NUMERICAL PERIODIC NORMALIZATION FOR CODIM 1 BIFURCATIONS OF LIMIT CYCLES *

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Abstract. Explicit computational formulas for the coefficients of the periodic normal forms for all codim 1 bifurcations of limit cycles in generic autonomous ODEs are derived. They include second-order coefficients for the fold (limit point) bifurcation, as well as third-order coefficients for the flip (period-doubling) and Neimark-Sacker (torus) bifurcations. The formulas are independent of the dimension of the phase space and involve solutions of certain boundary-value problems on the interval $[0,T]$, where T is the period of the critical cycle, as well as multilinear functions from the Taylor expansion of the right-hand sides near the cycle. The formulas allow to distinguish between sub- and super-critical bifurcations, in agreement with earlier asymptotic expansions of the bifurcating solutions. Our formulation makes it possible to use robust numerical boundary value algorithms based on orthogonal collocation, rather than shooting techniques, which greatly expands its applicability. The actual implementation is described in detail. We include three numerical examples, in which codim 2 singularities are detected along branches of codim 1 bifurcations of limit cycles as zeroes of the periodic normal form coefficients.

Key words. Normal forms, limit cycles, bifurcations

AMS subject classifications. 34C20, 37G15, 37M20, 65L07

1. Introduction. Isolated periodic orbits (limit cycles) of smooth differential equations

$$\dot{u} = f(u,\alpha), \quad u \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^m, \tag{1.1}$$

play an important role in applications. In generic systems of the form (1.1) depending on one control parameter (*i.e.*, with $m = 1$) a hyperbolic limit cycle exists for an open interval of parameter values α. At a boundary of such an interval, the limit cycle may not exist, degenerating into an orbit homoclinic to an equilibrium or another nonhyperbolic limit cycle (see, for example [23]). We do not consider such cases here, instead focusing on those where the cycle does exist at the boundary parameter values, but loses its hyperbolicity due to the presence of a nontrivial multiplier μ, with $|\mu| = 1$.

The codim 1 bifurcations of limit cycles in generic systems (1.1) are well-understood (see, for example, [1], [17], [23]). The standard approach to the theoretical and numerical analysis of local bifurcations of limit cycles is based on *Poincaré maps*: Given a transversal section Σ to the cycle Γ at $x(0)$, such a map assigns to each point y of Σ close to $x(0)$ another point $\mathcal{P}(y,\alpha)$, where the orbit of (1.1) starting at y intersects Σ again close to $x(0)$. In local coordinates in Σ, the Poincaré map will be represented by a smooth map $\mathcal{P} : \mathbb{R}^{n-1} \times \mathbb{R}^p \to \mathbb{R}^{n-1}$. The cycle corresponds to a fixed point of this map; the eigenvalues of its linearization at the fixed point are the

*Draft of July 14, 2004

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nontrivial multipliers of the periodic solution. Once the Poincaré map is introduced, the theory of local bifurcations of maps can be applied.

It is well known (see, for example, [1, 17, 19] for the general theory and [23] for computational formulas) that in generic smooth one-parameter families of maps

$$y \mapsto \mathcal{P}(y, \alpha), \quad y \in \mathbb{R}^{n-1}, \alpha \in \mathbb{R}^1, \tag{1.2}$$

only the following three bifurcations of fixed points occur:

(1) *The Fold*: The fixed point y_0 has a simple eigenvalue $\lambda_1 = 1$ and no other eigenvalues on the unit circle, while the restriction of (1.2) to a one-dimensional center manifold $\mathcal{W}^c(y_0)$ at the critical parameter value has the form $\xi \mapsto \xi + \tilde{b}\xi^2 + O(\xi^3)$, where $\tilde{b} \neq 0$. When the parameter crosses the critical value, two fixed points coalesce and disappear. This bifurcation is often called a *saddle-node bifurcation*, a *fold*, or a *limit point (LP)*, since two periodic solutions collide and disappear when the parameter passes the critical value in (1.1). If $\mathcal{A}v = \mathcal{P}_y v$ and $\mathcal{B}(u, v) = \mathcal{P}_{yy}[u, v]$ are evaluated at the critical fixed point y_0, then

$$\tilde{b} = \frac{1}{2}\langle q^*, \mathcal{B}(q, q)\rangle, \tag{1.3}$$

where $\mathcal{A}q = q$, $\mathcal{A}^{\mathrm{T}}q^* = q^*$, and $\langle q^*, q\rangle = 1$. Here and in what follows, $\langle u, v\rangle = u^{\mathrm{H}}v = \bar{u}^{\mathrm{T}}v$ is the standard scalar product in an appropriate complex (or real) finite-dimensional vector space; here: $\mathbb{R}^{n-1}$. It should also be noted that the coefficient $\tilde{b}$ is not uniquely defined but depends on the normalization of q. A similar remark holds for all other normal form coefficients.

(2) *The Flip*: The fixed point y_0 has a simple eigenvalue $\lambda_1 = -1$ and no other eigenvalues on the unit circle, while the restriction of (1.2) to a one-dimensional center manifold $\mathcal{W}^c(y_0)$ at the critical parameter value can be transformed to the normal form $\xi \mapsto -\xi + \tilde{c}\xi^3 + O(\xi^4)$, where $\tilde{c} \neq 0$. When the parameter crosses the critical value, a cycle of period 2 bifurcates from the fixed point. This is a *period-doubling (PD)* of the periodic solution, *i.e.*, there are nearby periodic solutions of approximately double (minimal) period. If $\mathcal{C}(u, v, w) = \mathcal{P}_{yyy}[u, v, w]$ is evaluated at y_0, then

$$\tilde{c} = \frac{1}{6}\langle p^*, \mathcal{C}(p, p, p) + 3\mathcal{B}(p, (I_{n-1} - \mathcal{A})^{-1}\mathcal{B}(p, p))\rangle, \tag{1.4}$$

where I_{n-1} is the $(n-1) \times (n-1)$ identity matrix, $\mathcal{A}p = -p$, $\mathcal{A}^{\mathrm{T}}p^* = -p^*$, and $\langle p^*, p\rangle = 1$.

(3) *The Neimark-Sacker (NS) bifurcation*: The fixed point y_0 has simple critical eigenvalues $\lambda_{1,2} = e^{\pm i\theta}$ and no other eigenvalues on the unit circle. Assume that

$$e^{iq\theta} - 1 \neq 0, \quad q = 1, 2, 3, 4 \ \text{ (no strong resonances)}.$$

Then the restriction of (1.2) to a two-dimensional center manifold $\mathcal{W}^c(y_0)$ at the critical parameter value can be transformed to the normal form $\eta \mapsto \eta e^{i\theta}(1 + \tilde{d}|\eta|^2) + O(|\eta|^4)$, where η is a complex variable and d is a complex number. Further assume that

$$\mathrm{Re}\ \tilde{d} \neq 0.$$

Under the above assumptions, a unique *closed invariant curve* around the fixed point appears when the parameter crosses the critical value. This curve corresponds to an

invariant torus, on which the flow of (1.1) contains periodic or quasi-periodic motions. One has the following expression for $\tilde{d}$:

$$\tilde{d} = \frac{1}{2} e^{-i\theta} \langle v^*, \mathcal{C}(v, v, \bar{v}) + 2\mathcal{B}(v, (I_{n-1} - \mathcal{A})^{-1} \mathcal{B}(v, \bar{v})) + \mathcal{B}(\bar{v}, (e^{2i\theta} I_{n-1} - \mathcal{A})^{-1} \mathcal{B}(v, v)) \rangle, \tag{1.5}$$

where $\mathcal{A}v = e^{i\theta} v$, $\mathcal{A}^{\mathrm{T}} v^* = e^{-i\theta} v^*$, and $\langle v^*, v \rangle = 1$.

Application of these theoretical results to the analysis of concrete ODEs in science and engineering is limited. One reason for this is the necessity to compute the Poincaré map (1.2) and its derivatives $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \ldots)$ numerically. Finite differences work for low-dimensional non-stiff systems (1.1), where they allow the approximation of the Jacobian matrix $\mathcal{A} = \mathcal{P}_y$ with reasonable accuracy. However, this approach fails, when the cycle is unstable or when (1.1) is stiff. In general, finite-difference approximations of higher-order partial derivatives (*i.e.*, $\mathcal{B}, \mathcal{C}, \mathcal{D}, \ldots$) have very low accuracy due to loss of significant digits, and are therefore unreliable.

AUTO97 and CONTENT 1.5 support the detection of codim 1 bifurcations along one-parameter families of limit cycles, but normal forms are not computed. AUTO can continue all codim 1 cycle bifurcations in two parameters, however, it provides no tools to detect and classify codim 2 singularities. CONTENT computes the normal form coefficients at codim 1 bifurcations of fixed points of maps and detects all eleven generic codim 2 bifurcations. However, application of these capabilities of CONTENT to limit cycle analysis is usually based on the numerical construction of the Poincaré map (1.2) and the computation of its partial derivatives via finite differences. Better results for computing $\mathcal{B}$ and higher-order derivative tensors can be achieved by numerical integration of the corresponding variational equations along the periodic solution. This method has been successfully used in [26] to compute normal form coefficients of the Poincaré map at the fold-flip bifurcation in a 4-dimensional atmosphere circulation model. The approach, unfortunately, does not work well for stiff systems. An interesting alternative to numerical integration of the variational equations is to compute the higher-order derivatives of the Poincaré map $\mathcal{P}$ by *automatic differentiation* [16] of the (for example, `C`-)code used to compute the Poincaré map. To our knowledge, this alternative approach in computing normal forms ("*normalization*") has been not been used yet; see, however, [18].

There are at least two approaches to the analysis of the limit cycles bifurcations that are not directly based on the Poincaré map. Since it is known which periodic or quasi-periodic solutions can bifurcate at generic codim 1 bifurcations of limit cycles, one can compute the Taylor series for the period $T(\varepsilon)$, for the corresponding parameter $\alpha(\varepsilon)$, and for the bifurcating solution itself, as functions of the solution amplitude ε. The solvability of the linear systems is guaranteed by the *Fredholm alternative*. This approach, which is conceptually similar to the *Lyapunov-Schmidt method*, has been successfully applied to all codim 1 bifurcations of limit cycles in [22, Chapter XI]. The resulting asymptotic expressions use the derivatives of the right-hand side of (1.1) with respect to u and α and involve solutions to linear boundary value problems (BVPs) (on the interval $[0, T]$ in the LPC (Limit Point of Cycles) and NS cases and on the interval $[0, 2T]$ for the PD bifurcation). They allow to distinguish between sub- and supercritical bifurcations. However, these formulas are rather involved; in particular, for the Neimark-Sacker case, where one has to distinguish between various sub-harmonic and quasi-periodic solutions, and to our knowledge they have not been implemented in bifurcation software.

There is another theoretical approach [1] for the analysis of limit cycle bifurcations

in (1.1), which avoids the Poincaré map reduction. First, in a neighborhood of Γ in $W^c(\Gamma)$, *normal coordinates* can be chosen so that the restricted system (1.1) becomes a non-autonomous T-*periodic* system in $\mathbb{R}^{n-1}$. This periodic system can be considered as an autonomous system with one cyclic variable (mod T). Near the bifurcation, this system can be restricted to an (n_c+1)-dimensional invariant *center manifold* $W^c(\Gamma)$, thus giving a periodic n_c-dimensional system of ODEs. One can then apply (in general, $2T$-)periodic coordinate transformations to this system, and write it as the sum of an autonomous n_c-dimensional normal form and higher-order periodic terms. The autonomous part of this *periodic normal form* allows one to study local and global bifurcations of (1.1) near the critical cycle. This approach is very useful for the theoretical analysis of limit cycle bifurcations (see [1] and [4] for normal forms for some codim 2 cases).

Since the late 1980s, an improvement of this approach is known [13], that combines the computation of the center manifold with the normalization of the ODEs restricted to this manifold. This technique leads to simple formulas for the computation of normal form coefficients in two codim 1 cases of equilibrium bifurcations in ODEs (derived earlier with the Lyapunov-Schmidt method), as well as in all five codim 2 cases (see [24]). Although a similar normalization technique was introduced in [12] for time-periodic systems and in [20, 21] for limit cycle bifurcations, it has remained mainly a theoretical tool up to now. There are no numerical algorithms for the computation of the coefficients of the normal forms on $W^c(\Gamma)$ that are based on this approach and that have been implemented in available bifurcation software.

Below we derive a powerful numerical normalization tool based on this technique. In a sense, we combine the periodic normal forms derived in [20] with the Fredholm alternative used in [22]. It should be noted that the idea to apply Fredholm's solvability condition to compute the normal form coefficients for time-periodic systems can be traced back to [12]. The main difference between our approach and that of [12] and [20] is that we avoid Fourier series solutions of the linear BVPs, instead solving them numerically, using orthogonal collocation as discretization. This leads to simple and explicit algorithms for the normal form coefficients. A further simplification occurs because we consider only the critical normal forms, and therefore we do not need derivatives of $f(u,\alpha)$ with respect to α. Our results fully agree with the asymptotic expansions for the bifurcating solutions derived in [22].

This paper is organized as follows. In Section 2 we fix notation and formulate the periodic normalization on the center manifold. In Sections 3–5 we apply this technique to derive explicit formulas to compute the critical normal form coefficients for fold, period-doubling and torus bifurcations of limit cycles. The formulas are independent of the dimension of the phase space and involve solutions to certain BVPs on the interval $[0,T]$, where T is the period of the critical cycle, as well as multilinear functions from the Taylor expansion of the right-hand sides of (1.1) near the cycle. In Section 6 we show that our algorithms fit very well into the BVP-collocation framework of existing continuation software. Three numerical examples are given in Section 7. Future work is discussed in Section 8.

2. The method. Write (1.1) at the critical parameter values as

$$\dot{u} = F(u), \tag{2.1}$$

and suppose that (2.1) has a periodic solution $u_0(t) = u_0(t+T)$, where $T > 0$ is its (minimal) period. Develop $F(u_0(t)+v)$ into the Taylor series

$$F(u_0(t)+v) = F(u_0(t)) + A(t)v + \frac{1}{2}B(t;v,v) + \frac{1}{6}C(t;v,v,v) + O(\|v\|^4), \qquad (2.2)$$

where $A(t) = F_u(u_0(t))$ and $B(t,v,v) = F_{uu}(u_0(t))[v,v]$, $C(t;v,v,v) = F_{uuu}(u_0(t))$ $[v,v,v]$. Recall, that the components of B and C are given by

$$B_i(t;x,y) = \sum_{j,k=1}^{n} \left.\frac{\partial^2 F_i(\xi)}{\partial\xi_j\partial\xi_k}\right|_{\xi=u_0(t)} x_j y_k, \qquad (2.3)$$

$$C_i(t;x,y,z) = \sum_{j,k,l=1}^{n} \left.\frac{\partial^3 F_i(\xi)}{\partial\xi_j\partial\xi_k\partial\xi_l}\right|_{\xi=u_0(t)} x_j y_k z_l, \qquad (2.4)$$

for $i = 1, 2, \ldots, n$. The multilinear forms A, B, and C are periodic in t with period T.

Consider the initial-value problem for the fundamental matrix solution $Y(t)$, namely

$$\frac{dY}{dt} = A(t)Y, \quad Y(0) = I_n, \qquad (2.5)$$

where I_n is the $n \times n$ identity matrix. The monodromy matrix $M = Y(T)$ always has a "trivial"eigenvalue $\mu_n = 1$. The cycle is hyperbolic if there are no other eigenvalues with $|\mu| = 1$, and nonhyperbolic otherwise.

The cycle has a fold bifurcation if $\mu_1 = 1$ is an eigenvalue of $Y(T)$, the corresponding Jordan block is two-dimensional, and there are no other critical eigenvalues of the monodromy matrix. The cycle has a period-doubling (flip) bifurcation if $\mu_1 = -1$ is simple and the only nontrivial critical eigenvalue of $Y(T)$. Finally, at a Neimark-Sacker (torus) bifurcation, there is a simple pair of nonreal eigenvalues $\mu_{1,2} = e^{\pm i\theta}$, such that $e^{iq\theta} \neq 1$ for $q = 1, 2, 3, 4$ (no strong resonances), and $Y(T)$ has no further critical multipliers other than 1. We will refer to these conditions as the *spectral assumptions*.

To describe the periodic normal forms for the three critical cases mentioned above, we parametrize the corresponding (n_c+1)-dimensional center manifold $W^c(\Gamma)$ near Γ by (τ,ξ), where $\tau \in [0,T]$ or $[0,2T]$, and ξ is a real or complex coordinate, depending on the bifurcation. It follows from [20] that it is possible to select the ξ-coordinates so that the restriction of (2.1) to the corresponding critical center manifold $W^c(\Gamma)$ will take one of the following *periodic normal forms*.

The periodic normal form at the Limit Point of Cycles (LPC) bifurcation is

$$\begin{cases} \dfrac{d\tau}{dt} &= 1 - \xi + a\xi^2 + \cdots, \\ \dfrac{d\xi}{dt} &= b\xi^2 + \cdots, \end{cases} \qquad (2.6)$$

where $\tau \in [0,T]$, ξ is a real coordinate on $W^c(\Gamma)$ that is transverse to Γ, $a, b \in \mathbb{R}$, and the dots denote nonautonomous T-periodic $O(\xi^3)$-terms. One can show that b and $\tilde{b}$ vanish together, where $\tilde{b}$ is obtained via the Poincaré map reduction and given by (1.3).

The periodic normal form at the Period Doubling (PD) bifurcation is

$$\left\{\begin{array}{rcl} \dfrac{d\tau}{dt} &=& 1+a\xi^2+\cdots, \\ \dfrac{d\xi}{dt} &=& c\xi^3+\cdots, \end{array}\right. \tag{2.7}$$

where $\tau \in [0,2T]$, ξ is a real coordinate on $W^c(\Gamma)$ that is transverse to Γ, $a,c \in \mathbb{R}$, and the dots denote nonautonomous $2T$-periodic $O(\xi^4)$-terms. The coefficient c determines the stability of the critical cycle; if $c \neq 0$ then

$$\text{sign } c = \text{sign } \tilde{c},$$

where $\tilde{c}$ is obtained via the Poincaré map reduction and given by (1.4).

The periodic normal form at the Neimark-Sacker (NS) bifurcation is

$$\left\{\begin{array}{rcl} \dfrac{d\tau}{dt} &=& 1+a|\xi|^2+\cdots, \\ \dfrac{d\xi}{dt} &=& \dfrac{i\theta}{T}\xi+d\xi|\xi|^2+\cdots, \end{array}\right. \tag{2.8}$$

where $\tau \in [0,T]$, ξ is a complex coordinate on $W^c(\Gamma)$ that is complementary to τ, $a \in \mathbb{R}, d \in \mathbb{C}$, and the dots denote nonautonomous T-periodic $O(|\xi|^4)$-terms. If $\text{Re } d \neq 0$ then

$$\text{sign}(\text{Re } d) = \text{sign}(\text{Re } \tilde{d}),$$

where $\tilde{d}$ is given by (1.5), obtained via the Poincaré map reduction.

In view of the above, we can assume that a parametrization of the center manifold $W^c(\Gamma)$ is selected so that the restriction of (2.1) to this manifold has one of the normal forms (2.6), (2.7), or (2.8). The Taylor expansions of T- or $2T$-periodic unknown functions involved in these parametrizations can be found by solving appropriate BVPs on $[0,T]$ or $[0,2T]$, respectively, so that (2.1) restricted to $W^c(\Gamma)$ will have the corresponding periodic normal form. The coefficients a, b, and c arise from the solvability conditions for the BVPs as integrals of scalar products over $[0,T]$, involving quadratic and cubic terms of (2.1) near the periodic solution u_0, as well as the critical eigenfunctions.

The following (or similar) construction will often be used below. Denote by $\mathcal{C}^k([a,b],\mathbb{R}^n)$ the space of k times continuously differentiable functions on $[a,b]$, with values in $\mathbb{R}^n$. Let $\varphi \in \mathcal{C}^1([0,T],\mathbb{R}^n)$ be the only solution of the BVP

$$\left\{\begin{array}{rcl} \dot{\varphi}(\tau)-A(\tau)\varphi(\tau) &=& 0,\ \tau\in[0,T], \\ \varphi(T)-\varphi(0) &=& 0, \\ \int_0^T \langle\varphi(\tau),\varphi(\tau)\rangle d\tau - 1 &=& 0, \end{array}\right.$$

and let $\varphi^* \in \mathcal{C}^1([0,T],\mathbb{R}^n)$ be a nontrivial solution of the adjoint BVP

$$\left\{\begin{array}{rcl} \dot{\varphi}^*(\tau)+A^{\mathrm{T}}(\tau)\varphi^*(\tau) &=& 0,\ \tau\in[0,T], \\ \varphi^*(T)-\varphi^*(0) &=& 0. \end{array}\right. \tag{2.9}$$

If $h \in \mathcal{C}^1([0,T],\mathbb{R}^n)$ is a solution of the singular BVP

$$\left\{\begin{array}{rcl} \dot{h}(\tau)-A(\tau)h(\tau) &=& g(\tau),\ \tau\in[0,T], \\ h(T)-h(0) &=& 0, \end{array}\right. \tag{2.10}$$

then $g \in \mathcal{C}^1([0,T],\mathbb{R}^n)$ satisfies

$$\int_0^T \langle \varphi^*(\tau), g(\tau)\rangle \; d\tau = 0. \tag{2.11}$$

Indeed, taking into account (2.9), we see that this integral equals

$$\int_0^T \langle \varphi^*(\tau), \dot{h}(\tau) - A(\tau)h(\tau)\rangle \; d\tau = -\int_0^T \langle \dot{\varphi}^*(\tau) + A^{\mathrm{T}}(\tau)\varphi^*(\tau), h(\tau)\rangle \; d\tau = 0.$$

We will refer to (2.11) as the *Fredholm solvability condition*. If (2.11) holds, then the problem (2.10) has a unique solution h, satisfying

$$\int_0^T \langle \varphi^*(\tau), h(\tau)\rangle \; d\tau = 0.$$

3. The fold bifurcation. The two-dimensional critical center manifold $W^c(\Gamma)$ at the LPC bifurcation can be parametrized locally by (τ,ξ) as

$$u = u_0(\tau) + \xi v(\tau) + H(\tau,\xi), \quad \tau \in [0,T], \; \xi \in \mathbb{R}, \tag{3.1}$$

where H satisfies $H(T,\xi) = H(0,\xi)$, and has the Taylor expansion

$$H(\tau,\xi) = \frac{1}{2}h_2(\tau)\xi^2 + O(\xi^3), \tag{3.2}$$

with $h_2(T) = h_2(0)$, while

$$\left\{ \begin{array}{rcl} \dot{v}(\tau) - A(\tau)v(\tau) - F(u_0(\tau)) & = & 0, \; \tau \in [0,T], \\ v(T) - v(0) & = & 0, \\ \int_0^T \langle v(\tau), F(u_0(\tau))\rangle d\tau & = & 0. \end{array} \right. \tag{3.3}$$

The function v exists due to LEMMA 2 of [20]. Note that (3.3) implies

$$\int_0^T \langle \varphi^*(\tau), F(u_0(\tau))\rangle \; d\tau = 0, \tag{3.4}$$

for any φ^* satisfying (2.9). Moreover, due to the spectral assumptions at the LPC-point, we can also assume that

$$\int_0^T \langle \varphi^*(\tau), v(\tau)\rangle d\tau = 1. \tag{3.5}$$

Therefore, φ^* is the unique solution of the BVP

$$\left\{ \begin{array}{rcl} \dot{\varphi}^*(\tau) + A^{\mathrm{T}}(\tau)\varphi^*(\tau) & = & 0, \; \tau \in [0,T], \\ \varphi^*(T) - \varphi^*(0) & = & 0, \\ \int_0^T \langle \varphi^*(\tau), v(\tau)\rangle d\tau - 1 & = & 0. \end{array} \right. \tag{3.6}$$

The function $h_2(\tau)$ can be found by solving an appropriate BVP, assuming that (2.1) restricted to $W^c(\Gamma)$ has the periodic normal form (2.6). The coefficient b arises from the solvability condition for the BVP as an integral over the interval $[0,T]$ of scalar products. Specifically, these scalar products involve the quadratic terms of

(1.1) near the periodic solution u_0, the (generalized) eigenfunction v, and the adjoint eigenfunction φ^* defined by (3.6).

Substitute (3.1) into (2.1), using (2.2), (2.6), and (3.2), as well as

$$\frac{du}{dt} = \frac{\partial u}{\partial \xi}\frac{d\xi}{dt} + \frac{\partial u}{\partial \tau}\frac{d\tau}{dt}.$$

This gives

$$\begin{aligned}\dot{u}_0 + \xi(\dot{v} - \dot{u}_0) + \xi^2\left(\frac{1}{2}\dot{h}_2 + a\dot{u}_0 + bv - \dot{v}\right) + O(\xi^3)\\ = F(u_0) + \xi A(\tau)v + \frac{1}{2}\xi^2(A(\tau)h_2 + B(\tau; v, v)) + O(\xi^3),\end{aligned}$$

where dots denote the derivatives with respect to τ.

Collecting the ξ^0-terms we get the identity

$$\dot{u}_0 = F(u_0),$$

since u_0 is the periodic solution of (2.1).

The ξ^1-terms provide another identity, namely,

$$\dot{v} - A(\tau)v - \dot{u}_0 = 0,$$

due to (3.3).

Finally, collecting the ξ^2-terms, we obtain the equation for h_2

$$\dot{h}_2 - A(\tau)h_2 = B(\tau; v, v) - 2a\dot{u}_0 + 2\dot{v} - 2bv, \tag{3.7}$$

to be solved in the space of vector-functions on $[0, T]$ satisfying $h_2(T) = h_2(0)$. The differential operator $\frac{d}{d\tau} - A(\tau)$ is singular in this space, with $\dot{u}_0$ as the eigenfunction corresponding to zero eigenvalue. The null-eigenfunction φ^* of the adjoint operator $-\frac{d}{d\tau} - A^{\mathrm{T}}(\tau)$ is defined by (3.6). Thus, the Fredholm solvability condition implies that

$$\int_0^T \langle \varphi^*(\tau), B(\tau; v(\tau), v(\tau)) - 2a\dot{u}_0(\tau) + 2\dot{v}(\tau) - 2bv(\tau)\rangle \, d\tau = 0.$$

Using (3.4) and (3.5), we get the expression

$$b = \frac{1}{2}\int_0^T \langle \varphi^*(\tau), B(\tau; v(\tau), v(\tau)) + 2A(\tau)v(\tau)\rangle \, d\tau. \tag{3.8}$$

Here v and φ^* are defined by (3.3) and (3.6), respectively. Therefore, the critical coefficient b in the periodic normal form for the LPC bifurcation has been computed. The bifurcation is nondegenerate if $b \neq 0$. Note that the coefficient a does not enter in (3.8) due to (3.4).

4. The period-doubling bifurcation. The two-dimensional critical center manifold $W^c(\Gamma)$ at the PD bifurcation can be parametrized locally by (τ, ξ) as

$$u = u_0(\tau) + \xi w(\tau) + H(\tau, \xi), \quad \tau \in [0, 2T], \ \xi \in \mathbb{R}, \tag{4.1}$$

where the function H satisfies $H(2T,\xi) = H(0,\xi)$. It has the Taylor expansion

$$H(\tau,\xi) = \frac{1}{2}h_2(\tau)\xi^2 + \frac{1}{6}h_3(\tau)\xi^3 + O(\xi^4), \tag{4.2}$$

with $h_j(2T) = h_j(0)$, while

$$w(\tau) = \begin{cases} v(\tau), & \tau \in [0,T], \\ -v(\tau - T), & \tau \in [T,2T], \end{cases} \tag{4.3}$$

with

$$\begin{cases} \dot{v}(\tau) - A(\tau)v(\tau) &= 0,\ \tau \in [0,T], \\ v(T) + v(0) &= 0, \\ \int_0^T \langle v(\tau), v(\tau)\rangle d\tau - 1 &= 0. \end{cases} \tag{4.4}$$

The function v exists due to Lemma 5 of [20].

The parametrization (4.1) provides a two-cover of $W^c(\Gamma)$ that is locally diffeomorphic to the Möbius band (see Fig. 4.1).

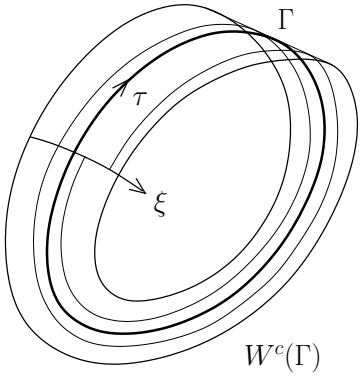


Fig. 4.1. *Center manifold $W^c(\Gamma)$ at the period-doubling bifurcation.*

The functions $h_2(\tau)$ and $h_3(\tau)$ can be found by solving appropriate BVPs, assuming that (2.1) restricted to $W^c(\Gamma)$ has the periodic normal form (2.7). The coefficients a and c arise from the solvability conditions for the BVPs as integrals of scalar products over the interval $[0,T]$. Specifically, these scalar products involve the quadratic and cubic terms of (1.1) near the periodic solution u_0, the eigenfunction v, and a similar adjoint eigenfunction v^* satisfying

$$\begin{cases} \dot{v}^*(\tau) + A^{\mathrm{T}}(\tau)v^*(\tau) &= 0,\ \tau \in [0,T], \\ v^*(T) + v^*(0) &= 0, \\ \int_0^T \langle v^*(\tau), v(\tau)\rangle d\tau - 1/2 &= 0. \end{cases} \tag{4.5}$$

Similarly to (4.3), define

$$w^*(\tau) = \begin{cases} v^*(\tau), & \tau \in [0,T], \\ -v^*(\tau - T), & \tau \in [T,2T], \end{cases} \tag{4.6}$$

Note that

$$\int_0^{2T} \langle w^*(\tau), w(\tau)\rangle d\tau = 1. \tag{4.7}$$

To derive the normal form coefficients, we proceed as in Section 3, namely, we substitute (4.1) into (2.1), and use (2.2), as well as (2.7) and (4.2). This gives

$$\begin{aligned}\dot{u}_0 + \xi\dot{w} + \xi^2\left(\frac{1}{2}\dot{h}_2 + a\dot{u}_0\right) + \xi^3\left(\frac{1}{6}\dot{h}_3 + a\dot{w} + cw\right) + O(\xi^4)\\ = F(u_0) + \xi A(\tau)w + \frac{1}{2}\xi^2(A(\tau)h_2 + B(\tau; w, w))\\ + \frac{1}{6}\xi^3(A(\tau)h_3 + 3B(\tau; w, h_2) + C(\tau; w, w, w)) + O(\xi^4),\end{aligned}$$

where dots denote the derivatives with respect to τ, for $\tau \in [0, 2T]$.

Collecting the ξ^0-terms we get the identity

$$\dot{u}_0 = F(u_0),$$

since u_0 is the T-periodic solution of (2.1).

The ξ^1-terms provide the identity

$$\dot{w} = A(\tau)w\ ,$$

due to (4.3) and (4.4).

Collecting the ξ^2-terms, we obtain the equation for h_2

$$\dot{h}_2 - A(\tau)h_2 = B(\tau; w, w) - 2a\dot{u}_0, \tag{4.8}$$

to be solved in the space of functions on $[0, 2T]$ satisfying $h_2(2T) = h_2(0)$. In this space, the differential operator $\frac{d}{d\tau} - A(\tau)$ is singular with two linearly-independent null-functions: $\psi = \dot{u}_0$ and w.

Thus, two Fredholm solvability conditions are involved, namely,

$$\int_0^{2T} \langle w^*(\tau), B(\tau; w(\tau), w(\tau)) - 2a\dot{u}_0(\tau)\rangle\ d\tau = 0,$$

which holds automatically for any a, due to the T-periodicity of the right-hand side of (4.8), and

$$\int_0^{2T} \langle \psi^*(\tau), B(\tau; w(\tau), w(\tau)) - 2a\dot{u}_0(\tau)\rangle\ d\tau = 0,$$

where ψ^* satisfies

$$\left\{\begin{array}{rcl}\dot{\psi}^*(\tau) + A^{\mathrm{T}}(\tau)\psi^*(\tau) &=& 0,\ \tau \in [0, T],\\ \psi^*(T) - \psi^*(0) &=& 0,\\ \int_0^T \langle \psi^*(\tau), F(u_0(\tau))\rangle\ d\tau - 1/2 &=& 0,\end{array}\right. \tag{4.9}$$

and is extended to $[T, 2T]$ by periodicity. Note that $\int_0^{2T} \langle \psi^*(\tau), F(u_0(\tau))\rangle\ d\tau \neq 0$, since 0 is a semi-simple eigenvalue of the differential operator $\frac{d}{d\tau} - A(\tau)$. This leads to the expression

$$a = \frac{1}{2}\int_0^{2T} \langle \psi^*(\tau), B(\tau; w(\tau), w(\tau)\rangle\ d\tau,$$

or, equivalently,

$$a = \int_0^T \langle \psi^*(\tau), B(\tau; v(\tau), v(\tau)) \rangle \, d\tau, \tag{4.10}$$

where v and ψ^* are defined by (4.4) and (4.9), respectively.

With a defined in this way, let h_2 be the unique solution of (4.8) in the space of functions on $[0, 2T]$ satisfying $h_2(0) = h_2(2T)$, as well as two orthogonality conditions:

$$\int_0^{2T} \langle w^*(\tau), h_2(\tau) \rangle \, d\tau = 0,$$

which holds automatically, due to the T-periodicity of h_2 ($h_2(0) = h_2(T)$), and

$$\int_0^{2T} \langle \psi^*(\tau), h_2(\tau) \rangle \, d\tau = 0,$$

which is equivalent to

$$\int_0^{T} \langle \psi^*(\tau), h_2(\tau) \rangle \, d\tau = 0.$$

Thus h_2 is the unique solution of the BVP

$$\left\{ \begin{array}{rcl} \dot{h}_2(\tau) - A(\tau)h_2(\tau) - B(\tau; v(\tau), v(\tau)) + 2aF(u_0(\tau)) & = & 0, \ \tau \in [0, T], \\ h_2(T) - h_2(0) & = & 0, \\ \int_0^T \langle \psi^*(\tau), h_2(\tau) \rangle \, d\tau & = & 0, \end{array} \right. \tag{4.11}$$

and extended by periodicity to $[T, 2T]$. Collecting the ξ^3-terms, we get the equation for h_3

$$\dot{h}_3 - A(\tau)h_3 = C(\tau; w, w, w) + 3B(\tau; w, h_2) - 6a\dot{w} - 6cw, \tag{4.12}$$

that again must be solved in the space of functions on $[0, 2T]$ satisfying $h_3(2T) = h_3(0)$. Its solvability implies

$$\int_0^{2T} \langle w^*(\tau), C(\tau; w(\tau), w(\tau), w(\tau)) + 3B(\tau; w(\tau), h_2(\tau)) - 6a\dot{w}(\tau) - 6cw(\tau) \rangle \, d\tau = 0.$$

Taking into account (4.7), we obtain

$$c = \frac{1}{6} \int_0^{2T} \langle w^*(\tau), C(\tau; w(\tau), w(\tau), w(\tau)) + 3B(\tau; w(\tau), h_2(\tau)) - 6aA(\tau)w(\tau) \rangle \, d\tau,$$

and finally

$$c = \frac{1}{3} \int_0^{T} \langle v^*(\tau), C(\tau; v(\tau), v(\tau), v(\tau)) + 3B(\tau; v(\tau), h_2(\tau)) - 6aA(\tau)v(\tau) \rangle \, d\tau, \tag{4.13}$$

where a is defined by (4.10), h_2 is the solution of (4.11), and v and v^* are defined by (4.4) and (4.5), respectively. Thus, the critical coefficient c in the periodic normal form for the PD bifurcation has been computed. The critical cycle is stable within the center manifold if $c < 0$ and is unstable if $c > 0$. In the former case, the bifurcation is supercritical, while in the latter case it is subcritical.

5. The torus bifurcation. The three-dimensional critical center manifold $W^c(\Gamma)$ at the NS bifurcation can be parametrized locally by (τ,ξ) as

$$u = u_0(\tau) + \xi v(\tau) + \bar{\xi}\bar{v}(\tau) + H(\tau,\xi,\bar{\xi}), \quad \tau \in [0,T],\ \xi \in \mathbb{C}, \tag{5.1}$$

where the real function H satisfies $H(T,\xi,\bar{\xi}) = H(0,\xi,\bar{\xi})$, and has the Taylor expansion

$$\begin{aligned} H(\tau,\xi,\bar{\xi}) = {} & \frac{1}{2}h_{20}(\tau)\xi^2 + h_{11}(\tau)\xi\bar{\xi} + \frac{1}{2}h_{02}(\tau)\bar{\xi}^2 \\ & + \frac{1}{6}h_{30}(\tau)\xi^3 + \frac{1}{2}h_{21}(\tau)\xi^2\bar{\xi} + \frac{1}{2}h_{12}(\tau)\xi\bar{\xi}^2 + \frac{1}{6}h_{03}(\tau)\bar{\xi}^3 \\ & + O(|\xi|^4), \end{aligned} \tag{5.2}$$

with $h_{ij}(T) = h_{ij}(0)$ and $h_{ij} = \bar{h}_{ji}$ so that h_{11} is real, while

$$\left\{ \begin{array}{rcl} \dot{v}(\tau) - A(\tau)v(\tau) + \dfrac{i\theta}{T}v(\tau) &=& 0,\ \tau \in [0,T], \\ v(T) - v(0) &=& 0, \\ \int_0^T \langle v(\tau), v(\tau)\rangle d\tau - 1 &=& 0. \end{array} \right. \tag{5.3}$$

The function v exists due to LEMMA 2 of [20]. Recall that $\langle u, v\rangle = u^{\mathrm{H}}v = \bar{u}^{\mathrm{T}}v$.

Note that

$$w(\tau) = \exp\left(\frac{i\theta\tau}{T}\right) v(\tau)$$

satisfies

$$\left\{ \begin{array}{rcl} \dot{w}(\tau) - A(\tau)w(\tau) &=& 0,\ \tau \in [0,T], \\ w(T) - e^{i\theta}w(0) &=& 0, \\ \int_0^T \langle w(\tau), w(\tau)\rangle d\tau - 1 &=& 0, \end{array} \right. \tag{5.4}$$

which is often used in the defining system for the NS bifurcation.

As in the previous cases, the functions $h_{ij}(\tau)$ can be found by solving appropriate BVPs, assuming that (2.1) restricted to $W^c(\Gamma)$ has the periodic normal form (2.8).

Also introduce the adjoint eigenfunction v^* that satisfies

$$\left\{ \begin{array}{rcl} \dot{v}^*(\tau) + A^{\mathrm{T}}(\tau)v^*(\tau) - \dfrac{i\theta}{T}v^*(\tau) &=& 0,\ \tau \in [0,T], \\ v^*(T) - v^*(0) &=& 0, \\ \int_0^T \langle v^*(\tau), v(\tau)\rangle d\tau - 1 &=& 0. \end{array} \right. \tag{5.5}$$

Substitute (5.1) into (2.1), using (2.2), (2.8), and (5.2), as well as

$$\frac{du}{dt} = \frac{\partial u}{\partial \xi}\frac{d\xi}{dt} + \frac{\partial u}{\partial \bar{\xi}}\frac{d\bar{\xi}}{dt} + \frac{\partial u}{\partial \tau}\frac{d\tau}{dt}.$$

This gives

$$\begin{aligned}
&\dot{u}_0 + \xi\left(\dot{v} + \frac{i\theta}{T}v\right) + \bar{\xi}\left(\dot{\bar{v}} - \frac{i\theta}{T}\bar{v}\right)\\
&\quad + \frac{1}{2}\left(\dot{h}_{20} + \frac{2i\theta}{T}h_{20}\right)\xi^2 + (a\dot{u}_0 + \dot{h}_{11})|\xi|^2 + \frac{1}{2}\left(\dot{h}_{02} - \frac{2i\theta}{T}h_{02}\right)\bar{\xi}^2\\
&\quad + \frac{1}{2}\left(\dot{h}_{21} + \frac{i\theta}{T}h_{21} + 2a\dot{v} + 2dv\right)\xi^2\bar{\xi} \;+\; \cdots\\
&\quad = F(u_0) + \xi A(\tau)v + \bar{\xi}A\bar{v}(\tau)\\
&\quad + \;\; \frac{1}{2}\xi^2\Big(\, Ah_{20} + B(\tau; v, v)\Big) + |\xi|^2\Big(\, Ah_{11} + B(\tau; v, \bar{v})\Big) + \frac{1}{2}\bar{\xi}^2\Big(\, A\bar{h}_{02} + B(\tau; \bar{v}, \bar{v})\Big)\\
&\quad + \;\; \frac{1}{2}\Big(\, 2B(\tau; h_{11}, v) + B(\tau; h_{20}, \bar{v}) + C(\tau; v, v, \bar{v})\Big)\xi^2\bar{\xi} \;+\; \cdots,
\end{aligned}$$

where among cubic terms only the $\xi^2\bar{\xi}$-terms are displayed. All non-displayed cubic terms and higher-order terms are irrelevant for the computation of d. Collecting coefficients, we get the following equations:

The ξ-independent terms give the usual identity

$$\dot{u}_0 = F(u_0).$$

The ξ-terms give another identity, namely,

$$\dot{v} - A(\tau)v + \frac{i\theta}{T}v = 0,$$

while the $\bar{\xi}$-terms lead to the corresponding complex-conjugate identity.

Collecting the coefficients of the ξ^2- or $\bar{\xi}^2$-terms leads to the equation

$$\dot{h}_{20} - A(\tau)h_{20} + \frac{2i\theta}{T}h_{20} = B(\tau; v, v), \tag{5.6}$$

or its complex-conjugate. This equation has a unique solution $h_{20}(\tau)$ satisfying $h_{20}(T) = h_{20}(0)$, since $e^{2i\theta}$ is not a multiplier of the critical cycle by the spectral assumptions. Thus, h_{20} can be found from the BVP

$$\left\{\begin{array}{rcl}
\dot{h}_{20}(\tau) - A(\tau)h_{20}(\tau) + \dfrac{2i\theta}{T}h_{20}(\tau) - B(\tau; v(\tau), v(\tau)) &=& 0,\ \tau \in [0, T],\\
h_{20}(T) - h_{20}(0) &=& 0.
\end{array}\right. \tag{5.7}$$

The $|\xi|^2$-terms give

$$\dot{h}_{11} - A(\tau)h_{11} = B(\tau; v, \bar{v}) - a\dot{u}_0, \tag{5.8}$$

where $h_{11}(T) = h_{11}(0)$. The differential operator $\frac{d}{d\tau} - A(\tau)$ has a nontrivial kernel spanned by $\dot{u}_0$. The null-eigenfunction of the adjoint operator $-\frac{d}{d\tau} - A^{\mathrm{T}}(\tau)$ is φ^* given by the equation

$$\left\{\begin{array}{rcl}
\dot{\varphi}^*(\tau) + A^{\mathrm{T}}(\tau)\varphi^*(\tau) &=& 0,\ \tau \in [0, T],\\
\varphi^*(T) - \varphi^*(0) &=& 0,\\
\int_0^T \langle \varphi^*(\tau), F(u_0(\tau))\rangle d\tau - 1 &=& 0.
\end{array}\right. \tag{5.9}$$

Note that $\int_0^T \langle \varphi^*(\tau), F(u_0(\tau))\rangle d\tau \neq 0$, since the trivial multiplier 1 is simple, due to the spectral assumptions. The Fredholm solvability condition implies

$$a = \int_0^T \langle \varphi^*(\tau), B(\tau; v(\tau), \bar{v}(\tau)\rangle \; d\tau. \tag{5.10}$$

With a defined in this way, let h_{11} be the unique solution of (5.8) satisfying $h_{11}(T) = h_{11}(0)$ and

$$\int_0^T \langle \varphi^*(\tau), h_{11}(\tau)\rangle \; d\tau = 0,$$

i.e.,

$$\left\{ \begin{array}{rcl} \dot{h}_{11}(\tau) - A(\tau)h_{11}(\tau) - B(\tau; v(\tau), \bar{v}(\tau)) + aF(u_0(\tau)) &=& 0, \; \tau \in [0,T], \\ h_{11}(T) - h_{11}(0) &=& 0, \\ \int_0^T \langle \varphi^*(\tau), h_{11}(\tau)\rangle \; d\tau &=& 0. \end{array} \right. \tag{5.11}$$

Finally, the coefficients of the $\xi^2\bar{\xi}$-terms give the singular equation

$$\dot{h}_{21} - Ah_{21} + \frac{i\theta}{T}h_{21} = 2B(\tau; h_{11}, v) + B(\tau; h_{20}, \bar{v}) + C(\tau; v, v, \bar{v}) - 2a\dot{v} - 2dv.$$

If one takes into account (5.5), the Fredholm solvability condition implies

$$\begin{aligned} d = \frac{1}{2}\int_0^T \langle v^*(\tau), B(\tau; h_{11}(\tau), v(\tau)) + B(\tau; h_{20}(\tau), \bar{v}(\tau)) + C(\tau; v(\tau), v(\tau), \bar{v}(\tau))\rangle \; d\tau \\ - \; a\int_0^T \langle v^*(\tau), A(\tau)v(\tau)\rangle \; d\tau + \frac{ia\theta}{T}, \end{aligned} \tag{5.12}$$

where a is defined by (5.10), h_{11} and h_{20} by (5.11) and (5.7), respectively, and v and v^* satisfy (5.3) and (5.5), respectively. Thus, the critical coefficient d in the periodic normal form for the NS bifurcation has been computed. The critical cycle is stable within the center manifold if Re $d < 0$ and is unstable if Re $d > 0$. In the former case, the Neimark-Sacker bifurcation is supercritical, while in the latter case it is subcritical.

6. Implementation issues. Numerical implementation of the formulas derived in the preceding sections requires the evaluation of integrals of scalar functions over $[0,T]$, and the solution of nonsingular linear BVPs with integral constraints. Such tasks can be carried out with continuation software such as AUTO [8], CONTENT [25], and MATCONT [6]. In these software packages, periodic solutions to (1.1) are computed with the method of *orthogonal collocation* with piecewise polynomials applied to properly formulated BVPs. The standard BVP for the periodic solutions is formulated on the unit interval $[0,1]$, so that the period T becomes a parameter, and it involves an integral phase condition:

$$\left\{ \begin{array}{rcl} \dot{x}(\tau) - Tf(x(\tau), \alpha) &=& 0, \; \tau \in [0,1], \\ x(0) - x(1) &=& 0, \\ \int_0^1 \langle x(\tau), \dot{\xi}(\tau)\rangle \; d\tau &=& 0, \end{array} \right. \tag{6.1}$$

where ξ is a previously calculated periodic solution, rescaled to $[0,1]$.

In the orthogonal collocation method [2], the problem (6.1) is discretized as follows. First the interval $[0,1]$ is subdivided into N smaller intervals:

$$0 = \tau_0 < \tau_1 < \cdots < \tau_N = 1.$$

In each of these intervals the solution $x(\tau)$ is approximated by a degree m vector-valued polynomial $x^{(i)}(\tau)$, by introducing $m+1$ equidistant mesh points on each interval, namely,

$$\tau_{i,j} = \tau_i + \frac{j}{m}(\tau_{i+1} - \tau_i) \ (j = 0, 1, \ldots, m),$$

and defining the polynomials $x^{(i)}(\tau)$ as

$$x^{(i)}(\tau) = \sum_{j=0}^{m} x_{i,j}\ell_{i,j}(\tau).$$

Here $x_{i,j}$ is the discretization of $x(\tau)$ at $\tau = \tau_{i,j}$. (We note that $x_{i,m} = x_{i+1,0}$.) The $\ell_{i,j}(\tau)$'s are the Lagrange basis polynomials

$$\ell_{i,j}(\tau) = \prod_{k=0,k\neq j}^{m} \frac{\tau - \tau_{i,k}}{\tau_{i,j} - \tau_{i,k}}.$$

In each interval $[\tau_i, \tau_{i+1}]$ we require the polynomials $x^{(i)}(\tau)$ to satisfy the BVP exactly at m collocation points $\zeta_{i,j}$ $(j = 1, \ldots, m)$. It is well-known that the best choice for collocation points is Gauss points [5], *i.e.*, the roots of the Legendre polynomial of degree m, relative to the interval $[\tau_i, \tau_{i+1}]$.

Now let $y(\tau)$ be a function defined in $[0,1]$, and assume that we want to integrate it over $[0,1]$. The total number of mesh points is $Nm+1$. Each mesh point $\tau_{i,j}$ in a mesh interval $[\tau_i, \tau_{i+1}]$ has a particular weight w_{j+1}, the Lagrange quadrature coefficient. Some mesh points belong to two mesh intervals. We set $t_i = \tau_i - \tau_{i-1}$, $(i = 1, \ldots, N)$. The integration weight $\sigma_{i,j}$ of $\tau_{i,j}$ is given by $w_{j+1}t_{i+1}$, for $0 \leq i \leq N-1$ and $0 < j < m$. For $i = 0, \ldots, N-2$, the integration weight of $\tau_{i,m}$ $(\tau_{i,m} = \tau_{i+1,0})$ is given by $\sigma_{i,m} = w_{m+1}t_{i+1} + w_1 t_{i+2}$, and the integration weights of τ_0 and τ_N are given by $w_1 t_1$ and $w_{m+1}t_N$, respectively. Thus, the integral $\int_0^1 y(\tau)d\tau$ can be approximated by $\sum_{i=0}^{N-1}\sum_{j=0}^{m-1} y(\tau_{i,j})\sigma_{i,j} + y(1)\sigma_{N,0}$.

Using the above procedure, we obtain the discretized version of the BVP (6.1), namely,

$$\left\{\begin{array}{rcl} \sum_{j=0}^{m} x_{i,j}\dot{\ell}_{i,j}(\zeta_{i,k}) - Tf(\sum_{j=0}^{m} x_{i,j}\ell_{i,j}(\zeta_{i,k}), \alpha) & = & 0, \\ x_{0,0} - x_{N-1,m} & = & 0, \\ \sum_{i=0}^{N-1}\sum_{j=0}^{m-1} \sigma_{i,j}\langle x_{i,j}, \dot{\xi}_{i,j}\rangle + \sigma_{N,0}\langle x_{N,0}, \dot{\xi}_{N,0}\rangle & = & 0. \end{array}\right. \tag{6.2}$$

The first equation actually represents Nm equations, one for each combination of $i = 0, 1, 2, ..., N-1$ and $k = 1, 2, ..., m$.

The numerical continuation of the solutions of the discretized BVP (6.2) leads to structured, sparse linear systems, which in AUTO [8] and CONTENT [25] are solved by an efficient, specially adapted elimination algorithm, that computes the multipliers as a by-product, without explicitly using the Poincaré map. To detect codim 1 bifurcations, one can specify test functions that are based on computing multipliers [11, 8] or on solving appropriate bordered linear BVPs [9].

Once a codim 1 bifurcation has been detected, one can compute the normal form coefficients using the formulas derived in the previous sections. All BVPs are reformulated on the unit interval $[0,1]$, and all integrals are scaled accordingly. Moreover, if the bordering methods from [9] are used to continue LPC, PD, and NS bifurcations of limit cycles, then the computation of the normal form coefficients requires little extra effort, since all necessary eigenfunctions have already been computed, either while evaluating the test functionals, or their gradients. These coefficients then serve as test functions for detecting codim 2 singularities of limit cycles due to *nonlinear degeneracies* of LPC, PD or NS bifurcations, *i.e.*, the cusp (CPC), the degenerate period-doubling (DP), and the Chenciner (CH) bifurcation. By-products of these computations are test functions for detecting certain codim 2 singularities of limit cycles due to *linear degeneracies*, namely, the strong 1:1 resonance (R1), the strong 1:2 resonance (R2), the fold-Neimark-Sacker bifurcation (FN), and the fold-flip bifurcation (FF).

6.1. Discretization symbols. All computed functions have to be discretized using the *same mesh* as in (6.2). For a given vector function $\eta \in \mathcal{C}^1([0,1],\mathbb{R}^n)$ we consider three different discretizations:

- $\eta_M \in \mathbb{R}^{(Nm+1)n}$ the vector of the function values at the mesh points;
- $\eta_C \in \mathbb{R}^{Nmn}$ the vector of the function values at the collocation points;
- $\eta_W = \begin{bmatrix} \eta_{W_1} \\ \eta_{W_2} \end{bmatrix} \in \mathbb{R}^{Nmn} \times \mathbb{R}^n$ where η_{W_1} is the vector of the function values at the collocation points multiplied by the Gauss-Legendre weights and the lengths of the corresponding mesh intervals, and $\eta_{W_2} = \eta(0)$.

Formally we also introduce the structured sparse matrix $L_{C\times M}$ that converts a vector η_M of function values at the mesh points into a vector η_C of its values at the collocation points, namely, $\eta_C = L_{C\times M}\eta_M$. This matrix is never formed explicitly; its entries are approximated by the $\ell_{i,j}(\zeta_{i,k})$-coefficients in (6.2). We also need a matrix $A_{C\times M}$ such that $A_{C\times M}\eta_M = (A(t)\eta(t))_C$. Again this matrix need not be formed explicitly. On the other hand, we do need the matrix $(D - TA(t))_{C\times M}$ explicitly; it is defined by $(D-TA(t))_{C\times M}\eta_M = (\dot\eta(t) - TA(t)\eta(t))_C$. Finally, let the tensors $B_{C\times M\times M}$ and $C_{C\times M\times M\times M}$ be defined by $B_{C\times M\times M}\eta_{1M}\eta_{2M} = (B(t;\eta_1(t),\eta_2(t)))_C$ and

$$C_{C\times M\times M\times M}\eta_{1M}\eta_{2M}\eta_{3M} = (C(t;\eta_1(t),\eta_2(t),\eta_3(t)))_C,$$

for all $\eta_i \in \mathcal{C}^1([0,1],\mathbb{R}^n)$. (These tensors are not formed explicitly.)

Let $f(t), g(t) \in \mathcal{C}^0([0,1],\mathbb{R})$ be two scalar functions. Then the integral $\int_0^1 f(t)dt$ is represented by $\sum_{i=0}^{N-1}\sum_{j=1}^{m}\omega_j(f_C)_{i,j}t_{i+1} = \sum_{i=0}^{N-1}\sum_{j=1}^{m}(f_{W_1})_{i,j}$, where $(f_C)_{i,j} = f(\zeta_{i,j})$ and ω_j is the Gauss-Legendre quadrature coefficient. The integral $\int_0^1 f(t)g(t)dt$ is approximated with Gauss-Legendre by $f_{W_1}^{\mathrm{T}} g_C = f_{W_1}^{\mathrm{T}} L_{C\times M} g_M$. For vector functions $f(t), g(t) \in \mathcal{C}^0([0,1],\mathbb{R}^n)$, the integral $\int_0^1 \langle f(t), g(t)\rangle\, dt$ is formally approximated by the same expression: $f_{W_1}^{\mathrm{T}} g_C = f_{W_1}^{\mathrm{T}} L_{C\times M} g_M$.

We now consider the LPC, PD and NS cases separately.

6.2. LPC bifurcation. The first task is to rescale the computed functions to the interval $[0,1]$. We start by defining $u_1(t) = u_0(Tt)$ for $t \in [0,1]$. The linear BVPs (3.3) and (3.6) are replaced by

$$\left\{\begin{array}{rcl} \dot v_1(t) - TA(t)v_1(t) - TF(u_1(t)) &=& 0,\ t\in[0,1], \\ v_1(0) - v_1(1) &=& 0, \\ \int_0^1 \langle v_1(t), F(u_1(t))\rangle dt &=& 0, \end{array}\right. \tag{6.3}$$

where $v(\tau) = v_1(\tau/T)$, and

$$\left\{ \begin{array}{rcl} \dot{\varphi}_1^*(t) + TA^{\mathrm{T}}(t)\varphi_1^*(t) & = & 0,\ t \in [0,1], \\ \varphi_1^*(0) - \varphi_1^*(1) & = & 0, \\ \int_0^1 \langle \varphi_1^*(t), \varphi_1^*(t)\rangle dt - 1 & = & 0, \end{array} \right. \tag{6.4}$$

respectively. We then compute $I = \int_0^1 \langle \varphi_1^*(t), v_1(t)\rangle dt$. If $I = 0$ then we have a strong 1:1 resonance (a limit cycle with two nontrivial multipliers equal to 1). If not, then we rescale φ^* so that $I = 1$. It then follows that $\varphi^*(\tau) = \varphi_1^*(\tau/T)/T$.

As an intermediate result, we obtain

$$b = \frac{1}{2}\int_0^1 \langle \varphi_1^*(t), B(t; v_1(t), v_1(t)) + 2A(t)v_1(t)\rangle\ dt. \tag{6.5}$$

We compute v_{1M} by solving the discretization of (6.3)

$$\left[\begin{array}{c} (D - TA(t))_{C\times M} \\ \delta_0 - \delta_1 \\ (g_{W_1})^{\mathrm{T}} L_{C\times M} \end{array} \right] v_{1M} = \left[\begin{array}{c} Tg_C \\ 0 \\ 0 \end{array} \right], \tag{6.6}$$

where $g(t) = F(u_1(t))$.

It is more efficient to compute φ_{1W}^* than φ_{1M}^*, since φ_1^* will be used only to compute integrals of the form $\int_0^1 \langle \varphi_1^*(t), \zeta(t)\rangle dt$. Moreover, φ_{1W}^* can be computed with the same matrix used in (6.6), thus saving factorization costs. Formally, the computation of φ_{1W}^* is based on Proposition A.1 from the Appendix. Instead of approximating φ_1^* by solving

$$\left[\begin{array}{c} (D + TA^{\mathrm{T}}(t))_{C\times M} \\ \delta_0 - \delta_1 \end{array} \right] \varphi_{1M}^* = 0,$$

we remark that $\left[\begin{array}{c} \varphi_1^* \\ \varphi_1^*(0) \end{array} \right]$ is orthogonal to the range of $\left[\begin{array}{c} D - TA(t) \\ \delta_0 - \delta_1 \end{array} \right]$. By discretization we obtain

$$(\varphi_{1W}^*)^{\mathrm{T}} \left[\begin{array}{c} (D - TA(t))_{C\times M} \\ \delta_0 - \delta_1 \end{array} \right] = 0.$$

To normalize $\varphi_{1W_1}^*$, we require

$$\sum_{i=0}^{N-1} \sum_{j=1}^{m} \left|(\varphi_{1W_1}^*)_{i,j}\right|_1 = 1. \tag{6.7}$$

Here $|.|_1$ denotes the 1-norm (sum of absolute values) of a vector. This choice if convenient for computational reasons. Then $\int_0^1 \langle \varphi_1^*(t), v_1(t)\rangle dt$ is approximated by $(\varphi_{1W_1}^*)^{\mathrm{T}} L_{C\times M} v_{1M}$ and if this quantity is nonzero, φ_{1W}^* is rescaled to ensure $\int_0^1 \langle \varphi_1^*(t), v_1(t)\rangle dt = 1$.

The expression (6.5) for the normal form coefficient b finally reduces to

$$b = \frac{1}{2}(\varphi_{1W_1}^*)^{\mathrm{T}}(B_{C\times M\times M} v_{1M} v_{1M} + 2A_{C\times M} v_{1M}). \tag{6.8}$$

6.3. PD bifurcation. Again, we rescale the computed quantities to the interval $[0,1]$. The linear BVPs (4.4) and (4.5) are replaced by

$$\left\{\begin{array}{rcl} \dot{v_1}(t) - TA(t)v_1(t) &=& 0,\ t\in[0,1],\\ v_1(0)+v_1(1) &=& 0,\\ \int_0^1 \langle v_1(t), v_1(t)\rangle dt - 1 &=& 0, \end{array}\right. \tag{6.9}$$

where $v(\tau) = v_1(\tau/T)/\sqrt{T}$, and

$$\left\{\begin{array}{rcl} \dot{v_1}^*(t) + TA^{\mathrm{T}}(t)v_1^*(t) &=& 0,\ t\in[0,1],\\ v_1^*(0)+v_1^*(1) &=& 0,\\ \int_0^1 \langle v_1^*(t), v_1^*(t)\rangle dt - 1/2 &=& 0, \end{array}\right. \tag{6.10}$$

respectively. We note that the last equation in (6.10) differs from the last equation in (4.5). We then compute $I = \int_0^1 \langle v_1^*(t), v_1(t)\rangle dt$. If $I = 0$ then we have a strong 1:2 resonance (a limit cycle with two multipliers equal to -1). If not, then we rescale v_1^* so that $I = 1/2$, which corresponds the normalization condition used in (6.10). It then follows that $v^*(\tau) = v_1^*(\tau/T)/\sqrt{T}$.

We also replace (4.9) by

$$\left\{\begin{array}{rcl} \dot{\psi_1^*}(t) + TA^{\mathrm{T}}(t)\psi_1^*(t) &=& 0,\ t\in[0,1],\\ \psi_1^*(0) - \psi_1^*(1) &=& 0,\\ \int_0^1 \langle \psi_1^*(t), \psi_1^*(t)\rangle\ dt - 1 &=& 0. \end{array}\right. \tag{6.11}$$

Again, the last equation in (6.11) differs from the last equation in (4.9). We then compute $I = \int_0^1 \langle \psi_1^*(t), F(u_1(t))\rangle dt$. If $I = 0$ then we have a fold-flip bifurcation. If not, then we rescale ψ_1^* so that $I = 1$. It then follows that $\psi^*(\tau) = \psi_1^*(\tau/T)/T$.

This leads to the expression

$$a_1 = \int_0^1 \langle \psi_1^*(t), B(t; v_1(t), v_1(t)\rangle\ dt\ , \tag{6.12}$$

where $a_1 = aT$.

With a_1 defined this way, let $h_{2,1}$ be the unique solution of the BVP

$$\left\{\begin{array}{rcl} \dot{h}_{2,1}(t) - TA(t)h_{2,1}(t) - B(t; v_1(t), v_1(t)) + 2a_1F(u_1(t)) &=& 0,\ t\in[0,1],\\ h_{2,1}(0) - h_{2,1}(1) &=& 0,\\ \int_0^1 \langle \psi_1^*(t), h_{2,1}(t)\rangle\ dt &=& 0, \end{array}\right. \tag{6.13}$$

where $h_2(\tau) = h_{2,1}(\tau/T)$.

Therefore we obtain

$$\begin{array}{rl} c = \dfrac{1}{3}\displaystyle\int_0^1 \langle v_1^*(t), \frac{1}{T}C(t; v_1(t), v_1(t), v_1(t)) + 3B(t; v_1(t), h_{2,1}(t))\rangle\ dt & \\ \qquad -\dfrac{2a_1}{T}\displaystyle\int_0^1 \langle v_1^*(t), A(t)v_1(t)\rangle\ dt. & \end{array} \tag{6.14}$$

We compute v_{1M} by solving

$$\left[\begin{array}{c} (D - TA(t))_{C\times M} \\ \delta_0 + \delta_1 \end{array}\right] v_{1M} = 0. \tag{6.15}$$

We normalize v_{1M} by requiring $\sum_{i=0}^{N-1}\sum_{j=0}^{m}\sigma_j\langle(v_{1M})_{i,j},(v_{1M})_{i,j}\rangle = 1$, where σ_j is the Lagrange quadrature coefficient.

As in the LPC-case, it is more efficient to compute v_{1W}^*, rather than v_M^*, since v_1^* will be used only to compute integrals of the form $\int_0^1\langle v_1^*(t),\zeta(t)\rangle dt$. Moreover, v_{1W}^* can be computed with the same matrix in (6.15), thus saving factorization costs. Formally, the computation of v_{1W}^* is based on Proposition A.2 (see the Appendix). Instead of approximating v_1^* by solving

$$\left[\begin{array}{c}(D+TA^{\mathrm{T}}(t))_{C\times M}\\ \delta_0+\delta_1\end{array}\right]v_{1M}^*=0,$$

we observe that $\left[\begin{array}{c}v_1^*\\ v_1^*(0)\end{array}\right]$ is orthogonal to the range of $\left[\begin{array}{c}D-TA(t)\\ \delta_0+\delta_1\end{array}\right]$. By discretization we obtain

$$(v_{1W}^*)^{\mathrm{T}}\left[\begin{array}{c}(D-TA(t))_{C\times M}\\ \delta_0+\delta_1\end{array}\right]=0.$$

To normalize $v_{1W_1}^*$, we require $\sum_{i=0}^{N-1}\sum_{j=1}^{m}\left|(v_{1W_1}^*)_{i,j}\right|_1=1$. Then $\int_0^1\langle v_1^*(t),v_1(t)\rangle dt$ is approximated by $(v_{1W_1}^*)^{\mathrm{T}}L_{C\times M}v_{1M}$. If this quantity is nonzero then v_{1W}^* is rescaled so that $\int_0^1\langle v_1^*(t),v_1(t)\rangle dt=1/2$.

From Proposition A.1 it follows that we can approximate ψ_1^* like v_1^*. Namely, we compute ψ_{1W}^* by solving

$$(\psi_{1W}^*)^{\mathrm{T}}\left[\begin{array}{c}(D-TA(t))_{C\times M}\\ \delta_0-\delta_1\end{array}\right]=0$$

and normalize $\psi_{1W_1}^*$ by requiring

$$\sum_{i=0}^{N-1}\sum_{j=1}^{m}\left|(\psi_{1W_1}^*)_{i,j}\right|_1=1.$$

Then $\int_0^1\langle\psi_1^*(t),F(u_1(t))\rangle dt$ is approximated by $(\psi_{1W_1}^*)^{\mathrm{T}}(F(u_1(t)))_C$ and if this quantity is nonzero, ψ_{1W}^* is rescaled so that $\int_0^1\langle\psi_1^*(t),F(u_1(t))\rangle dt=1$.

Having found v_{1M} and ψ_{1W}^*, a_1 can be computed using (6.12) as

$$a_1=(\psi_{1W_1}^*)^{\mathrm{T}}B_{C\times M\times M}v_{1M}v_{1M}.$$

Next, $(h_{2,1})_M$ is found by solving the discretization of (6.13), namely,

$$\left[\begin{array}{c}(D-TA(t))_{C\times M}\\ \delta_0-\delta_1\\ (\psi_{W_1}^*)^{\mathrm{T}}L_{C\times M}\end{array}\right](h_{2,1})_M=\left[\begin{array}{c}B_{C\times M\times M}v_{1M}v_{1M}+2a_1g_C\\ 0\\ 0\end{array}\right],$$

where $g_C=(F(u_1(t)))_C$.

Finally, the expression (6.14) for the normal form coefficient c becomes

$$\begin{aligned}c=\frac{1}{3T}(v_{1W_1}^*)^{\mathrm{T}}\,&(C_{C\times M\times M\times M}v_{1M}v_{1M}v_{1M}+3TB_{C\times M\times M}v_{1M}(h_{2,1})_M)\\ &-\frac{2a_1}{T}(v_{1W_1}^*)^{\mathrm{T}}A_{C\times M}v_{1M}.\end{aligned}\tag{6.16}$$

6.4. Torus bifurcation. As before, we first rescale the time variable to the unit time-interval. The linear BVPs (5.3) and (5.5) are replaced by

$$\left\{\begin{array}{rcl} \dot{v_1}(t) - TA(t)v_1(t) + i\theta v_1(t) &=& 0,\ t\in[0,1],\\ v_1(0) - v_1(1) &=& 0,\\ \int_0^1 \langle v_1(t), v_1(t)\rangle dt - 1 &=& 0, \end{array}\right. \tag{6.17}$$

where $v(\tau) = v_1(\tau/T)/\sqrt{T}$, and

$$\left\{\begin{array}{rcl} \dot{v_1}^*(t) + TA^{\mathrm{T}}(t)v_1^*(t) - i\theta v_1^*(t) &=& 0,\ t\in[0,1],\\ v_1^*(0) - v_1^*(1) &=& 0,\\ \int_0^1 \langle v_1^*(t), v_1^*(t)\rangle dt - 1 &=& 0, \end{array}\right. \tag{6.18}$$

respectively. Note that the last equation in (6.18) differs from the last equation in (5.5). To rescale v_1^* we first compute $I = \int_0^1 \langle v_1^*(t), v_1(t)\rangle dt$. If $I \neq 0$ then we rescale v_1^* so that $I = 1$. (The case $I = 0$ corresponds to a bifurcation of codimension three or higher.) It then follows that $v^*(\tau) = v_1^*(\tau/T)/\sqrt{T}$.

We also replace (5.9) by

$$\left\{\begin{array}{rcl} \dot{\varphi_1}^*(t) + TA^{\mathrm{T}}(t)\varphi_1^*(\tau) &=& 0,\ t\in[0,1],\\ \varphi_1^*(0) - \varphi_1^*(1) &=& 0,\\ \int_0^1 \langle \varphi_1^*(t), \varphi_1^*(t)\rangle dt - 1 &=& 0. \end{array}\right. \tag{6.19}$$

Again, note that the last equation in (6.19) differs from the last equation in (5.9). Now compute $I = \int_0^1 \langle \varphi_1^*(t), F(u_1(t))\rangle dt$. If $I = 0$ then we have a fold–Neimark-Sacker bifurcation. If $I \neq 0$ then we rescale φ_1^* so that $I = 1$. It follows that $\varphi^*(\tau) = \varphi_1^*(\tau/T)/T$. (5.7) is replaced by

$$\left\{\begin{array}{rcl} \dot{h}_{20,1}(t) - A(t)h_{20,1}(t) + 2i\theta h_{20,1}(t) - B(t; v_1(t), v_1(t)) &=& 0,\ t\in[0,1],\\ h_{20,1}(0) - h_{20,1}(1) &=& 0. \end{array}\right. \tag{6.20}$$

where $h_{20}(\tau) = h_{20,1}(\tau/T)$. This leads to the expression

$$a_1 = \int_0^1 \langle \varphi_1^*(\tau), B(t; v_1(t), \bar{v}_1(t))\rangle\ dt, \tag{6.21}$$

where $a = a_1/T$.

With a_1 defined in this way, let $h_{11,1}$ be the unique solution of the BVP

$$\left\{\begin{array}{rcl} \dot{h}_{11,1}(t) - A(t)h_{11,1}(t) - B(t; v_1(t), \bar{v}_1(t)) + a_1F(u_1(t)) &=& 0,\ t\in[0,1],\\ h_{11,1}(0) - h_{11,1}(1) &=& 0,\\ \int_0^1 \langle \varphi_1^*(t), h_{11,1}(t)\rangle\ dt &=& 0, \end{array}\right. \tag{6.22}$$

where $h_{11}(\tau) = h_{11,1}(\tau/T)$.

Finally we obtain

$$\begin{array}{rcl} d &=& \dfrac{1}{2}\displaystyle\int_0^1 \langle v_1^*(t), B(t; h_{11,1}(t), v_1(t)) + B(t; h_{20,1}(t), \bar{v}_1(t))\rangle\ dt\\ &+& \dfrac{1}{2T}\displaystyle\int_0^1 \langle v_1^*(t), C(t; v_1(t), v_1(t), \bar{v}_1(t))\rangle\ dt - \dfrac{a_1}{T}\displaystyle\int_0^1 \langle v_1^*(t), A(t)v_1(t)\rangle\ dt + \dfrac{ia_1\theta}{T^2} \end{array} \tag{6.23}$$

We compute v_{1M} by solving

$$\left[\begin{array}{c} (D - TA(t) + i\theta I_n)_{C\times M} \\ \delta_0 - \delta_1 \end{array}\right] v_{1M} = 0, \tag{6.24}$$

where $(D - TA(t) + i\theta I_n)_{C\times M}$ is defined like $(D - TA(t))_{C\times M}$. We normalize v_{1M} by requiring that $\sum_{i=0}^{N-1}\sum_{j=0}^{m}\sigma_j\langle(v_{1M})_{i,j},(v_{1M})_{i,j}\rangle = 1$, where σ_j is the Lagrange quadrature coefficient. Again, it is more efficient to compute v_{1W}^* than v_M^*, since v_1^* will be used only to compute integrals of the form $\int_0^1\langle v_1^*(t),\zeta(t)\rangle dt$. Moreover, v_{1W}^* can be computed with the same matrix in (6.23). Formally, the computation of v_{1W}^* is based on Proposition A.3 from the Appendix. Instead of approximating v_1^* by solving

$$\left[\begin{array}{c} (D + TA^{\mathrm{T}}(t) - i\theta I_n)_{C\times M} \\ \delta_0 + \delta_1 \end{array}\right] v_{1M}^* = 0,$$

we remark that $\left[\begin{array}{c} v_1^* \\ v_1^*(0) \end{array}\right]$ is orthogonal to the range of $\left[\begin{array}{c} D - TA(t) + i\theta \\ \delta_0 + \delta_1 \end{array}\right]$. By discretization we obtain

$$(v_{1W}^*)^{\mathrm{H}}\left[\begin{array}{c} (D - TA(t) + i\theta I_n)_{C\times M} \\ \delta_0 + \delta_1 \end{array}\right] = 0.$$

To normalize $v_{1W_1}^*$ we require that $\sum_{i=0}^{N-1}\sum_{j=1}^{m}\left|(v_{1W_1}^*)_{i,j}\right|_1 = 1$. Then $\int_0^1\langle v_1^*(t),v_1(t)\rangle dt$ is approximated by $(v_{1W_1}^*)^{\mathrm{T}}L_{C\times M}v_{1M}$. If this quantity is nonzero then v_{1W}^* is rescaled so that $\int_0^1\langle v_1^*(t),v_1(t)\rangle dt = 1$.

From Proposition A.1 it follows that we can approximate φ_1^* like v_1^*. To be precise, we compute φ_{1W}^* by solving

$$(\varphi_{1W}^*)^{\mathrm{T}}\left[\begin{array}{c} (D - TA(t))_{C\times M} \\ \delta_0 - \delta_1 \end{array}\right] = 0\ ,$$

and we normalize $\varphi_{1W_1}^*$ by requiring that $\sum_{i=0}^{N-1}\sum_{j=1}^{m}\left|(\varphi_{1W_1}^*)_{i,j}\right|_1 = 1$. Then the integral $\int_0^1\langle\varphi_1^*(t),F(u_1(t))\rangle dt$ is approximated by $(\varphi_{1W_1}^*)^{\mathrm{T}}(F(u_1(t)))_C$. If this quantity is nonzero then φ_{1W}^* is rescaled, so that $\int_0^1\langle\varphi_1^*(t),F(u_1(t))\rangle dt = 1$. We compute $(h_{20,1})_M$ by solving

$$\left[\begin{array}{c} (D - TA(t) + 2i\theta I_n)_{C\times M} \\ \delta_0 - \delta_1 \end{array}\right](h_{20,1})_M = \left[\begin{array}{c} B_{C\times M\times M}v_{1M}v_{1M} \\ 0 \end{array}\right].$$

The coefficient a_1 can be computed using (6.21) as

$$a_1 = (\varphi_{W_1}^*)^{\mathrm{T}}B_{C\times M\times M}v_{1M}\bar{v}_{1M},$$

while $(h_{11,1})_M$ is found by solving the discretization of (6.22),

$$\left[\begin{array}{c} (D - TA(t))_{C\times M} \\ \delta_0 - \delta_1 \\ (\varphi_{W_1}^*)^{\mathrm{T}}L_{C\times M} \end{array}\right](h_{11,1})_M = \left[\begin{array}{c} B_{C\times M\times M}v_{1M}\bar{v}_{1M} - a_1(F(u_1(t)))_C \\ 0 \\ 0 \end{array}\right].$$

The expression (6.23) for the normal form coefficient d becomes

$$\begin{aligned} d \;=\; & \frac{1}{2}(v_{1W_1}^*)^{\mathrm{T}}(B_{C\times M\times M}(h_{11,1})_M v_{1M} + B_{C\times M\times M}(h_{20,1})_M\bar{v}_{1M}) \\ & + \frac{1}{2T}(v_{1W_1}^*)^{\mathrm{T}}C_{C\times M\times M\times M}v_{1M}v_{1M}\bar{v}_{1M} - \frac{a_1}{T}(v_{1W_1}^*)^{\mathrm{T}}A_{C\times M}v_{1M} + \frac{ia_1\theta}{T^2}. \end{aligned} \tag{6.25}$$

7. Examples.

7.1. The LPC normal form coefficient in the ABC-reaction. We have computed the normal form coefficient b of (2.6) in a model of a continuously stirred tank reactor, with consecutive A $\to$ B $\to$ C reactions, as studied by Doedel & Heinemann [10]. It has three state variables, u_1, u_2, u_3, and five parameters, p_1, p_2, p_3, p_4, and p_5 :

$$\begin{cases} \dot{u_1} &= -u_1 + p_1(1-u_1)e^{u_3}, \\ \dot{u_2} &= -u_2 + p_1 e^{u_3}(1-u_1-p_5 u_2), \\ \dot{u_3} &= -u_3 - p_3 u_3 + p_1 p_4 e^{u_3}(1-u_1+p_2 p_5 u_2). \end{cases} \tag{7.1}$$

This model is used as a demo in the AUTO manual [8]. In the notation of [10], we have $u_1 = y$, where $1-y$ is the concentration of reactant A, $u_2 = z$, the concentration of reactant B, $u_3 = \theta$, the temperature, $p_1 = D$, the Damkohler number, $p_2 = \alpha$, the ratio of reaction heats, $p_3 = \beta$, the heat transfer coefficient, $p_4 = B$, the adiabatic temperature rise, and $p_5 = \sigma$, the selectivity ratio.

In Fig. 7.1 the equilibrium curve computed with MATCONT is represented. The parameter values are $p_2 = 1$, $p_3 = 1.5$, $p_4 = 8$, $p_5 = 0.04$, with free parameter p_1, starting from the equilibrium at $p_1 = 0.1$, for which $u_1 = 0.13304$, $u_2 = 0.13223$, $u_3 = 0.42833$. The curve of equilibria contains four Hopf points, denoted, from left to right, H_1, H_2, H_3, H_4, respectively. As shown in [10], in the case $p_2 = 1$, the Hopf

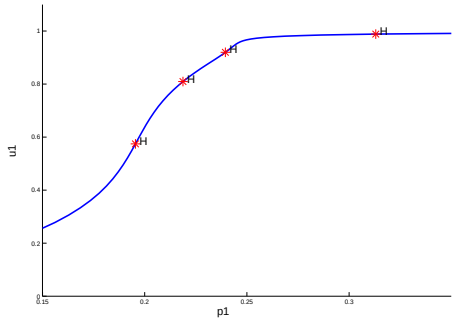


FIG. 7.1. *Equilibrium curve of the $A \to B \to C$ reaction for $p_2 = 1$.*

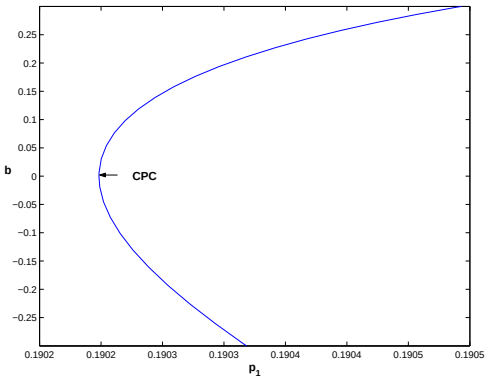
points H_1 and H_4 are connected by a family of periodic solutions, and H_2 and H_3 are similarly connected. The family of solutions that connects H_1 to H_4, contains three fold bifurcations of periodic solutions, as also observed in [10]. We continue the first fold bifurcation of periodic solutions numerically in two parameters p_1 and p_2. This family contains a cusp point of periodic orbits (CPC) detected in MATCONT as a zero of the coefficient b computed with (6.8). In Fig. 7.2(a)-(c) we present the normal form coefficient b, the first component u_1 of the state variables vector, and p_2, respectively, as functions of p_1.

7.2. The PD normal form coefficient in a feedback control system. We have used (6.16) to compute the PD normal form coefficient c of (2.7) in a feedback control system, described in [14], [15] and further used in [23] (Example 5.4, p. 178):

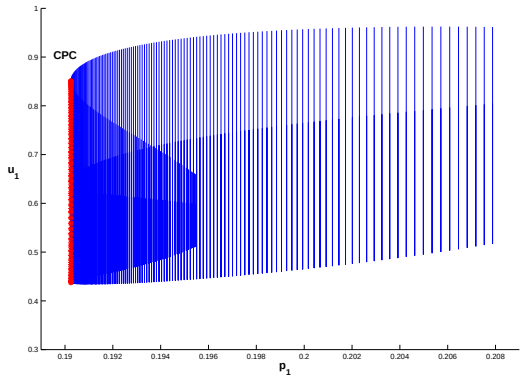
$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -\alpha z - \beta y - x + x^2. \end{cases} \tag{7.2}$$

Due to special structure of this system, a good approximation to the PD curve can be found by the harmonic balance method, cf. [28], [29].

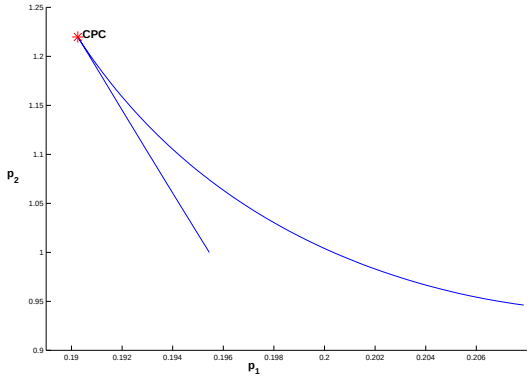
We have computed a family of periodic orbits numerically, as described in the Cl_Matcont manual and in [7], starting from the Hopf point for $\alpha = 1$ and $\beta = 1$ at $(0, 0, 0)$. We used $N = 20$ (mesh intervals) and $m = 4$ (collocation points) for the discretization. We detected two period doubling points with period 6.36407 at $\alpha = \pm 0.6303020$, respectively. The noncritical multipliers at the first PD point are



(a) *The behavior of b near a CPC point on a LPC curve in the* $A \to B \to C$ *reaction.*

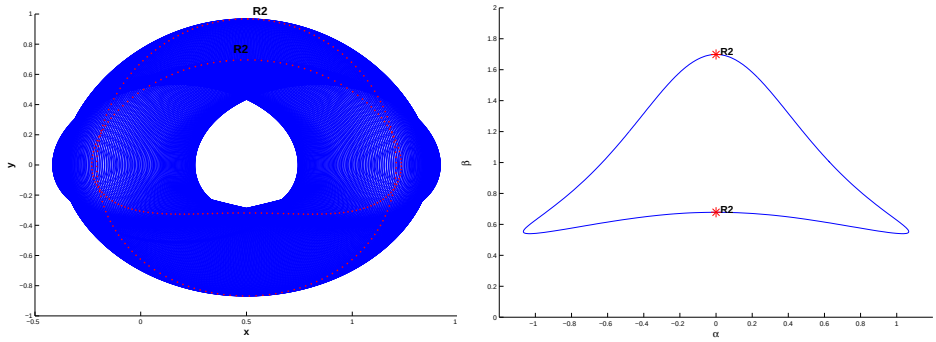


(b) *LPC curve in the* (p_1, u_1)*-space.*



(c) *LPC curve in the* (p_1, p_2)*-space.*

FIG. 7.2.

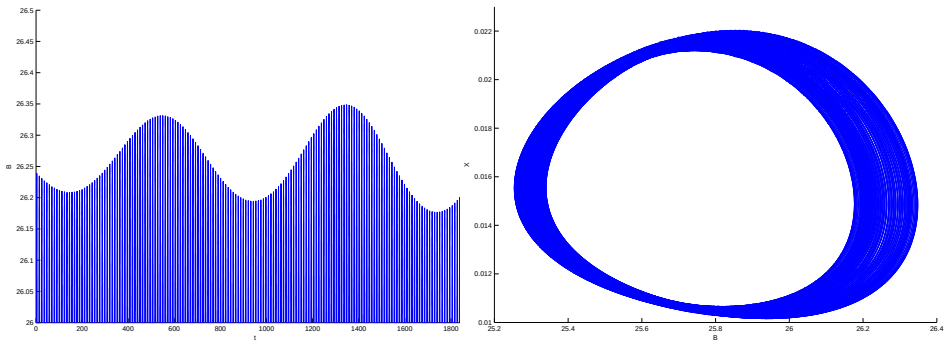


(a) *PD curve with two R2 points for* $\alpha = 0$.

(b) *The family in the* (α, β)*-space.*

FIG. 7.3.

inside the unit circle, so the periodic orbit could be stable. At the second PD point there is one multiplier outside the unit circle, and therefore the orbit is unstable. At the PD points the normal form coefficient c were computed. At the first PD point we find that $c = -0.04267737 < 0$. Therefore, the critical periodic orbit at the first PD point is stable, and a stable limit cycle with approximately double period exists for nearby parameter values. This was confirmed by computing the periodic orbit and its multipliers. At the second PD point the normal form coefficient is $c = 0.04268605 > 0$. Hence the periodic orbit with double period is unstable in the center manifold. By computing the orbit with doubled period and monitoring the multipliers near this second PD point, we found that it has indeed two multipliers outside the unit circle. From the first PD point we computed the branch of PD-cycles. The normal form coefficient is used as a test function. We also use $I = \int_0^1 \langle v_1^*(t), v_1(t)\rangle dt$ as another test function. We detected two strong 1:2 resonances on this curve, for $\alpha = 0$. By monitoring the multipliers at those R2 points, we found that in both cases there are two multipliers equal to -1. This family is represented in Fig. 7.3.



(a) *Modulated oscillations.*

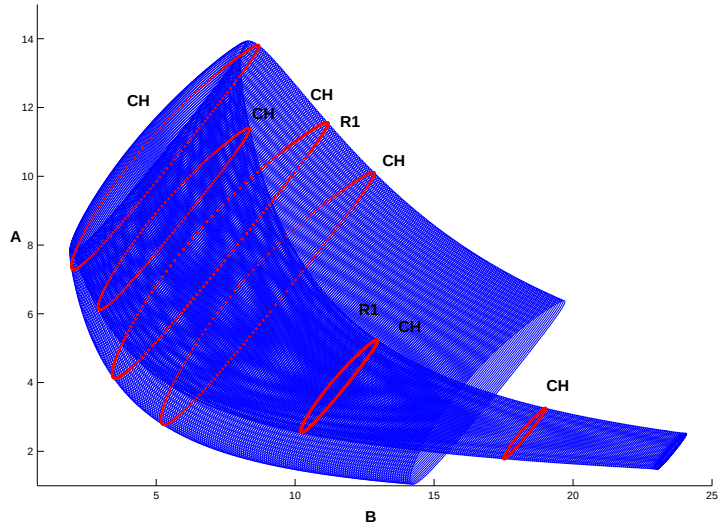
(b) *Orbits on a stable 2-torus.*

FIG. 7.4.

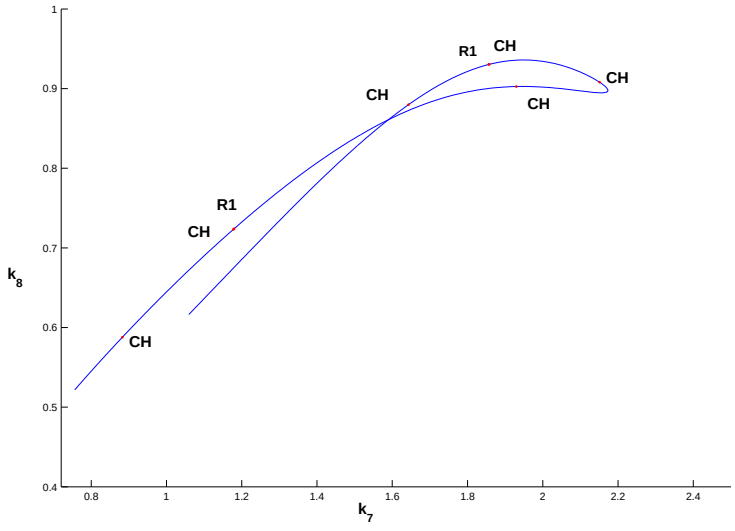
7.3. The NS normal form coefficient in a chemical model. The following model of the peroxidase-oxidase reaction was studied by Steinmetz and Larter [27]:

$$\begin{cases} \dot{A} &= -k_1ABX - k_3ABY + k_7 - k_{-7}A, \\ \dot{B} &= -k_1ABX - k_3ABY + k_8, \\ \dot{X} &= k_1ABX - 2k_2X^2 + 2k_3ABY - k_4X + k_6, \\ \dot{Y} &= -k_3ABY + 2k_2X^2 - k_5Y, \end{cases} \tag{7.3}$$

where A, B, X, Y are state variables and k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, and k_{-7} are parameters. The following values correspond to an unstable equilibrium in (7.3):



(a) *NS (torus) bifurcation curve.*



(b) *Codim 2 bifurcation points on the NS curve.*

FIG. 7.5.

Variable	*Value*	*Parameter*	*Value*
A	31.78997	k_1	0.1631021
B	1.45468	k_2	1250
X	0.01524586	k_3	0.046875
Y	0.1776113	k_4	20
		k_5	1.104
		k_6	0.001
		k_7	4.235322
		k_8	0.5
		k_{-7}	0.1175

We continued this equilibrium with decreasing k_7, keeping all other parameters fixed. We found a Hopf point at $k_7 \approx 0.712475$, where the first Lyapunov coefficient is negative. We then computed the family of stable limit cycles that bifurcates from the Hopf point. At $k_7 \approx 0.716434$ a *torus* (NS) bifurcation occurs. The real part of normal form coefficient d of (2.8) at this point is $\mathrm{Re}\, d = -1.405999 \cdot 10^{-6}$, and therefore the emanating tori would be stable, locally. If we start a time integration from a point on the critical limit cycle, with a slightly increased parameter value, namely $k_7 = 0.7167$, then after a transient period the orbit exhibits modulated oscillations with two frequencies near the limit cycle (see Fig. 7.4). This is a motion on a stable two-dimensional torus that arises from the Neimark-Sacker bifurcation. The NS point can be used as a starting point for the 2-parameter continuation of the corresponding codim 1 bifurcation, using k_7 and k_8 as control parameters. We monitored $\mathrm{Re}\, d$ of the normal form coefficient d, computed with (6.25), during this continuation; it vanishes in a Chenciner bifurcation point (CH). The computed bifurcation curve is presented in Fig. 7.5(a) and Fig. 7.5(b), in the (A, B)-plane and in the (k_7, k_8)-plane, respectively. The NS curve contains two additional codim 2 points, where a triple multiplier $\mu = 1$ is present (also counting the trivial multiplier). These are 1:1 strong resonance points [23]. Between the 1:1 points, the NS curve is a *neutral saddle cycle* curve. Near such codim 2 points complicated homoclinic structures exist.

It should be noted that the algorithm for the NS continuation, as implemented in MATCONT, is sufficiently robust to pass through the 1:1 resonance points (within a 10^{-3} parameter-range).

8. Discussion. The formulas for the normal form coefficients derived in this paper allow numerical verification of the nondegeneracy conditions (see [23]) for all codim 1 limit cycle bifurcations. In particular, the coefficients for the period-doubling and torus bifurcations allow one to distinguish between sub- and supercritical cases. These coefficients serve as test functions for detecting codim 2 bifurcations of limit cycles.

The new algorithms fit very well into the BVP-framework (see [3, 9]) of AUTO [8], CONTENT [25], and, particularly, MATCONT [6], which contains our current proto-type implementation.

The underlying technique can also be used to derive the coefficients of the periodic normal forms for codim 2 singularities of limit cycles. Although periodic normal forms are known in most of these codim 2 cases (see [1, 4]), substantial work remains to be done on the derivation and implementation of formulas for their coefficients. When implemented, such formulas will allow one to verify the nondegeneracy conditions for the codim 2 bifurcations.

A comparison of the numerical periodic normalization in the current paper to the computation of normal form coefficients of the Poincaré map via automatic differentiation, is also a task in future work.

Appendix A. Kernels of some differential-difference operators.

In Section 6 we used the following inner product: If $\zeta_1, \zeta_2 \in \mathcal{C}^0([0,1], \mathbb{C}^n)$ and $\eta_1, \eta_2 \in \mathbb{C}^n$, then

$$\langle \begin{bmatrix} \zeta_1 \\ \eta_1 \end{bmatrix}, \begin{bmatrix} \zeta_2 \\ \eta_2 \end{bmatrix} \rangle = \int_0^1 \langle \zeta_1(t), \zeta_2(t) \rangle \, dt + \langle \eta_1, \eta_2 \rangle = \int_0^1 \zeta_1^{\mathrm{H}}(t) \zeta_2(t) dt + \eta_1^{\mathrm{H}} \eta_2.$$

If this inner product vanishes then we write

$$\begin{bmatrix} \zeta_1 \\ \eta_1 \end{bmatrix} \perp \begin{bmatrix} \zeta_2 \\ \eta_2 \end{bmatrix}.$$

In Section 6 we also used the following Propositions:

PROPOSITION A.1. *Consider two differential-difference operators*

$$\phi_{1,2} : \mathcal{C}^1([0,1], \mathbb{R}^n) \to \mathcal{C}^0([0,1], \mathbb{R}^n) \times \mathbb{R}^n,$$

where

$$\phi_1(\zeta) = \begin{bmatrix} \dot{\zeta} - TA\zeta \\ \zeta(0) - \zeta(1) \end{bmatrix}, \quad \phi_2(\zeta) = \begin{bmatrix} \dot{\zeta} + TA^{\mathrm{T}}\zeta \\ \zeta(0) - \zeta(1) \end{bmatrix}.$$

If $\zeta \in \mathcal{C}^1([0,1], \mathbb{R}^n)$*, then* $\zeta \in \mathrm{Ker}(\phi_1)$ *if and only if*

$$\begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \perp \phi_2(\mathcal{C}^1([0,1], \mathbb{R}^n)),$$

and $\zeta \in \mathrm{Ker}(\phi_2)$ *if and only if*

$$\begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \perp \phi_1(\mathcal{C}^1([0,1], \mathbb{R}^n)).$$

Proof.
If ζ is in the kernel of ϕ_1 then $\dot{\zeta} - TA(t)\zeta = 0$ and $\zeta(0) - \zeta(1) = 0$.
For all $g \in \mathcal{C}^1([0,1], \mathbb{R}^n)$ we have

$$\begin{array}{rcll} & \int_0^1 g(t)^{\mathrm{T}} \dot{\zeta}(t) dt - \int_0^1 T g(t)^{\mathrm{T}} A(t) \zeta(t) dt & = & 0, \\ \Rightarrow & g(t)^{\mathrm{T}} \zeta(t)|_0^1 - \int_0^1 \dot{g}(t)^{\mathrm{T}} \zeta(t) dt - \int_0^1 T g(t)^{\mathrm{T}} A(t) \zeta(t) dt & = & 0, \\ \Rightarrow & g(1)^{\mathrm{T}} \zeta(1) - g(0)^{\mathrm{T}} \zeta(0) - \int_0^1 (\dot{g}(t) + TA(t)^{\mathrm{T}} g(t))^{\mathrm{T}} \zeta(t) dt & = & 0, \\ \Rightarrow & -(g(0) - g(1))^{\mathrm{T}} \zeta(0) - \int_0^1 (\dot{g}(t) + TA(t)^{\mathrm{T}} g(t))^{\mathrm{T}} \zeta(t) dt & = & 0, \\ \Rightarrow & \langle \begin{bmatrix} \dot{g} + TA^{\mathrm{T}} g \\ g(0) - g(1) \end{bmatrix}, \begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \rangle & = & 0. \end{array}$$

Conversely, assume that $\langle \begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix}, \begin{bmatrix} \dot{g} + TA^{\mathrm{T}} g \\ g(0) - g(1) \end{bmatrix} \rangle = 0$ for all $g \in \mathcal{C}^1([0,1], \mathbb{R}^n)$.

$$\begin{array}{rcll} \Rightarrow & \int_0^1 \zeta^{\mathrm{T}}(t)(\dot{g}(t) + TA(t)^{\mathrm{T}} g(t)) dt + \zeta^{\mathrm{T}}(0)(g(0) - g(1)) & = & 0, \\ \Rightarrow & \zeta(1)^{\mathrm{T}} g(1) - \zeta(0)^{\mathrm{T}} g(0) + \zeta(0)^{\mathrm{T}}(g(0) - g(1)) - \int_0^1 (\dot{\zeta}(t) - TA(t)\zeta(t))^{\mathrm{T}} g(t) dt & = & 0, \\ \Rightarrow & -(\zeta(0) - \zeta(1))^{\mathrm{T}} g(1) - \int_0^1 (\dot{\zeta}(t) - TA(t)\zeta(t))^{\mathrm{T}} g(t) dt & = & 0. \end{array}$$

If $\dot{\zeta}(t) - TA(t)\zeta(t) \neq 0$, then there exists a $g(t)$ with $g(1) = 0$ such that

$$\int_0^1 (\dot{\zeta}(t) - TA(t)\zeta(t))^{\mathrm{T}} g(t) dt \neq 0.$$

This is impossible, so $\dot{\zeta}(t) + TA(t)^{\mathrm{T}}\zeta(t) = 0$. Hence $(\zeta(0) - \zeta(1))^{\mathrm{T}} g(1) = 0$ for all g, hence $\zeta(0) - \zeta(1) = 0$. From $\dot{\zeta}(t) - TA(t)\zeta(t) = 0$ and $\zeta(0) = \zeta(1)$ follows that $\zeta \in \mathrm{Ker}(\phi_1)$.

The proof of the second part is similar. □

PROPOSITION A.2. *Consider* $\phi_{1,2} : \mathcal{C}^1([0,1],\mathbb{R}^n) \to \mathcal{C}^0([0,1],\mathbb{R}^n) \times \mathbb{R}^n$, *where*

$$\phi_1(\zeta) = \begin{bmatrix} \dot{\zeta} - TA\zeta \\ \zeta(0) + \zeta(1) \end{bmatrix}, \phi_2(\zeta) = \begin{bmatrix} \dot{\zeta} + TA^{\mathrm{T}}\zeta \\ \zeta(0) + \zeta(1) \end{bmatrix}.$$

If $\zeta \in \mathcal{C}^1([0,1],\mathbb{R}^n)$, *then* $\zeta \in \mathrm{Ker}(\phi_1)$ *if and only if*

$$\begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \perp \phi_2(\mathcal{C}^1([0,1],\mathbb{R}^n)),$$

and $\zeta \in Ker(\phi_2)$ *if and only if*

$$\begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \perp \phi_1(\mathcal{C}^1([0,1],\mathbb{R}^n)).$$

Proof. As in Proposition A.1. □

PROPOSITION A.3. *Consider* $\phi_{1,2} : \mathcal{C}^1([0,1],\mathbb{C}^n) \to \mathcal{C}^0([0,1],\mathbb{C}^n) \times \mathbb{C}^n$, *where*

$$\phi_1(\zeta) = \begin{bmatrix} \dot{\zeta} - TA\zeta + i\theta I_n \\ \zeta(0) - \zeta(1) \end{bmatrix}, \phi_2(\zeta) = \begin{bmatrix} \dot{\zeta} + TA^{\mathrm{T}}\zeta - i\theta I_n \\ \zeta(0) - \zeta(1) \end{bmatrix}.$$

If $\zeta \in \mathcal{C}^1([0,1],\mathbb{C}^n)$, *then* $\zeta \in \mathrm{Ker}(\phi_1)$ *if and only if*

$$\begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \perp \phi_2(\mathcal{C}^1([0,1],\mathbb{C}^n)),$$

and $\zeta \in \mathrm{Ker}(\phi_2)$ *if and only if*

$$\begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \perp \phi_1(\mathcal{C}^1([0,1],\mathbb{C}^n)).$$

Proof. As in Proposition A.1. □

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