\ell-1} \varepsilon_k \lesssim 1,$$

which thus hold uniformly in j and ℓ.

As a consequence of $I = \sum_{j=j_0}^\infty (Q_j - Q_{j-1})$ on $\mathcal{H}_s(\Gamma)$ by Theorem 5.3, we have $\operatorname{clos}_{H_s(\Gamma)} \cup_{j \geq j_0} \mathcal{S}(\Phi_j) = \mathcal{H}_s(\Gamma)$. Since for any $u \in \mathcal{H}_s(\Gamma)$, $j \leq k \leq \ell$ and $u_k \in \mathcal{S}(\Phi_k)$, $\breve{Q}_j^{(\ell)} Q_\ell u = \breve{Q}_j^{(k)} u_k + \breve{Q}_j^{(\ell)} Q_\ell(u - u_k)$, from $\|\breve{Q}_j^{(\ell)} Q_\ell\|_{s \to s} \lesssim 1$ we infer that for any $j \geq j_0$, $(\breve{Q}_j^{(\ell)} Q_\ell u)_{\ell \geq j}$ is a Cauchy-sequence in $\mathcal{H}_s(\Gamma)$, and we set $\breve{Q}_j u = \lim_{\ell \to \infty} \breve{Q}_j^{(\ell)} Q_\ell u$. We conclude that $\breve{Q}_j : \mathcal{H}_s(\Gamma) \to \mathcal{H}_s(\Gamma)$ is uniformly bounded, with $\Im(\breve{Q}_j) = \mathcal{S}(\Phi_j)$ and $\Im(\breve{Q}_{j+1} - \breve{Q}_j) = \mathcal{S}(\breve{\Psi}_j)$.

The Jackson estimate (4.4) and $\|\breve{Q}_j\|_{L_2(\Gamma) \to L_2(\Gamma)} \lesssim 1$ show $\|I - \breve{Q}_j\|_{d \to 0} \lesssim 2^{-dj}$, and so using interpolation that $\|I - \breve{Q}_j\|_{s \to 0} \lesssim 2^{-sj}$ ($s \in [0, d]$). From the Bernstein inequality (4.5), for any $s \in [0, \gamma)$, $s \leq d$, and thus for any $s \in (-\min\{\tilde{\gamma}, \tilde{d}\}, \min\{\gamma, d\})$, we conclude that $\|\breve{Q}_{j+1} - \breve{Q}_j\|_{s \to s} \lesssim 1$.

(V) Let $s \in (-\min\{\tilde{\gamma}, \tilde{d}\}, \min\{\gamma, d\})$. With $\breve{Q}_{j_0 - 1} := 0$, we are going to prove that

$$\sum_{j=j_0}^\infty 4^{sj} \|(\breve{Q}_j - \breve{Q}_{j-1})u\|_{L_2(\Gamma)}^2 \lesssim \|u\|_s^2, \quad (u \in \mathcal{H}_s(\Gamma)). \tag{A.4}$$

Let $t \in (-\min\{\tilde{\gamma}, \tilde{d}\}, s)$ with $t \leq 0$. For $u \in \mathcal{H}_s(\Gamma)$, $w_\ell := (Q_\ell - Q_{\ell-1})u$, Theorem 6.1 shows that $u = \sum_{\ell=j_0}^\infty w_\ell$, $\|u\|_s^2 \eqsim \sum_{\ell=j_0}^\infty 4^{s\ell} \|w_\ell\|_{L_2(\Gamma)}^2$, and $\|w_\ell\|_t \lesssim 2^{t\ell} \|w_\ell\|_{L_2(\Gamma)}$. Since $\|\breve{Q}_j\|_{t \to 0} \lesssim 2^{-tj}$, which follows from $\|\breve{Q}_j\|_{t \to t} \lesssim 1$ and the extended Bernstein inequality Lemma 4.2, and $\breve{Q}_j - \breve{Q}_{j-1} = 0$ on $\mathcal{S}(\Phi_{j-1})$, we arrive at

$$\begin{aligned}
&\sum_{j=j_0}^\infty 4^{sj} \|(\breve{Q}_j - \breve{Q}_{j-1})u\|_{L_2(\Gamma)}^2 = \sum_{j=j_0}^\infty \sum_{\ell,\ell'=j_0}^\infty 4^{sj} \langle (\breve{Q}_j - \breve{Q}_{j-1})w_\ell, (\breve{Q}_j - \breve{Q}_{j-1})w_{\ell'} \rangle_{L_2(\Gamma)} \\
&= \sum_{\ell,\ell'=j_0}^\infty \sum_{j=j_0}^{\min\{\ell,\ell'\}} 4^{sj} \langle (\breve{Q}_j - \breve{Q}_{j-1})w_\ell, (\breve{Q}_j - \breve{Q}_{j-1})w_{\ell'} \rangle_{L_2(\Gamma)} \\
&\lesssim \sum_{\ell,\ell'=j_0}^\infty \sum_{j=j_0}^{\min\{\ell,\ell'\}} 4^{sj} 4^{-tj} \|w_\ell\|_t \|w_{\ell'}\|_t \lesssim \sum_{\ell,\ell'=j_0}^\infty 4^{(s-t)\min\{\ell,\ell'\}} \|w_\ell\|_t \|w_{\ell'}\|_t \\
&\lesssim \sum_{\ell,\ell'=j_0}^\infty 4^{(s-t)\min\{\ell,\ell'\}} 2^{(t-s)(\ell+\ell')} (2^{s\ell} \|w_\ell\|_{L_2(\Gamma)})(2^{s\ell'} \|w_{\ell'}\|_{L_2(\Gamma)}) \\
&\lesssim \sum_{\ell=j_0}^\infty 4^{s\ell} \|w_\ell\|_{L_2(\Gamma)}^2 \eqsim \|u\|_s^2,
\end{aligned}$$

where we have used that the infinite matrix $[2^{(s-t)(2\min\{\ell,\ell'\}-\ell-\ell')}]_{\ell,\ell'\geq j_0}$ is bounded.

(VI) For $s \in [-\tilde{d}, \gamma)$, it holds that

$$\|\!|\cdot|\!\|_s \lesssim 2^{sj}\|\cdot\|_{L_2(\Gamma)} \quad \text{on } \Im(\breve{Q}_j - \breve{Q}_{j-1}).$$

Indeed for $s > 0$ this is just the Bernstein inequality (4.5), whereas for $s < 0$ it is a consequence of the extended Bernstein inequality Lemma 4.2 and (A.2). Now let $s \in (-\tilde{d}, \gamma)$, and let $\varepsilon > 0$ be such that $s \pm \varepsilon \in [-\tilde{d}, \gamma)$. Then for any $\breve{w}_j \in \Im(\breve{Q}_j - \breve{Q}_{j-1})$, with $\sum_{j=j_0}^{\infty} 4^{sj}\|\breve{w}_j\|_{L_2(\Gamma)}^2 < \infty$, it holds that

$$\begin{aligned}\|\!|\sum_{j=j_0}^{\infty} \breve{w}_j|\!\|_s^2 &= \sum_{j,j'=j_0}^{\infty} \langle\!\langle \breve{w}_j, \breve{w}_{j'}\rangle\!\rangle_s \lesssim \sum_{j=j_0}^{\infty}\sum_{j'\geq j}^{\infty} \|\!|\breve{w}_j|\!\|_{s+\varepsilon}\|\!|\breve{w}_{j'}|\!\|_{s-\varepsilon} \\ &\lesssim \sum_{j=j_0}^{\infty}\sum_{j'\geq j}^{\infty} 2^{\varepsilon j}2^{-\varepsilon j'}(2^{sj}\|\breve{w}_j\|_{L_2(\Gamma)})(2^{sj'}\|\breve{w}_{j'}\|_{L_2(\Gamma)}) \lesssim \sum_{j=j_0}^{\infty} 4^{sj}\|\breve{w}_j\|_{L_2(\Gamma)}^2.\end{aligned}$$

Since, when $s \in (-\tilde{\gamma}, d)$, (A.4) shows that $\sum_{j=j_0}^{\infty} 4^{sj}\|\breve{w}_j\|_{L_2(\Gamma)}^2 \lesssim \|\!|\sum_{j=j_0}^{\infty} \breve{w}_j|\!\|_s^2$, and Φ_{j_0} is an $L_2(\Gamma)$-Riesz basis for $\Im(\breve{Q}_{j_0})$, and for $j > j_0$, $\breve{\Psi}_{j-1}$ is a uniform $L_2(\Gamma)$-Riesz basis for $\Im(\breve{Q}_j - \breve{Q}_{j-1})$, we conclude that $\Phi_{j_0} \cup \cup_{j\geq j_0} 2^{-sj}\breve{\Psi}_j$ is a Riesz system in $\mathcal{H}_s(\Gamma)$. Finally, since, as follows from Theorem 5.3, $\text{clos}_{H_s(\Gamma)} \cup_{j\geq 0} \mathcal{S}(\Phi_j) = \mathcal{H}_s(\Gamma)$, we conclude it is even Riesz basis for this space, with which the proof of Theorem 6.1 is completed.

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