GEOMETRICAL SPINES OF LENS MANIFOLDS

S. Anisov

Abstract. We introduce the concept of "geometrical spine" for 3-manifolds with natural metrics, in particular, for lens manifolds. We show that any spine of $L_{p,q}$ that is close enough to its geometrical spine contains at least $E(p,q)-3$ vertices, which is exactly the conjectured value for the complexity $c(L_{p,q})$. As a byproduct, we find the minimal rotation distance (in the Sleator–Tarjan–Thurston sense) between a triangulation of a regular p-gon and its image under rotation.

§1. Introduction

One can try to measure the complexity of a 3-manifold M by the minimal number of tetrahedra $n = n(M)$ such that M can be cut into n tetrahedra. This approach has some drawbacks, in particular, this number n is not necessarily additive under connected summation. S. Matveev has defined the complexity $c(M)$ of a 3-manifold as the minimal number of vertices of an almost simple spine of M, see [6] and definitions below. The complexity $c(M)$ is equal to $n(M)$ for many 3-manifolds (including lenses) but is free of some of disadvantages of $n(M)$. Still, it is very difficult to find $c(M)$ or even to estimate it from below, while reasonable upper bounds are frequently quite straigthforward.

Given a metric on M, one can consider the cut locus (see [4]) of a point $x \in M$. These cut loci are spines of M, though not necessarily almost simple, especially if the metric admits many isometries. Almost simple spines can, however, be obtained by small perturbations of the cut loci (or of the metric). This approach to constructing spines seems not to be systematically elaborated earlier, but it frequently yields spines with the smallest known numbers of vertices. We show, in particular, that this is the case for lens manifolds.

Since $L_{p,q}$ is a quotient space of $S^3 \in \mathbb{C}^2$ by a unitary action of a cyclic p-element group $\mathbb{Z}_p$ (the generator of $\mathbb{Z}_p$ takes a unit vector $(z,w) \in \mathbb{C}^2$ to $(\xi z, \xi^q w)$, where $\xi = \mathrm{e}^{2\pi i/p}$), it carries a natural metric of constant curvature $+1$ and volume $2\pi^2/p$. In this paper we study spines of $L_{p,q}$ that are close to cut loci of points $x \in L_{p,q}$ with respect to this natural metric.

Recall that the best known spines of $L_{p,q}$ contain $E(p,q)-3$ vertices (see [6]; a different construction of those spines can be found in [1]), where $E(p,q)$ is the number of subtractions that the Euclid algorithm needs to convert an unordered pair of positive integers (p,q) into $(d,0)$, where $d = \text{g.c.d.}(p,q)$ (of course, $d=1$ if (p,q) encodes a lens); in other words, $E(p,q)$ is equal to the sum of the elements of the continued fraction representing p/q.

Conjecture 1 [6, 7]. *Let $p \geq 3$. Then*

$$c(L_{p,q}) = E(p,q) - 3. \tag{1}$$

In particular, for $q = 1$ we have $E(p, 1) = p$, and equation (1) yields $c(L_{p,1}) = p - 3$. A possible explanation for the mysterious summand -3 in (1) is that the number of diagonals of a p-gon required to triangulate it equals $p - 3$, see also the proof of Theorem 4.

The paper is organized as follows. In §2 we prove a number-theoretic property of the function $E(p, q)$ (Theorem 1), which is later used in Theorem 4. The proof of Theorem 1 is based on the properties of the Farey tesselation of the hyperbolic plane; these properties are discussed in §2 and in Appendix. In §3 we recall the definitions of simple and special polyhedra, spines and the complexity of 3-manifolds. Geometrical spines are introduced in §4. In §5 we discuss the structure of geometrical spines of lens manifolds; in particular, we show the duality between spines and convex hulls (Lemma 3) and prove that the combinatorial type of a simple geometrical spine of a lens does not depend on the choice of "base point" involved in the construction (Lemma 4). In §6 we prove (see Theorem 3) that any simple spine P of a lens manifold $L_{p,q}$ has at least $E(p, q) - 3$ vertices provided that P is a small perturbation of a geometric spine. Theorem 3 is reduced to a combinatorial statement (Theorem 4) about the rotation distance [9] between triangulations of a regular p-gon. Finally, in §7 we discuss sharpness of lower bounds given by Theorems 3 and 4.

§2. Digression: a sanity test for Conjecture 1

Before going any further, let us check that $E(p, q)$ (where $(p, q) = 1$ and $0 < q < p$) is a well defined function of a lens manifold $L_{p,q}$. It is well known (see, e.g., [3]) that lenses $L_{p,q}$ and $L_{p,p-q}$ are homeomorphic, as well as lenses $L_{p,q}$ and $L_{p,r}$, where $r = q^{-1}$ modulo p. This suggests the following property of the Euclid algorithm.

Theorem 1. *The following relations hold*:

a) $E(p, q) = E(p, p - q)$;
b) $E(p, q) = E(p, r)$ *whenever* $0 < r < p$ *and* $rq = 1$ *modulo* p (p *and* q *are coprime in this case*).

If these relations were false, Conjecture 1 would be automatically false, too.

The first subtraction of the Euclid algorithm converts the pair $(p, p - q)$ into $(q, p - q)$. By definition of $E(p, q)$, we get $E(p, p - q) = E(q, p - q) + 1$, and similarly $E(p, q) = E(q, p - q) + 1$, which proves the first relation. The second relation is proved later in this section; the proof is based on some properties of the *Farey tesselation* of the hyperbolic plane H^2.

Consider the ideal triangle in H^2 with vertices at 0, 1, and ∞. Take its mirror images in its sides. This gives the triangles $(-1, 0, \infty)$, $(0, 1/2, 1)$, and $(1, 2, \infty)$, where (a, b, c) denotes the ideal triangle with vertices a, b, and c. On the next step, construct the images of the triangles obtained in the previous step under reflections in their sides that are not sides of triangles obtained earlier. Continuing this way, we get a tesselation of H^2 into equal ideal triangles (of course, all ideal triangles in H^2 are congruent; what is special about this tesselation is its symmetry in any of its edges), see Fig. 1. It is called the *Farey tesselation.* Centers of the triangles on Fig. 1 are the vertices of the graph Γ dual to the Farey tesselation. This graph, which is the infinite binary tree embedded in H^2, is shown on Fig. 1 by dotted lines.

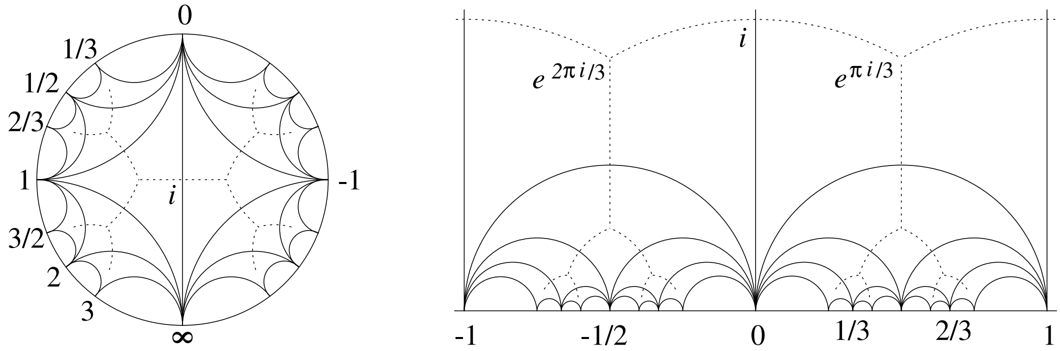

FIGURE 1. Farey tesselation of hyperbolic plane

Algebraic properties of the Farey tesselation are collected in Lemma 1 below; its geometric properties are discussed in §4.

Lemma 1.

a) *A segment* $(m/n, p/q)$, *where* (m, n) *and* (p, q) *are pairs of coprime integers, occurs as an edge of the Farey tesselation if and only if* $mq - np = \pm 1$ (*we agree that* $\infty = 1/0$);

b) $E(p, q)$ *equals the number of Farey triangles that are cut by the geodesic segment in* H^2 *connecting* p/q *with* i (*or with any other interior point* ti, $t > 0$, *on the edge* $(0, \infty)$ *of the Farey tesselation*). *If* $|p| > |q|$, *then* $E(p, q)$ *equals the number of Farey triangles cut by the geodesic segment in* H^2 *connecting* p/q *with* 0;

c) *the modular group* $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\{\pm I\}$ *acting on the upper half-plane by fractional linear transformations* $z \mapsto \frac{az+b}{cz+d}$, *preserves the Farey tesselation*;

d) *the third vertices of two Farey triangles incident to an edge* $(m/n, p/q)$ *are* $\frac{m+p}{n+q}$ *and* $\frac{m-p}{n-q}$;

e) *vertices of the Farey triangles obtained after* n *steps of the construction of the tesselation and belonging to* $[0, 1]$ *form the Farey sequence* (*see* [10]) *of depth* n.

For the proof of Lemma 1 see Appendix.

Proof of Theorem 1. The symmetry of H^2 in the line $(0, \infty)$, $z \mapsto -\bar{z}$, followed by a modular mapping $z \mapsto \frac{z}{z+1}$, is an isometry of H^2 preserving the Farey tesselation. This isometry is nothing but the symmetry $z \mapsto \frac{\bar{z}}{\bar{z}-1}$ of H^2 in the line $(0, 2)$. It takes 0 to 0 and p/q to $p/(p - q)$, thus a geodesic line $(0, p/q)$ gets mapped to $(0, p/(p - q))$. Theorem 1a) follows now (once again) from Lemma 1b).

If $0 < r < p$ and $rq = 1$ modulo p, we have $rq = pk + 1$. The symmetry of H^2 in $(0, \infty)$, $z \mapsto -\bar{z}$, followed by a modular map $z \mapsto \frac{rz+p}{kz+q}$, is an isometry of H^2 preserving the Farey tesselation. This isometry $z \mapsto \frac{r\bar{z}-p}{k\bar{z}-q}$ takes 0 to p/q and p/r to 0, so it takes the geodesic line $(0, p/r)$ to the geodesic $(0, p/q)$. The second statement of Theorem 1 follows now from Lemma 1b). $\square$

The mapping used in the proof of part b) of the theorem has a transparent topological meaning. Like a solid torus is obtained from $T^2 \times [0, 1]$ by contracting the meridians of $T^2 \times \{0\}$, lens manifolds can be constructed from $T^2 \times [0, 1]$ by contracting a family of curves of some rational slope α in $T^2 \times \{0\}$ and a family of curves of rational slope β in $T^2 \times \{1\}$. Choose a basis in the lattice $\pi_1(T^2) = \mathbb{Z}^2$ so that $\alpha = 0$. If $\beta = p/q$ in this basis, we get $L_{p,q}$. To interchange the roles of $T^2 \times \{0\}$ and $T^2 \times \{1\}$ in this construction, we use a new basis in $\pi_1(T^2)$ related to the previous one by an orientation reversing linear transformation with the matrix $\begin{pmatrix} q & -k \\ p & -r \end{pmatrix}$. This linear map takes a vector with slope y/x to a vector with slope $\frac{r(y/x)-p}{k(y/x)-q}$. In particular, it takes the slope 0 to p/q and p/r to 0, thus showing that the interchanging of $T^2 \times \{0\}$ and $T^2 \times \{1\}$ in the construction above leads to encoding $L_{p,q}$ as $L_{p,r}$.

Alternative proofs of Theorem 1 can be found in the preprints [1, pp. 5–6] and [5, pp. 29–30]. Yet another proof of Theorem 1 b is based on the identity $n_k + 1/(n_{k-1} + 1/(n_{k-2} + \ldots + 1/n_1)\ldots) = p/s$ with $s \equiv (-1)^{k-1} r$ modulo p, where $p/q = n_1 + 1/(n_2 + 1/(n_3 + \ldots + 1/n_k)\ldots)$ is the continued fraction representing p/q. This identity holds whenever $n_1 \geq 2$ and $n_k \geq 2$.

§3. Spines and complexity of 3-manifolds

Let us recall some definitions (we follow [6, 7] here). By K denote the 1-dimensional skeleton of the tetrahedron, which is just the clique (that is, the complete graph) with 4 vertices; K is homeomorphic to a circle with three radii.

Definition 1. A compact 2-dimensional polyhedron is called *almost simple* if the link of its every point can be embedded in K. An almost simple polyhedron P is said to be *simple* if the link of each point of P is homeomorphic to either a circle or a circle with a diameter or the whole graph K. A point of an almost simple polyhedron is *non-singular* if its link is homeomorphic to a circle, it is said to be a *triple point* if its link is homeomorphic to a circle with a diameter, and it is called a *vertex* if its link is homeomorphic to K. The set of singular points of a simple polyhedron P (i.e., the union of the vertices and the triple lines) is called its *singular graph* and is denoted by SP.

It is easy to see that any compact subpolyhedron of an almost simple polyhedron is almost simple as well. Neighborhoods of non-singular and triple points of a simple polyhedron are shown in Fig. 2 a, b; Fig. 2 c–f represents four equivalent ways of looking at vertices; in particular, Fig. 2 e shows the cone over the 1-dimensional skeleton of the tetrahedron and Fig. 2 f shows the Voronoi diagram of the vertices of a regular tetrahedron.

Definition 2. A simple polyhedron P with at least one vertex is said to be *special* if it contains no closed triple lines (without vertices) and every connected component of $P \setminus SP$ is a 2-dimensional cell.

Definition 3. A polyhedron $P \subset \operatorname{Int} M$ is called a *spine* of a compact 3-dimensional manifold M if $M \setminus P$ is homeomorphic to $\partial M \times (0,1]$ (if $\partial M \neq 0$) or to an open 3-cell (if $\partial M = 0$). In other words, P is a spine of M if a manifold M with boundary (or a closed manifold M punctured at one point) can be collapsed onto P. A spine P of a 3-manifold M is said to be *almost simple*, *simple*, or *special* if it is an almost simple, simple, or special polyhedron, respectively.

Given a special spine P of a compact manifold M^3, one can construct a dual singular triangulation of M^3 with one vertex (lying in the middle of the 3-cell $M \setminus P$), see Fig. 2 f; if M is a manifold with connected boundary, a similar construction

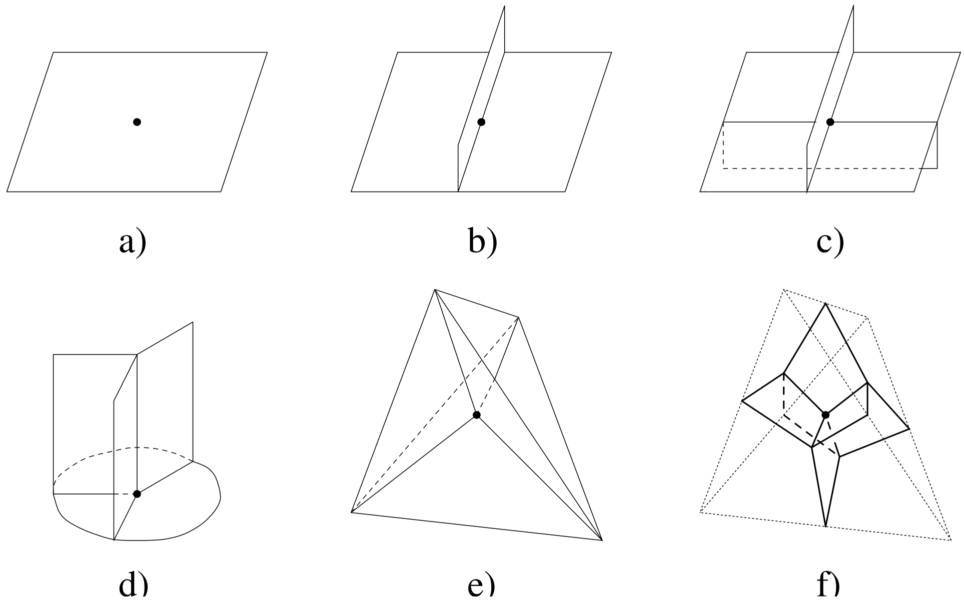

FIGURE 2. Nonsingular (a) and triple (b) points; ways of looking at vertices (c–f)

gives a triangulation of the one-point compactification of $M \setminus \partial M$. In both cases, there is a one-to-one correspondence between vertices of P and tetrahedra of the triangulation.

Definition 4. The *complexity* $c(M)$ of a compact 3-manifold M is the minimal possible number of vertices of an almost simple spine of M. An almost simple spine with the smallest possible number of vertices is said to be a *minimal* spine of M.

Theorem 2 [6]**.** *Let M be an orientable irreducible 3-manifold with incompressible (or empty) boundary and without essential annuli. If $c(M) > 0$ (that is, if M is different from (possibly punctured) S^3, $\mathbb{R}P^3$, and $L_{3,1}$), then any minimal spine of M is special.*

Note that lenses $L_{p,q}$ with $p > 3$ satisfy the conditions of Theorem 2. Thus any minimal spine of $L_{p,q}$ is special and corresponds to a decomposition of $L_{p,q}$ into $n = c(L_{p,q})$ tetrahedra.

§4. Cut loci and Voronoi diagrams

We are going to use relations between spines, cut loci, and Voronoi diagrams. For the definition and some properties of cut loci, see [4, Chapter VIII, §7]. Here we only recall that given a point $x \in M$ of a complete Riemannian manifold (M, g), the *cut locus* of x is the closure (in M) of the set of points $y \in M$ such that the shortest geodesic between x and y is not unique. If M is compact, then the preimage $\exp^{-1}(C(x)) \subset T_xM$ of a cut locus $C(x)$ under the exponential map $\exp\colon T_xM \to M$ is homeomorphic to a sphere and $M \setminus C(x)$ is homeomorphic to a ball [4].

Now we have the following method of constructing spines of compact 3-manifolds: choose a metric g on M and a point $x \in M$. Then the cut locus $C(x)$ is a spine of M, though not necessarily special. Contrary, for any spine P of M, there is a metric g such that P is isotopic to the cut locus $C(x)$ for some $x \in M$ with respect to g (take a standard metric on a unit ball homeomorphic to $M \setminus P$, transfer it to M and smoothen it around P).

Examples. 1. The cut locus of the sphere S^n (with the standard metric) with respect to the North pole N is the South pole S. So a single point is a spine of S^3.

2. The cut locus of a "rectangular" flat torus T^2 with respect to a point $x = (\varphi_0, \psi_0)$ is the union of the parallel $\psi = \psi_0 + \pi$ and the meridian $\varphi = \varphi_0 + \pi$ opposite to x. The complement $T^2 \setminus C(x)$ is a flat rectangle.

3. The cut locus of a generic flat torus T^2 with respect to any point $x \in T^2$ is the union of three nonhomotopic geodesic segments connecting the two local maxima of d_x, the distance to x. The complement $T^2 \setminus C(x)$ is a flat centrally symmetric hexagon. The preimage of $C(x)$ under the universal covering $\mathbb{R}^2 \to T^2$ is shown on Fig. 3.

4. The cut locus of the torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ (where $\mathbb{R}^3$ carries a standard metric and $\mathbb{Z}^3$ is the integer span of an orthogonal basis) with respect to the point $(1/2, 1/2, 1/2)$ is the union of three "coordinate" tori T^2. This spine is not special, since it contains, in particular, lines of transversal intersection of two surfaces. However, the cut locus of a point in a generic flat T^3 is a special spine with 6 vertices, which is a minimal spine, because $c(T^3) = 6$ [7]. See also Fig. 12 in [1].

Voronoi diagrams are, roughly speaking, cut loci with respect to many points. Originally they were defined in [11] for Euclidean plane (or, more generally, for $\mathbb{R}^n$) with finitely many nodes $A_1, \ldots, A_k \in \mathbb{R}^n$ as the set of points where the nearest node is not unique. A Voronoi diagram divides $\mathbb{R}^n$ into Voronoi domains $U_1, \ldots, U_k$, where U_i consists of the points of $\mathbb{R}^n$ that are closer to A_i than to A_j with any $j \neq i$. This admits an obvious generalization for locally finite subsets of complete Riemannian manifolds. The following examples 5–8 are parallel to the examples 1–4 above.

Examples. 5. The Voronoi diagram of the sphere S^n with respect to a single point N is empty; the whole S^n forms the single Voronoi domain. The Voronoi diagram of S^n with respect to two antipodal points N and S (North and South poles) is the equator $S^{n-1} \subset S^n$; Northern and Southern hemispheres form the Voronoi domains (of course, we use here and below the standard metric on S^n).

6. The Voronoi diagram of the square lattice $\{(m, n) \mid m, n \in \mathbb{Z}\}$ in $\mathbb{R}^2$ is the square grid formed by lines $\{x\} = 1/2$ and $\{y\} = 1/2$, where $\{\cdot\}$ denotes the fractional part. The Voronoi domains are unit squares centered at points with integer coordinates.

7. The Voronoi diagram of the lattice $\{(m + n/2, \sqrt{3}n/2) \mid m, n \in \mathbb{Z}\} \subset \mathbb{R}^2$ generated by the vectors $(1, 0)$ and $(1/2, \sqrt{3}/2)$ looks like a honeycomb, see Fig. 3, left. The Voronoi domains are regular hexagons. The Voronoi diagram of a generic lattice in $\mathbb{R}^2$ is also hexagonal, see Fig. 3, right.

8. The Voronoi diagram of the cubic lattice $\{(k, l, m) \mid k, l, m \in \mathbb{Z}\}$ in $\mathbb{R}^3$ is formed by the planes $\{x\} = 1/2$, $\{y\} = 1/2$, and $\{z\} = 1/2$. The Voronoi domains are unit cubes. For a generic rank 3 lattice in $\mathbb{R}^3$, the Voronoi domains are centrally symmetric polyhedra with 14 facets, 36 edges, and 24 vertices.

The next two examples are central for the rest of the paper.

Example 9: universal covering. Let $p\colon \widetilde{M} \to M$ be the universal covering of M, where $\widetilde{M}$ is a constant curvature space H^n, $\mathbb{R}^n$, or S^n (but M itself is different

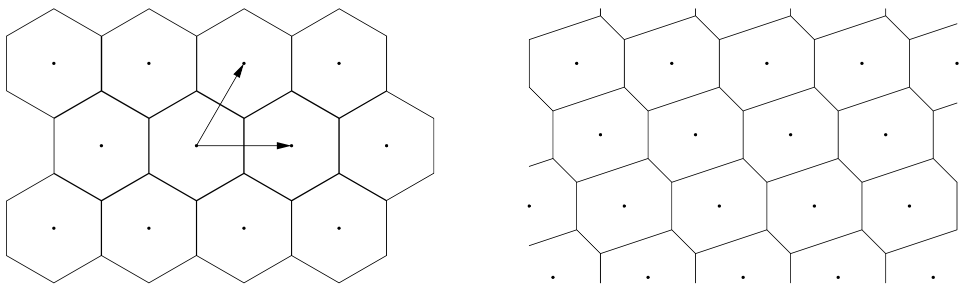

FIGURE 3. Voronoi diagrams of generic lattices in $\mathbb{R}^2$

from S^n); assume that M inherits a constant curvature metric from $\widetilde{M}$. Choose $x \in M$, set $X = \{x_1, x_2, \dots\} = p^{-1}(x) \subset \widetilde{M}$, and consider the Voronoi diagram in $\widetilde{M}$ with respect to X. In this situation all Voronoi domains are contractible and the projection of the Voronoi diagram in $\widetilde{M}$ is the cut locus of M with respect to x. If M is $\mathbb{R}P^3$, we get Example 5. Other examples above also are particular cases of this construction.

Example 10: Farey tesselation. The Teichmüller space for T^2 is the hyperbolic plane $H^2 = \{z = x + iy \in \mathbb{C} \mid y > 0\}$: T^2_z can be thought of as the quotient space of $\mathbb{R}^2$ over the lattice $\{m \cdot 1 + n \cdot z \mid m, n \in \mathbb{Z}\} \subset \mathbb{C}$. Let $X \subset H^2$ be the set of all parameters z corresponding to the tori with three equally short shortest geodesics (i.e., tori glued from a regular hexagon). Then the Farey tesselation is nothing but the Voronoi diagram of H^2 with respect to X. The vertices of the triangles represent the slopes of the elements of $\pi_1(T^2)$, and the modular group action on H^2 (restricted to the absolute) corresponds to coordinate changes in $\pi_1(T^2)$. The intersection points of solid and dashed lines on Fig. 1 represent "square" tori, and other points on the solid lines represent "rectangular" tori. The cut locus θ of a flat torus (generically it is a graph with two vertices and three pairwise nonhomotopic edges) changes isotopically as its parameter z varies inside of a Farey triangle, and bifurcates when z crosses Farey edges. Three vertices of a Farey triangle containing z are the slopes of three cycles in T^2_z formed by pairs of edges of θ. We omit rather straightforward proofs of these statements; see also [2].

§5. Application to lens manifolds

According to Example 9 above, a spine of a lens manifold $L_{p,q}$ can be obtained as the covering projection image of the Voronoi diagram (in $S^3 \subset \mathbb{R}^4$) of the $\mathbb{Z}_p$-orbit $\mathbb{Z}_p(z, w) = \{(\xi^k z, \xi^{kq} w) \mid k = 0, 1, \dots, p-1,\ \xi = \mathrm{e}^{2\pi i/p}\}$ of a point $(z, w) \in \mathbb{C}^2$. Then the ambient space $\mathbb{R}^4$ is divided into p congruent convex polyhedral cones with a common vertex at the origin. The Voronoi domains in S^3 are described by the following statement.

Lemma 2. *Let $\{A_1, \dots, A_k\} \subset S^{n-1}$ be a finite set of points on the unit sphere in $\mathbb{R}^n$. This set defines Voronoi domains in S^{n-1} and in $\mathbb{R}^n$. Then the Voronoi domains in S^{n-1} are the intersections of the Voronoi domains in $\mathbb{R}^n$ with S^n.*

Proof. This is obvious for a two-point set $\{A_1, A_2\}$. The general case follows from the case of two points. $\square$

Lemma 3. *Let $A = \{A_1, \dots, A_k\} \subset S^{n-1}$ be a finite set of points on the unit sphere in $\mathbb{R}^n$. Suppose that this set is not contained in a hyperplane. By* $\mathrm{Conv}(A)$

denote the convex hull of $A \subset \mathbb{R}^n$. Then the Voronoi diagram $\mathrm{VD}(A) \subset S^{n-1}$ *is dual to* $\mathrm{Conv}(A)$: *its vertices are unit outer normals to the facets of* $\mathrm{Conv}(A)$ *etc. In particular, the combinatorial type of* $\mathrm{VD}(A)$ *is determined by the combinatorial type of* $\mathrm{Conv}(A)$.

Proof. This follows from the definitions and from Lemma 2. $\square$

Consider the circles $\widetilde{S}^1_z = \{(\mathrm{e}^{i\varphi}, 0) \mid 0 \le \varphi < 2\pi\}$ and $\widetilde{S}^1_w = \{(0, \mathrm{e}^{i\varphi}) \mid 0 \le \varphi < 2\pi\}$ in S^3. They cover two *core circles* S^1_z and S^1_w in $L_{p,q}$. Take a point $x = (1, 0)$ (or any other point of $\widetilde{S}^1_z$). Its orbit $\mathbb{Z}_p x$ is a regular p-gon in the plane $w = 0$. The Voronoi diagram for $\mathbb{Z}_p x$ in $\mathbb{R}^4$ looks like "an orange slice times $\mathbb{R}^2$" and consists of p copies of $\mathbb{R}^3_+$ glued together along the plane $z = 0$ so that all dihedral angles are equal to $2\pi/p$. The Voronoi diagram for $\mathbb{Z}_p x$ in S^3 consists of p hemispheres $S^2_+ = S^2 \cap \mathbb{R}^3_+$ glued together at equal angles along the circle $\widetilde{S}^1_w$.

The covering mapping takes each of these hemispheres to a disk so that its boundary goes p times along S^1_w. The resulting spine of $L_{p,q}$ consists of this disk and S^1_w. Locally, its transversal (to S^1_w) section looks like the set $Y_p = \{r\,\mathrm{e}^{2\pi k i/p} \mid 0 \le r < 1,\ k = 0, 1, \ldots, p-1\}$ (Y_3 looks like Y, Y_4 like $\times$, Y_5 like $\star$, Y_6 like $*$ etc.). A neighborhood of S^1_w in this spine fibers over S^1_w with fiber Y_p. This fiber bundle is nontrivial: monodromy is a positive (clockwise as the parameter φ on S^1_w grows) rotation by $2\pi r/p$, where $r = q^{-1}$ modulo p.[1]

If $p = 3$, this construction yields a simple spine of $L_{3,1}$ without vertices (thus showing that $c(L_{3,1}) = 0$). However, for $p > 3$ the spines obtained this way are not almost simple because of the line (the core circle S^1_w) of multiplicity p. Simple spines can be obtained by small perturbation of these cut loci or, in many cases, by choosing the point x outside of the core circles.

Lemma 4. *If a cut locus $C(x)$ is a simple spine of $L_{p,q}$, then its combinatorial type is independent of the choice of x in $L_{p,q} \setminus \{S^1_w, S^1_z\}$.*

Proof. Suppose that $C(x)$ is a simple spine of $L_{p,q}$. Let (z_0, w_0) be one of p preimages of x under the covering $S^3 \to L_{p,q}$. Then the set of all preimages of x is the orbit $\mathbb{Z}_p(z_0, w_0)$, and its Voronoi diagram in S^3 is a simple polyhedron, because it covers $C(x)$. By Lemma 3, the combinatorial type of $C(x)$ is determined by the combinatorial type of the convex hull $\mathrm{Conv}(\mathbb{Z}_p(z_0, w_0))$. Note that $z_0 \ne 0$ and $w_0 \ne 0$; otherwise $C(x)$ is not a simple polyhedron.

Let (z_1, w_1) be any point of $S^3 \setminus \{\widetilde{S}^1_w, \widetilde{S}^1_z\}$, i.e., $z_1^2 + w_1^2 = 1$, $z_1 \ne 0$, $w_1 \ne 0$. Consider a linear transformation of $\mathbb{C}^2$ defined by a diagonal matrix $\mathrm{diag}(z_1/z_0, w_1/w_0)$. This is an invertible linear transformation that takes the orbit $\mathbb{Z}_p(z_0, w_0)$ to $\mathbb{Z}_p(z_1, w_1)$. It also takes $\mathrm{Conv}(\mathbb{Z}_p(z_0, w_0))$ to $\mathrm{Conv}(\mathbb{Z}_p(z_1, w_1))$. Then these two convex hulls have the same combinatorial type. Therefore, the combinatorial type of $C(x)$ does not depend on the choice of $x \in L_{p,q} \setminus \{S^1_w, S^1_z\}$. $\square$

If $q = \pm 1$ modulo p, then the $\mathbb{Z}_p$-orbit of any point is a (flat) regular p-gon. In this case $C(x)$ cannot be a simple spine of $L_{p,q}$ unless $p = 3$. If $q \ne \pm 1$ modulo p, then $C(x)$, where $x \in L_{p,q} \setminus \{S^1_w, S^1_z\}$, is a simple spine of $L_{p,q}$ with $E(p,q) - 3$ vertices, so the assumption of simplicity of $C(x) \subset L_{p,q}$ is satisfied, see [2].

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[1] The same construction with a point $y = (0, 1)$ instead of $x = (1, 0)$ gives a Y_p-fibration over S^1_z with rotation by $2\pi q/p$ as the monodromy.

§6. Perturbations of $C(x)$ and spines of $L_{p,q}$

Recall (see §4) that any spine $P \subset M^3$ is (up to isotopy) a cut locus $C(x)$ for some pair (x, g), where $x \in M$ and g is a Riemanian metric. We say that a spine P' is a *small perturbation* of P if P' is a cut locus for a pair (x', g') that is a small perturbation of (x, g).

Theorem 3. *If an almost simple spine P of $L_{p,q}$ is a small perturbation of a cut locus $C(x)$ (defined by the standard metric on $L_{p,q}$), then P contains at least $E(p, q) - 3$ vertices.*

Proof. It is sufficient to prove the theorem for the case $x \in S^1_z$. Indeed, the case $x \in S^1_w$ is similar to the case $x \in S^1_z$, and if x' does not lie on a core circle, the combinatorial type of $C(x')$ (and of its small perturbations) does not depend on x', so we can assume that x' lies very close to S^1_z; then $C(x')$ is a small perturbation of $C(x)$ with $x \in S^1_z$, and small perturbations of $C(x')$ also are small perturbations of $C(x)$.

A C^∞-small perturbation of the metric leads to a small perturbation of $C(x)$. This perturbation is a local isotopy in a neighborhood of any nonsingular point of $C(x)$, of any point on a triple line of $C(x)$, and in a neighborhood of any vertex, see Fig. 1. (Thus a small perturbation of a simple polyhedron is a simple polyhedron of the same combinatorial type, in particular, with the same number of vertices.)

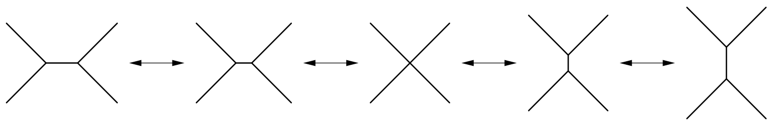

Figure 4. A flip

Consider a cut locus $C(x) \subset L_{p,q}$, where $x \in S^1_z$; it was described in the paragraph following Lemma 3. If a perturbation is generic, multiple line (in our case S^1_w) splits in a number of triple lines, which may end in new vertices. Consider a section of $C(x)$ by a transversal to S^1_w at $y \in S^1_w$. For a generic y, the effect of the perturbation is the splitting of a degree p vertex of the graph Y_p into $p-2$ trivalent vertices; denote the perturbed section by $Y_p(y)$. As $y \in S^1_w$ varies, *flips* (see Fig. 4) may occur in $Y_p(y)$. Note that flips occuring in transversal sections of a simple polyhedron P correspond to vertices of P; see Fig. 5, where transversal sections of a neighborhood of a vertex in a simple polyhedron by three parallel planes are shown.

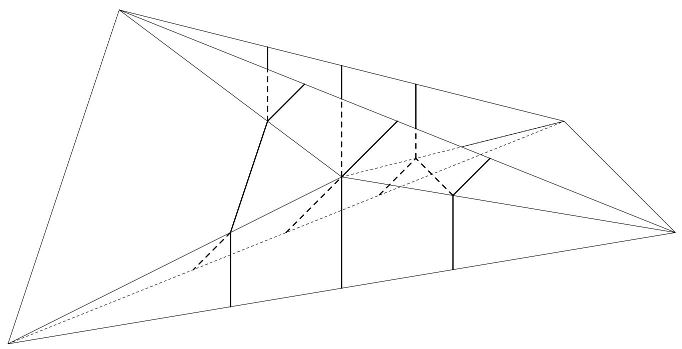

Figure 5. Vertices correspond to flips

Let us move y along the circle S^1_w. Then $Y_p(y)$ undergoes isotopy and flips, and the sequence of flips converts $Y_p(0)$ into $Y_p(2\pi)$; due to monodromy (described in §5), the latter graph is the former one rotated by the angle $2\pi r/p$, where $r = q^{-1}$ modulo p. We have to estimate from below the number of flips required to convert $Y_p(0)$ into its image under rotation.

There is a natural one-to-one correspondence between the isotopy classes of trivalent resolutions of Y_p (with fixed p boundary points) and the triangulations of the regular p-gon: for any triangulation, its dual graph is an isotopy class of trivalent resolutions of Y_p, and vise versa. A *flip* in a triangulation is replacing of two triangles ABC and ACD having a common side AC by two triangles BCD and ABD with a common side BD; in other words, a flip switches the diagonal in a triangulated quadrilateral. Flips in trivalent resolutions of Y_p correspond to flips of dual triangulations. Taking into account Theorem 1 b, Theorem 3 follows from Theorem 4 below. $\square$

Theorem 4. *Let Δ_1, Δ_2 be two triangulations of a regular p-gon, $p \geq 3$. By $d(\Delta_1, \Delta_2)$ denote the rotation distance (see [9]) between Δ_1 and Δ_2, that is, the minimal number of flips required to convert Δ_1 into Δ_2. If Δ_2 is Δ_1 rotated by $2\pi q/p$, where $p > q > 0$ and $(p, q) = 1$, then $d(\Delta_1, \Delta_2) \geq E(p, q) - 3$.*

Remarks. 1. The lower bound given by Theorem 4 is exact, i.e., there always exists a triangulation Δ_1 such that $d(\Delta_1, \Delta_2) = E(p, q) - 3$.
2. The condition $(p, q) = 1$ is not necessary for the inequality $d(\Delta_1, \Delta_2) \geq E(p, q) - 3$. The equality $d(\Delta_1, \Delta_2) = \max(0, E(p, q) - 3)$ holds for all p, q such that $p \geq q \geq 0$ and $p \geq 3$. Nevertheless, in the proof below we assume, for the sake of simplicity, that $(p, q) = 1$.

Proof. Without loss of generality, we may suppose that $p > 2q$. Indeed, clockwise rotation by $2\pi q/p$ is the same as counterclockwise rotation by $2\pi(p - q)/p$, and $E(p, q) = E(p, p - q)$ by Theorem 1 a. The number $E(p, q)$ appears in this proof as the sum of the elements of the continued fraction expansion of p/q.

We start with the case $q = 1$; then $E(p, q) = E(p, 1) = p$. A triangulation of a p-gon involves $p - 3$ diagonals. Any of them intersects its own image under the "minimal" rotation by $2\pi/p$, therefore every of them needs to be moved by a flip, and this requires at least as many flips as diagonals, that is, at least $p-3$ flips. This proves the theorem for $q = 1$, i.e., for the case of the shortest continued fraction.

Two simple ideas have been used here: first, $d(\Delta_1, \Delta_2)$ is not less than the number of diagonals of Δ_1 that are not mapped to other diagonals of Δ_1 by the rotation, so it is sufficient to estimate the number of those diagonals, and, second, long diagonals cannot survive small rotations.

Define the *length* of a diagonal to be the number of sides of the polygon in the shorter arc bounded by the endpoints of the diagonal; thus, the length of a diagonal is at least 2 and at most $p/2$. Note that any diagonal longer than x intersects its image under a rotation by $2\pi x/p$.

Lemma 5. *Let x be an integer such that $2 \leq x \leq p/2$. Then any triangulation of a regular p-gon contains at least $\lceil p/x \rceil - 3$ diagonals that are longer than x.*

Proof. Cut the polygon along every short diagonal of the triangulation (that is, along every diagonal of length at most x). Consider the piece of the p-gon containing its center. This piece has at least $\lceil p/x \rceil$ sides and is triangulated by at least $\lceil p/x \rceil - 3$ diagonals, all of which are longer than x. $\square$

Now consider the case of a two-term continued fraction, $p/q = n_1 + 1/n_2$. Then $p = n_1n_2 + 1$, $q = n_2$, $E(p,q) = n_1 + n_2$, and the number of diagonals in a triangulation is $p - 3 = n_1n_2 - 2$. A (clockwise) rotation by $2\pi q/p$, repeated n_1 times, is equivalent to a (counterclockwise) rotation by $2\pi/p$, which none of the $p-3$ diagonals can survive. Therefore, at least $\left\lceil \frac{p-3}{n_1} \right\rceil$ diagonals are destroyed by a single rotation by $2\pi q/p$; this number can be equal to n_2 or $n_2 - 1$. However, one can give a better estimate.

By Lemma 5, the triangulation contains at least $\lceil p/q \rceil - 3 = n_1 - 2$ diagonals that are longer than q. A rotation by $2\pi q/p$ destroys all of them and at least $1/n_1$ fraction of the other diagonals (because otherwise some of the latter diagonals survive a rotation by $2\pi q/p$ repeated n_1 times), and we get $d(\Delta_1, \Delta_2) \geq (n_1 - 2) + \left\lceil \frac{(p-3)-(n_1-2)}{n_1} \right\rceil = (n_1 - 2) + \left\lceil \frac{n_1n_2 - n_1}{n_1} \right\rceil = (n_1 - 2) + (n_2 - 1) = E(p,q) - 3$.

In the general case we have $p/q = n_1 + 1/(n_2 + 1/(n_3 + \ldots + 1/n_k)\cdots)$. The ith *convergent* of this continued fraction is $p_i/q_i = n_1 + 1/(n_2 + \ldots + 1/n_i)\cdots)$, $1 \leq i \leq k$; we also set $p_0 = 1$ and $q_0 = 0$. Further, set $r_0 = p$, $r_1 = q$ and define the numbers $r_2, \ldots, r_k$ as the remainders in the Euclid algorithm: the relations $r_{i-1} = n_ir_i + r_{i+1}$ define the r_i recurrently. Note that $r_k = (p,q) = 1$. It is convenient to set $r_{k+1} = 0$.

Lemma 6. *For $i = 1, 2, \ldots, k-1$, we have $p/r_i > p_i$.*

Proof. This follows from the relation $p = p_ir_i + p_{i-1}r_{i+1}$, where $i = 1, \ldots, k-1$. This relation is proved by induction on i. For $i = 1$ it takes the form $p = p_1r_1 + p_0r_2 = n_1q + r_2$, which is the first step of the Euclid algorithm. The induction step is the equation $p_ir_i + p_{i-1}r_{i+1} = p_{i+1}r_{i+1} + p_ir_{i+2}$. The relations $p_{i+1} = p_in_{i+1} + p_{i-1}$ and $r_i = n_{i+1}r_{i+1} + r_{i+2}$ show that both sides are equal to $n_{i+1}p_ir_{i+1} + p_{i-1}r_{i+1} + p_ir_{i+2}$. $\square$

Consider a rotation by $2\pi q/p$ repeated p_i times, $i = 0, 1, \ldots, k$. The following lemma shows that this gives a (positive or negative) rotation by $2\pi r_{i+1}/p$.

Lemma 7. *For $i = 0, 1, \ldots, k$, the congruence $p_iq \equiv (-1)^ir_{i+1} \mod p$ holds.*

Proof. Induction on i. The base, $i = 0$, is obvious (recall that $p_0 = 1$ and $r_1 = q$). Induction step: $p_{i+1}q = (n_{i+1}p_i + p_{i-1})q \equiv (-1)^in_{i+1}r_{i+1} + (-1)^{i-1}r_i = (-1)^{i+1}r_{i+2} \mod p$, because $r_{i+2} = r_i - n_{i+1}r_{i+1}$. $\square$

Lemma 8. *If $2 \leq j \leq k$, then the following relation holds*:

$$(n_1 - 2) + \sum_{i=2}^{j-1} p_{i-1}n_i + p_{j-1}(n_j - 1) = p_j - 3. \tag{2}$$

Proof. Since $p_0 = 1$, equation (2) is equivalent to $\sum_{i=1}^{j} p_{i-1}n_i = p_j + p_{j-1} - 1$. This is easy to prove by induction, because $p_i = n_ip_{i-1} + p_{i-2}$, see, e.g., [10]. $\square$

Let us return to the proof of Theorem 4. By S_i, $i = 1, \ldots, k$, denote the set of diagonals of Δ_1 that are longer than r_i but not longer than r_{i-1}; set $s_i = \mathrm{Card}(S_i)$. Lemma 5 implies that $s_1 + \ldots + s_i \geq \lceil p/r_i \rceil - 3$; therefore, by Lemma 6,

$$s_1 + \ldots + s_i > p_i - 3 \qquad \text{for } \ i = 1, \ldots, k-1 \tag{3_i}$$

$$s_1 + \ldots + s_k = p - 3. \tag{3}$$

Lemma 9. *Let $s_1, \ldots, s_k$ be integers. Let j be an integer such that $2 \le j \le k$.*

a) *Suppose that (3_i) holds for any $i = 1, \ldots, j-1$, and $s_1 + \ldots + s_j = p_j - 3$. Then*

$$s_1 + \lceil s_2/p_1 \rceil + \ldots + \lceil s_j/p_{j-1} \rceil \ge n_1 + \ldots + n_j - 3. \qquad (4_j)$$

b) *Under the assumptions of* a), *the inequality (4_j) is an equality if and only if $s_1 = n_1 - 2$, $s_l = p_{l-1}n_l$ for $l = 2, \ldots, j-1$, and $s_j = p_{j-1}(n_j - 1)$.*

c) *Suppose that (3_i) holds for any $i = 1, \ldots, j-1$, and $s_1 + \ldots + s_j \ge p_j - 3 + dp_{j-1}$, where d is a positive integer. Then*

$$s_1 + \lceil s_2/p_1 \rceil + \ldots + \lceil s_j/p_{j-1} \rceil \ge n_1 + \ldots + n_j - 3 + d. \qquad (5_j)$$

d) *Under the assumptions of* c), *the inequality (5_j) is an equality if and only if $s_1 = n_1 - 2$, $s_l = p_{l-1}n_l$ for $l = 2, \ldots, j-1$, and $s_j = p_{j-1}(n_j - 1 + d)$.*

Proof. If $s_1 = n_1 - 2$, $s_l = p_{l-1}n_l$ for $l = 2, \ldots, j-1$, and $s_j = p_{j-1}(n_j - 1)$, then the assumptions of the part a) are satisfied by virtue of Lemma 8. The "if" part of the statement b) is obvious. Other statements are proved by induction on j. The idea is as follows: given the sum of the s_i, the left hand side of (4) and (5) would be minimized if, roughly speaking, as much as possible were divided by the biggest denominator (that is, by the last one), but the constraints (3_i) and (3) do not allow us to overload the last summand s_j; the ceiling function $\lceil \cdot \rceil$ is responsible for the clause "roughly speaking" in the above.

Base of induction: $j = 2$. Under the assumptions of a), if $s_2 = p_1(n_2 - 1)$, then $s_1 = (s_1 + s_2) - s_2 = p_2 - 3 - s_2 = n_2p_1 + p_0 - 3 - p_1(n_2 - 1) = n_1 - 2 = p_1 - 2$ and $s_1 + \lceil s_2/p_1 \rceil = n_1 + n_2 - 3$. If $s_2 > p_1(n_2 - 1)$, then $s_1 = p_2 - s_2 < p_1 - 2$, which contradicts (3_1). If $s_2 < p_1(n_2 - 1)$, then $s_2 = p_1(n_2 - 1) - t$ and $s_1 = n_1 - 2 + t$ for some $t > 0$. Then $s_1 + \lceil s_2/p_1 \rceil = n_1 + n_2 - 3 + t + \lceil -t/p_1 \rceil > n_1 + n_2 - 3$, because $p_1 = n_1 \ge 2$ (due to the assumption $p > 2q$). This proves statements a) and b) for $j = 2$. The proof of the statements c) and d) for $j = 2$ is similar.

Induction step: from $j = l - 1$ to $j = l$. Under the assumptions of a), if $s_l = p_{l-1}(n_l - 1)$, then $s_1 + \ldots + s_{l-1} = p_l - 3 - s_l = n_lp_{l-1} + p_{l-2} - 3 - (n_l - 1)p_{l-1} = p_{l-1} - 3 + p_{l-2}$. Now (5_{j-1}) with $d = 1$ implies (4_j) provided that $s_l = p_{l-1}(n_l - 1)$, and in this case equality in (4_j) requires that $s_1 = n_1 - 2$ and $s_l = p_{l-1}n_l$ for $l = 2, \ldots, j-1$.

Further, if $s_l > p_{l-1}(n_l - 1)$, then $\lceil s_l/p_{l-1} \rceil \ge n_l$, while the inequality $s_1 + \lceil s_2/p_1 \rceil + \ldots + \lceil s_{l-1}/p_{l-2} \rceil \ge n_1 + \ldots + n_{l-1} - 2$ follows from (3_{l-1}), (5_{l-1}), and statement d) for $j = l - 1$. Summing the last two inequalities, we get $s_1 + \lceil s_2/p_1 \rceil + \ldots + \lceil s_l/p_{l-1} \rceil \ge n_1 + \ldots + n_l - 2$, which is stronger than (4_l).

Finally, suppose that $s_l < p_{l-1}(n_l - 1)$. Define a positive integer d by $\lceil s_l/p_{l-1} \rceil = n_l - d$. If $d = 1$, we still have $\lceil s_l/p - l - 1 \rceil = n_l - 1$, but $s_1 + \ldots + s_{l-1} = p_l - 3 - s_l > p_{l-1} - 3 + p_{l-2}$, and the inequality (5_{j-1}) with $d = 1$, combined with the statement d) for $j = l - 1$, implies that $s_1 + \lceil s_2/p_1 \rceil + \ldots + \lceil s_l/p_{l-1} \rceil > n_1 + \ldots + n_{l-1} - 2 + n_l - 1$, which is stronger than (4_l). If $d > 1$, then $s_1 + \ldots + s_{l-1} = p_l - 3 - s_l \ge dp_{l-1} - 3 + p_{l-2} > p_{l-1} - 3 + dp_{l-2}$, and then $s_1 + \lceil s_2/p_1 \rceil + \ldots + \lceil s_{l-1}/p_{l-2} \rceil > n_1 + \ldots + n_{j-1} - 3 + d$. Together with $\lceil s_l/p_{l-1} \rceil = n_l - d$, this yields $s_1 + \lceil s_2/p_1 \rceil + \ldots + \lceil s_l/p_{l-1} \rceil > n_1 + \ldots + n_{l-1} - 2 + n_l - 1$, which is stronger than (4_l). This proves statements a) and b) for $j = l$. The proof of the statements c) and d) for $j = l$ is similar. Lemma 9 is proven. $\square$

By Lemma 7, a rotation of Δ_1 by $2\pi q/p$ repeated p_{i-1} times (where $i = 1, \dots, k$), being equivalent to a rotation by $\pm 2\pi r_i/p$, destroys all diagonals of Δ_1 that are longer than r_i. Repeating the argument from the case $k = 2$ above (the case of a two-term continued fraction), we conclude that a rotation of Δ_1 by $2\pi q/p$ destroys at least $\lceil s_i/p_{i-1} \rceil$ diagonals in any length group S_i. Therefore, it destroys at least $s_1 + \lceil s_2/p_1 \rceil + \ldots + \lceil s_k/p_{k-1} \rceil$ diagonals of Δ_1. Setting $j = k$ in Lemma 9 a), we get the statement of Theorem 4, provided that $k > 1$, which is equivalent to $q \neq 1$. The case $q = 1$ was considered above. The proof of Theorem 4 is completed. $\square$

One can show (see [2] and §7 below) that the lower bound given by Theorem 4 is exact. The proof of Theorem 4 shows that the numbers of diagonals of the triangulation in the length group S_i in this case are $s_1 = n_1 - 2$, $s_i = p_{i-1}n_i$ for $i = 2, \dots, k-1$, and $s_k = p_{k-1}(n_k - 1)$ (provided that $k > 1$), which are the numbers given by Lemma 9 b. Furthermore, the rotation by $2\pi q/p$ destroys only $(1/p_{i-1})$th fraction of the s_i diagonals belonging to S_i, and the contributions to $E(p,q) - 3$ from the groups $S_1, \dots, S_k$ are $n_1 - 2, n_2, \dots, n_{k-1}, n_k - 1$ respectively.

§7. Optimal triangulations of the regular p-gon

We show in this section that the bounds of Theorems 3 and 4 are sharp.

Theorem 5. *For any $x \in L_{p,q} \setminus \{S^1_w, S^1_z\}$, the cut locus $C(x)$ is a simple spine of $L_{p,q}$ with $E(p,q) - 3$ vertices, provided that $q > 1$.*

Proof. This follows from the description of the convex hull of the p-element preimage of x under the universal covering $S^3 \to L_{p,q}$, see [2], and from the results of §5 and §6 above. $\square$

Theorem 6. *Let p and q be coprime positive integers, $p \geq 3$.*

a) *There exists a triangulation Δ_1 of a regular p-gon that can be converted into Δ_2 by just $E(p,q) - 3$ flips, where Δ_2 is Δ_1 rotated by $2\pi q/p$.*
b) *Consider the cut locus $C(x) \subset L_{p,q}$ of an arbitrary point $x \in L_{p,q}$ with respect to the standard metric on $L_{p,q}$. There exists a small perturbation P of $C(x)$ such that P is a simple spine of $L_{p,q}$ with exactly $E(p,q) - 3$ vertices.*

Proof. As in the proof of Theorem 4, we assume that $q < p/2$. The case $p = 3$ is trivial. Below we assume that $p \geq 4$.

First consider the case $q = 1$. By $A_1, A_2, \dots, A_p$ denote the consecutive vertices of the p-gon. Let Δ_1 consist of the diagonals A_1A_i, where $i = 3, 4, \dots, n-1$; then Δ_2 consists of the diagonals A_2A_i, where $i = 4, \dots, n$. Take Δ_1 and make a flip along A_1A_3 (for any diagonal, a unique flip involving it is possible), then along A_1A_4, and so forth. After the first flip we get A_2A_4, after the second one, A_2A_5, and so forth. In $p - 3$ flips we get Δ_2. This proves statement a) for $q = 1$.

If $q = 1$, then the combinatorial type of $C(x) \subset L_{p,q}$ does not depend on x; moreover, there exists an isometry of $L_{p,1}$ taking x_1 to x_2 and $C(x_1)$ to $C(x_2)$ for any $x_1, x_2 \in L_{p,1}$. The 1-dimensional stratum of $C(x)$ is a closed line S^1 of multiplicity p, and a section of $C(x)$ transversal to S^1, denoted by Y_p, gets rotated by $2\pi/q$ by monodromy along S^1, see §5. A sequence of $p - 3$ flips converting Δ_1 to Δ_2 corresponds to a simple spine P of $L_{p,1}$ with $p - 3$ vertices, see the proof of Theorem 3. Thus, statement b) of the theorem in the case $q = 1$ follows from the statement a).

Now suppose that $q > 1$. If $x \in L_{p,q}$ does not lie on the core circles S^1_w, S^1_z (see §5), then the statement b) follows from Theorem 5; a perturbation of $C(x)$ is not necessary at all in this case. If x lies on a core circle, it is enough to shift x slightly out of S^1_w or S^1_z. The statement a) is deduced from the statement b) by repeating a part of the proof of Theorem 3. Thus, Theorem 6 follows from Theorem 5. $\square$

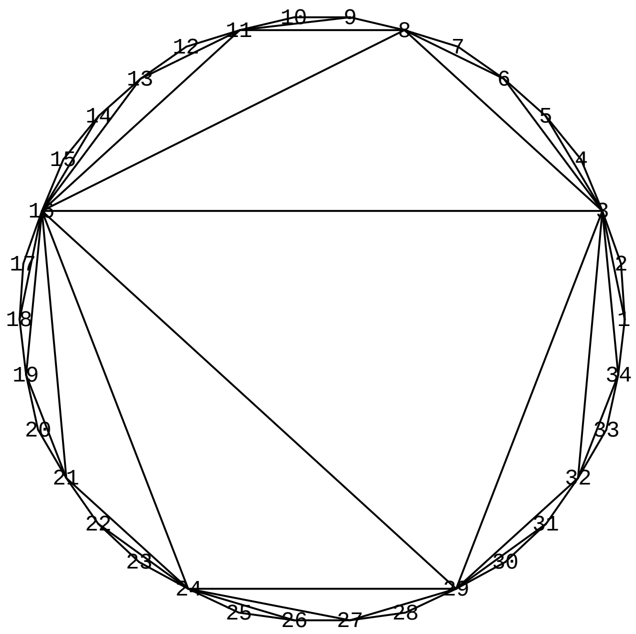

Figure 6. Optimal triangulation for $(p, q) = (34, 13)$

The following Mathematica® script[2] produces the optimal (in the sense of Theorem 6) triangulation Δ_1 for the case $p = 34$, $q = 13$ (then $k = 6$, $n_1 = 2$, $n_2 = n_3 = n_4 = n_5 = 1$, $n_6 = 2$, and $E(p, q) - 3 = 5$). The best choice of the eccentricity `e` depends on the pair (p, q).

```
<<DiscreteMath`ComputationalGeometry`

e=.007; p=34; q=13;
points=Table[(1+e Sin[(2k Pi)(q/p)])
          {Cos[(2k Pi)/p],Sin[(2k Pi)/p]},{k,0,p-1}];
d=DiagramPlot[points, TrimPoints->3];
triangulation=PlanarGraphPlot[points];
Show[{d,Graphics[RGBColor[1,0,0]],triangulation},
          PlotRange->{{-1.2,1.2},{-1.2,1.2}}];
```

The output is shown on Fig. 6.

Appendix: properties of the Farey tesselation

Proof of Lemma 1. The modular group has two generators, $S\colon z \mapsto -\frac{1}{z}$ (central symmetry of H^2 at i) and $T\colon z \mapsto z + 1$, see [8]. By construction of the Farey tesselation, both S and T preserve it. This implies statement c).

[2]Special thanks to Roderik Lindenbergh and Martijn van Manen

Statement b) is proved by induction over $E(p,q)$. It holds if $E(p,q) \le 1$ (in this case $p/q = 0$ or $p/q = \infty$ or $p/q = \pm 1$). Let $E(p,q) > 1$. Since the symmetries in the lines $(0,\infty)$ and $(1,-1)$ preserve the Farey tesselation, we can suppose that $p > q > 0$. By the induction hypothesis, a geodesic segment $(ti, (p-q)/q)$ intersects (the interiors of) $E(p-q,q)$ Farey triangles. The isometry $z \mapsto z+1$ preserves the tesselation and takes ti to $1+ti$ and $(p-q)/q$ to p/q, so that a geodesic line $(ti,(p-q)/q)$ gets mapped into a geodesic line $(1+ti, p/q)$ intersecting the same number $E(p-q,q)$ of Farey triangles. Since $p > q > 0$, a geodesic line $(ti, p/q)$ intersects the Farey edge $(1,\infty)$ at $1+t'i$ (where ${t'}^2 = p/q + 1 + t^2(1-q/p)$). The segment $(ti, 1+t'i)$ of this line intersects the triangle $(0,1,\infty)$ and the remaining part of it intersects $E(p,q)-1$ other Farey triangles, so the line $(ti,p/q)$ cuts $E(p,q)$ Farey triangles. This applies also for the geodesic line $(0,p/q)$. Statement b) follows.

To prove statement a), note that the modular group action actually defines the Farey tesselation, not only preserves it. Indeed, maps S, T, and $ST^{-1}S$ take the triangle $(0,1,\infty)$ to its images under reflections in its sides. Now, if a triangle Δ is obtained from $(0,1,\infty)$ by a modular transformation F, one gets its three neighbors from $(0,1,\infty)$ by a modular transformation $F^{-1}GF$ with $G = S$ or T or $ST^{-1}S$. Further, modular maps do not change $mq - np$, because $m'q' - n'p' = \det\begin{pmatrix} m' & p' \\ n' & q' \end{pmatrix} = \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} m & p \\ n & q \end{pmatrix}\right) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\det\begin{pmatrix} m & p \\ n & q \end{pmatrix} = mq - np$: since the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of a modular map has determinant 1, it keeps m' coprime with n' and p' coprime with q'.

Thus any Farey edge can be mapped to an edge of the triangle $(0,1,\infty)$ by a modular map, which preserves $mq - np$, but this amounts to ± 1 for any edge of $(0,1,\infty)$. This proves the "only if" part of statement a). A mapping $z \mapsto \frac{mz+p}{nz+q}$, which is modular if $mq - np = 1$, takes an edge $(0,\infty)$ to $(m/n, p/q)$, which is a Farey edge as well by statement c). If $mq - np = -1$, consider the map $z \mapsto \frac{pz+m}{qz+n}$. Statement a) follows.

In the assumptions of statement d) we have $mq - np = \pm 1$ by virtue of a). Then $m(n+q) - n(m+p) = \pm 1$, and statement a) implies that $\left(\frac{m+p}{n+q}, \frac{m}{n}\right)$ is an edge of the Farey tesselation. Similarly, $\left(\frac{m+p}{n+q}, \frac{p}{q}\right)$, $\left(\frac{m-p}{n-q}, \frac{m}{n}\right)$, and $\left(\frac{m-p}{n-q}, \frac{p}{q}\right)$ are edges of the Farey tesselation, which thus contains triangles $\left(\frac{m}{n}, \frac{p}{q}, \frac{m+p}{n+q}\right)$ and $\left(\frac{m}{n}, \frac{p}{q}, \frac{m-p}{n-q}\right)$; this proves statement d).

Statement e) is an immediate consequence of d), by construction of the Farey series. By the way, this explains the term "Farey tesselation". $\square$

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Dept. of Mathematics, Utrecht University, P.O. Box 80.010, 3508 TA Utrecht, the NETHERLANDS

E-mail address: `anisov@math.uu.nl`