WAVELETS WITH PATCHWISE CANCELLATION PROPERTIES

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Abstract. We construct wavelets on general n-dimensional domains or manifolds via a domain decomposition technique, resulting in so-called composite wavelets. With this construction, wavelets with supports that extend to more than one patches are only continuous over the patch interfaces. Normally, this limited smoothness restricts the possibility for matrix compression, and with that the application of these wavelets in (adaptive) methods for solving operator equations. By modifying the scaling functions on the interval, and with that on the n-cube that serves as parameter domain, we obtain composite wavelets that have *patchwise cancellation properties* of any required order, meaning that the restriction of any wavelet to each patch is again a wavelet. This is also true when the wavelets are required to satisfy zeroth order homogeneous Dirichlet boundary conditions on (part of) the boundary. As a result, compression estimates now depend only on the patchwise smoothness of the wavelets that one may choose. Taking also stability into account, our composite wavelets have all properties for the application to the (adaptive) solution of well-posed operator equations of orders $2t$ for $t \in (-\frac{1}{2}, \frac{3}{2})$.

1. Motivation and background

For some $n' \geq n \geq 1$, let Ω be an n-dimensional manifold in $\mathbb{R}^{n'}$. We are interested in approximating the solution of an equation $Lu = f$, where for some Hilbert space H of functions on Ω, typically being a Sobolev space, with dual H', $L : H \to H'$ is boundedly invertible, and $f \in H'$. When Ω is a domain in $\mathbb{R}^n$, we think of the equation as being the result of a variational formulation of a boundary value problem, and when it is a true manifold, we have in mind an integral equation formulated on the boundary of an $(n+1)$-dimensional domain.

Now let us assume that we have available a *Riesz basis* Ψ for H of *wavelet* type, where each wavelet is assumed to have the *cancellation property* of a certain order, meaning that, possibly after making some smooth transformation of coordinates, it is orthogonal to all polynomials of that order.

Thinking of strongly elliptic problems, for any $V \subset H$ spanned by some finite subset of the wavelets, we can approximate u by the Galerkin solution from V. This approach has two attractive features. First, since Ψ is a Riesz basis, the stiffness matrix with respect to the wavelet basis is well-conditioned uniformly in V, allowing an efficient iterative solution. Second, for L being a singular integral operator, the cancellation property of a sufficiently high order of the wavelets allows to compress the stiffness matrix, which in this case is dense, to a sparse one without qualitatively affecting the discretization error. The compression, i.e., the dropping of certain entries, applies, in different forms, to two types of entries. The so-called *first compression* applies to pairs of wavelets that have supports with a sufficiently large mutual distance, whereas, thinking of piecewise smooth wavelets, the *second compression* applies to pairs of wavelets that have overlapping supports, but for which the support of one wavelet living on the higher level is sufficiently far away from the singular support of the other one living on a lower level (see [Sch98, DHS02]). Together, the well-conditionedness of the stiffness matrix, and, for singular integral operators, the compression of this matrix allow to find an approximate solution of the Galerkin system in $\mathcal{O}(\#V)$ operations

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with an error that, up to some absolute constant factor, is as good as that of the exact solution of this system.

Instead of computing Galerkin approximations from fixed, finite dimensional subspaces V, as exposed in [CDD01, CDD02], the availability of a wavelet basis Ψ opens a way to approximate the solution u using an *adaptive scheme*. Since Ψ is a Riesz basis for H, the equation $Lu = f$ has an equivalent formulation as an infinite, well-posed (in ℓ_2-metric) matrix-vector system, which is formally the Galerkin system with "V" equal to H, equipped with the wavelet basis. Now, coarsely speaking, the idea is to apply a simple iterative scheme, like Richardson iteration, directly to this infinite system, where in each iteration the application of the matrix is replaced by that of an adaptively compressed matrix. Apart from the second compression, and for integral operators, the first compression, now it is necessary to consider a *third compression* that applies to pairs of wavelets for which the support of the wavelet on the higher level intersects the singular support of the wavelet on the lower level. Indeed, note that, even for differential operators, without either the second or third compression one is left with infinitely many non-zero entries in each column. The decay of the entries as meant in the third compression, as function of the difference in levels of the wavelets involved, relies not only on the cancellation property of the wavelet on the higher level, but also on a sufficient *global smoothness* of the wavelet on the lower level. Essentially only for *spline wavelets*, which have maximal smoothness in relation to their approximation orders, the compression error can be shown to be sufficiently small so that the adaptive wavelet method has *optimal computational complexity*, in the sense that the approximations yielded by this algorithm converge with the same rate as that of so-called *best N-term approximations*, taking only a number of arithmetic operations that is proportional to the vector length (see [Ste04b], and, for the computation of the required entries using quadrature, see [GS04, Gan04]).

In view of above applications, it is by now well-known how to construct wavelets on the line ([CDF92]), and with additional efforts on the interval ([DKU99]), and so using tensor products, on n-cubes, that, properly scaled, generate Riesz bases for a range of Sobolev spaces, have the cancellation property of any required order, and that are sufficiently smooth in relation to their approximation order. The challenge is to construct such wavelets on more *general domains* or *manifolds* Ω. The most well-known method is that via a *domain decomposition* approach (see [DS99a], and [CTU99, CM00] for related techniques). With this approach, Ω is written as $\overline{\Omega} = \cup_{q=1}^{M} \overline{\Omega}_q$, where $\Omega_q \cap \Omega_{q'} = \emptyset$ for $q \neq q'$, $\Omega_q = \kappa_q(\square)$, where the $\kappa_q : \mathbb{R}^n \to \mathbb{R}^{n'}$ are smooth, regular parametrizations, and $\square = (0,1)^n$. Primal and dual scaling functions constructed on $\square$ are lifted to the Ω_q, and are continuously connected over the interfaces. The resulting global primal and dual scaling functions are biorthogonal with respect to the *modified $L_2(\Omega)$-scalar product* $\langle\!\langle u, v\rangle\!\rangle = \sum_q \int_\square (u \circ \kappa_q)(x)(v \circ \kappa_q)(x)dx$. The "composite" wavelets are now obtained by subtracting the biorthogonal projection from collections of functions spanning "initial" complement spaces between any two successive spaces in the primal multiresolution analysis, which collections are simply obtained by lifting such collections on $\square$. Although this construction realizes wavelets on general domains or manifolds that, properly scaled, generate Riesz bases for a range of Sobolev spaces, it also has some limitations:

- wavelets with supports that extend to more than one patch generally have no cancellation property with respect to the *canonical L_2-scalar product*,
- the wavelets generate Riesz bases for Sobolev spaces H^s generally only for $s > -\frac{1}{2}$ (and $s < \frac{3}{2}$),
- wavelets with supports that extend to more than one patch are only continuous,

where the first two limitations are a consequence of the fact that biorthogonality is realized with respect to the modified L_2-scalar product.

These limitations were already recognized in [DS99a], and in [DS99b], the same authors developed an elegant approach to construct wavelets on general domains or manifolds that, properly scaled, generate Riesz bases for H^s for in principal any s, if not restricted by the regularity of the manifold, and that have the cancellation property of any desired order. Unfortunately, so far with this approach it seems not easy to construct wavelets that have competitive quantitative properties. However, new results in this direction have been reported recently in [KS04].

By an adaptation of the construction from [DS99a], in [Ste04a] we constructed composite wavelets that generate Riesz bases for the Sobolev spaces H^s for the full range of s that is allowed by the continuous gluing of functions over the patch interfaces (i.e., $|s| < \frac{3}{2}$), if not restricted by the regularity of the manifold, and that all have the cancellation property of any desired order. These wavelets have all properties required for the earlier mentioned application of solving Galerkin systems resulting from differential- or singular integral operators with orders $2t$ when $|t| < \frac{3}{2}$. For the application in the adaptive wavelet method, however, the limited smoothness of wavelets over patch interfaces has an adverse affect on the third compression, and optimality of that scheme can only be shown for wavelets with relatively small approximations orders $d < \frac{3n/2-t}{n-1}$ ([Ste04b, Remark 2.4]).

In the present paper, we construct composite wavelets that have the stronger *patchwise cancellation property* of any desired order, meaning that the restriction of any wavelet to any patch Ω_q has the cancellation property of that order. This property will also hold when, in case the manifold has a boundary, at the primal side a zeroth order homogeneous Dirichlet boundary condition is prescribed on (part of) this boundary. Although also with this construction, wavelets with supports that intersect patch interfaces are only continuous, thanks to the patchwise cancellation property, in any case for differential operators, this limited smoothness does not affect the third compression. Indeed, for such an operator, say of order $2t$ with $t < \frac{3}{2}$, an entry in the infinite stiffness matrix corresponding to wavelets ψ_λ, ψ_μ is of the form $\int_\Omega \sum_{|\alpha|,|\beta|\leq t} a_{\alpha,\beta}\partial^\alpha\psi_\lambda\partial^\beta\psi_\mu = \sum_q \int_{\Omega_q} \sum_{|\alpha|,|\beta|\leq t} a_{\alpha,\beta}\partial^\alpha\psi_\lambda\partial^\beta\psi_\mu$. Now, in principal for any order of the wavelets, above right-hand side shows that with only a sufficient *patchwise smoothness*, as with (lifted) splines, and with a patchwise cancellation property of a sufficiently high order, the appropriate decay of the entry is ensured as function of the difference $||\lambda| - |\mu||$ in levels, and with that, optimality of the adaptive wavelet scheme.

The patchwise cancellation property, say of order $\tilde{d}$, also induces a limitation. Thinking of a domain Ω being simply the union of n-cubes Ω_q, this property means that any wavelet ψ_λ, as always except those with level $|\lambda| = 0$ which are scaling functions, restricted to each Ω_q is orthogonal to all polynomials of degree $\tilde{d}-1$. Assuming, as will be ensured, that the collection $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$ is a Riesz basis for $L_2(\Omega)$, with dual basis $\tilde{\Psi}$, then for any $v \in \prod_q P_{\tilde{d}-1}(\Omega_q) \subset L_2(\Omega)$, we have $v = \sum_{\lambda\in\Lambda}\langle v, \psi_\lambda\rangle\tilde{\psi}_\lambda = \sum_{\lambda\in\Lambda,|\lambda|=0}\langle v, \psi_\lambda\rangle\tilde{\psi}_\lambda$, so that apparently $\prod_q P_{\tilde{d}-1}(\Omega_q)$ is in the span of the dual scaling functions on the lowest level. So necessarily, all dual spaces contain functions that have jumps across interfaces between patches, from which we infer, regardless whether biorthogonality is realized with respect to the canonical or the modified L_2-scalar product, that the (primal) wavelets cannot generate Riesz bases for H^s when $s \leq -\frac{1}{2}$. We will ensure, however, that, properly scaled, our wavelets generated Riesz bases for H^s for $s \in (-\frac{1}{2}, \frac{3}{2})$, which, apart from the application in the adaptive wavelet scheme, makes them suitable for the solution of Galerkin systems resulting from differential- or singular integral operators of orders $2t$, with $t \in (-\frac{1}{2}, \frac{3}{2})$.

This paper is organized as follows. In Sect. 2, we collect the assumptions on the primal and dual multiresolution analyses on the interval. We construct modified scaling function that eventually will lead to composite wavelets with patchwise cancellation properties. In Sect. 3, we construct multiresolution analyses on the n-cube simply by means of tensor products. Following the approach from [DS99a], in Sect. 4, we construct composite wavelets on general domains or manifolds, that, thanks to the modification of the scaling functions, have patchwise cancellation properties. Finally, in Sect. 5, we illustrate our approach by constructing on a simple L-shaped domain, wavelets of order 2 that have patchwise cancellation properties of order 2.

In order to avoid the repeated use of generic but unspecified constants, in this paper by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

For H being a separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and for a countable collection Σ of functions in H, which we formally view as a (column) vector, and for $\mathbf{c}=(c_\sigma)_{\sigma\in\Sigma}$ a vector of scalars, with $\mathbf{c}^T\Sigma$ we will mean the expansion $\sum_{\sigma\in\Sigma} c_\sigma\sigma$. The span of Σ will be denoted as $\mathcal{S}(\Sigma)$. For $x\in H$, with $\langle\Sigma,x\rangle$ and $\langle x,\Sigma\rangle$ we will mean the column- and row-vectors with coefficients $\langle\sigma,x\rangle$ and $\langle x,\sigma\rangle$, $\sigma\in\Sigma$. When $\tilde\Sigma$ is another countable collection in H, with $\langle\Sigma,\tilde\Sigma\rangle$ we denote the matrix $(\langle\sigma,\tilde\sigma\rangle)_{\sigma\in\Sigma,\tilde\sigma\in\tilde\Sigma}$. For $V\subset H$ being a dense, continuously embedded Banach space, as usual we will use $\langle\cdot,\cdot\rangle$ sometimes also to denote the duality pairing $\langle\cdot,\cdot\rangle_{V\times V'}$, which, with the aforementioned meaning, can also be applied to collections from V and/or V'.

On the spaces of (possibly infinite) scalar vectors or matrices, we will exclusively use the ℓ_2-scalar product, ℓ_2-norm or the resulting operator norm, that we therefore simply denote by $\langle\cdot,\cdot\rangle$ or $\|\cdot\|$, respectively. A collection Σ is called a *Riesz system* when $\|\mathbf{c}^T\Sigma\|\eqsim\|\mathbf{c}\|$, i.e., when $\langle\Sigma,\Sigma\rangle$ is boundedly invertible, and Σ is called a *Riesz basis* when it is in addition a basis for H. When Σ depends on a parameter, we will speak about *uniform Riesz systems* (or bases) when the above equivalence holds uniformly over the values this parameter may attain.

2. Biorthogonal multiresolution on $(0,1)$

2.1. **Assumptions.** Throughout this section, let $\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle_{L_2(0,1)}$ and $\|\cdot\|=\|\cdot\|_{L_2(0,1)}$. For some fixed $m\in\mathbb{Z}$, and all $j\in\mathbb{Z}_m:=\mathbb{Z}\cap[m,\infty)$, we assume an index set $\{0,1\}\subset I_j\subset[0,1]$ with $x\in I_j$ if and only if $1-x\in I_j$, and $\sup_{k\in\mathbb{N}_0}\#(I_j\cap[k2^{-j},(k+1)2^{-j}])\lesssim 1$. For $j\in\mathbb{Z}_m$, we assume collections of uniformly $L_2(0,1)$-bounded primal and dual scaling functions $\Phi_j=[\phi_{j,x}]_{x\in I_j}$ and $\tilde\Phi_j=[\tilde\phi_{j,x}]_{x\in I_j}$ that satisfy

$$(\mathcal{L})\qquad \phi_{j,x}(y)=0 \text{ if not } |x-y|\lesssim 2^{-j},$$

$$(\mathcal{S})\qquad \phi_{j,x}(\cdot)=\phi_{j,1-x}(1-\cdot),$$

$$(\mathcal{J})\qquad \text{for some } d\in\mathbb{N},\ P_{d-1}(0,1)\subset\mathcal{S}(\Phi_j),$$

$$(\mathcal{N})\qquad \mathcal{S}(\Phi_j)\subset\mathcal{S}(\Phi_{j+1}),$$

$$(\mathcal{B})\qquad \text{for some } \gamma>0, \text{ and any } s\in[0,\gamma),$$

$$\|\cdot\|_{H^s(0,1)}\lesssim 2^{sj}\|\cdot\| \quad \text{on } \mathcal{S}(\Phi_j),$$

with analogous conditions on $(\tilde\Phi_j)_j$, denoted as $(\tilde{\mathcal{L}})$, $(\tilde{\mathcal{S}})$, $(\tilde{\mathcal{J}})$, $(\tilde{\mathcal{N}})$, and $(\tilde{\mathcal{B}})$, with (d,γ) replaced by generally different parameters $(\tilde d,\tilde\gamma)$. We assume that the *primal* scaling functions are *continuous*, and satisfy

$$(\mathcal{V})\qquad \phi_{j,x}(0)=0 \text{ for } x\in I_j\backslash\{0\}.$$

Finally, we assume that the primal and dual scaling functions are *biorthogonal*, i.e., that

$$(\mathcal{D})\qquad \langle\Phi_j,\tilde\Phi_j\rangle=I.$$

Although in applications it is important to select the minimal level m as small as possible (see [BF01] for an approach), for convenience, in order to be not forced to handle exceptional cases corresponding to the coarsest levels, we will assume that m is sufficiently large .

From the assumptions on I_j, the uniform boundedness of all scaling functions, $(\mathcal{L})$, $(\tilde{\mathcal{L}})$, and $(\mathcal{D})$, it follows that both Φ_j and $\tilde\Phi_j$ are *uniform* $L_2(0,1)$-*Riesz systems* (see e.g. [DKU99, Lemma 2.1]).

In addition, for $j\in\mathbb{Z}_{m+1}$, we assume an index set $J_j\subset(0,1)$ with $J_j\cap I_j=\emptyset$, $x\in J_j$ if and only if $1-x\in J_j$, and $\sup_{k\in\mathbb{N}_0}\#(J_j\cap[k2^{-j},(k+1)2^{-j}])\lesssim 1$. For $j\in\mathbb{Z}_{m+1}$, we assume collections

$$\Xi_j=[\xi_{j,x}]_{x\in J_j}\subset\mathcal{S}(\Phi_{j+1})$$

that satisfy corresponding conditions $(\mathcal{L})$, $(\mathcal{S})$, and $(\mathcal{V})$, the latter meaning that all $\xi_{j,x}$ vanish at $\{0,1\}$, such that

$$\Upsilon_{j+1}:=\begin{bmatrix}\Phi_j^T & \Xi_j^T\end{bmatrix}^T \text{ is a } \textit{uniform } L_2(0,1)\textit{-Riesz basis for } \mathcal{S}(\Phi_{j+1}),$$

and

$$\langle \Upsilon_{j+1}, \tilde{\Phi}_{j+1}\rangle^{-1} \text{ is } \textit{uniformly local}, \tag{2.1}$$

with which we mean that only entries of this matrix indexed by (x, y) with $|x - y| \lesssim 2^{-j}$ might be nonzero. The last condition is only needed if one is interested in having also locally supported *dual* wavelets. By $\Upsilon_{j+1} = \langle \Upsilon_{j+1}, \tilde{\Phi}_{j+1}\rangle \Phi_{j+1}$, note that $\langle \Upsilon_{j+1}, \tilde{\Phi}_{j+1}\rangle^{-T}$ represents the transformation from Φ_{j+1} to the two-level basis Υ_{j+1}.

Remark 2.1. In [DS99a], the matrices $\langle \Upsilon_{j+1}, \tilde{\Phi}_{j+1}\rangle^T$ and $\langle \Upsilon_{j+1}, \tilde{\Phi}_{j+1}\rangle^{-T}$ are denoted as $\breve{\mathbf{M}}_j$ and $\breve{\mathbf{G}}_j$, respectively. The completion of Φ_j by Ξ_j to a uniform L_2-Riesz basis for $\mathcal{S}(\Phi_{j+1})$, or equivalently, the completion of the $\#I_{j+1} \times \#I_j$ matrix $\langle \Phi_j, \tilde{\Phi}_{j+1}\rangle^T$ by the $\#I_{j+1} \times \#J_j$ matrix $\langle \Xi_j, \tilde{\Phi}_{j+1}\rangle^T$ to a uniformly well-conditioned matrix $\langle \Upsilon_{j+1}, \tilde{\Phi}_{j+1}\rangle^T$ is known as an (initial) *stable completion* (see [CDP96]).

For any $2 \le d \le \tilde{d}$ with $d + \tilde{d}$ even, collections Φ_j, $\tilde{\Phi}_j$ that satisfy all above assumptions with $\gamma = d - \frac{1}{2}$ and $\tilde{\gamma}$ growing proportionally with $\tilde{d}$ were constructed in [DKU99]. For some parameter $\mathbb{N} \ni r \ge d - 1$ that one may choose, the primal collections Φ_j span the standard spline space of order d with respect to the knot sequence

$$(\underbrace{0, \ldots, 0}_{d \text{ times}}, r2^{-j}, r2^{-j} + 2^{-j}, \ldots, 1 - r2^{-j}, \underbrace{1, \ldots, 1}_{d \text{ times}}). \tag{2.2}$$

As collections Ξ_j one may take bases for the biorthogonal complement spaces $\mathcal{S}(\Phi_{j+1}) \cap \mathcal{S}(\Phi_j)^{\perp_{L_2(0,1)}}$, i.e., collections of biorthogonal wavelets on the interval as constructed in [DKU99], which, in view of $(\mathcal{V})$, have to be modified by subtracting from those wavelets that do not vanish at 0 or 1 a suitable multiple of $\phi_{j-1,0}$ or $\phi_{j-1,1}$ (cf. [DS99a, Remark 2.4.2]). These biorthogonal wavelets themselves were constructed by subtracting biorthogonal projections from simpler initial stable completions Ξ_j. Alternatively, just these completions can be applied, which, for $n > 1$, actually leads to composite wavelets with smaller (singular) supports (cf. [Har01, HS04]).

2.2. **Modified scaling functions.** We are going to construct sequences of collections of modified primal and dual scaling functions $(\Phi_j^{\text{new}})_{j \in \mathbb{Z}_m}$ and $(\tilde{\Phi}_j^{\text{new}})_{j \in \mathbb{Z}_m}$, spanning the original spaces, which satisfy $(\mathcal{D})$, and all of $(\mathcal{L})$–$(\mathcal{V})$ or $(\tilde{\mathcal{L}})$–$(\tilde{\mathcal{B}})$, respectively, and, *in addition*, for which

$$P_{\tilde{d}-1}(0,1) \subset \mathcal{S}(\tilde{\Phi}_j^{\text{new}} \backslash \{\tilde{\phi}_{j,0}^{\text{new}}, \tilde{\phi}_{j,1}^{\text{new}}\}), \tag{2.3}$$

and, with ∂ being each of $\{0\}$, $\{1\}$ or $\{0, 1\}$,

$$\mathcal{S}(\{\tilde{\phi}_{j,x}^{\text{new}} : x \in I_j \backslash \partial\}) \subset \mathcal{S}(\{\tilde{\phi}_{j+1,x}^{\text{new}} : x \in I_{j+1} \backslash \partial\}). \tag{2.4}$$

We will search the modified collections in the form

$$\tilde{\Phi}_j^{\text{new}} = \mathbf{B}_j \tilde{\Phi}_j, \quad \Phi_j^{\text{new}} = \mathbf{B}_j^{-T} \Phi_j, \tag{2.5}$$

with, for some vector $\alpha_j = [\alpha_{j,x}]_{x \in I_j \backslash \{0,1\}}$ of uniformly bounded scalars with a uniformly bounded number of non-zeros, and $\alpha_j^{\updownarrow} := [\alpha_{j,1-x}]_{x \in I_j \backslash \{0,1\}}$,

$$\mathbf{B}_j := \left[\begin{array}{c|c|c} 1 & \cdots 0 \cdots & 0 \\ \hline \alpha_j & I & \alpha_j^{\updownarrow} \\ \hline 0 & \cdots 0 \cdots & 1 \end{array}\right], \quad \text{and thus} \quad \mathbf{B}_j^{-T} = \left[\begin{array}{c|c|c} 1 & -\alpha_j^T & 0 \\ \hline \vdots & & \vdots \\ 0 & I & 0 \\ \vdots & & \vdots \\ \hline 0 & -(\alpha_j^{\updownarrow})^T & 1 \end{array}\right]. \tag{2.6}$$

Since these transformations do not change the spans, and retain $(\mathcal{D})$, and $(\mathcal{L})$, $(\mathcal{S})$, $(\mathcal{V})$, and $(\tilde{\mathcal{L}})$, $(\tilde{\mathcal{S}})$, the only properties to verify are (2.3) and (2.4).

Under some mild additional conditions, the following theorem shows that suitable coefficients $\alpha_{j,x}$ can always be found.

Theorem 2.2. *Assume, for $j \in \mathbb{Z}_m$, that $\langle \phi_{j,0}, \tilde{\phi}_{j+1,1}\rangle = 0$ and*

$$\langle \tilde{\phi}_{j,x}, \phi_{j+1,0}\rangle = 0 \quad \text{for } x \in I_j \backslash \{0\}, \tag{2.7}$$

and that

$$\sup_{j\in\mathbb{Z}_m} \langle \phi_{j,0}, \tilde{\phi}_{j+1,0}\rangle\langle \tilde{\phi}_{j,0}, \phi_{j+1,0}\rangle < 1. \tag{2.8}$$

Then, with

$$\alpha_{j+1,x} := \frac{\langle \tilde{\phi}_{j,0}, \phi_{j+1,0}\rangle}{1 - \langle \phi_{j,0}, \tilde{\phi}_{j+1,0}\rangle\langle \tilde{\phi}_{j,0}, \phi_{j+1,0}\rangle} \langle \phi_{j,0}, \tilde{\phi}_{j+1,x}\rangle, \quad (x \in I_{j+1}\backslash\{0,1\}),$$

we have

$$\mathcal{S}(\tilde{\Phi}_j) \subset \mathcal{S}(\tilde{\Phi}^{\text{new}}_{j+1} \backslash \{\tilde{\phi}^{\text{new}}_{j+1,0}, \tilde{\phi}^{\text{new}}_{j+1,1}\}) \tag{2.9}$$

which implies both (2.4) *and, after replacing m by $m+1$, by* $(\tilde{\mathcal{J}})$ *also* (2.3).

Before proving this theorem, note that the condition $\langle \phi_{j,0}, \tilde{\phi}_{j+1,1}\rangle = 0$ follows from $(\mathcal{L})$, $(\tilde{\mathcal{L}})$ when m is sufficiently large. When the dual scaling functions are continuous at 0, (2.7) follows from

$$\tilde{\phi}_{j,x}(0) = 0 \text{ for } x \in I_j\backslash\{0\},\ j \in \mathbb{Z}_m, \tag{2.10}$$

which is the analogue of $(\mathcal{V})$ at the dual side, since, by $(\tilde{\mathcal{J}})$, then $\tilde{\phi}_{j,0}(0) \neq 0$ ($j \in \mathbb{Z}_m$). A modification of the collections Φ_j, $\tilde{\Phi}_j$ from [DKU99] such that (2.10) is valid is introduced in [DS99a]. The condition (2.8) is satisfied when the boundary primal and dual scaling functions on all levels are generated from one pair by means of dyadic dilation, in which case $\langle \phi_{j,0}, \tilde{\phi}_{j+1,0}\rangle\langle \tilde{\phi}_{j,0}, \phi_{j+1,0}\rangle = \frac{1}{2}$.

Proof. By $(\mathcal{L})$, $(\tilde{\mathcal{L}})$, the number of nonzero coefficients $\alpha_{j+1,x}$ is uniformly bounded.

Because of $(\mathcal{N})$, $(\tilde{\mathcal{N}})$, for $v \in \mathcal{S}(\tilde{\Phi}_j)$ we have

$$\begin{aligned} v = \langle v, \Phi_j\rangle\tilde{\Phi}_j &= \Big\langle v, \langle \Phi_j, \tilde{\Phi}_{j+1}\rangle \Phi_{j+1}\Big\rangle \langle \tilde{\Phi}_j, \Phi_j\rangle \tilde{\Phi}_{j+1} = \\ &= \langle v, \Phi_{j+1}\rangle\langle \Phi_j, \tilde{\Phi}_{j+1}\rangle^T \langle \tilde{\Phi}_j, \Phi_{j+1}\rangle\tilde{\Phi}_{j+1}, \end{aligned}$$

and also

$$v = \langle v, \Phi_{j+1}\rangle\tilde{\Phi}_{j+1}. \tag{2.11}$$

Comparing the coefficient in front of $\tilde{\phi}_{j+1,0}$ in both expressions for v using (2.7), we infer that

$$\begin{aligned} &\Big(1 - \langle \phi_{j,0}, \tilde{\phi}_{j+1,0}\rangle\langle \tilde{\phi}_{j,0}, \phi_{j+1,0}\rangle\Big) \langle v, \phi_{j+1,0}\rangle = \\ &\qquad \sum_{x\in I_{j+1}\backslash\{0\}} \langle \phi_{j,0}, \tilde{\phi}_{j+1,x}\rangle\langle \tilde{\phi}_{j,0}, \phi_{j+1,0}\rangle\langle v, \phi_{j+1,x}\rangle, \end{aligned}$$

or, by definition of $\alpha_{j+1,x}$ and the assumption $\langle \phi_{j,0}, \tilde{\phi}_{j+1,1}\rangle = 0$, that

$$\langle v, \phi_{j+1,0}\rangle = \sum_{x\in I_{j+1}\backslash\{0,1\}} \alpha_{j+1,x}\langle v, \phi_{j+1,x}\rangle,$$

and so by $(\tilde{\mathcal{S}})$ that $\langle v, \phi_{j+1,1}\rangle = \sum_{x\in I_{j+1}\backslash\{0,1\}} \alpha_{j+1,1-x}\langle v, \phi_{j+1,x}\rangle$. By substituting these expressions into (2.11), by definition of $\tilde{\Phi}^{\text{new}}_{j+1}$ we find that

$$v = \sum_{x\in I_{j+1}\backslash\{0,1\}} \langle v, \phi_{j+1,x}\rangle\tilde{\phi}^{\text{new}}_{j+1,x} \quad \Big(= \sum_{x\in I_{j+1}\backslash\{0,1\}} \langle v, \phi^{\text{new}}_{j+1,x}\rangle\tilde{\phi}^{\text{new}}_{j+1,x} \Big)$$

which completes the proof. $\square$

In Theorem 2.2 the vector α_j was constructed such that (2.9) is valid, which property generally is stronger than the combination of (2.3) and (2.4), and therefore generally requires more nonzero coefficients than is strictly necessary, resulting in modified scaling functions with larger supports. In the following theorem, under some additional conditions, however without (2.7), we present a construction that realizes (2.3) and (2.4) with a generally minimal number of nonzero coefficients $\alpha_{j,x}$.

Theorem 2.3. *For $j \in \mathbb{Z}_m$, let $\bar{I}_j \subset I_j \backslash \{0,1\}$ with $\#\bar{I}_j = \tilde{d}$, $\sup\{x : x \in \bar{I}_j\} \lesssim 2^{-j}$, and*

$$\inf_{0 \neq p \in P_{\tilde{d}-1}(0,1)} \sup_{0 \neq v \in \mathcal{S}(\{\phi_{j,x} : x \in \bar{I}_j\})} \frac{|\langle p, v \rangle|}{\|p\| \|v\|} \gtrsim 1. \tag{2.12}$$

With $Q_j : L_2(0,1) \to \mathcal{S}(\{\phi_{j,x} : x \in \bar{I}_j\})$ being the projector with $\Im(I - Q_j) \perp_{L_2(0,1)} P_{\tilde{d}-1}(0,1)$, whose existence follows from (2.12), let

$$\langle (I - Q_{j+1})\phi_{j+1,0}, \tilde{\phi}_{j,x} \rangle = 0 \quad \text{for all } x \in I_j \backslash (\bar{I}_j + \{0\}). \tag{2.13}$$

Then for each $j \in \mathbb{Z}_m$, there exists a unique vector $\alpha_j = [\alpha_{j,x}]_{x \in I_j \backslash \{0,1\}}$ with $\alpha_{j,x} = 0$ when $x \notin \bar{I}_j$, whose coefficients are uniformly bounded, such that Φ_j^{new} and $\tilde{\Phi}_j^{\text{new}}$ defined by (2.5), (2.6) satisfy (2.3) and (2.4).

Proof. We solve $\{\alpha_{j,x} : x \in \bar{I}_j\}$ from the condition

$$\phi_{j,0}^{\text{new}} := \phi_{j,0} + \sum_{x \in \bar{I}_j} \alpha_{j,x} \phi_{j,x} \perp_{L_2(0,1)} P_{\tilde{d}-1}(0,1),$$

which, after selecting some basis $\{p_0, \ldots, p_{\tilde{d}-1}\}$ for $P_{\tilde{d}-1}(0,1)$, amounts to inverting the matrix $[\langle \phi_{j,x}, p_\ell \rangle]_{x \in \bar{I}_j, 0 \leq \ell \leq \tilde{d}-1}$. Because of (2.12), there is a unique, uniformly bounded solution. Note that $\phi_{j,0}^{\text{new}} = (I - Q_j)\phi_{j,0}$.

Knowing $P_{\tilde{d}-1}(0,1) \subset \mathcal{S}(\tilde{\Phi}_j) = \mathcal{S}(\tilde{\Phi}_j^{\text{new}})$, ($\mathcal{S}$) and ($\mathcal{D}$), (2.3) is equivalent to $\phi_{j,0}^{\text{new}} \perp_{L_2(0,1)} P_{\tilde{d}-1}(0,1)$, which holds by construction.

Knowing $\mathcal{S}(\tilde{\Phi}_j^{\text{new}}) \subset \mathcal{S}(\tilde{\Phi}_{j+1}^{\text{new}})$, ($\mathcal{S}$), ($\tilde{\mathcal{S}}$) and ($\mathcal{D}$), (2.4) is equivalent to

$$\phi_{j+1,0}^{\text{new}} \perp_{L_2(0,1)} \mathcal{S}(\tilde{\Phi}_j^{\text{new}} \backslash \{\tilde{\phi}_{j,0}^{\text{new}}\}). \tag{2.14}$$

By construction, for $x \in I_j \backslash (\bar{I}_j + \{0\})$ we have $\tilde{\phi}_{j,x}^{\text{new}} \subset \mathcal{S}(\{\tilde{\phi}_{j,x} : x \in I_j \backslash (\bar{I}_j + \{0\})\})$, and so the assumption (2.13) shows that

$$\langle \phi_{j+1,0}^{\text{new}}, \tilde{\phi}_{j,x}^{\text{new}} \rangle = 0 \quad \text{for all } x \in I_j \backslash (\bar{I}_j \cup \{0\}). \tag{2.15}$$

From (2.3) we know that $p_\ell = \sum_{x \in I_j \backslash \{0,1\}} \langle p_\ell, \phi_{j,x}^{\text{new}} \rangle \tilde{\phi}_{j,x}^{\text{new}}$ for $0 \leq \ell \leq \tilde{d}-1$. From $\phi_{j+1,0}^{\text{new}} \perp_{L_2(0,1)} P_{\tilde{d}-1}(0,1)$ and (2.15), we infer that $\phi_{j+1,0}^{\text{new}} \perp_{L_2(0,1)} \sum_{x \in \bar{I}_j} \langle p_\ell, \phi_{j,x}^{\text{new}} \rangle \tilde{\phi}_{j,x}^{\text{new}}$. Since $\phi_{j,x}^{\text{new}} = \phi_{j,x}$ for $x \in I_j \backslash \{0,1\}$, and $[\langle p_\ell, \phi_{j,x} \rangle]_{0 \leq \ell \leq \tilde{d}-1, x \in \bar{I}_j}$ is an invertible system because of (2.12), this means that $\langle \phi_{j+1,0}^{\text{new}}, \tilde{\phi}_{j,x}^{\text{new}} \rangle = 0$ also for $x \in \bar{I}_j$, which completes the proof of (2.14) and thus of the theorem. □

In the following, for a particular choice of $\bar{I}_j$, we verify the conditions of Theorem 2.3 for any pair of biorthogonal scaling functions constructed in [DKU99]. By definition and ($\mathcal{V}$), $\tilde{\phi}_{j,0}$ is the dual of the unique primal scaling function $\phi_{j,0}$ that does not vanish at 0. We now take $\bar{I}_j$ to be the index set of those $\tilde{d}$ dual scaling functions other than $\tilde{\phi}_{j,0}$ for which the minima of their supports are closest to 0. Inside this collection, there is one $\tilde{\phi}_{j,z}$ such that $(\phi_{j,z}, \tilde{\phi}_{j,z})$ is a biorthogonal pair of scaling functions on the line as constructed in [CDF92]. The set $I_j^\circ := \{0\} + I_j \backslash \{z\}$ (denoted as $\tilde{I}_j^{\text{L}}$ in [DKU99]) is the index set of the left boundary adapted scaling functions. By construction of the scaling functions in [DKU99], it holds that $\langle \phi_{j+1,y}, \tilde{\phi}_{j,x} \rangle = 0$ for all $y \in \bar{I}_{j+1}$, $x \in I_j \backslash (\bar{I}_j + \{0\})$, so that (2.13) is valid.

Since all $\{\phi_{j,x} : x \in \bar{I}_j\}$ are constructed from one set by means of dyadic dilation, the quantity at the left-hand side of (2.12) is independent of j. By construction of the boundary adapted scaling functions in [DKU99], there exists a basis $\{p_x : x \in I_j^\circ\}$ for $P_{\tilde{d}-1}(0,1)$ such that $p_x = \tilde{\phi}_{j,x} + \sum_{v \in I_j \backslash I_j^\circ} \langle p_x, \phi_{j,v} \rangle \tilde{\phi}_{j,v}$. By applying this basis, using biorthogonality we infer that

(2.12) is equivalent to invertibility of $[\langle\tilde\phi_{j,x}+\langle p_x,\phi_{j,z}\rangle\tilde\phi_{j,z},\phi_{j,y}\rangle]_{x\in I_j^\circ,y\in\bar I_j}$. Since, modulo invertible transformations, $\{\tilde\phi_{j,x}+\langle p_x,\phi_{j,z}\rangle\tilde\phi_{j,z}:x\in I_j^\circ\}$ is the collection of left boundary adapted dual scaling functions that one obtains by increasing the parameter r from (2.2) by 1, and $\{\phi_{j,y}:y\in\bar I_j\}$ is the collection of left boundary adapted primal scaling functions from which $\phi_{j,0}$ is removed, and to which the interior primal scaling function succeeding $\phi_{j,z}$ is added, Theorem 4.2 of [DS98] shows that the aforementioned matrix is indeed invertible, and thus that (2.12) is valid.

Remark 2.4. For the collections $\Phi_j^{\rm new}$, $\tilde\Phi_j^{\rm new}$ yielded by Theorem 2.3, it holds that the functions from $\{\phi_{j,x}^{\rm new}:x\in I_j\backslash\{0,1\}\}$ vanish at $\{0,1\}$, and that $P_{\tilde d-1}(0,1)\subset\mathcal{S}(\{\tilde\phi_{j,x}^{\rm new}:x\in I_j\backslash\{0,1\}\})$ meaning that these reduced collections satisfy so-called *complementary boundary conditions* of order zero. Starting with any pair $(\Phi_j)_j$, $(\tilde\Phi_j)_j$ from [DKU99], one may verify that these reduced collections, if not equal, span the *same spaces* as the collections with zeroth order complementary boundary conditions constructed in [DS98]. In this paper, however, the *full* collections $\Phi_j^{\rm new}$ and $\tilde\Phi_j^{\rm new}$ will be needed. Under appropriate conditions, transformations as in Theorems 2.2 or 2.3 can be applied repeatedly, starting with $\{\phi_{j,x}^{\rm new}:x\in I_j\backslash\{0,1\}\}$ and $\{\tilde\phi_{j,x}^{\rm new}:x\in I_j\backslash\{0,1\}\}$, yielding biorthogonal collections with complementary boundary conditions of higher order.

As a final ingredient on $[0,1]$, we define collections of functionals $\Lambda_j=[\lambda_{j,x}]_{x\in I_j}\subset C[0,1]'$, $\tilde\Lambda_j=[\tilde\lambda_{j,x}]_{x\in I_j}\subset L_2(0,1)'$ by $\lambda_{j,0}(v)=v(0)$, $\lambda_{j,1}(v)=v(1)$, $\tilde\lambda_{j,0}(v)=\tilde\lambda_{j,1}(v)=0$, and, for $x\in I_j\backslash\{0,1\}$, $\lambda_{j,x}(v)=\langle v,\tilde\phi_{j,x}^{\rm new}\rangle$ and $\tilde\lambda_{j,x}(v)=\langle v,\phi_{j,x}^{\rm new}\rangle$. Note that, with the common notation as discussed at the end of the §1,

$$\langle\Phi_j^{\rm new},\Lambda_j\rangle=I,\quad\langle\tilde\Phi_j^{\rm new},\tilde\Lambda_j\rangle=\operatorname{diag}([0,1,\cdots,1,0]).$$

In the next two sections, *we will drop the superscript "new", so that with Φ_j and $\tilde\Phi_j$ collections are meant, as constructed in Theorems 2.2 or 2.3, that satisfy* ($\mathcal{D}$), ($\mathcal{L}$)–($\mathcal{V}$) *or* ($\tilde{\mathcal{L}}$)–($\tilde{\mathcal{B}}$), *respectively, and, in addition,* (2.3) *and* (2.4). Using these collections, we will construct composite wavelet bases on general domains or manifolds following the construction introduced in [DS99a]. We will present the main steps, and refer for proofs to [DS99a] or [Ste04a].

3. Biorthogonal multiresolution on $(0,1)^n$

With $\square:=(0,1)^n$, let $\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle_{L_2(\square)}$ throughout this section. With Π we will denote the collection of all affine mappings from $\square$ onto $\square$, which consists of any composition of reflections of type $x\mapsto(x_1,\ldots,x_{i-1},1-x_i,x_{i+1},\ldots,x_n)$ $(1\le i\le n)$ and permutations of the n Cartesian coordinates. With a *face* of $\square$, we mean a complete, closed face of any dimension $0\le k\le n-1$, i.e., for $n=3$, it is either a vertex, an edge or a facet.

For $j\in\mathbb{N}_0$, we define index sets $I_j^\square=(I_j)^n$, $J_j^\square=(J_j\cup I_j)^n\backslash I_j^\square$; collections of functions $\Phi_j^\square=[\phi_{j,x}^\square]_{x\in I_j^\square}$, $\tilde\Phi_j^\square=[\tilde\phi_{j,x}^\square]_{x\in I_j^\square}$ and $\Xi_{j,x}^\square=[\xi_{j,x}^\square]_{x\in J_j^\square}$, by $\phi_{j,x}^\square(y)=\prod_{i=1}^n\phi_{j,x_i}(y_i)$, $\tilde\phi_{j,x}^\square(y)=\prod_{i=1}^n\tilde\phi_{j,x_i}(y_i)$, and $\xi_{j,x}^\square(y)=\prod_{i=1}^n\omega_{j,x_i}(y_i)$ with $\omega_{j,x_i}:=\xi_{j,x_i}$ when $x_i\in J_j$ and $\omega_{j,x_i}:=\phi_{j,x_i}$ otherwise; and collections of functionals $\Lambda_j^\square=[\lambda_{j,x}]_{j\in I_j^\square}\subset C(\overline{\square})'$, $\tilde\Lambda_j^\square=[\tilde\lambda_{j,x}]_{j\in I_j^\square}\subset L_2(\square)'$ by, for univariate functions v_i, $\lambda_{j,x}^\square(\prod_i v_i)=\prod_i\lambda_{j,x_i}(v_i)$, $\tilde\lambda_{j,x}^\square(\prod_i v_i)=\prod_i\tilde\lambda_{j,x_i}(v_i)$ which determines $\lambda_{j,x}^\square$, $\tilde\lambda_{j,x}^\square$ on the whole of $C(\overline{\square})$, $L_2(\square)$, respectively. Clearly, we have $\langle\Phi_j^\square,\tilde\Phi_j^\square\rangle=I$, $\langle\Phi_j^\square,\Lambda_j^\square\rangle=I$, and $\langle\tilde\Phi_j^\square,\tilde\Lambda_j^\square\rangle=\operatorname{diag}(\{d_{j,x}:x\in I_j^\square\})$ with $d_{j,x}=0$ if $x\in\partial\square$, and $d_{j,x}=1$ otherwise.

For a proof of the next proposition we refer to [DS99a, Proposition 3.1], where for the projector at the dual side (2.3) has to be used.

Proposition 3.1. *For the projectors $P_j^\square:v\mapsto\langle v,\Lambda_j^\square\rangle\Phi_j^\square$, $\tilde P_j^\square:v\mapsto\langle v,\tilde\Lambda_j^\square\rangle\tilde\Phi_j^\square$* onto $\mathcal{S}(\Phi_j^\square)$ *and to $\mathcal{S}(\tilde\Phi_j^\square)$, respectively, we have*

$$\|(I-\tilde P_j^\square)v\|_{L_2(\square)}\lesssim2^{-\tilde dj}|v|_{H^{\tilde d}(\square)}\quad(v\in H^{\tilde d}(\square)),$$

and, when $d>\frac n2$,

$$\|(I-P_j^\square)v\|_{L_2(\square)}\lesssim2^{-dj}|v|_{H^{\tilde d}(\square)}\quad(v\in H^{d}(\square)).$$

Moreover, writing $P_j^\square v=\sum_{x\in I_j^\square}\mathbf{c}_{j,x}\phi_{j,x}^\square$, $\tilde{P}_j^\square v=\sum_{x\in I_j^\square}\tilde{\mathbf{c}}_{j,x}\tilde{\phi}_{j,x}^\square$, *it holds that if* v *vanishes on a face of* $\square$, *then* $\mathbf{c}_{j,x}=0$ *for all* x *on that face, whereas* $\tilde{\mathbf{c}}_{j,x}=0$ *for all* $x\in\partial\square$ *anyway.*

Remark 3.2. The (primal and dual) wavelets that we are going to construct on general domains or manifolds Ω will depend on $\mathcal{S}(\Phi_j^\square)$ and $\mathcal{S}(\tilde{\Phi}_j^\square)$, but not on the selection of their bases $\Phi_j^\square$ and $\tilde{\Phi}_j^\square$. On the other hand, they do depend on the collection $\Xi_j^\square$. We have constructed this collection by taking tensor products of functions from Φ_j and Ξ_j. However, $\Xi_j^\square=\{\xi_{j,x}^\square : x\in J_j^\square\}$ can be any collection that satisfies $\xi_{j,x}^\square(y)=0$ if not $|x-y|\lesssim 2^{-j}$; $\xi_{j,x}^\square$ vanishes on any face of $\square$ that does not contain x; $(\xi_{j,x}^\square)|_{\partial\square}=(\xi_{j,\pi(x)}^\square\circ\pi)|_{\partial\square}$ for all $\pi\in\Pi$; $\Upsilon_{j+1}^\square := \big[(\Phi_j^\square)^T \quad (\Xi_j^\square)^T\big]^T$ is a uniform $L_2(\square)$-Riesz basis for $\mathcal{S}(\Phi_{j+1}^\square)$; and, but only if one is interested in having dual wavelets that are also locally supported, for which $\langle\Upsilon_{j+1}^\square,\tilde{\Phi}_{j+1}^\square\rangle^{-1}$ is uniformly local. Taking an alternative collection $\Xi_j^\square$ can yield wavelets with smaller supports, or, which still has to be investigated, it might result in quantitatively better conditioned wavelet bases.

4. Biorthogonal wavelets on Ω

4.1. **Setting.** For some $n'\geq n\geq 1$, let Ω be a n-dimensional bounded manifold in $\mathbb{R}^{n'}$, with or without a boundary. We assume that Ω is given as

$$\overline{\Omega}=\cup_{q=1}^M\overline{\Omega_q},\ \text{with } \Omega_q\cap\Omega_{q'}=\emptyset \text{ when } q\neq q',\ \text{and } \Omega_q=\kappa_q(\square),$$

where $\kappa_q:\mathbb{R}^n\to\mathbb{R}^{n'}$ are some smooth, regular parametrizations. We assume that the splitting of Ω into the patches Ω_q is *conforming* in the sense for any $q\neq q'$, $\overline{\Omega_q}\cap\overline{\Omega_q'}$ is either empty, or

$$\kappa_q^{-1}(\overline{\Omega_q}\cap\overline{\Omega_q'}) \text{ is a face of } \square,$$

and, in addition, that the parametrizations can be chosen such that the following *matching condition* is satisfied: there exists a $\pi\in\Pi$ with

$$(\mathcal{M}) \qquad \kappa_{q'}\circ\pi\circ\kappa_q^{-1}=I \quad \text{on } \overline{\Omega_q}\cap\overline{\Omega_q'}.$$

Note that our setting allows Ω to be a bounded domain in $\mathbb{R}^n$, as well as an open or closed bounded manifold in $\mathbb{R}^{n'}$ for some $n'>n$.

We equip $L_2(\Omega)$ with the *modified* $L_2(\Omega)$*-scalar product*

$$\langle\!\langle u,v\rangle\!\rangle := \sum_{q=1}^M \langle u\circ\kappa_q, v\circ\kappa_q\rangle_{L_2(\square)}, \tag{4.1}$$

which defines a norm that is equivalent to the canonical $L_2(\Omega)$-norm.

We include the possibility that homogeneous, zeroth order Dirichlet boundary conditions are prescribed on some part $\partial\Omega_D\subset\overline{\Omega}\backslash\Omega$, for which, for all $1\leq q\leq M$,

$$\kappa_q^{-1}(\partial\Omega_D\cap\overline{\Omega}_q) \text{ is a, possibly empty, union of faces of } \square. \tag{4.2}$$

4.2. **Biorthogonal multiresolution on Ω.** We define the index sets $I_j^\Omega\subset\overline{\Omega}\backslash\partial\Omega_D$, and analogously J_j^Ω, by

$$I_j^\Omega=(\cup_{q=1}^M\kappa_q(I_j^\square))\cap(\overline{\Omega}\backslash\partial\Omega_D). \tag{4.3}$$

By $(\mathcal{M})$ and $\pi(I_j^\square)=I_j^\square$ $(\pi\in\Pi)$, for any $1\leq q,q'\leq M$ with $\overline{\Omega_q}\cap\overline{\Omega_{q'}}\neq\emptyset$, the sets $\kappa_q(I_j^\square)$ and $\kappa_{q'}(I_j^\square)$ restricted to this interface coincide.

Setting, for $x\in\overline{\Omega}$,

$$k(x)=\#\{q : x\in\overline{\Omega_q}\},$$

for $j\in\mathbb{N}_0$ we define the collections $\Phi_j^\Omega=[\phi_{j,x}^\Omega]_{x\in I_j^\Omega}$, $\tilde{\Phi}_j^\Omega=[\tilde{\phi}_{j,x}^\Omega]_{x\in I_j^\Omega}$, $\Xi_j^\Omega=[\xi_{j,x}^\Omega]_{x\in J_j^\Omega}$ and $\Upsilon_{j+1}^\Omega := \big[(\Phi_j^\Omega)^T \quad (\Xi_j^\Omega)^T\big]^T$ of functions on Ω by

$$\phi_{j,x}^\Omega(y)=\frac{1}{\sqrt{k(x)}}\cdot\begin{cases}\phi_{j,\kappa_q^{-1}(x)}^\square(\kappa_q^{-1}(y)) & \text{when } x\in\overline{\Omega_q},,\ y\in\Omega_q \text{ for some } 1\leq q\leq M,\\ 0 & \text{elsewhere,}\end{cases} \tag{4.4}$$

with analogous definitions of $\tilde{\phi}^{\Omega}_{j,x}$ and $\xi^{\Omega}_{j,x}$. Note that by ($\mathcal{S}$), ($\mathcal{V}$) and (4.2), $\phi_{j,x}$ and $\xi_{j,x}$ extend to *continuous* functions on $\overline{\Omega}$, that vanish on $\partial\Omega_D$. However, since $\tilde{\Phi}_j$ does not satisfy the analogue of ($\mathcal{V}$) at the dual side, the collection $\{\tilde{\phi}^{\Omega}_{j,x} : x \in \cup_{q=1}^M \kappa_q(I_j^{\square} \cap \square)\} \subset \tilde{\Phi}^{\Omega}_j$, that is, the collection of dual scaling functions that are not 'glued' over the patch interfaces, contains functions that do not vanish on all interfaces, and which thus are *discontinuous*, as well as functions that do *not vanish on* $\partial\Omega_D$. By assumption ($\mathcal{L}$) or ($\tilde{\mathcal{L}}$), the collections Φ^{Ω}_j, $\tilde{\Phi}^{\Omega}_j$, Ξ^{Ω}_j are *uniformly local.* With this we mean that $x \in \operatorname{supp} \phi_{j,x}$, and that $d_\Omega(x,y) \lesssim 2^{-j}$ for any $y \in \operatorname{supp} \phi_{j,x}$, where $d_\Omega(x,y)$ denotes the geodesic distance of x and y over Ω, i.e., the length of the shortest curve on Ω connecting x and y.

Setting $\mathbf{E}_{j,q} : \ell_2(I_j^{\square}) \to \ell_2(I_j^{\Omega})$, and analogously $\mathbf{F}_{j,q} : \ell_2(J_j^{\square}) \to \ell_2(J_j^{\Omega})$, by

$$(\mathbf{E}_{j,q}\mathbf{c}_j^{\square})_x = \frac{1}{\sqrt{k(x)}} \cdot \begin{cases} \mathbf{c}^{\square}_{j,\kappa_q^{-1}(x)} & x \in \overline{\Omega_q}, \\ 0 & \text{otherwise}, \end{cases} \tag{4.5}$$

by construction of Φ^{Ω}_j from $\Phi^{\square}_j$, and the fact that the latter collection is a uniform $L_2(\square)$-Riesz system, we have

$$\langle\!\langle \Phi^{\Omega}_j, \Phi^{\Omega}_j \rangle\!\rangle = \sum_{q=1}^M \mathbf{E}_{j,q} \langle \Phi^{\square}_j, \Phi^{\square}_j \rangle_{L_2(\square)} \mathbf{E}^T_{j,q} \eqsim \sum_{q=1}^M \mathbf{E}_{j,q}\mathbf{E}^T_{j,q} = I,$$

meaning that Φ^{Ω}_j, and analogously $\tilde{\Phi}^{\Omega}_j$ and Υ^{Ω}_j are *uniform* $L_2(\Omega)$*-Riesz systems.* Moreover, we have *biorthogonality*, i.e.,

$$\langle\!\langle \Phi^{\Omega}_j, \tilde{\Phi}^{\Omega}_j \rangle\!\rangle = \sum_{q=1}^M \mathbf{E}_{j,q} \langle \Phi^{\square}_j, \tilde{\Phi}^{\square}_j \rangle_{L_2(\square)} \mathbf{E}^T_{j,q} = \sum_{q=1}^M \mathbf{E}_{j,q}\mathbf{E}^T_{j,q} = I.$$

Setting $\Lambda^{\Omega}_j = [\lambda^{\Omega}_{j,x}]_{x \in I^{\Omega}_j}$, $\tilde{\Lambda}^{\Omega}_j = [\tilde{\lambda}^{\Omega}_{j,x}]_{x \in I^{\Omega}_j}$ by

$$\lambda^{\Omega}_{j,x}(u) = \sqrt{k(x)}\, \lambda^{\square}_{j,\kappa_q^{-1}(x)}(u \circ \kappa_q) \quad \text{when } x \in \overline{\Omega}_q,$$

with an analogous definition of $\tilde{\lambda}^{\Omega}_{j,x}$, for $P^{\Omega}_j : C(\overline{\Omega}) \to \mathcal{S}(\Phi^{\Omega}_j) : v \mapsto \langle\!\langle v, \Lambda^{\Omega}_j \rangle\!\rangle \Phi^{\Omega}_j$ we have

$$((I - P^{\Omega}_j)v) \circ \kappa_q = (I - P^{\square}_j)(v \circ \kappa_q), \quad (1 \le q \le M), \tag{4.6}$$

with the analogous statement for $\tilde{P}^{\Omega}_j : L_2(\Omega) \to \mathcal{S}(\tilde{\Phi}^{\Omega}_j) : v \mapsto \langle\!\langle v, \Lambda^{\Omega}_j \rangle\!\rangle \Phi^{\Omega}_j$.

Because of ($\mathcal{V}$), at the primal side we have the analogue of (2.3), i.e., for ∂ being each of $\{0\}$, $\{1\}$ or $\{0,1\}$, it holds that

$$\mathcal{S}(\{\phi_{j,x} : x \in I_j \backslash \partial\}) \subset \mathcal{S}(\{\phi_{j+1,x} : x \in I_{j+1} \backslash \partial\}),$$

and also

$$\mathcal{S}(\{v_{j+1,x} : x \in (I_j + J_j) \backslash \partial\}) = \mathcal{S}(\{\phi_{j+1,x} : x \in I_{j+1} \backslash \partial\}).$$

As a consequence, despite the possible exclusion of functions because of the boundary conditions, cf. (4.3), we have

$$\mathcal{S}(\Phi^{\Omega}_j) \subset \mathcal{S}(\Phi^{\Omega}_{j+1}), \quad \mathcal{S}(\tilde{\Phi}^{\Omega}_j) \subset \mathcal{S}(\tilde{\Phi}^{\Omega}_{j+1}), \quad \text{and} \quad \mathcal{S}(\Upsilon^{\Omega}_{j+1}) = \mathcal{S}(\Phi^{\Omega}_{j+1}).$$

In particular, the representations of the inclusions (the "refinement matrices") or basis transformations can be expressed in terms of their counterparts on the cube, and with that in terms of their counterparts on the interval (however, cf. Remark 3.2), as follows:

$$\begin{aligned}
\langle\!\langle \Phi_j^\Omega, \tilde{\Phi}_{j+1}^\Omega \rangle\!\rangle^T &= \sum_{q=1}^M \mathbf{E}_{j+1,q} \langle \Phi_j^\square, \tilde{\Phi}_{j+1}^\square \rangle_{L_2(\square)}^T \mathbf{E}_{j,q}^T, \\
\langle\!\langle \tilde{\Phi}_j^\Omega, \Phi_{j+1}^\Omega \rangle\!\rangle^T &= \sum_{q=1}^M \mathbf{E}_{j+1,q} \langle \tilde{\Phi}_j^\square, \Phi_{j+1}^\square \rangle_{L_2(\square)}^T \mathbf{E}_{j,q}^T, \\
\langle\!\langle \Upsilon_{j+1}^\Omega, \tilde{\Phi}_{j+1}^\Omega \rangle\!\rangle^T &= \sum_{q=1}^M \mathbf{E}_{j+1,q} \langle \Upsilon_{j+1}^\square, \tilde{\Phi}_{j+1}^\square \rangle_{L_2(\square)}^T \left[\begin{array}{c|c} \mathbf{E}_{j,q}^T & 0 \\ \hline 0 & \mathbf{F}_{j,q}^T \end{array}\right], \\
\langle\!\langle \Upsilon_{j+1}^\Omega, \tilde{\Phi}_{j+1}^\Omega \rangle\!\rangle^{-T} &= \sum_{q=1}^M \left[\begin{array}{c|c} \mathbf{E}_{j,q} & 0 \\ \hline 0 & \mathbf{F}_{j,q} \end{array}\right] \langle \Upsilon_{j+1}^\square, \tilde{\Phi}_{j+1}^\square \rangle_{L_2(\square)}^{-T} \mathbf{E}_{j+1,q}^T,
\end{aligned}$$

where, because of (2.1), as the other three matrices, $\langle\!\langle \Upsilon_{j+1}^\Omega, \tilde{\Phi}_{j+1}^\Omega \rangle\!\rangle^{-T}$ is uniformly local, meaning that only entries indexed by (x,y) with $d_\Omega(x,y) \lesssim 2^{-j}$ might be nonzero.

4.3. **Wavelets.** Since Φ_j^Ω, $\tilde{\Phi}_j^\Omega$ are biorthogonal, uniformly local, uniform $L_2(\Omega)$-Riesz systems, the collections of *primal and dual wavelets* Ψ_j^Ω, $\tilde{\Psi}_j^\Omega$ defined by

$$\Psi_j^\Omega = \Xi_j^\Omega - \langle\!\langle \Xi_j^\Omega, \tilde{\Phi}_j^\Omega \rangle\!\rangle \Phi_j^\Omega, \quad \tilde{\Psi}_j^\Omega = \left[\begin{array}{c|c} 0 & I \end{array}\right] \langle\!\langle \Upsilon_{j+1}^\Omega, \tilde{\Phi}_{j+1}^\Omega \rangle\!\rangle^{-T} \tilde{\Phi}_{j+1}^\Omega$$

are biorthogonal (with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$), uniformly local, uniform $L_2(\Omega)$-Riesz bases for the spaces $\mathcal{S}(\Phi_{j+1}^\Omega) \cap (\mathcal{S}(\tilde{\Phi}_j^\Omega))^{\perp_{\langle\!\langle \cdot,\cdot \rangle\!\rangle}}$ and $\mathcal{S}(\tilde{\Phi}_{j+1}^\Omega) \cap (\mathcal{S}(\Phi_j^\Omega))^{\perp_{\langle\!\langle \cdot,\cdot \rangle\!\rangle}}$, respectively. Here also $\langle\!\langle \Xi_j^\Omega, \tilde{\Phi}_j^\Omega \rangle\!\rangle$ can be expressed in terms of its counterpart on the cube, via

$$\begin{aligned}
\langle\!\langle \Xi_j^\Omega, \tilde{\Phi}_j^\Omega \rangle\!\rangle &= \sum_{q=1}^M \mathbf{F}_{j,q} \langle \Xi_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square)} \mathbf{E}_{j,q}^T \\
&= \sum_{q=1}^M \mathbf{F}_{j,q} \langle \Xi_j^\square, \tilde{\Phi}_{j+1}^\square \rangle_{L_2(\square)} \langle \tilde{\Phi}_j^\square, \Phi_{j+1}^\square \rangle_{L_2(\square)}^T \mathbf{E}_{j,q}^T.
\end{aligned}$$

For $s \geq 0$, we set

$$\mathcal{H}^s(\Omega) = \begin{cases} H^s_{0,\partial\Omega_D}(\Omega) & \text{when } s \leq 1, \\ H^s(\Omega) \cap H^1_{0,\partial\Omega_D}(\Omega) & \text{when } s > 1, \end{cases} \qquad \tilde{\mathcal{H}}^s(\Omega) = H^s(\Omega),$$

$\mathcal{H}^{-s}(\Omega) = \tilde{\mathcal{H}}^s(\Omega)'$, and $\tilde{\mathcal{H}}^{-s}(\Omega) = \mathcal{H}^s(\Omega)'$, where, in the true manifold case, we restrict ourselves to those s for which the definition of the Sobolev spaces is permitted by the regularity of the manifold. Furthermore, we set $\Psi_{-1}^\Omega = \Phi_0^\Omega$ and $\tilde{\Psi}_{-1}^\Omega = \tilde{\Phi}_0^\Omega$. Then, together, the nesting of primal and dual spaces, the fact that they can be equipped with biorthogonal uniform $L_2(\Omega)$-Riesz bases, the Jackson estimates that can be deduced from Proposition 3.1 and (4.6), the Bernstein estimates implied by ($\mathcal{B}$) at primal and dual side and the continuous/discontinuous gluing of basis function over the interfaces between patches at primal and dual side, imply that for $s \in (-\min\{\frac{1}{2}, \tilde{\gamma}\}, \min\{\frac{3}{2}, \gamma\})$ or $s \in (-\min\{\frac{3}{2}, \gamma\}, \min\{\frac{1}{2}, \tilde{\gamma}\})$, the collections

$$\cup_{j \geq -1} 2^{-s(j+1)} \Psi_j^\Omega \quad \text{or} \quad \cup_{j \geq -1} 2^{-s(j+1)} \tilde{\Psi}_j^\Omega$$

are *Riesz bases for* $\mathcal{H}^s(\Omega)$ *or* $\tilde{\mathcal{H}}^s(\Omega)$, respectively (cf. [DS98]), assuming that, in the true manifold case, s is such that this Sobolev space is defined.

The crucial property of the primal wavelets, as reflected in the title of this work, is given by the following result, which is a direct consequence of, for $j \geq 0$, $\mathcal{S}(\Psi_j^\Omega) \perp_{\langle\!\langle \cdot,\cdot \rangle\!\rangle} \mathcal{S}(\tilde{\Phi}_j^\Omega)$, (2.3), and $\tilde{\Phi}_j^\Omega \supset \{\tilde{\phi}_{j,x}^\Omega : x \in \cup_{q=1}^M \kappa_q(I_j^\square \cap \square)\}$.

Proposition 4.1. *For all $j \in \mathbb{N}_0$, $x \in J_j^\Omega$ and $1 \leq q \leq M$,*

$$\psi_{j,x}^\Omega|_{\Omega_q} \circ \kappa_q \perp_{L_2(\square)} Q_{\tilde{d}-1}(\square).$$

So the primal wavelets have the *patchwise cancellation property of order* $\tilde{d}$, which has the attractive consequences concerning *matrix compression* as discussed at the beginning of this paper.

5. Illustration

We choose the simplest case from [DKU99] to illustrate the proposed construction, namely the case $d = \tilde{d} = 2$ (and $r = 1$, cf. (2.2)). Actually, we take the collections Φ_j, $\tilde{\Phi}_j$ from [DS99a], which differ from those from [DKU99] by basis transformations. These transformations have no influence on the new primal and dual scaling functions, also not after "gluing", and so not on our wavelets, but they were essential for the construction of the composite wavelets in [DS99a], since they guarantee the analogue of $(\mathcal{V})$ at the dual side. The collection Φ_j spans the space of continuous piecewise linears with respect to an equidistant subdivision of the interval $[0,1]$ into 2^j intervals. With $I_j := \{2^{-jk} : k = 0, 1, \ldots, 2^j\}$, the primal scaling functions are defined by $\phi_{j,x}(y) = \delta_{xy} 2^{j/2}$ $(y \in I_j)$, i.e., they are the familiar "hat functions", except for the boundary function

$$\phi_{j,0}(x) = 2^{j/2} \begin{cases} 1 - 2^j \cdot 7x/6, & 0 \le x \le 2^{-j}, \\ 2^j \cdot x/6 - 1/3, & 2^{-j} \le x \le 2 \cdot 2^{-j}, \\ 0, & \text{elsewhere,} \end{cases}$$

and likewise $\phi_{j,1}$. Applying a canonical ordering of the scaling functions, the primal and dual refinement relations are given by

$$\langle \Phi_j, \tilde{\Phi}_{j+1} \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 7/12 & -1/6 & -1/12 & & & \\ & 1/2 & 1 & 1/2 & & & \\ & & & 1/2 & 1 & 1/2 & \\ & & & & & & \ddots \end{bmatrix},$$

$$\langle \tilde{\Phi}_j, \Phi_{j+1} \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 3/2 & -3/4 & & & & & \\ & 1/2 & 3/2 & 1/2 & -1/4 & & & \\ & & -1/4 & 1/2 & 3/2 & 1/2 & -1/4 & \\ & & & & & & & \ddots \end{bmatrix}.$$

To find suitably modified scaling functions, we can either apply Theorem 2.2 or 2.3, that result in an update of $\phi_{j,0}$ with a linear combination of 2 or 3 neighbouring scaling functions, respectively. We apply Theorem 2.3, which amounts to solving $\alpha_{j,2^{-j}}$ and $\alpha_{j,2\cdot 2^{-j}}$ such that $\phi_{j,0} + \alpha_{j,2^{-j}} \phi_{j,2^{-j}} + \alpha_{j,2\cdot 2^{-j}} \phi_{j,2\cdot 2^{-j}} \perp_{L_2(0,1)} P_1(0,1)$. From the discussion following the proof of Theorem 2.3 we know in advance that a unique solution exists. We found $\alpha_{j,2^{-j}} = 2/3$ and $\alpha_{j,2\cdot 2^{-j}} = -1/3$, so that the new boundary function is given by

$$\phi_{j,0}^{\text{new}}(x) = 2^{j/2} \begin{cases} 1 - 2^j \cdot 11x/6, & 0 \le x \le 2^{-j}, \\ 2^j \cdot 7/6x - 2, & 2^{-j} \le x \le 2 \cdot 2^{-j}, \\ 1 - 2^j x/3 & 2 \cdot 2^{-j} \le x \le 3 \cdot 2^{-j}, \\ 0, & \text{elsewhere,} \end{cases}$$

and likewise $\phi_{j,1}^{\text{new}}$. Making the corresponding transformation at the dual side indicated in (2.5) and (2.6), the new refinement matrices are given by

$$\langle \Phi_j^{\text{new}}, \tilde{\Phi}_{j+1}^{\text{new}} \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 11/12 & -7/6 & -1/4 & 1/3 & 1/6 & \\ & 1/2 & 1 & 1/2 & & & \\ & & & 1/2 & 1 & 1/2 & \\ & & & & & & \ddots \end{bmatrix},$$

$$\langle \tilde{\Phi}_j^{\text{new}}, \Phi_{j+1}^{\text{new}} \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -1/4 & 3/2 & -3/4 & & & & & & & \\ & 3/2 & 1 & 1/2 & -1/4 & & & & & \\ & -1/2 & 0 & 1/2 & 3/2 & 1/2 & -1/4 & & & \\ & & & & -1/4 & 1/2 & 3/2 & 1/2 & -1/4 & \\ & & & & & & & & & \ddots \end{bmatrix}.$$

The new scaling functions closest to the left boundary of $[0,1]$ are shown in Figure 5, where

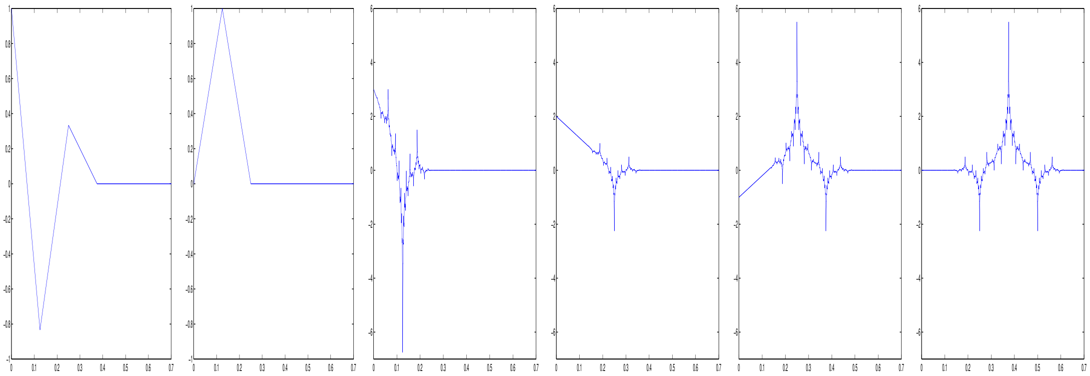

FIGURE 1. The first two primal (left) and the first four dual (right) new scaling functions.

the second primal and the fourth dual scaling function correspond to the stationary functions on the line from [CDF92]. Note that there are three new dual scaling functions that do not vanish at 0, whereas there is only one function in $\tilde{\Phi}_j$ with that property. The sets $\Phi_j \setminus \{\phi_{j,0}^{\text{new}}, \phi_{j,1}^{\text{new}}\}$ and $\tilde{\Phi}_j \setminus \{\tilde{\phi}_{j,0}^{\text{new}}, \tilde{\phi}_{j,1}^{\text{new}}\}$ are identical to the scaling functions constructed in [DS98] to obtain wavelets with complementary boundary conditions (cf. Remark 2.4).

As a first application of the modified scaling functions to construct composite wavelets, we consider the one-dimensional domain $\Omega = [0,2]$ with patches $[0,1]$ and $[1,2]$, where we impose Dirichlet boundary conditions at $x = 2$. The scaling functions are glued according to (4.4). Then, setting $J_j := I_{j+1} \setminus I_j$ and using the canonical hierarchical basis as initial complementary basis Ξ_j, we obtain the wavelet functions. In Figure 5 one finds the left boundary wavelet, one of both wavelets with supports across the interface, and the right boundary wavelet satisfying the Dirichlet boundary condition. Notice that the dual wavelet with support across the interface $x = 1$ exhibits there a jump.

We consider next the two-dimensional L-shaped domain $\Omega = ([0,2] \times [0,2.5]) \setminus ([1,2] \times [0,1.5])$ which we represent by three patches according to Figure 5. On the faces indicated by the black bullets we prescribe Dirichlet boundary conditions. As set $\Xi_j^{\square}$ on the reference domain $\square = (0,1)^2$, we employ the standard choice $[\Phi_j^{\text{new}} \otimes \Xi_j, \Xi_j \otimes \Phi_j^{\text{new}}, \Xi_j \otimes \Xi_j]$ (cf. Remark 3.2).

In Figure 5 we plotted two primal wavelets and their corresponding duals which live near the re-entrant corner $(x,y) = (1,1.5)$. Even though we have mixed boundary conditions in a neighbourhood of this vertex, these primal wavelets satisfy homogeneous Dirichlet boundary conditions everywhere. In Figure 5 we plotted two other wavelets, where one satisfies Dirichlet boundary conditions and the other does not. All duals wavelets do not satisfy Dirichlet boundary conditions.

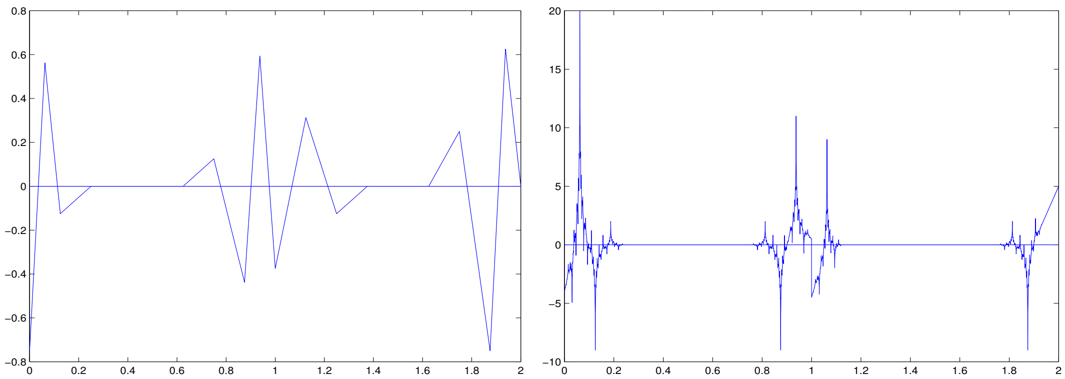

FIGURE 2. Primal (left) and corresponding dual (right) wavelets on $\Omega = [0, 2]$ composed from the patches $[0, 1]$ and $[1, 2]$.

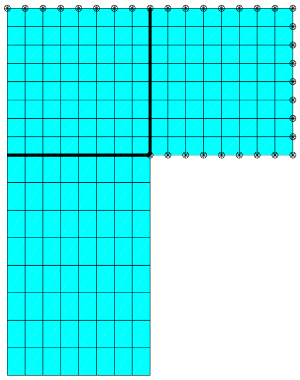

FIGURE 3. The L-shaped domain as a union of three patches.

The implementation of the wavelets follows the same lines as that of the (standard) composite wavelets introduced in [DS99a], see [Har01, HS04] for details. However, those composite wavelets that "meet" Dirichlet boundary conditions or, due to the jump in the Jacobians of the parametrizations, those with supports across the interface between the rectangle $[0, 1] \times [0, 1.5]$ and the square $[0, 1] \times [1.5, 2.5]$ have no cancellation properties with respect to the canonical $L_2(\Omega)$-scalar product. All our primal wavelets satisfy patchwise cancellation properties of order 1, and so in particular cancellation properties of order 1 with respect to the canonical $L_2(\Omega)$-scalar product. On the other hand, it is fair to say that, in this example, the $L_2(\Omega)$-condition number of our primal wavelets was about four times the $L_2(\Omega)$-condition number of the corresponding composite wavelets. For completeness, note that, due to the different dual spaces, the collection of wavelets on any given level constructed in this paper span a different space than the collection of standard composite wavelets on that level.

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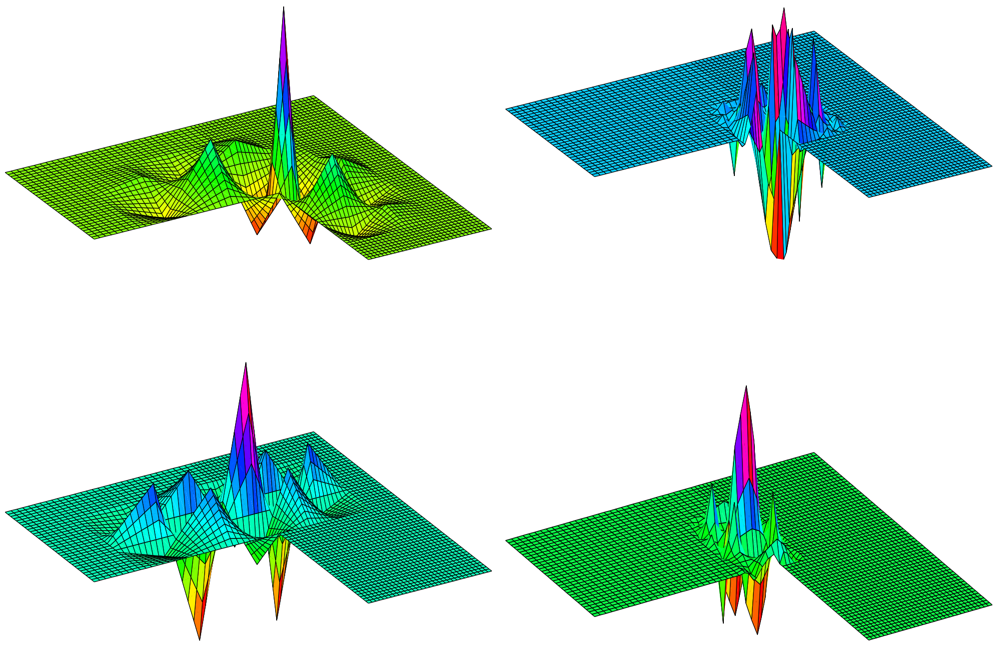

FIGURE 4. Primal (left) and corresponding dual (right) wavelets on the L-shaped domain near the re-entrant corner.

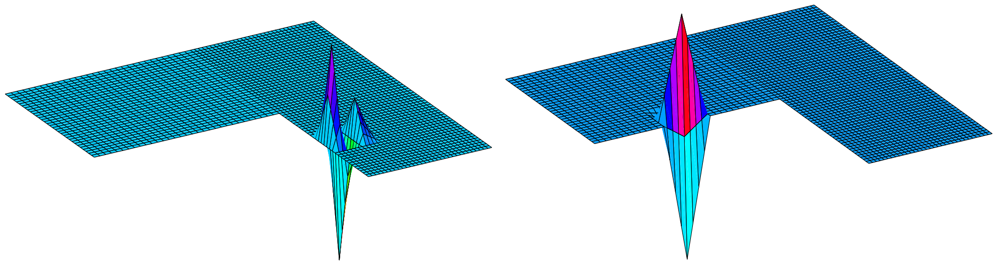

FIGURE 5. Wavelets with (left) and without (right) Dirichlet boundary conditions.

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