COMPUTATION OF SINGULAR INTEGRAL OPERATORS IN WAVELET COORDINATES

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Abstract. With respect to a wavelet basis, singular integral operators can be well approximated by sparse matrices, and in [*Found. Comput. Math.*, 2 (2002), pp. 203–245] and [*SIAM J. Math. Anal.*, 35 (2004), pp. 1110–1132], this property was used to prove certain optimal complexity results in the context of adaptive wavelet methods. These results, however, were based upon the assumption that, on average, each entry of the approximating sparse matrices can be computed at unit cost. In this paper, we confirm this assumption by carefully distributing computational costs over the matrix entries in combination with choosing efficient quadrature schemes.

1. Introduction

Boundary integral methods reduce elliptic boundary value problems in domains to integral equations formulated on the boundary of the domain. Although the dimension of the underlying manifold decreases by one, the finite element discretization of the resulting boundary integral equations gives densely populated stiffness matrices, causing serious obstructions to accurate numerical solution processes. In order to overcome this difficulty, various successful approaches for approximating the stiffness matrix by sparse ones have been developed, such as multipole expansions, panel clustering, and wavelet compression, see e.g. [Atk97, Hac95]. We will restrict ourselves here to the latter approach.

In [BCR91], Beylkin, Coifman and Rokhlin first observed that wavelet bases give rise to almost sparse stiffness matrices for the Galerkin discretization of singular integral operators, meaning that the stiffness matrix has many small entries that can be discarded without reducing the order of convergence of the resulting solution. This result ignited the development of efficient compression techniques for boundary integral equations based upon wavelets. In [vPS97, Sch98, DHS02] it was shown that for a wide class of boundary integral operators a wavelet basis can be chosen so that the full accuracy of the Galerkin discretization can be retained at a computational work of the order N (possibly with a logarithmic factor in some studies), where N is the number of degrees of freedom used in the discretization. First nontrivial implementations of these algorithms and their performance tests are reported in [LS99, Har01].

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The main reason why a stiffness matrix entry is small is that the kernel of the involved integral operator is increasingly smooth away from its diagonal, and that the wavelets have vanishing moments, i.e., wavelets are L_2-orthogonal to polynomials up to a certain degree. Another advantage of the Galerkin-wavelet discretization is that the diagonally scaled stiffness matrices are well-conditioned uniformly in their sizes, guaranteeing a uniform convergence rate of iterative methods for the linear systems. Moreover, recent developments suggest a natural use of wavelets in adaptive discretization methods that approximate the solution using, up to a constant factor, as few degrees of freedom as possible. In the following, we will consider the adaptive wavelet method from [CDD02].

Let $H^t(\Gamma)$ be the usual Sobolev space defined on a sufficiently smooth n-dimensional manifold $\Gamma \subset I\!R^{n+1}$, and let $H^{-t}(\Gamma)$ be its dual space. Then we consider the problem of finding the solution $u \in H^t(\Gamma)$ of

$$Lu = g,$$

where $L : H^t(\Gamma) \to H^{-t}(\Gamma)$ is a boundedly invertible linear operator, and $g \in H^{-t}(\Gamma)$. We will think of this problem as being the result of a variational formulation of a strongly elliptic boundary integral equation of order $2t$. With $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$ being a Riesz basis for $H^t(\Gamma)$, an equivalent infinite matrix-vector problem reads as

$$\mathbf{M}\mathbf{u} = \mathbf{g}, \tag{1.1}$$

where, with $\langle \, , \, \rangle$ denoting the duality product on $H^t(\Gamma) \times H^{-t}(\Gamma)$, $\mathbf{M} := \langle \Psi, L\Psi \rangle : \ell_2(\Lambda) \to \ell_2(\Lambda)$ is boundedly invertible, $\mathbf{g} := \langle \Psi, g \rangle \in \ell_2(\Lambda)$, and $u = \mathbf{u}^T\Psi$.

Considering Ψ to be a wavelet basis, in [CDD02] an iterative adaptive method has been developed for approximating the solution of this infinite dimensional problem by a finitely supported vector within any given tolerance. Roughly speaking, the method consists of the application of a simple iterative scheme to the infinite matrix-vector problem, where each application of the infinite stiffness matrix $\mathbf{M}$ is replaced by an *inexact* version. To assess the quality of the method, the $\ell_2(\Lambda)$-error of the obtained approximation after spending $\mathcal{O}(N)$ operations is compared with that of a *best N-term approximation* for $\mathbf{u}$, i.e., a vector $\mathbf{u}_N$ with at most N non-zero coefficients that has $\ell_2(\Lambda)$-distance to $\mathbf{u}$ less or equal to that of any vector with a support of that size.

In any case for wavelets that are sufficiently smooth, the theory of non-linear approximation ([DeV98]) shows that if *both*

$$0 < s < \tfrac{d-t}{n},$$

where d is the order of the wavelets, *and* the solution u is in the Besov space $B^{sn+t}_\tau(L_\tau(\Gamma))$ with $\frac{1}{\tau} = \frac{1}{2} + s$, then $\mathbf{u} \in \mathcal{A}^s$, meaning that

$$\sup_{N \in I\!N} N^s \|\mathbf{u} - \mathbf{u}_N\| < \infty. \tag{1.2}$$

Here $\| \, \|$ denotes the standard norm on $\ell_2(\Lambda)$, and later, on other occasions, the standard norm on the space of linear operators from $\ell_2(\Lambda)$ to $\ell_2(\Lambda)$. Note that for any $\mathbf{v} \in \ell_2(\Lambda)$, $\|u - \mathbf{v}^T\Psi\|_{H^t(\Gamma)} \eqsim \|\mathbf{u} - \mathbf{v}\|$. In order to avoid the repeated use of generic but unspecified constants, in this paper by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$. The attractive feature of these best N-term approximations is the fact that the condition involving Besov regularity is much milder than the corresponding condition $u \in H^{sn+t}(\Gamma)$ involving Sobolev regularity that would be needed to guarantee the same rate of convergence with approximation from the fixed, i.e., non-adaptive spaces spanned by N wavelets on the coarsest scales. Note that with wavelets of order d, the maximal rate that can be expected is $\frac{d-t}{n}$.

The efficiency of the adaptive method from [CDD02] hinges on the cost of the approximate matrix-vector product, which depends how well $\mathbf{M}$ can be approximated by a computable sparse matrix. We will use the following definition.

Definition 1.1. $\mathbf{M}$ is called *s^*-computable*, when for each $j \in \mathbb{N}_0$, we can construct an infinite matrix $\mathbf{M}_j^*$ having in each column $\mathcal{O}(2^j)$ non-zero entries, *whose computation takes $\mathcal{O}(2^j)$ operations*, such that for any $s < s^*$, $\|\mathbf{M} - \mathbf{M}_j^*\| \lesssim 2^{-js}$.

The main theorem from [CDD02] now says that if $\mathbf{u} \in \mathcal{A}^s$ for some s, *and* $\mathbf{M}$ is s^*-computable for an $s^* > s$, then the number of arithmetic operations and storage locations used by the adaptive wavelet algorithm for computing an approximation for $\mathbf{u}$ within tolerance ε is of the order $\varepsilon^{-1/s}$. Since in view of (1.2) the same order of storage locations is generally needed to approximate $\mathbf{u}$ within this tolerance using best N-term approximations, assuming these would be available, this result shows that this solution method has *optimal computational complexity*.

Remark 1.2. Actually, instead of being s^*-computable, in [CDD02] it was assumed that $\mathbf{M}$ is "s^*-compressible". Apart from our addition that each column of $\mathbf{M}_j^*$ should not only have $\mathcal{O}(2^j)$ entries, but also that, on average, the computation of each of these entries should take $\mathcal{O}(1)$ operations, it is easily seen that the definition of "s^*-compressible" from [CDD02] is equivalent to our definition of s^*-computable (cf. [Ste04a, Remark 2.4]). In [CDD02] the average unit cost assumption was mentioned separately afterwards (in Assumption 2).

Remark 1.3. In Definition 1.1, we may allow the computational cost and the number of non-zeroes in each column of $\mathbf{M}_j^*$ to be $\mathcal{O}(j^c 2^j)$ with any fixed constant $c \in \mathbb{R}$. Indeed, in the spirit of Remark 2.4 of [Ste04a], one can show that this results in an equivalent definition.

To conclude optimality of the adaptive wavelet method, it is necessary to show that $\mathbf{M}$ is s^*-computable for some $s^* \geq \frac{d-t}{n}$, since otherwise for a solution u that has sufficient Besov regularity, the computability will be the limiting factor. On the other hand, since, for wavelets of order d, by imposing whatever smoothness conditions $\mathbf{u} \in \mathcal{A}^s$ can only be guaranteed for $s < \frac{d-t}{n}$, showing s^*-computability for some $s^* \geq \frac{d-t}{n}$ is also a sufficient condition for optimality of the adaptive wavelet method.

Assuming the average unit cost property, s^*-computability for some $s^* \geq \frac{d-t}{n}$ has been demonstrated in [Ste04a] for both differential and singular integral operators, and piecewise polynomial wavelets that are sufficiently smooth and have sufficiently many vanishing moments. More precisely, under such conditions it was proven that for some $s^* \geq \frac{d-t}{n}$, the infinite stiffness matrix $\mathbf{M}$ is s^*-compressible, a concept that, different than in [CDD02], we define as follows.

Definition 1.4. $\mathbf{M}$ is called *s^*-compressible*, when for each $j \in \mathbb{N}_0$, there exists an infinite matrix $\mathbf{M}_j$, *constructed by dropping entries from* $\mathbf{M}$, such that in each column it has $\mathcal{O}(2^j)$ non-zero entries, and that for any $s < s^*$, $\|\mathbf{M} - \mathbf{M}_j\| \lesssim 2^{-js}$.

Only in the special case of a differential operator with constant coefficients, entries of $\mathbf{M}$ can be computed exactly, in $\mathcal{O}(1)$ operations, so that s^*-compressibility immediately implies s^*-computability. In general, numerical quadrature is required to approximate the entries. In this paper, considering singular integral operators resulting from the boundary integral method, we will show that $\mathbf{M}$ is s^*-computable for the same value of s^* as is was shown to be s^*-compressible. The case of differential operators is treated in the paper [GS04]. We split the task into two parts. First we derive a criterion on the accuracy-work balance of a numerical quadrature scheme to approximate any entry of $\mathbf{M}$, such that, for a suitable choice of the work invested in approximating the entries of the compressed matrix $\mathbf{M}_j$ as function of both wavelets involved, we obtain an approximation $\mathbf{M}_j^*$ of which the computation of each column requires $\mathcal{O}(j^c 2^j)$ operations with a fixed constant c (cf. Remark 1.3), and $\|\mathbf{M}_j - \mathbf{M}_j^*\| \leq 2^{-js^*}$, meaning that $\mathbf{M}$ is s^*-computable. Second, we show that we can fulfill the above criterion by the application of certain quadrature rules of variable order.

This paper is organized as follows. We collect some error estimates for numerical quadrature in Section 2. In Section 3, assumptions are formulated on the singular integral operator and the wavelets, and the result concerning s^*-compressibility is recalled from [Ste04a]. Then in Section 4, rules for the numerical approximation of the entries of the stiffness matrix are derived, with which s^*-computability for some $s^* \geq \frac{d-t}{n}$ will be demonstrated.

At the end of this introduction, we fix a few more notations. A monomial of n variables is conveniently written using a *multi-index* $\alpha \in \mathbb{N}_0^n$ as $x^\alpha := x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Likewise we write partial differentiation operators, that is, $\partial^\alpha := \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$. We set $|\alpha| := \alpha_1 + \ldots + \alpha_n$, and the relation $\alpha \leq \beta$ is defined as $\alpha_i \leq \beta_i$ for all $i \in \overline{1,n}$. We have $|\alpha \pm \beta| = |\alpha| \pm |\beta|$ provided that $\alpha - \beta \in \mathbb{N}_0^n$ in case of subtraction. Binomial coefficients are naturally defined as $\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \ldots \binom{\alpha_n}{\beta_n}$.

2. Error estimates for numerical quadrature

In this section, we recall some quadrature error estimates, referring to e.g. [GS04] for detailed proofs. We define the *radius* of a star-shaped domain Ω by

$$\operatorname{rad}(\Omega) := \min_{y \in S(\Omega)} \max_{x \in \partial\Omega} |x - y|, \tag{2.1}$$

where $S(\Omega) := \operatorname{clos}\{y \in \Omega : \Omega \text{ is star-shaped w.r.t. } y\}$. Apparently, we always have $\operatorname{rad}(\Omega) \leq \operatorname{diam}(\Omega)$, and the radius of a convex domain equals the radius of its smallest circumscribed sphere.

On a star-shaped domain Ω, let us now consider quadrature rules of the form $Q : f \mapsto \sum_j w_j f(x_j)$ to approximate $I : f \mapsto \int_\Omega f$. We will only consider rules that are *internal* meaning that all $x_j \in \operatorname{clos}\Omega$. The quadrature *error functional* is defined as $E := I - Q$.

Proposition 2.1. *For a rule Q of* order p*, meaning that $E(f) = 0$ for all $f \in P_{p-1}(\Omega)$, and any $f \in W_\infty^p(\Omega)$ we have*

$$|E(f)| \leq \left(1 + \frac{\sum_j |w_j|}{\operatorname{vol}(\Omega)}\right) \cdot \frac{n^p}{p!} \cdot \operatorname{rad}(\Omega)^p \cdot \operatorname{vol}(\Omega) \cdot |f|_{W_\infty^p(\Omega)}. \tag{2.2}$$

Note that for a rule that is *positive*, meaning that all $w_j > 0$, and that has order $p > 0$, we have $\frac{\sum_j |w_j|}{\operatorname{vol}(\Omega)} = 1$.

Let us now consider a collection $\mathcal{O}$ of disjoint star-shaped Lipschitz subdomains $\Omega' \subset \Omega$, the latter not necessarily being star-shaped, such that $\operatorname{clos}\Omega = \cup_{\Omega'\in\mathcal{O}} \operatorname{clos}\Omega'$, which collection we will refer to as being a *quadrature mesh.* Writing $I(f)$ as $\sum_{\Omega'\in\mathcal{O}} \int_{\Omega'} f$, on each subdomain Ω' we employ a quadrature rule $Q_{\Omega'}(f) = \sum_j w_j^{\Omega'} f(x_j^{\Omega'})$ of order p, defining a *composite* quadrature rule Q of *rank* $N := \#\mathcal{O}$ (and order p) by $Q(f) := \sum_{\Omega'\in\mathcal{O}} Q_{\Omega'}(f)$.

Proposition 2.2. *For the error functional $E = I - Q$ of this composite quadrature rule, and $f \in W^p_\infty(\Omega)$ we have*

$$|E(f)| \leq \left(1 + \sup_{\Omega'\in\mathcal{O}} \frac{\sum_j |w_j^{\Omega'}|}{\operatorname{vol}(\Omega')}\right) \cdot \sup_{\Omega'\in\mathcal{O}} \left(\frac{N^{1/n}\operatorname{rad}(\Omega')}{\operatorname{diam}(\Omega)}\right)^p$$
$$\times N^{-p/n} \cdot \frac{n^p}{p!} \cdot \operatorname{diam}(\Omega)^p \cdot \operatorname{vol}(\Omega) \cdot |f|_{W^p_\infty(\Omega)}.$$

In view of above estimate, as well as to control the number of function evaluations that are required, in this paper we will consider families $(Q_p)_{p\in\mathbb{N}}$ of composite quadrature rules $Q_p : f \mapsto \sum_{\Omega'\in\mathcal{O}} \sum_j w_j^{p,\Omega'} f(x_j^{p,\Omega'})$ of order p with a fixed mesh $\mathcal{O}$, that are *admissible* meaning that they satisfy

$$\sup_{p\in\mathbb{N},\Omega'\in\mathcal{O}} \max\left\{\frac{\sum_j |w_j^{p,\Omega'}|}{\operatorname{vol}(\Omega')}, \frac{\# x_j^{p,\Omega'}}{p^n}\right\} < \infty.$$

Note that the bound on the number of abscissae in each subdomain is reasonable because the space of polynomials of total degree $p-1$ has $\binom{p-1+n}{n} \leq p^n$ degrees of freedom. Moreover, for a quadrature mesh $\mathcal{O}$ we define the following quantity

$$C_{\mathcal{O}} := \sup_{\Omega'\in\mathcal{O}} \frac{(\#\mathcal{O})^{1/n}\operatorname{rad}(\Omega')}{\operatorname{diam}(\Omega)}. \tag{2.3}$$

Finally in this section, we consider *product quadrature rules* which are generally applied on Cartesian product domains. Let A and B be domains of possibly different dimensions, equipped with the quadrature rules $Q^{(A)} : g \mapsto \sum_j w_j g(x_j)$ and $Q^{(B)} : h \mapsto \sum_k v_k h(y_k)$ to approximate $I^{(A)} : g \mapsto \int_A g$ and $I^{(B)} : h \mapsto \int_B h$, respectively. For simplicity, in this setting we will always assume that these rules are *positive* and have *strictly positive orders.* Now with the product rule $Q^{(A)} \times Q^{(B)}$ we mean the mapping $f \mapsto \sum_{jk} w_j v_k f(x_j, y_k)$ to approximate $I : f \mapsto \int_{A\times B} f$.

Lemma 2.3. *With error functionals $E^{(A)} := I^{(A)} - Q^{(A)}$ and $E^{(B)} := I^{(B)} - Q^{(B)}$, the product rule $Q := Q^{(A)} \times Q^{(B)}$ satisfies*

$$|I(f) - Q(f)| \leq \operatorname{vol}(A) \sup_{x\in A} |E^{(B)}(f(x,\cdot))| + \operatorname{vol}(B) \sup_{y\in B} |E^{(A)}(f(\cdot,y))|, \tag{2.4}$$

as long as both $E^{(A)}(f(\cdot,y))$ and $E^{(B)}(f(x,\cdot))$ make sense for all $y \in B$ and $x \in A$, respectively.

As an application of this lemma, we have the following result for product quadrature rules on rectangular domains.

Proposition 2.4. *Consider the rectangular domain $\square := (0,l_1) \times \ldots \times (0,l_n)$ and define $l := \max_i l_i$. For the i-th coordinate direction, let $Q_M^{(i)}$ be a composite quadrature rule of order p with respect to a quadrature mesh on $(0,l_i)$ of M equally* *sized subintervals. Then for the product quadrature rule* $Q := Q_M^{(1)} \times \ldots \times Q_M^{(n)}$ *to approximate* $I : f \mapsto \int_\Box f$, *and* f *such that* $\partial_i^p f \in L_\infty(\Box)$, $i \in \overline{1,n}$, *we have*

$$|I(f) - Q(f)| \leq n \frac{2^{1-p}}{p!} M^{-p} \cdot l^{n+p} \cdot \max_{i \in \overline{1,n}} \|\partial_i^p f\|_{L_\infty(\Box)}. \tag{2.5}$$

In particular, this quadrature rule is exact on $Q_{p-1}(\Box) := P_{p-1}(0, l_1) \times \ldots \times P_{p-1}(0, l_n)$.

3. Compressibility

For some $\mu \in I\!N$, let Γ be a patchwise smooth, compact n-dimensional, globally $C^{\mu-1,1}$ manifold in $I\!R^{n+1}$. Following [DS99b], we assume that $\Gamma = \cup_{q=1}^M \overline{\Gamma_q}$, with $\Gamma_q \cap \Gamma_{q'} = \emptyset$ when $q \neq q'$, and that for each $1 \leq q \leq M$, there exists

- a domain $\Omega_q \subset I\!R^n$, and a C^∞-parametrization $\kappa_q : I\!R^n \to I\!R^{n+1}$ with $\mathrm{Im}(\kappa_q|_{\Omega_q}) = \Gamma_q$,
- a domain $I\!R^n \supset \hat{\Omega}_q \supset\supset \Omega_q$, and an extension of $\kappa_q|_{\Omega_q}$ to a $C^{\mu-1,1}$ parametrization $\hat{\kappa}_q : \hat{\Omega}_q \to \mathrm{Im}(\hat{\kappa}_q) \subset \Gamma$.

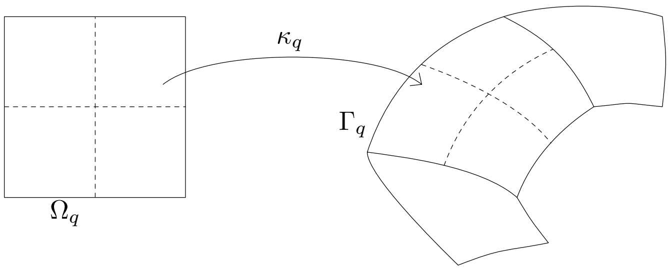

Figure 1. Parametrization of the manifold.

Formally supposing that the domains Ω_q are pairwise disjoint, for notational convenience we introduce the invertible mapping $\kappa : \cup_q \Omega_q \to \cup_q \Gamma_q \subset \Gamma$ via

$$\kappa(x) := \kappa_q(x) \quad \text{with } q \text{ such that } x \in \Omega_q.$$

For $|s| \leq \mu$, the Sobolev spaces $H^s(\Gamma)$ are well-defined, where for $s < 0$, $H^s(\Gamma)$ is the dual of $H^{-s}(\Gamma)$. Let

$$\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$$

be a *Riesz basis for* $H^t(\Gamma)$ of wavelet type. The index λ encodes both the level, denoted by $|\lambda| \in I\!N_0$, and the location of the wavelet ψ_λ. We will assume that the wavelets are *local* and *piecewise smooth* with respect to nested subdivisions in the following sense. We assume that there exists a sequence $(\mathcal{O}_\ell)_{\ell \in I\!N_0}$ of collections $\mathcal{O}_\ell$ of disjoint "uniformly" Lipschitz domains $\Theta \in \mathcal{O}_\ell$, with

$$\mathrm{diam}(\Theta) \eqsim 2^{-\ell} \qquad \text{and} \qquad \mathrm{vol}(\Theta) \eqsim 2^{-n\ell}, \tag{3.1}$$

and where each $\Theta \in \mathcal{O}_\ell$ is contained in some Ω_q, and its closure is the union of the closures of a uniformly bounded number of subdomains from $\mathcal{O}_{\ell+1}$. For a precise definition of a collection of sets to be a collection of uniformly Lipschitz domains, we refer to [Ste04a, Remark 2.1]. Defining the collections of *panels*

$$\mathcal{G}_\ell := \{\kappa(\Theta) : \Theta \in \mathcal{O}_\ell\}, \qquad (\ell \in I\!N_0),$$

we assume that $\Gamma = \cup_{\Pi \in \mathcal{G}_\ell} \overline{\Pi}$, ($\ell \in I\!N_0$), and that for each $\lambda \in \Lambda$ there exists a subcollection $\mathcal{G}_\lambda \subset \mathcal{G}_{|\lambda|}$ with

$$\sup_{\lambda \in \Lambda} \#\mathcal{G}_\lambda < \infty \quad \text{and} \quad \sup_{\ell \in I\!N_0, \Pi \in \mathcal{G}_\ell} \#\{\lambda : |\lambda| = \ell,\, \Pi \in \mathcal{G}_\lambda\} < \infty,$$

such that $\operatorname{supp} \psi_\lambda = \cup_{\Pi \in \mathcal{G}_\lambda} \operatorname{clos} \Pi$, being a connected set, and that on each $\Theta \in \kappa^{-1}(\mathcal{G}_\lambda)$, the pull-back $\hat{\psi}_{\lambda,\Theta} := (\psi_\lambda \circ \kappa)|_\Theta$ is smooth with

$$\sup_{x \in \Theta} |\partial^\beta \hat{\psi}_{\lambda,\Theta}(x)| \lesssim 2^{(|\beta| + \frac{n}{2} - t)|\lambda|} \qquad \text{for } \beta \in N_0^n. \tag{3.2}$$

We assume that the wavelets have the so-called *cancellation property of order* $\tilde{d} \in I\!N$, saying that there exists a constant $\eta > 0$, such that for any $p \in [1, \infty]$, for all continuous, patchwise smooth functions v and $\lambda \in \Lambda$,

$$|\langle v, \psi_\lambda \rangle| \lesssim 2^{-|\lambda|(\frac{n}{2} - \frac{n}{p} + t + \tilde{d})} \max_{1 \leq q \leq M} |v|_{W_p^{\tilde{d}}(B(\operatorname{supp} \psi_\lambda; 2^{-|\lambda|}\eta) \cap \Gamma_q)}, \tag{3.3}$$

where for $A \subset I\!R^{n+1}$ and $\varepsilon > 0$, $B(A; \varepsilon) := \{y \in I\!R^{n+1} : \operatorname{dist}(A, y) < \varepsilon\}$.

Furthermore, for some $k \in I\!N_0 \cup \{-1\}$, with $k < \mu$ and

$$\gamma := k + \tfrac{3}{2} > t, \tag{3.4}$$

we assume that all $\psi_\lambda \in C^k(\Gamma)$, where $k = -1$ means no global continuity condition, and that for all $r \in [-\tilde{d}, \gamma)$, $s < \gamma$, necessarily with $|s|, |r| \leq \mu$,

$$\| \cdot \|_{H^r(\Gamma)} \lesssim 2^{\ell(r-s)} \| \cdot \|_{H^s(\Gamma)} \qquad \text{on } W_\ell := \operatorname{span}\{\psi_\lambda : |\lambda| = \ell\}. \tag{3.5}$$

Inside a patch, a similar property can be required for larger ranges: For all $q \in \overline{1, M}$, and $r \in [-\tilde{d}, \gamma)$, $s < \gamma$, we assume that

$$\| \cdot \|_{H^r(\Gamma_q)} \lesssim 2^{\ell(r-s)} \| \cdot \|_{H^s(\Gamma_q)} \quad \text{on } \operatorname{span}\{\psi_\lambda : |\lambda| = \ell, B(\operatorname{supp} \psi_\lambda; 2^{-\ell}\eta) \subset \overline{\Gamma_q}\}. \tag{3.6}$$

Remark 3.1. Wavelets that satisfy the assumptions in principle for any d, $\tilde{d}$ and smoothness permitted by both d and the regularity of the manifold were constructed in [DS99b]. Apart from this construction, all known approaches based on non-overlapping domain decompositions yield wavelets which over the interfaces between patches are only continuous. With the constructions from [DS99a, CTU99, CM00], biorthogonality was realized with respect to a modified $L_2(\Gamma)$-scalar product. As a consequence, with the interpretation of functions as functionals via the Riesz mapping with respect to the standard $L_2(\Gamma)$ scalar product, for negative t the wavelets only generate a Riesz basis for $H^t(\Gamma)$ when $t > -\frac{1}{2}$, and likewise wavelets with supports that extend to more than one patch generally have no cancellation properties in the sense of (3.3). Recently in [Ste04b], this difficulty was overcame, and wavelets were constructed that all have the cancellation property of the full order, and that generate Riesz bases for the full range of Sobolev spaces $H^t(\Gamma)$ that is allowed by continuous gluing of functions over the patch interfaces *and* the regularity of the manifold.

For some $|t| \leq \mu$, let L be a bounded operator from $H^t(\Gamma) \to H^{-t}(\Gamma)$, where we have in mind a singular integral operator of order $2t$. We assume that the operator L is defined by

$$Lu(z) = \int_\Gamma K(z, z') u(z') d\Gamma_{z'}, \quad (z \in \Gamma), \tag{3.7}$$

and that its *local kernel function*

$$\hat{K}(x,x') := K(\kappa(x),\kappa(x')) \cdot |\partial\kappa(x)| \cdot |\partial\kappa(x')|$$

satisfies for all $x, x' \in \cup_{1\leq q\leq M}\Omega_q$, and $\alpha, \beta \in \mathbb{N}_0^n$,

$$|\partial_x^\alpha \partial_{x'}^\beta \hat{K}(x,x')| \lesssim \frac{|\alpha+\beta|!}{\varsigma^{|\alpha+\beta|}} \cdot \mathrm{dist}(\kappa(x),\kappa(x'))^{-(n+2t+|\alpha+\beta|)}, \tag{3.8}$$

with a constant $\varsigma > 0$ (cf. [Har01, DHS02]), provided that $n + 2t + |\alpha+\beta| > 0$. If the kernel function $K(z, z')$ contains non-integrable singularities, the integral (3.7) has to be understood in the *finite part* sense of Hadamard, see e.g. [SW92, SL00]. Following [DHS02], we emphasize that (3.8) requires patchwise smoothness but no global smoothness of Γ. Only assuming global Lipschitz continuity of Γ, the local kernel of any standard boundary integral operator of order $2t$ can be shown to satisfy (3.8).

We assume that for some $\sigma \in (0, \mu - |t|]$, both L and its adjoint L' are bounded from $H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$.

Remark 3.2. If Γ is a C^∞-manifold, then these boundary integral operators are known to be pseudo-differential operators, meaning that for any $\sigma \in \mathbb{R}$ they define bounded mappings from $H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$. For Γ being only Lipschitz continuous, for the classical boundary integral equations it is known that $L : H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$ is bounded for the maximum possible value $\sigma = 1 - |t|$ (cf. [Cos88]). With increasing smoothness of Γ one may expect this boundedness for larger values of σ. Results in this direction can be found in [MS04].

Furthermore, with $\tilde{H}^s(\Gamma_q) := \begin{cases} H^s(\Gamma_q) & \text{when } s \geq 0, \\ (H_0^{-s}(\Gamma_q))' & \text{when } s < 0, \end{cases}$ we assume that there exists a $\tau \in (0, \mu - |t|]$ such that

$$L : H^{t+\tau}(\Gamma) \to \tilde{H}^{-t+\tau}(\Gamma_q) \quad \text{is bounded for all } 1 \leq q \leq M. \tag{3.9}$$

Remark 3.3. Since for any $|s| \leq \mu$, the restriction of functions on Γ to Γ_q is a bounded mapping from $H^s(\Gamma)$ to $\tilde{H}^s(\Gamma_q)$, from the boundedness of $L : H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$, it follows that in any case (3.9) is valid for $\tau = \sigma$. So for example for Γ being a C^∞-manifold, (3.9) is valid for any $\tau \in \mathbb{R}$. Yet, in particular when $t < 0$, for Γ being less smooth it might happen that (3.9) is valid for a τ that is strictly larger than any σ for which $L : H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$ is bounded.

In the following theorem, we recall the main result on compressibility for boundary integral operators from [Ste04a].

Theorem 3.4. *For Ψ being a Riesz basis for $H^t(\Gamma)$ as described above with $t + \tilde{d} > 0$, and $\tilde{d} > \gamma - 2t$, let $\mathbf{M} = \langle \Psi, L\Psi \rangle$.*

Let $\alpha \in (\frac{1}{2}, 1)$ and $b_i := (1+i)^{-1-\varepsilon}$ for some $\varepsilon > 0$. Choose k satisfying

$$\begin{aligned} &k = \frac{1}{n-1} && \textit{when } n > 1, \\ &k > \frac{\min\{t+\tilde{d}, \tau\}}{\gamma - t} \quad \textit{and} \quad k \geq \max\{1, \frac{\min\{t+\tilde{d},\tau\}}{\min\{t+\mu,\sigma\}}\} && \textit{when } n = 1. \end{aligned} \tag{3.10}$$

We define the infinite matrix $\mathbf{M}_j$ *for* $j \in \mathbb{N}$ *by replacing all entries* $\mathbf{M}_{\lambda,\lambda'} = \langle \psi_\lambda, L\psi_{\lambda'}\rangle$ *by zeros when*

$$\big||\lambda| - |\lambda'|\big| > jk, \quad \text{or} \tag{3.11}$$

$$\big||\lambda| - |\lambda'|\big| \le j/n \quad \text{and} \quad \delta(\lambda,\lambda') \ge \max\{3\eta, 2^{\alpha(j/n - ||\lambda|-|\lambda'||)}\}, \quad \text{or} \tag{3.12}$$

$$\begin{aligned} &\big||\lambda| - |\lambda'|\big| > j/n \quad \text{and} \\ &\quad \tilde\delta(\lambda,\lambda') \ge \max\{2^{n(j/n - ||\lambda|-|\lambda'||)} b_{||\lambda|-|\lambda'||-j/n}, 2\eta 2^{-||\lambda|-|\lambda'||}\}, \end{aligned} \tag{3.13}$$

where

$$\delta(\lambda,\lambda') := 2^{\min\{|\lambda|,|\lambda'|\}} \mathrm{dist}(\mathrm{supp}\,\psi_\lambda, \mathrm{supp}\,\psi_{\lambda'}), \tag{3.14}$$

and

$$\tilde\delta(\lambda,\lambda') := 2^{\min\{|\lambda|,|\lambda'|\}} \times \begin{cases} \mathrm{dist}(\mathrm{supp}\,\psi_\lambda, \mathrm{sing\,supp}\,\psi_{\lambda'}) & \text{when } |\lambda| > |\lambda'|, \\ \mathrm{dist}(\mathrm{sing\,supp}\,\psi_\lambda, \mathrm{supp}\,\psi_{\lambda'}) & \text{when } |\lambda| < |\lambda'|, \end{cases}$$

and η *is from* (3.3).

Then the number of non-zero entries in each column of $\mathbf{M}_j$ *is of order* 2^j*, and for any*

$$s \le \min\left\{\tfrac{t+\tilde d}{n}, \tfrac{\tau}{n}\right\}, \ \text{with } s < \tfrac{\gamma-t}{n-1},\ s \le \tfrac{\sigma}{n-1} \text{ and } s \le \tfrac{\mu+t}{n-1} \text{ when } n > 1,$$

it holds that $\|\mathbf{M} - \mathbf{M}_j\| \lesssim 2^{-js}$*. We conclude that* $\mathbf{M}$ *is* s^**-compressible, as defined in Definition 1.4, with* $s^* = \min\{\frac{t+\tilde d}{n}, \frac{\tau}{n}, \frac{\sigma}{n-1}, \frac{\gamma-t}{n-1}, \frac{\mu+t}{n-1}\}$ *when* $n > 1$*, and* $s^* = \min\{t + \tilde d, \tau\}$ *when* $n = 1$*.*

From this theorem we infer that if $\tilde d \ge d - 2t$, $\tau \ge d - t$ and, when $n > 1$, $\frac{\min\{\gamma-t,\sigma,t+\mu\}}{n-1} \ge \frac{d-t}{n}$, then $s^* \ge \frac{d-t}{n}$ as required. For $n > 1$, the condition involving γ is satisfied for instance for spline wavelets, where $\gamma = d - \frac12$, in case $\frac{d-t}{n} \ge \frac12$.

If each entry of $\mathbf{M}$ can be exactly computed in $\mathcal{O}(1)$ operations, then s^*-compressibility implies s^*-computability, as defined in Definition 1.1, and so, when indeed $s^* \ge \frac{d-t}{n}$, it implies the optimal computational complexity of the adaptive wavelet scheme from [CDD02]. In general, one is not able to compute the matrix entries exactly. What is more, it is far from obvious how to compute the entries of $\mathbf{M}_j$ sufficiently accurate while keeping the average computational expense per entry in each column uniformly bounded. In the next section, additionally assuming that the wavelets are essentially *piecewise polynomials*, we will show that it is possible to arrange quadrature schemes which admit s^*-computability of $\mathbf{M}$.

4. Computability

In this section, we will present a numerical integration scheme which computes an approximation $\mathbf{M}_j^*$ of $\mathbf{M}_j$ such that, for some specified constant c, by spending $\mathcal{O}(j^c 2^j)$ computational work per column of $\mathbf{M}_j^*$, the approximation error satisfies $\|\mathbf{M}_j - \mathbf{M}_j^*\| \lesssim 2^{-js^*}$ with s^* given by Theorem 3.4, implying that $\mathbf{M}$ is s^*-computable.

Let us consider the computation of individual entries

$$\mathbf{M}_{\lambda,\lambda'} = \int_\Gamma \psi_\lambda(z) \left(\int_\Gamma K(z,z')\psi_{\lambda'}(z') d\Gamma_{z'} \right) d\Gamma_z \tag{4.1}$$

of $\mathbf{M}$. Unless explicitly stated otherwise, throughout this section we assume that

$$|\lambda| \geq |\lambda'|.$$

We start with an assumption.

Assumption 4.1. For any $\Xi \in \mathcal{G}_\lambda$, $\Xi' \in \mathcal{G}_{|\lambda|}$ with $\Xi' \subset \operatorname{supp} \psi_{\lambda'}$, in the following we assume that the integral

$$\int_\Xi \int_{\Xi'} K(z, z') \psi_\lambda(z) \psi_{\lambda'}(z') d\Gamma_z d\Gamma_{z'}$$

is well-defined.

This assumption obviously holds in case of proper or improper integrals. However, it requires an appropriate interpretation of the integrals in case of strongly- or hyper-singular kernels. For strongly singular kernels on surfaces in $\mathbb{R}^3$ the assumption was confirmed in [HS93].

As a consequence of the assumption, we may write

$$\mathbf{M}_{\lambda,\lambda'} = \sum_{\Pi \in \mathcal{G}_\lambda} \sum_{\Pi' \in \mathcal{G}_{\lambda'}} I_{\lambda\lambda'}(\Pi, \Pi'), \tag{4.2}$$

with, for $\Pi \in \mathcal{G}_\lambda$ and $\Pi' \in \mathcal{G}_{\lambda'}$,

$$I_{\lambda\lambda'}(\Pi, \Pi') := \sum_{\{\Xi' \in \mathcal{G}_{|\lambda|} : \Xi' \subset \Pi'\}} \int_\Pi \int_{\Xi'} K(z, z') \psi_\lambda(z) \psi_{\lambda'}(z') d\Gamma_z d\Gamma_{z'}. \tag{4.3}$$

We assume that for each $\Pi \in \mathcal{G}_\lambda$, $\Pi' \in \mathcal{G}_{\lambda'}$ an approximation of the integral $I_{\lambda\lambda'}(\Pi, \Pi')$ is obtained by some numerical scheme dependent on j, and using (4.2), that these approximations are used to assemble the matrix $\mathbf{M}_j^*$. The following theorem defines a criterion on the computational cost in relation to the accuracy of computing the integrals $I_{\lambda\lambda'}(\Pi, \Pi')$ so that s^*-compressibility implies s^*-computability.

Theorem 4.2. *Let $s^* > 0$ be any given constant, and $\mathbf{M}$, $\mathbf{M}_j$ be as in Theorem 3.4. Let $\sigma : \cup_\ell \mathcal{G}_\ell \to \mathbb{R}$ be some fixed function such that*

$$\sigma(\Xi) \eqsim \operatorname{diam}(\Xi) \qquad \textit{for } \Xi \in \cup_\ell \mathcal{G}_\ell, \tag{4.4}$$

and let $d^*, e^* \in \mathbb{R}$ and $\varrho > 1$ be fixed constants. Assume that for any $p \in \mathbb{N}$, an approximation $I^*_{\lambda\lambda'}(\Pi, \Pi')$ of the integral $I_{\lambda\lambda'}(\Pi, \Pi')$ can be computed such that by spending the number of

$$W \lesssim p^{2n}(1 + ||\lambda| - |\lambda'||) \tag{4.5}$$

arithmetical operations, the error satisfies

$$\begin{aligned} |E_{\lambda\lambda'}(\Pi, \Pi')| \lesssim\ & \varrho^{-p} 2^{||\lambda| - |\lambda'|| d^*} \\ & \times \max\left\{1, \frac{\operatorname{dist}(\Pi, \Pi')}{\varrho \max\{\sigma(\Pi), \sigma(\Pi')\}}\right\}^{e^* - p}. \end{aligned} \tag{4.6}$$

Then for any fixed $\vartheta \geq 0$, and for parameters θ and τ with

$$\theta \geq s^* / \log_2 \varrho \quad \textit{and} \quad \tau > (n/2 + d^*)/\log_2 \varrho, \tag{4.7}$$

by choosing p for the computation of $I^*_{\lambda\lambda'}(\Pi, \Pi')$ as the smallest positive integer satisfying

$$p > e^* + n \quad \textit{and} \quad p \geq j\theta + \tau||\lambda| - |\lambda'|| - \vartheta, \tag{4.8}$$

the so computed approximation $\mathbf{M}_j^*$ *of* $\mathbf{M}_j$ *satisfies* $\|\mathbf{M}_j - \mathbf{M}_j^*\| \lesssim 2^{-js^*}$, *where the work for computing each column of* $\mathbf{M}_j^*$ *is* $\mathcal{O}(j^{2n+1}2^j)$.

By taking s^* *as given in Theorem 3.4, we conclude that the matrix* $\mathbf{M}$ *is* s^**-computable for the same value of* s^* *as it was shown to be* s^**-compressible.*

The proof will use Schur's lemma that we recall here for the reader's convenience.

Lemma 4.3 (Schur's lemma)**.** *If for a matrix* $\mathbf{A} = (a_{\lambda,\lambda'})_{\lambda,\lambda'\in\Lambda}$, *there is a sequence* $w_\lambda > 0$, $\lambda \in \Lambda$, *and a constant* C *such that*

$$\sum_{\lambda'\in\Lambda} w_{\lambda'}|a_{\lambda\,\lambda'}| \le w_\lambda C, \quad (\lambda\in\Lambda), \quad \text{and} \quad \sum_{\lambda\in\Lambda} w_\lambda |a_{\lambda\,\lambda'}| \le w_{\lambda'} C, \quad (\lambda'\in\Lambda),$$

then $\|\mathbf{A}\| \le C$.

Proof of Theorem 4.2. Since $\#\mathcal{G}_\lambda$, $\#\mathcal{G}_{\lambda'} \lesssim 1$, it is sufficient to give the proof pretending that $\#\mathcal{G}_\lambda = \#\mathcal{G}_{\lambda'} = 1$.

With the matrix $(\Delta_{\lambda,\lambda'})_{\lambda,\lambda'\in\Lambda}$ defined by

$$\Delta_{\lambda,\lambda'} := \max\left\{1, \frac{\operatorname{dist}(\Pi,\Pi')}{\varrho\max\{\sigma(\Pi),\sigma(\Pi')\}}\right\}, \quad \Pi\in\mathcal{G}_\lambda,\ \Pi'\in\mathcal{G}_{\lambda'},$$

for each $\lambda\in\Lambda$, $\ell'\in\mathbb{N}_0$, and $\beta > n$, we can verify that

$$\sum_{|\lambda'|=\ell'} \Delta_{\lambda,\lambda'}^{-\beta} \lesssim 2^{n\max\{0,\ell'-|\lambda|\}}, \tag{4.9}$$

using the locality of the wavelets and the fact that $\sigma(\Pi') \eqsim \operatorname{diam}(\Pi') \eqsim 2^{-|\lambda'|}$ and that $\operatorname{vol}(\Pi') \eqsim 2^{-|\lambda'|n}$.

Denoting the entry (λ,λ') of the error matrix $\mathbf{M}_j - \mathbf{M}_j^*$ by $\varepsilon_{j,\lambda\lambda'}$, and by substituting $p \ge j\theta + \tau||\lambda| - |\lambda'|| - \vartheta$ into (4.6), we infer that

$$\varepsilon_{j,\lambda\lambda'} \quad \lesssim \quad 2^{-j\theta\log_2\varrho}2^{-||\lambda|-|\lambda'||(\tau\log_2\varrho - d^*)}\Delta_{\lambda,\lambda'}^{-(p-e^*)} \tag{4.10}$$

Recall that $\sigma := \tau\log_2\varrho - d^* > n/2$ and $p - e^* > n$. Applying Schur's lemma to the error matrix $\mathbf{M}_j - \mathbf{M}_j^*$ with weights $w_\lambda = 2^{-|\lambda|n/2}$, we have

$$\begin{aligned}
w_\lambda^{-1}\sum_{\lambda'} w_{\lambda'}|\varepsilon_{j,\lambda\lambda'}| &\lesssim 2^{-j\theta\log_2\varrho}2^{|\lambda|n/2}\sum_{\ell'\ge 0}2^{-\ell' n/2}2^{-(|\lambda|-\ell')\sigma}\cdot\sum_{|\lambda'|=\ell'}\Delta_{\lambda,\lambda'}^{-(p-e^*)} \\
&\lesssim 2^{-j\theta\log_2\varrho}2^{|\lambda|n/2}\sum_{0\le\ell'\le|\lambda|}2^{-\ell' n/2}2^{-(|\lambda|-\ell')\sigma}\cdot 1 \\
&\quad + 2^{-j\theta\log_2\varrho}2^{|\lambda|n/2}\sum_{\ell'>|\lambda|}2^{-\ell' n/2}2^{-(\ell'-|\lambda|)\sigma}\cdot 2^{(\ell'-|\lambda|)n} \\
&\lesssim 2^{-j\theta\log_2\varrho},
\end{aligned}$$

where we used (4.9) in the second step. Now by the symmetry of the estimate (4.10) in λ and λ', we conclude that the error in the computed matrix $\mathbf{M}_j^*$ satisfies

$$\|\mathbf{M}_j - \mathbf{M}_j^*\| \lesssim 2^{-j\theta\log_2\varrho} \le 2^{-js^*}.$$

The work for computing the entry $(\mathbf{M}_j^*)_{\lambda,\lambda'}$ is of order

$$p(j,\lambda,\lambda')^{2n}(1 + ||\lambda| - |\lambda'||) \lesssim (j\theta + \tau||\lambda| - |\lambda'||)^{2n}(1 + ||\lambda| - |\lambda'||).$$

Since $\mathbf{M}_j^*$ contains nonzero entries only for $||\lambda| - |\lambda'|| \le jk$, we can bound the work for computing each element $(\mathbf{M}_j^*)_{\lambda,\lambda'}$ by a constant multiple of j^{2n+1}. Now using the fact that each column of $\mathbf{M}_j$ contains $\mathcal{O}(2^j)$ nonzero entries, we conclude the computational work per column is $\mathcal{O}(j^{2n+1}2^j)$. $\square$

By applying the error estimates from Section 2, we will now show how numerical quadrature schemes satisfying (4.5) and (4.6) can be realized. We will consider variable order quadrature rules, meaning that constants absorbed by the "$\lesssim$" symbol will not depend on the quadrature order. To this end, we consider a general finite subdivision $\Upsilon \subset (\cup_\ell \mathcal{G}_\ell)^2$ of the integration domain $\Pi \times \Pi'$ such that $\{\Xi \times \Xi' \in \Upsilon : \operatorname{dist}(\Xi, \Xi') = 0\} \subset \mathcal{G}^2_{|\lambda|}$. Then in view of Assumption 4.1, we can split the integral (4.3) as

$$I_{\lambda\lambda'}(\Pi, \Pi') = \sum_{\Xi \times \Xi' \in \Upsilon} I_{\lambda\lambda'}(\Xi, \Xi'), \tag{4.11}$$

with

$$I_{\lambda\lambda'}(\Xi, \Xi') := \int_\Xi \int_{\Xi'} K(z, z')\psi_\lambda(z)\psi_{\lambda'}(z')d\Gamma_z d\Gamma_{z'}.$$

First we will study the numerical evaluation of an individual integral $I(\Xi, \Xi')$ for the case that $\operatorname{dist}(\Xi, \Xi') > 0$. We can write the integral $I(\Xi, \Xi')$ in local coordinates

$$I_{\lambda\lambda'}(\Xi, \Xi') = \int_\Theta \int_{\Theta'} \hat{K}(x, x')\hat{\psi}_{\lambda,\kappa^{-1}(\Pi)}(x)\hat{\psi}_{\lambda',\kappa^{-1}(\Pi')}(x')dxdx', \tag{4.12}$$

where $\Theta = \kappa^{-1}(\Xi)$ and $\Theta' = \kappa^{-1}(\Xi')$.

Definition 4.4. The wavelet basis Ψ is said to be of *P-type of order e* when for all $\lambda \in \Lambda$ and $\Theta \in \mathcal{O}_{|\lambda|}$, $\hat{\psi}_{\lambda,\Theta} \in P_{e-1}(\Theta)$. Similarly, Ψ is of *Q-type of order e* when for all $\lambda \in \Lambda$ and $\Theta \in \mathcal{O}_{|\lambda|}$, Θ is an n-rectangle and $\hat{\psi}_{\lambda,\Theta} \in Q_{e-1}(\Theta)$.

Lemma 4.5. *Assume that the wavelet basis Ψ is of P-type of order e and that $\operatorname{dist}(\kappa(\Theta), \kappa(\Theta')) > 0$. For the domains Θ and Θ', we employ composite quadrature rules from admissible families (uniformly in Θ, Θ') of orders p and fixed ranks N, and apply the product of these quadrature rules to approximate the non-singular integral $I_{\lambda\lambda'}(\kappa(\Theta), \kappa(\Theta'))$ from* (4.12). *We define*

$$\sigma(\kappa(\tilde{\Theta})) := \frac{nC}{\varsigma N^{1/n}} \operatorname{diam}(\tilde{\Theta}) \qquad \textit{for all } \tilde{\Theta} \in \cup_\ell \mathcal{O}_\ell, \tag{4.13}$$

where $\varsigma > 0$ is the constant involved in the Calderon-Zygmund estimate (3.8), *and C is an upper bound on the quantity* (2.3) *for quadrature meshes on $\tilde{\Theta} \in \cup_\ell \mathcal{O}_\ell$. Then with*

$$\omega := \frac{\operatorname{dist}(\kappa(\Theta), \kappa(\Theta'))}{\max\{\sigma(\kappa(\Theta)), \sigma(\kappa(\Theta'))\}}, \tag{4.14}$$

for any $p \geq \max\{e - 2t - n, e - 1\}$, the quadrature error $E(\kappa(\Theta), \kappa(\Theta'))$ satisfies

$$\begin{aligned} |E(\Xi, \Xi')| &\lesssim 2^{||\lambda|-|\lambda'||(n/2-t)}\omega^{-(n+p)} \max\{1, \omega\}^{e-1} \\ &\quad \times \min\{\sigma(\kappa(\Theta)), \sigma(\kappa(\Theta'))\}^n \operatorname{dist}(\kappa(\Theta), \kappa(\Theta'))^{-2t}. \end{aligned} \tag{4.15}$$

Proof. Since there will be no risk of confusion, we will write $\hat{\psi}_\lambda$ and $\hat{\psi}_{\lambda'}$ instead of $\hat{\psi}_{\lambda,\kappa^{-1}(\Pi)}$ and $\hat{\psi}_{\lambda',\kappa^{-1}(\Pi')}$, respectively. By Lemma 2.3, the error of the product quadrature is

$$|E(\kappa(\Theta), \kappa(\Theta'))| \leq \operatorname{vol}(\Theta') \cdot \sup_{x' \in \Theta'} |E(x')| + \operatorname{vol}(\Theta) \cdot \sup_{x \in \Theta} |E'(x)|, \tag{4.16}$$

where we denoted by $E(x')$ the error of the quadrature over the domain Θ with the integrand $x \mapsto \hat{K}(x,x')\hat{\psi}_\lambda(x)\hat{\psi}_{\lambda'}(x')$. Analogously $E'(x)$ denotes the error of the quadrature over Θ'. Using Proposition 2.2 to bound $E(x')$, we have

$$|E(x')| \lesssim \frac{n^p}{p!} C^p N^{-p/n} \operatorname{vol}(\Theta) \cdot \operatorname{diam}(\Theta)^p \cdot |\hat{\psi}_{\lambda'}(x')| \cdot |\hat{K}(\cdot,x')\hat{\psi}_\lambda|_{W^p_\infty(\Theta)}. \tag{4.17}$$

The partial derivatives with $|\eta| = p$, satisfy

$$\begin{aligned}
\left|\partial_x^\eta \left(\hat{K}(x,x')\hat{\psi}_\lambda(x)\right)\right| &= \left|\sum_{\xi\le\eta} \binom{\eta}{\xi} \partial_x^{\eta-\xi}\hat{K}(x,x')\partial_x^\xi \hat{\psi}_\lambda(x)\right| \\
&\le \sum_{\{\xi\le\eta: |\xi|\le e-1\}} \binom{\eta}{\xi} \left|\partial_x^{\eta-\xi}\hat{K}(x,x')\partial_x^\xi\hat{\psi}_\lambda(x)\right|,
\end{aligned}$$

since $\partial^\xi \hat{\psi}_\lambda$ can only be nonzero when $|\xi| \le e-1$ because $\hat{\psi}_\lambda \in P_{e-1}$. Applying the estimates (3.2) and (3.8) we have, with $\delta := \operatorname{dist}(\kappa(\Theta), \kappa(\Theta'))$

$$\begin{aligned}
|\hat{K}(\cdot,x')\hat{\psi}|_{W^p_\infty(\Theta)} &\lesssim \max_{|\eta|=p} \sum_{\{\xi\le\eta:|\xi|\le e-1\}} \binom{\eta}{\xi} \frac{(p-|\xi|)!}{\varsigma^{p-|\xi|}} \delta^{-(n+2t+p-|\xi|)} 2^{(|\xi|+n/2-t)|\lambda|} \\
&\lesssim 2^{|\lambda|(n/2-t)}\delta^{-(n+2t+p)} \\
&\quad \times \max_{|\eta|=p} \sum_{\{\xi\le\eta:|\xi|\le e-1\}} \binom{\eta}{\xi} \frac{(p-|\xi|)!}{\varsigma^{p-|\xi|}} \left(2^{|\lambda|}\delta\right)^{|\xi|} \\
&\lesssim \frac{p!}{\varsigma^p} \cdot 2^{|\lambda|(n/2-t)}\delta^{-(n+2t+p)} \cdot \max\{1, 2^{|\lambda|}\delta\}^{e-1},
\end{aligned}$$

where $\binom{\eta}{\xi}(p-|\xi|)! \le p!$ was used. By substituting this result into (4.17), setting $c := nC/(\varsigma N^{1/n})$, and using $\operatorname{vol}(\Theta) \lesssim \operatorname{diam}(\Theta)^n$, $\operatorname{vol}(\Theta') \lesssim \operatorname{diam}(\Theta')^n$, and again (3.2), we get

$$\begin{aligned}
\operatorname{vol}(\Theta') \cdot \sup_{x'\in\Theta'} |E(x')| &\lesssim \operatorname{diam}(\Theta')^n c^p \operatorname{diam}(\Theta)^{n+p} \cdot 2^{(|\lambda|+|\lambda'|)(n/2-t)} \\
&\quad \times \delta^{-(n+2t+p)} \max\{1, 2^{|\lambda|}\delta\}^{e-1} \\
&= \operatorname{diam}(\Theta')^n \operatorname{diam}(\Theta)^{n+p} \cdot 2^{(|\lambda|+|\lambda'|)(n/2-t)} c^{-n}\delta^{-2t}\omega^{-n-p} \\
&\quad \times \max\{\operatorname{diam}(\Theta), \operatorname{diam}(\Theta')\}^{-n-p} \max\{1, 2^{|\lambda|}\delta\}^{e-1} \\
&= c^{-n} 2^{(|\lambda|+|\lambda'|)(n/2-t)}\delta^{-2t}\omega^{-n-p} \min\{\operatorname{diam}(\Theta), \operatorname{diam}(\Theta')\}^n \\
&\quad \times \left(\frac{\operatorname{diam}(\Theta)}{\max\{\operatorname{diam}(\Theta), \operatorname{diam}(\Theta')\}}\right)^p \max\{1, 2^{|\lambda|}\delta\}^{e-1},
\end{aligned}$$

by definition of ω. For the expression in the last row, employing the inequalities

$$\left(\frac{\operatorname{diam}(\Theta)}{\max\{\operatorname{diam}(\Theta), \operatorname{diam}(\Theta')\}}\right)^p \le 1,$$

and

$$\left(\frac{\operatorname{diam}(\Theta)}{\max\{\operatorname{diam}(\Theta),\operatorname{diam}(\Theta')\}}\right)^{p}\left(2^{|\lambda|}\delta\right)^{e-1}=\left(\frac{\operatorname{diam}(\Theta)}{\max\{\operatorname{diam}(\Theta),\operatorname{diam}(\Theta')\}}\right)^{p-e+1}$$
$$\times\left(\frac{\delta}{\max\{\operatorname{diam}(\Theta),\operatorname{diam}(\Theta')\}}\right)^{e-1}\left(\frac{\operatorname{diam}(\Theta)}{2^{-|\lambda|}}\right)^{e-1}\lesssim\omega^{e-1},$$

and taking the maximum over these two, the assertion of the lemma is proven for the first term in (4.16). The remaining second term in (4.16) can be estimated exactly in the same fashion by interchanging the roles of λ and λ'. □

Obviously, if Ψ is of Q-type of order e, then it is also of P-type of order $n(e-1)+1$. In the next lemma, however, we will see that product quadrature rules are quantitatively more efficient for Q-type wavelets.

Lemma 4.6. *Assume that the wavelet basis Ψ is of Q-type of order e and that* $\operatorname{dist}(\kappa(\Theta),\kappa(\Theta'))>0$. *For the domains Θ and Θ', we employ composite product quadrature rules of orders p and fixed ranks N as in Corollary 2.4, and apply the product of these quadrature rules to approximate the non-singular integral* $I_{\lambda\lambda'}(\kappa(\Theta),\kappa(\Theta'))$ *from* (4.12). *We define*

$$\sigma(\kappa(\tilde\Theta)):=\frac{1}{2\varsigma N^{1/n}}\tilde l\qquad\text{for all }\tilde\Theta\in\cup_\ell\mathcal{O}_\ell,\tag{4.18}$$

where $\tilde l$ is the maximum edge length of $\tilde\Theta$, and ς is the constant involved in the Calderon-Zygmund estimate (3.8). *Then with*

$$\omega:=\frac{\operatorname{dist}(\kappa(\Theta),\kappa(\Theta'))}{\max\{\sigma(\kappa(\Theta)),\sigma(\kappa(\Theta'))\}},\tag{4.19}$$

for any $p\geq\max\{e-2t-n,e-1\}$, the quadrature error $E(\kappa(\Theta),\kappa(\Theta'))$ satisfies

$$\begin{aligned}|E(\Xi,\Xi')|&\lesssim 2^{||\lambda|-|\lambda'||(n/2-t)}\omega^{-(n+p)}\max\{1,\omega\}^{e-1}\\&\qquad\times\min\{\sigma(\kappa(\Theta)),\sigma(\kappa(\Theta'))\}^n\operatorname{dist}(\kappa(\Theta),\kappa(\Theta'))^{-2t}.\end{aligned}\tag{4.20}$$

Proof. Adopting the notations from the previous proof, we use Corollary 2.4 to estimate $E(x')$.

$$|E(x')|\leq n\frac{2^{1-p}}{p!}N^{-p/n}l^{n+p}\cdot|\hat\psi_{\lambda'}(x')|\cdot\max_{j=\overline{1,n}}\left\|\partial^p_{x_j}\left(\hat K(x,x')\hat\psi_\lambda(x)\right)\right\|_{L_\infty(\Theta)}.$$

The partial derivative of order p along the j-th coordinate direction satisfies

$$\begin{aligned}\left|\partial^p_{x_j}\left(\hat K(x,x')\hat\psi_\lambda(x)\right)\right|&=\left|\sum_{k=0}^{p}\binom{p}{k}\partial^{p-k}_{x_j}\hat K(x,x')\partial^k_{x_j}\hat\psi_\lambda(x)\right|\\&\leq\sum_{k=0}^{\min\{p,e-1\}}\binom{p}{k}\left|\partial^{p-k}_{x_j}\hat K(x,x')\partial^k_{x_j}\hat\psi_\lambda(x)\right|,\end{aligned}$$

since $\partial_{x_j}^k \hat{\psi}_\lambda(x)$ can only be nonzero when $k \leq e-1$ because $\hat{\psi}_\lambda \in Q_{e-1}$. Applying the estimates (3.2) and (3.8) we have, with $\delta := \mathrm{dist}(\kappa(\Theta), \kappa(\Theta))$

$$\max_{j=\underline{1,n}} \|\hat{K}(\cdot, x')\hat{\psi}\|_{L_\infty(\Theta)} \lesssim 2^{|\lambda|(n/2-t)}\delta^{-(n+2t+p)} \cdot \sum_{k=0}^{\min\{p,e-1\}} \binom{p}{k} \frac{(p-k)!}{\varsigma^{p-k}} \left(2^{|\lambda|}\delta\right)^k$$
$$\lesssim \frac{p!}{\varsigma^p} \cdot 2^{|\lambda|(n/2-t)}\delta^{-(n+2t+p)} \cdot \max\{1, 2^{|\lambda|}\delta\}^{e-1}.$$

Further we can proceed as in the preceding proof. □

We now turn back to the computation of the integral $I_{\lambda\lambda'}(\Pi, \Pi')$ in (4.3). From Lemmata 4.5 and 4.6, we see that convergence of the quadrature rule as a function of the order p depends on the quantity ω, which is in essence the distance between the panels in terms of the size of the bigger panel. For panels Π and Π' that have a sufficiently large mutual distance, namely, when $\mathrm{dist}(\Pi, \Pi') > \max\{\sigma(\Pi), \sigma(\Pi')\}$ and thus $\omega > 1$, it makes sense to apply quadrature directly on the domain $\Pi \times \Pi'$, that is, not to apply a further splitting as in (4.11).

For the integrals with $0 < \mathrm{dist}(\Pi, \Pi') \leq \max\{\sigma(\Pi), \sigma(\Pi')\}$, however, the subdivision Υ has to be nontrivial. By subdividing the integration domain $\Pi \times \Pi'$ in such a way that $\omega > 1$ for all individual integrals $I_{\lambda\lambda'}(\Xi, \Xi')$, we will ensure convergence of the numerical integration also for these integrals.

Finally, for the case that $\mathrm{dist}(\Pi, \Pi') = 0$, quadrature methods developed for standard Galerkin boundary elements cannot be applied directly in the wavelet setting, because the panels Π and Π' can have very different sizes. Therefore, our strategy here will be to split the bigger panel into smaller panels such that the resulting singular integrals are over panels of the same level, *and* such that the nonsingular integrals are arranged so that $\omega > 1$ for each of them. In view of these considerations, we consider the following algorithm for producing a subdivision of the product domain $\Pi \times \Pi'$.

Algorithm 4.7. Let $\rho > 0$ be given, and $\sigma : \cup_l \mathcal{G}_l \to I\!\!R$ be a function satisfying

$$\sigma(\Xi) \eqsim \mathrm{diam}(\Xi) \quad \text{uniformly in } \Xi \in \cup_l \mathcal{G}_l. \tag{4.21}$$

Let a pair of elements $\Pi \in \mathcal{G}_\ell$ and $\Pi' \in \mathcal{G}_{\ell'}$ with $\ell \geq \ell'$ be given.

1. Set $\Upsilon := \emptyset$, $\Xi := \Pi$, $\Xi' := \Pi'$, and $\tilde{\ell} := \ell$, $\tilde{\ell}' := \ell'$.
2. If the pair Ξ and Ξ' does satisfy one of the conditions

$$\mathrm{dist}(\Xi, \Xi') \geq \rho \cdot \max\{\sigma(\Xi), \sigma(\Xi')\}, \tag{4.22}$$

or

$$\mathrm{dist}(\Xi, \Xi') = 0 \quad \text{and} \quad \Xi = \Pi,\, \Xi' \in \mathcal{G}_\ell, \tag{4.23}$$

accept the pair: $\Upsilon := \Upsilon \cup \{\Xi \times \Xi'\}$. If not, go to either step 3 or 4.

3. If $\tilde{\ell}' \leq \tilde{\ell}$, subdivide Ξ' into next level elements $\Xi'_i \in \mathcal{G}_{\tilde{\ell}'+1}$, and perform step 2 with $\tilde{\ell}' = \tilde{\ell}' + 1$, $\Xi' = \Xi'_i$ for each Ξ'_i.
4. If $\tilde{\ell}' > \tilde{\ell}$, subdivide Ξ into next level elements $\Xi_i \in \mathcal{G}_{\tilde{\ell}+1}$, and perform step 2 with $\tilde{\ell} = \tilde{\ell} + 1$, $\Xi = \Xi_i$ for each Ξ_i.

Remark 4.8. Algorithm 4.7 can already be found in, e.g., [Har01, LS99, vPS97] with $\rho = 1$ and $\sigma(\Xi) = \mathrm{diam}(\Xi)$. This nonuniform subdivision effectively distributes the "strength" of the nearly singular behavior of the integrand over individual subdomains. Later we will see that the parameter ρ can be used to control the convergence rate of quadrature schemes based on the subdivision generated by Algorithm 4.7.

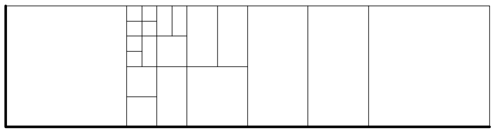

FIGURE 2. A possible subdivision of $\Pi \times \Pi'$ generated by Algorithm 4.7: $n = 1$, $\operatorname{dist}(\Pi, \Pi') = 0$ and $\Pi \cap \Pi' = \emptyset$.

Remark 4.9. Since the manifold is Lipschitz, and the subdivisions are nested and satisfy (3.1), one can verify that for any pair $\Xi, \Xi' \in \cup_\ell \mathcal{G}_\ell$ with $\operatorname{dist}(\Xi, \Xi') > 0$,

$$\operatorname{dist}(\Xi, \Xi') \geq c_\Gamma \min\{\operatorname{diam}\Xi, \operatorname{diam}\Xi'\},$$

with the constant c_Γ depending only on the manifold Γ and its parametrization.

Theorem 4.10. *For any $\Pi \times \Pi' \in \mathcal{G}_\ell \times \mathcal{G}_{\ell'}$ with $\ell \geq \ell'$, Algorithm 4.7 terminates. We have $\cup_{\Xi\times\Xi'\in\Upsilon} \Xi \times \Xi' = \Pi \times \Pi'$ and the number of elements in Υ can be bounded by*

$$\#\Upsilon \lesssim (\rho^n + 1)(\ell - \ell') + \rho^{2n} + 1, \tag{4.24}$$

with the constant absorbed by the "$\lesssim$" symbol not depending on Π, Π', and ρ.

Proof. In each two successive subdivisions the maximum diameter of the "current" panels decreases by a constant factor, while the minimum distance between the "current" pairs does not decrease. Furthermore, thinking of a pair of panels that have distance zero, if the panels of a current pair live on different levels, then the difference in levels is decreased by a subdivision. Therefore the conditions (4.22) or (4.23) will eventually be satisfied starting from any pair, implying that the algorithm will terminate.

To avoid some technicalities, we prove here the estimate (4.24) for the simple case that the manifold Γ is $\mathbb{R}^n$, and that $\sigma(\tilde{\Xi}) = \operatorname{diam}(\tilde{\Xi}) = 2^{-\tilde{\ell}}$ for all $\tilde{\Xi} \in \mathcal{G}_{\tilde{\ell}}$, $\tilde{\ell} \in \mathbb{N}_0$. For the general case an analogous proof is obtained by using the fact that Γ is Lipschitz and that $\sigma(\tilde{\Xi}) \eqsim \operatorname{diam}(\tilde{\Xi}) \eqsim 2^{-\tilde{\ell}}$ for all $\tilde{\Xi} \in \mathcal{G}_{\tilde{\ell}}$, $\tilde{\ell} \in \mathbb{N}_0$.

Let $N_{\tilde{\ell}}$ denote the number of pairs $\Xi \times \Xi' \in \Upsilon$ such that $\Xi' \in \mathcal{G}_{\tilde{\ell}}$. Then we can estimate the total number of pairs by estimating the numbers $N_{\tilde{\ell}}$ and summing over all $\tilde{\ell}$. It is obvious that if $\operatorname{dist}(\Pi, \Pi') > 0$, the number of pairs $\Xi \times \Xi' \in \Upsilon$ that satisfy (4.23) is zero, and if $\operatorname{dist}(\Pi, \Pi') = 0$, this number is uniformly bounded. Since in (4.24) this number is absorbed by the term 1 at the right hand side, in the remainder we will only count pairs of type (4.22).

In case $\tilde{\ell} \leq \ell$, we have $\Xi = \Pi$ for any $\Xi' \in \mathcal{G}_{\tilde{\ell}}$ with $\Xi \times \Xi' \in \Upsilon$. When, moreover $\tilde{\ell} > \ell'$ we have $\operatorname{dist}(\Pi, \Xi') \leq (2\rho+2)2^{-\tilde{\ell}}$. Indeed, if not, then the "parent" $\Xi'' \in \mathcal{G}_{\tilde{\ell}-1}$ of Ξ' would have satisfied $\operatorname{dist}(\Pi, \Xi'') > 2\rho 2^{-\tilde{\ell}} = \max\{\sigma(\Pi), \sigma(\Xi'')\}$ and so Ξ' would never have been created by the algorithm. We conclude that for $\ell' < \tilde{\ell} \leq \ell$, $N_{\tilde{\ell}} \lesssim \left((2\rho + 2)2^{-\tilde{\ell}} + 2^{-\ell}\right)^n / 2^{-\tilde{\ell}n} \lesssim \rho^n + 1$.

Now we consider $\Xi \times \Xi' \in \Upsilon$ with $\Xi' \in \mathcal{G}_{\tilde{\ell}}$ and $\tilde{\ell} > \ell$ (and such that $\Xi \times \Xi'$ satisfies (4.22)). By construction of the algorithm, we have either $\Xi \in \mathcal{G}_{\tilde{\ell}}$ or $\Xi \in \mathcal{G}_{\tilde{\ell}-1}$.

Similar arguments as have been used above show that for fixed Ξ, the number of such pairs is bounded by a constant multiple of $\rho^n + 1$. Since the number of such Ξ is bounded by a constant multiple of $2^{(\tilde{\ell}-\ell)n}$, we conclude that for $\tilde{\ell} > \ell$, $N_{\tilde{\ell}} \lesssim (\rho^n + 1)2^{(\tilde{\ell}-\ell)n}$.

Employing Remark 4.9, it is easy to see that the smallest subelements generated by this algorithm will belong to the level $\ell_{\max}$ that satisfies $\rho 2^{-\ell_{\max}} \gtrsim 2^{-\ell}$, implying that $2^{(\ell_{\max}-\ell)n} \lesssim \rho^n$. Therefore, we conclude that the number of elements in the subdivision Υ is bounded by a constant multiple of

$$1+\sum_{\tilde{\ell}=\ell'+1}^{\ell_{\max}} N_{\tilde{\ell}} \lesssim 1+\sum_{\tilde{\ell}=\ell'+1}^{\ell}(\rho^n+1)+\sum_{\tilde{\ell}=\ell+1}^{\ell_{\max}}(\rho^n+1)2^{(\tilde{\ell}-\ell)n} \lesssim (\rho^n+1)(\ell-\ell')+\rho^{2n}+1.$$

□

From the condition (4.23), we have that the singular integrals corresponding to the subdivision Υ are always over pairs of panels on the same level. In this paper, we make the following Assumption 4.11 on quadrature schemes for computing those singular integrals. For completeness, in the appendix we confirm this assumption for the simple case of the single layer kernel on polyhedral surfaces in $\mathbb{R}^3$. In any case for weakly- and strongly singular integrals, using the quadrature schemes from e.g. [Sau96, SS04], we expect that Assumption 4.11 can be verified generally.

Assumption 4.11. We assume that there exist $d_0^* \in \mathbb{R}$ and $\varrho_0 > 1$ such that for any $\lambda, \lambda' \in \Lambda$ with $|\lambda| \geq |\lambda'|$, $\Xi, \Xi' \in \mathcal{G}_{|\lambda|}$ with $\operatorname{dist}(\Xi,\Xi') = 0$, and for any order $p \in \mathbb{N}$, an approximation $I^*_{\lambda\lambda'}(\Xi,\Xi')$ of $I_{\lambda\lambda'}(\Xi,\Xi')$ can be computed within $W \lesssim p^{2n}$ arithmetical operations, having an error

$$|I_{\lambda\lambda'}(\Xi,\Xi') - I^*_{\lambda\lambda'}(\Xi,\Xi)'| \lesssim \varrho_0^{-p} 2^{||\lambda|-|\lambda'||d_0^*}. \tag{4.25}$$

Now we are ready to present an algorithm how to compute the integral (4.11) with the help of a generally non-uniform subdivision of the integration domain $\Pi \times \Pi'$.

Algorithm 4.12. Assume that Ψ is of P-type of order e, and choose the function $\sigma(\cdot)$ as in Lemma 4.5, and fix a value of $\rho > 1$. Then for any $p \in \mathbb{N}$ the following algorithm approximates the integral $I_{\lambda\lambda'}(\Pi,\Pi')$.

1. Apply Algorithm 4.7 with the above ρ and $\sigma(\cdot)$ to get the subdivision Υ of $\Pi \times \Pi'$;
2. For each subdomain $\Xi \times \Xi' \in \Upsilon$ apply either step 3 or 4;
3. If $\operatorname{dist}(\Xi,\Xi') > 0$, apply the quadrature scheme of order p from Lemma 4.5;
4. If $\operatorname{dist}(\Xi,\Xi') = 0$, apply the computational scheme of order p from Assumption 4.11.

Remark 4.13. For Q-type wavelets, the above algorithm can be redefined by replacing "Lemma 4.5" by "Lemma 4.6".

Theorem 4.14. *Let Ψ be of P-type of order e, and assume that an approximation $I^*_{\lambda\lambda'}(\Pi,\Pi')$ of $I_{\lambda\lambda'}(\Pi,\Pi')$ is computed by using Algorithm 4.12. Assume that $n \geq 2t$. Then, in case that*

$$\operatorname{dist}(\Pi,\Pi') \geq \rho \max\{\sigma(\Pi),\sigma(\Pi')\}, \tag{4.26}$$

with $e^* = e - 1 - 2t - n$, the error of the numerical integration satisfies

$$|E_{\lambda\lambda'}(\Pi,\Pi')| \lesssim \rho^{-p} 2^{-||\lambda|-|\lambda'||(t+n/2)} \left(\frac{\operatorname{dist}(\Pi,\Pi')}{\rho \max\{\sigma(\Pi),\sigma(\Pi')\}}\right)^{e^*-p}, \tag{4.27}$$

and the work for computing $I^*_{\lambda\lambda'}(\Pi,\Pi')$ is bounded by a constant multiple of p^{2n}, provided that $p \geq \max\{e-1, e^*+1\}$. In case that (4.26) does not hold, for any $d_1^* \geq |t| - n/2$, with $d_1^* > -n/2$ when $t = 0$, the error satisfies

$$|E_{\lambda\lambda'}(\Pi,\Pi')| \lesssim \rho^{-p} 2^{||\lambda|-|\lambda'||d_1^*} + \varrho_0^{-p} 2^{||\lambda|-|\lambda'||d_0^*}, \tag{4.28}$$

and the work is bounded by a constant multiple of $p^{2n}(1 + ||\lambda| - |\lambda'||)$, provided that $p \geq \max\{e-1, e^*+1\}$. In view of Remark 4.13, these results also hold for Q-type wavelets of order e.

By taking $\varrho := \min\{\varrho_0, \rho\}$ and $d^* := \max\{d_0^*, d_1^*\}$, we conclude that the criteria (4.5) and (4.6) for s^*-computability from Theorem 4.2 are satisfied.

Proof. Without loss of generality, we assume that $|\lambda| \geq |\lambda'|$. First, we will consider the case that (4.26) holds. In this case, we have the subdivision $\Upsilon = \{\Pi \times \Pi'\}$, and so the computational work is of order of p^{2n}. Applying Lemma 4.5 with $\Theta = \kappa^{-1}(\Pi)$ and $\Theta' = \kappa^{-1}(\Pi')$, taking into account the definition of ω, and using the fact that $\omega \geq \rho > 1$ and that $\min\{\sigma(\Pi),\sigma(\Pi')\} \lesssim 2^{-|\lambda|}$, we get

$$\begin{aligned}
|E_{\lambda\lambda'}(\Pi,\Pi')| &\lesssim 2^{(|\lambda|-|\lambda'|)(n/2-t)} \omega^{-(n+p)} \max\{1,\omega\}^{e-1} \\
&\qquad \times \min\{\sigma(\Pi),\sigma(\Pi')\}^n \operatorname{dist}(\Pi,\Pi')^{-2t} \\
&\lesssim 2^{-|\lambda|(t+n/2)+|\lambda'|(t-n/2)} \omega^{e-1-n-p} \omega^{-2t} \max\{\sigma(\Pi),\sigma(\Pi')\}^{-2t}.
\end{aligned}$$

Now using the estimate $\max\{\sigma(\Pi),\sigma(\Pi')\} \eqsim 2^{-|\lambda'|}$ and $n \geq 2t$, we have

$$\begin{aligned}
|E_{\lambda\lambda'}(\Pi,\Pi')| &\lesssim 2^{-(|\lambda|-|\lambda'|)(t+n/2)-|\lambda'|(n-2t)} \omega^{e^*-p} \\
&\lesssim \rho^{-p} 2^{-(|\lambda|-|\lambda'|)(t+n/2)} (\omega/\rho)^{e^*-p},
\end{aligned}$$

proving the first part of the theorem.

Let us now consider the case that (4.26) does not hold. Since ρ is fixed, the number of subdomains of the subdivision Υ is of order $1 + ||\lambda| - |\lambda'||$, and thus we get the work bound. By Assumption 4.11, the sum of the errors made in the approximations for $I_{\lambda\lambda'}(\Xi,\Xi')$ with $\Xi \times \Xi' \in \Upsilon$ and $\operatorname{dist}(\Xi,\Xi') = 0$ is responsible for the last term in (4.28).

We need to estimate the portion of the total error $E_{\lambda\lambda'}(\Pi,\Pi')$ that corresponds to the integrals $I_{\lambda\lambda'}(\Xi,\Xi')$ with $\Xi \times \Xi' \in \Upsilon$ and $\operatorname{dist}(\Xi,\Xi') > 0$. We denote by I_1 the sum of all these integrals arising from the subdivision Υ, and by I_1^* the computed approximation for I_1. Since by construction for any $\Xi \times \Xi' \in \Upsilon$ with $\operatorname{dist}(\Xi,\Xi') > 0$ it holds that $\frac{\operatorname{dist}(\Xi,\Xi')}{\max\{\sigma(\Xi),\sigma(\Xi')\}} \geq \rho > 1$, Lemma 4.5 gives

$$\begin{aligned}
|I_1 - I_1^*| &\lesssim \sum_{\{\Xi\times\Xi' \in \Upsilon : \operatorname{dist}(\Xi,\Xi')>0\}} 2^{(|\lambda|-|\lambda'|)(n/2-t)} \rho^{e-1-n-p} \\
&\qquad \times \min\{\sigma(\Xi),\sigma(\Xi')\}^n \operatorname{dist}(\Xi,\Xi')^{-2t} \\
&\lesssim \rho^{-p} 2^{-|\lambda|(t+n/2)+|\lambda'|(t-n/2)} \sum_{\{\Xi\times\Xi' \in \Upsilon : \operatorname{dist}(\Xi,\Xi')>0\}} \operatorname{dist}(\Xi,\Xi')^{-2t},
\end{aligned} \tag{4.29}$$

where we have used that $\min\{\sigma(\Xi),\sigma(\Xi')\} \lesssim 2^{-|\lambda|}$.

From the proof of Lemma 4.10, recall that for the number $N_{\tilde{\ell}}$ of $\Xi \times \Xi' \in \Upsilon$ with $\Xi' \in \mathcal{G}_{\tilde{\ell}}$, we have $N_{\tilde{\ell}} = 0$ for $\tilde{\ell} > \ell_{\max}$ where, since ρ is a fixed constant, $\ell_{\max} - |\lambda| \lesssim 1$, and furthermore $N_{\tilde{\ell}} \lesssim 1$ for $|\lambda'| \leq \tilde{\ell} \leq \ell_{\max}$. Since for $\Xi \times \Xi' \in \Upsilon$ with $\mathrm{dist}(\Xi, \Xi') > 0$ and $\Xi' \in \mathcal{G}_{\tilde{\ell}}$, $\mathrm{dist}(\Xi, \Xi') \eqsim 2^{-\tilde{\ell}}$, we may bound the sum in the last row of (4.29) on a constant multiple of

$$\sum_{\tilde{\ell}=|\lambda'|}^{\ell_{\max}} 2^{\tilde{\ell}\cdot 2t} \lesssim \begin{cases} 1 + ||\lambda| - |\lambda'|| & \text{if } t = 0, \\ 2^{|\lambda'|\cdot 2t} & \text{if } t < 0, \\ 2^{|\lambda|\cdot 2t} & \text{if } t > 0. \end{cases}$$

By substituting this result into (4.29), the proof is completed. □

Appendix A. Quadrature for singular integrals

In this appendix, we confirm Assumption 4.11 for the simple case of the single layer kernel on polyhedral surfaces in $\mathbb{R}^3$.

We assume that the manifold Γ is the surface of a three dimensional polyhedron, and that the subdivisions $\mathcal{G}_\ell$, ($\ell \in \mathbb{N}$), are generated by dyadic refinements of $\mathcal{G}_0$, being an initial conforming triangulation of Γ.

We take the operator L to be the single layer operator (thus $t = -\frac{1}{2}$) having the kernel

$$K(z, z') = \frac{1}{4\pi|z - z'|} \qquad z \neq z', \tag{A.1}$$

and assume that the wavelet basis Ψ is of P-type of order e. Let $\lambda, \lambda' \in \Lambda$ be indices with $|\lambda| \geq |\lambda'|$. Then in view of Assumption 4.11, we are ultimately interested in computing the integral

$$I := \int_\Xi \int_{\Xi'} K(z, z')\psi_\lambda(z)\psi_{\lambda'}(z')d\Gamma_z d\Gamma_{z'}, \tag{A.2}$$

where $\Xi, \Xi' \in \mathcal{G}_{|\lambda|}$ and $\mathrm{dist}(\Xi, \Xi') = 0$. With

$$T := \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < x_1 < 1\},$$

we can find affine bijections $\chi_\Xi : T \to \Xi$, and $\chi_{\Xi'} : T \to \Xi'$, thus with Jacobians $J_\Xi := |\partial\chi_\Xi| \eqsim 2^{-2|\lambda|}$, and $J_{\Xi'} := |\partial\chi_{\Xi'}| \eqsim 2^{-2|\lambda|}$, such that

$$I = \int_T \int_T \frac{g(x, y)}{|r(x, y)|} dx dy, \tag{A.3}$$

where $g(x, y) := (4\pi)^{-1} J_\Xi J_{\Xi'} \psi_\lambda(\chi_\Xi(x))\psi_{\lambda'}(\chi_{\Xi'}(y))$ and $r(x, y) := \chi_{\Xi'}(y) - \chi_\Xi(x)$. Taking into account that $n = 2$ and $t = -\frac{1}{2}$, from (3.2) we derive the following estimates

$$|\partial_x^\beta g| \lesssim 2^{-\frac{5}{2}|\lambda| + \frac{3}{2}|\lambda'|} \quad \text{and} \quad |\partial_y^\beta g| \lesssim 2^{-\frac{5}{2}|\lambda| + \frac{3}{2}|\lambda'|} 2^{(|\lambda'| - |\lambda|)|\beta|} \quad \text{for } \beta \in \mathbb{N}_0^2. \tag{A.4}$$

We present here a slight variation of the quadrature scheme developed in e.g. [Sau96, vPS97, SS04], see also [SL00]. The idea is to apply a degenerate coordinate transformation which is a generalization of the so called *Duffy's triangular coordinates*, effectively removing the singularity of the integrand while preserving a polyhedral shape of the integration domain. The coordinate transformations introduced here are somewhat simpler than the ones in the above mentioned papers, and we expect that the presentation is geometrically more intuitive.

To this end, we need to partition the integration domain $T\times T$ into several pyramides, which is necessary for us to use Duffy's transformations in order to remove the singularities, cf. [Sau96, vPS97]. Denote the vertices of the triangle T by $A_0 = (0,0)$, $A_1 = (0,1)$, and $A_2 = (1,1)$. Then obviously, $T\times T$ has nine vertices $A_{ik} := A_i \times A_k$ for $i,k = 0,1,2$. Note that $A_{00} = O$.

We break $T\times T$ up into two pyramides $D_1 := \{(x,y) \in T\times T : x_1 > y_1\}$ and $D_2 := \{(x,y) \in T\times T : x_1 < y_1\}$. One can verify that D_1 is the pyramid with vertex O and base $B_1 = A_{10}A_{11}A_{12}A_{20}A_{21}A_{22}$, being a triangular prism, and that D_2 is the pyramid with vertex O and base B_2 $B_2 = A_{01}A_{11}A_{21}A_{02}A_{12}A_{22}$, being also a triangular prism. Moreover, these prisms can be described as $B_1 = \{1\} \times (0,1) \times T$ and $B_2 = T \times \{1\} \times (0,1)$. Introducing the reflection with respect to the plane $x = y$ by $\mathcal{R} : (x,y) \mapsto (y,x)$, we notice the symmetry $B_2 = \mathcal{R}B_1$ and so $D_2 = \mathcal{R}D_1$.

By subdividing the prism B_1 into tetrahedra, we can get a simplicial partitioning of $T\times T$, because any simplicial partitioning of B_1 induces a simplicial partitioning of D_1, and by taking the image under the mapping $\mathcal{R}$, a simplicial partitioning of D_2. Our choice of such a partitioning is depicted in Figure 3.

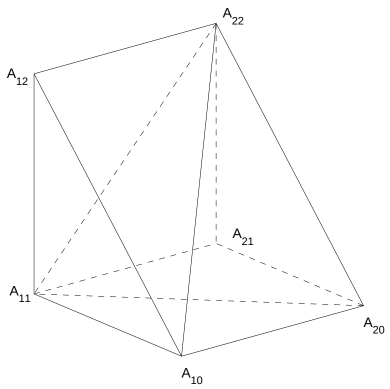

FIGURE 3. A simplicial partitioning of the prism B_1.

Consequently, the domain $T\times T$ is subdivided into the following simplices described by their vertices.

$$D_1 \begin{cases} S_1 = OA_{10}A_{11}A_{12}A_{22}, \\ S_2 = OA_{10}A_{11}A_{20}A_{22}, \\ S_3 = OA_{11}A_{20}A_{21}A_{22}, \end{cases} \quad \text{and} \quad D_2 \begin{cases} S_4 = OA_{01}A_{11}A_{21}A_{22}, \\ S_5 = OA_{01}A_{11}A_{02}A_{22}, \\ S_6 = OA_{11}A_{02}A_{12}A_{22}. \end{cases}$$

We notice the symmetry $S_i = \mathcal{R}S_{i+3}$ for $i = 1,2,3$. The above partitionings of $T\times T$ will be used in quadrature schemes for the integral (A.3).

In the following we will distinguish three basic cases:

- *Coincident panels:* $\Xi = \Xi'$, that is, the case of identical panels;
- *Edge adjacent panels:* $\overline{\Xi}$ and $\overline{\Xi'}$ share one common edge;
- *Vertex adjacent panels:* $\overline{\Xi}$ and $\overline{\Xi'}$ share one common vertex.

In view of (A.3), we need to integrate a singular function over a four dimensional polyhedral domain $T \times T$. The singularity of the function is located on different dimensional sets in different situations: whereas the singularity occurs at a point for vertex adjacent integrals, it occurs all along an edge in case the integral is edge adjacent, and for coincident integrals, the singularity is on a two dimensional "diagonal" of the domain. Therefore in each of the three cases, we first characterize the singularity in terms of the distance to the singularity set, and then introduce special coordinate transformations that annihilate the singularity.

A.1. **Case of identical panels.** First we will discuss the case of identical panels $\Xi = \Xi'$. In this case, the difference $r = \chi_\Xi(y) - \chi_\Xi(x)$ is zero if and only if $t := y - x = 0$. Since χ_Ξ is affine, we can write

$$r = 2^{-|\lambda|} l_1(t) = 2^{-|\lambda|} l_1(y_1 - x_1, y_2 - x_2),$$

where $l_1 : I\!R^2 \mapsto I\!R^3$ is a linear function depending only on the shape of Ξ. Noting that any panel Ξ is similar to a panel from the initial triangulation, we only have to deal with finitely many functions l_1. Introducing polar coordinates (ρ, θ) in $I\!R^2$ by $\rho = |t|$ and $\theta = t/|t| \in S^1$, being the unit circle in $I\!R^2$, this difference r reads as

$$r = 2^{-|\lambda|} \rho l_1(\theta).$$

Our goal is now to obtain an expression for $|r|^{-1}$, because this quantity essentially determines the singular behavior of the local kernel. Since r is defined on some complete neighborhood of $t = 0$, the function $l_1(\theta)$ has to be nonzero for any $\theta \in S^1$, and so we have

$$|r|^{-1} = 2^{|\lambda|} \rho^{-1} a(\theta)$$

with $a(\theta) := |l_1(\theta)|^{-1}$ which is analytic in a neighborhood of S^1. Now the integrand of (A.3) can be written as

$$|r(x, y)|^{-1} g(x, y) = 2^{|\lambda|} \rho^{-1} a(\theta) g(x, y). \tag{A.5}$$

It is time to use the above described simplicial partitioning of the integration domain $T \times T$, in combination with special coordinate transformations for the purpose of removing the singularity of the integrand. Introducing the notation $\mathcal{P} := T \times (0, 1) \times (0, 1)$, we define the transformations $\phi_i : \mathcal{P} \rightarrow S_i : (\eta, \zeta, \xi) \mapsto (x, y)$ for $i \in \overline{1, 6}$.

$$\phi_1(\eta,\zeta,\xi) = \begin{pmatrix} (1-\xi)\eta_1 + \xi \\ (1-\xi)\eta_2 \\ (1-\xi)\eta_1 + \xi\zeta \\ (1-\xi)\eta_2 + \xi\zeta \end{pmatrix}, \quad \phi_2(\eta,\zeta,\xi) = \begin{pmatrix} (1-\xi)\eta_1 + \xi \\ (1-\xi)\eta_2 + \xi\zeta \\ (1-\xi)\eta_1 \\ (1-\xi)\eta_2 \end{pmatrix},$$

$$\phi_3(\eta,\zeta,\xi) = \begin{pmatrix} (1-\xi)\eta_1 + \xi \\ (1-\xi)\eta_2 + \xi \\ (1-\xi)\eta_1 + \xi\zeta \\ (1-\xi)\eta_2 \end{pmatrix}, \tag{A.6}$$

and $\phi_{i+3} := \mathcal{R} \circ \phi_i$ for $i = 1, 2, 3$. The Jacobian of each transformation ϕ_i is given by $\xi(1 - \xi)^2$. Recall that ρ^{-1} characterizes the singularity of the integrand (A.5). In this regard, for each transformation ϕ_i one can show that

$$\rho = \xi f_i(\zeta), \quad \text{with an analytic } f_i(\zeta) \geq \tfrac{1}{\sqrt{2}} \quad \text{for any } \zeta \in [0, 1].$$

For instance, for ϕ_1 we have

$$\rho^2 = \xi^2(\zeta^2 + (1 - \zeta)^2) \geq \xi^2 \cdot \tfrac{1}{2},$$

since $\zeta^2 + (1 - \zeta)^2 \geq \frac{1}{2}$ for any $\zeta \in I\!R$. Moreover, for each ϕ_i one can verify that $\theta = \vartheta_i(\zeta)$ for some analytic function $\vartheta_i : [0, 1] \to S^1$.

In all, the Jacobian of the mapping ϕ_i annihilates the singularity in the integrand (A.5), meaning that the integral I in (A.3) now can be written as the following proper integral

$$(A.7)\qquad \begin{aligned} I &= \int_0^1 \int_0^1 \int_T \xi(1-\xi)^2 \sum_{i=1}^{6} \frac{g(\phi_i(\eta,\zeta,\xi))}{|r(\phi_i(\eta,\zeta,\xi))|} d\eta d\zeta d\xi \\ &= 2^{|\lambda|} \int_0^1 \int_0^1 \int_T (1-\xi)^2 \sum_{i=1}^{6} \frac{a(\vartheta_i(\zeta)) g(\phi_i(\eta,\zeta,\xi))}{f_i(\zeta)} d\eta d\zeta d\xi. \end{aligned}$$

Therefore we will be able to use standard quadrature schemes to approximate the integral I. Note that in numerical quadrature we can use the first expression in (A.7) for the integral I. The functions f_i and $a \circ \vartheta_i$ are introduced here merely for the analysis purpose.

Since the integrand in (A.7) is polynomial with respect to the variables ξ and η, we can always choose exact quadrature rules for integrations over those variables.

Proposition A.1. *Approximate the integral* (A.7) *by a product quadrature rule* $Q_\xi \times Q_\zeta \times Q_\eta$*, where* Q_ξ *and* Q_η *are quadrature rules exact for the integration over the variables* $\xi \in (0,1)$ *and* $\eta \in T$*, respectively, and* Q_ζ *is a composite quadrature rule for the integration over* $\zeta \in (0,1)$ *of varying order* p *and fixed rank* N*. Then there exist a constant* $\delta > 0$ *such that the quadrature error satisfies*

$$(A.8)\qquad |E(\Xi,\Xi')| \lesssim 2^{-\frac{3}{2}(|\lambda|-|\lambda'|)}(\delta N)^{-p}.$$

Choosing N *such that* $\delta N > 1$*, we conclude that in this case Assumption 4.11 is fulfilled with* $d_0^* = -\frac{3}{2}$*.*

Proof. In view of Lemma 2.3, it suffices to consider the integration over ζ. Using the analyticity of $\zeta \mapsto \frac{a(\vartheta_i(\zeta))}{f_i(\zeta)}$ one derives

$$\sup_{\zeta\in[0,1]} \left| \partial_\zeta^k \frac{a(\vartheta_i(\zeta))}{f_i(\zeta)} \right| \lesssim \frac{k!}{\delta^k} \quad \text{for } k \in \mathbb{N}_0,\ i \in \overline{1,6},$$

for some constant $\delta > 0$. From (A.4) and (A.6) we have for each $i \in \overline{1,6}$ that $g \circ \phi_i$ is a polynomial of order e and

$$|\partial_\zeta^k (g \circ \phi_i)| \lesssim 2^{-\frac{5}{2}|\lambda| + \frac{3}{2}|\lambda'|} \quad \text{for } k \in \overline{1,e-1}.$$

Now using Proposition 2.4 the proof is obtained. □

A.2. **Case of edge adjacent panels.** Now we will discuss the case when $\overline{\Xi}$ and $\overline{\Xi'}$ share exactly one common edge. Without loss of generality, we assume that $\chi_\Xi(x) = \chi_{\Xi'}(x)$ for all $x \in (0,1) \times \{0\}$. Then, the difference $r = \chi_{\Xi'}(y) - \chi_\Xi(x)$ is zero if and only if

$$t = (t_1, t_2, t_3) := (y_1 - x_1, x_2, y_2)$$

equals zero. Since χ_Ξ and $\chi_{\Xi'}$ are affine, we can write

$$r = \chi_{\Xi'}(x_1 + t_1, t_3) - \chi_\Xi(x_1, t_2) = 2^{-|\lambda|} l_1(t),$$

where $l_1 : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear function depending only on the shapes of Ξ and Ξ'. Introducing polar coordinates (ρ, θ) in $\mathbb{R}^3$ by $\rho = |t|$ and $\theta = t/|t| \in S^2$, being the unit sphere in $\mathbb{R}^3$, this difference r reads as

$$r = r(\rho, \theta) = 2^{-|\lambda|} \rho l_1(\theta).$$

Since r is defined on a complete neighborhood of $t = 0$ in $\mathbb{R} \times \mathbb{R}^2_{\geq 0}$, the function $l_1(\theta) \neq 0$ for any $\theta \in S^2$ with $\theta_2, \theta_3 \geq 0$, allowing us to write

$$|r|^{-1} = 2^{|\lambda|}\rho^{-1}b(\theta)$$

with $b(\theta) := |l_1(\theta)|^{-1}$ which is analytic in a neighborhood of $S^2 \cap (\mathbb{R} \times \mathbb{R}^2_{\geq 0})$. Then the integrand of (A.3) can be written as

$$|r(x,y)|^{-1}g(x,y) = 2^{|\lambda|}\rho^{-1}b(\theta)g(x,y). \tag{A.9}$$

Now we define the transformations $\phi_i : \mathcal{P} \to S_i : (\eta, \zeta, \xi) \mapsto (x, y)$ for $i \in \overline{1,6}$.

$$\phi_1(\eta,\zeta,\xi) = \begin{pmatrix} (1-\xi)\zeta + \xi \\ \xi\eta_2 \\ (1-\xi)\zeta + \xi\eta_1 \\ \xi\eta_1 \end{pmatrix}, \quad \phi_2(\eta,\zeta,\xi) = \begin{pmatrix} (1-\xi)\zeta + \xi \\ \xi\eta_1 \\ (1-\xi)\zeta + \xi\eta_2 \\ \xi\eta_2 \end{pmatrix},$$

$$\phi_3(\eta,\zeta,\xi) = \begin{pmatrix} (1-\xi)\zeta + \xi \\ \xi \\ (1-\xi)\zeta + \xi\eta_1 \\ \xi\eta_2 \end{pmatrix}, \tag{A.10}$$

and $\phi_{i+3} := \mathcal{R} \circ \phi_i$ for $i = 1, 2, 3$. For each transformation ϕ_i one can show that the Jacobian equals $\xi^2(1-\xi)$, and that

$$\rho = \xi f_i(\eta), \quad \text{with an analytic } f_i(\eta) \geq \tfrac{1}{\sqrt{2}} \quad \text{for any } \eta \in \overline{T}.$$

For instance, for ϕ_1 we have

$$\rho^2 = \xi^2(\eta_1^2 + (1-\eta_1)^2 + \eta_2^2) \geq \xi^2 \cdot \tfrac{1}{2}.$$

Moreover, for each ϕ_i one can verify that $\theta = \vartheta_i(\eta)$ with some analytic function $\vartheta_i : \overline{T} \to S^2$.

In all, the Jacobian of the mapping ϕ_i annihilates the singularity in the integrand (A.9), meaning that the integral I in (A.3) now can be written as the following proper integral

$$\begin{aligned} I &= \int_0^1 \int_0^1 \int_T \xi^2(1-\xi) \sum_{i=1}^{6} \frac{g(\phi_i(\eta,\zeta,\xi))}{|r(\phi_i(\eta,\zeta,\xi))|} d\eta d\zeta d\xi \\ &= 2^{|\lambda|} \int_0^1 \int_0^1 \int_T \xi(1-\xi) \sum_{i=1}^{6} \frac{b(\vartheta_i(\eta))g(\phi_i(\eta,\zeta,\xi))}{f_i(\eta)} d\eta d\zeta d\xi, \end{aligned} \tag{A.11}$$

and thus the standard quadrature schemes on $\mathcal{P}$ can be applied.

Proposition A.2. *Approximate the integral* (A.11) *by a product quadrature rule $Q_\xi \times Q_\zeta \times Q_\eta$, where Q_ξ and Q_ζ are quadrature rules exact for the integration over the variables $\xi, \zeta \in (0,1)$, respectively, and Q_η is a composite quadrature rule for the integration over $\eta \in T$ of varying order p and fixed rank N. Then there exist a constant $\delta > 0$ such that the quadrature error satisfies*

$$|E(\Xi, \Xi')| \lesssim 2^{-\frac{3}{2}(|\lambda|-|\lambda'|)}(\delta N)^{-p}. \tag{A.12}$$

Choosing N such that $\delta N > 1$, we conclude that in this case Assumption 4.11 is fulfilled with $d_0^* = -\frac{3}{2}$.

The proof is obtained similarly to the proof of Proposition A.1.

A.3. **Case of vertex adjacent panels.** Let $\overline{\Xi}$ and $\overline{\Xi'}$ share exactly one common point. Without loss of generality, we assume that $\chi_\Xi(0) = \overline{\Xi} \cap \overline{\Xi'} = \chi_{\Xi'}(0)$. Then obviously, the difference $r = r(x, y) = \chi_{\Xi'}(y) - \chi_\Xi(x)$ is zero if and only if $t := (x, y)$ equals zero. Since χ_Ξ and $\chi_{\Xi'}$ are affine, we can write

$$r(x, y) = 2^{-|\lambda|} l_1(x, y),$$

where $l_1 : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear function depending only on the shapes of Ξ and Ξ'. Introducing polar coordinates (ρ, θ) in $\mathbb{R}^4$ by $\rho = |t|$ and $\theta = t/|t| \in S^3$, being the unit sphere in $\mathbb{R}^4$, this difference r reads as

$$r = r(\rho, \theta) = 2^{-|\lambda|} \rho l_1(\theta).$$

Since r is defined on a complete neighborhood of $t = 0$ in $\{t \in \mathbb{R}^4 : t_1 \geq t_2 \geq 0,\ t_3 \geq t_4 \geq 0\}$, the function $l_1(\theta) \neq 0$ for any $\theta \in S^3$ with $\theta_1 \geq \theta_2 \geq 0$ and $\theta_3 \geq \theta_4 \geq 0$, allowing us to write

$$|r|^{-1} = 2^{|\lambda|} \rho^{-1} c(\theta)$$

with $c(\theta) := |l_1(\theta)|^{-1}$ which is analytic in a neighborhood of $\{\theta \in S^3 : \theta_1 \geq \theta_2 \geq 0,\ \theta_3 \geq \theta_4 \geq 0\}$. Then the integrand of (A.3) can be written as

$$|r(x, y)|^{-1} g(x, y) = 2^{|\lambda|} \rho^{-1} c(\theta) g(x, y). \tag{A.13}$$

We define the transformations ϕ_1 and ϕ_2 that map the coordinates $(\eta, \zeta, \xi) \in \mathcal{P}$ onto the four dimensional pyramides D_1 and D_2 respectively.

$$\phi_1(\eta, \zeta, \xi) = \xi(1, \zeta, \eta_1, \eta_2), \quad \text{and} \quad \phi_2(\eta, \zeta, \xi) = \xi(\eta_1, \eta_2, 1, \zeta). \tag{A.14}$$

Notice that $\phi_1 = \mathcal{R} \circ \phi_2$ with $\mathcal{R}$ being the reflection $x \leftrightarrow y$. For both of the transformations, the Jacobian equals ξ^3, and we have

$$\rho = \xi f(\eta, \zeta) \quad \text{with } f(\eta, \zeta) = \sqrt{1 + {\eta_1}^2 + {\eta_2}^2 + \zeta^2}.$$

Moreover, for the transformation ϕ_1 we have $\theta = \vartheta_1(\eta, \zeta) := f(\eta, \zeta)^{-1}(1, \zeta, \eta_1, \eta_2)$, and for the transformation ϕ_2 we have $\theta = \vartheta_2(\eta, \zeta) := \mathcal{R}\vartheta_1(\eta, \zeta)$.

In all, the Jacobian of the mapping ϕ_i annihilates the singularity in the integrand (A.13), meaning that the integral I in (A.3) now can be written as the following proper integral

$$\begin{aligned} I &= \int_0^1 \int_0^1 \int_T \xi^3 \sum_{i=1}^{2} \frac{g(\phi_i(\eta, \zeta, \xi))}{|r(\phi_i(\eta, \zeta, \xi))|} d\eta d\zeta d\xi \\ &= 2^{|\lambda|} \int_0^1 \int_0^h \int_T \xi^2 \sum_{i=1}^{2} \frac{c(\vartheta_i(\eta, \zeta)) g(\phi_i(\eta, \zeta, \xi))}{f(\eta, \xi)} d\eta d\zeta d\xi, \end{aligned} \tag{A.15}$$

and thus the standard quadrature schemes on $\mathcal{P}$ can be applied.

Proposition A.3. *Approximate the integral* (A.15) *by a product quadrature rule $Q_\xi \times Q_\zeta \times Q_\eta$, where Q_ξ is a quadrature rule exact for the integration over $\xi \in (0, 1)$, and Q_ζ and Q_η are composite quadrature rules for the integration over $\zeta \in (0, 1)$ and $\eta \in T$, respectively, of varying order p and fixed rank N. Then there exist a constant $\delta > 0$ such that the quadrature error satisfies*

$$|E(\Xi, \Xi')| \lesssim 2^{-\frac{3}{2}(|\lambda| - |\lambda'|)} (\delta N)^{-p}. \tag{A.16}$$

Choosing N such that $\delta N > 1$, we conclude that in this case Assumption 4.11 is fulfilled with $d_0^* = -\frac{3}{2}$.

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