

# Replicator dynamics with mutations for games with a continuous strategy space

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## Abstract

A partial differential equation is derived, describing the replicator dynamics with mutations of games with a continuous strategy space. This equation is then applied to continuous versions of symmetric 2x2 games, such as the Prisoners Dilemma, Hawk-Dove and Coordination games and to the Ultimatum Game. In the latter case, we find solutions which differ significantly from the expected solution.

## 1 Introduction

Evolutionary game dynamics is a fast developing field, with applications in biology, economics, sociology and anthropology. Background material and countless references can be found in the monographs by Weibull [1], Fudenberg & Levine [2], Samuelson [3], Hofbauer & Sigmund [4], Cressman [5] and Vincent & Brown [6] or in the survey paper by Hofbauer & Sigmund [7]. The standard ingredients of evolutionary game dynamics are a population of players, an  $n$ -person game, a set of strategies and a rule to update the distribution of strategies from one generation to the next. Within this general setting several variations are possible: time can be discrete or continuous, populations can be finite or infinite, the game can have a finite or infinite number of strategies. Also, there are several choices for the updating rule, the most popular of which are Adaptive Dynamics (See e.g. Diekmann [8]) and Replicator Dynamics, introduced by Taylor & Jonker [9]. In this paper we consider a model with continuous time, an infinite population and a 2-person game, where each participant has a continuum of strategies to choose from. The update rule we use is the replicator dynamics, with deterministic mutations. Although this model has been alluded to by Dieckmann [10], where a hierarchy of evolutionary models is presented, its details have not been worked out before.

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The strategy space that we consider is a subset of  $R^n$ , which in all applications is compact. In section 2 we first consider replicator dynamics with a deterministic mutation term on a finite set of strategies. Using a procedure that is familiar in Statistical Mechanics, we make a transition to an infinite, continuous, strategy-space. The state-space of the model is then no longer discrete, but consists of distributions over the strategies. We derive a partial differential equation, together with appropriate boundary conditions, that describes the time-evolution of this distribution, given an initial distribution. Without the mutation term, the equation we derive was previously studied by Cressman [13] and Oechsler & Riedel [14]. Our addition of a mutation term is new and leads to completely different dynamics.

In section 3, the equation is applied to symmetric 2x2 games, where we need to extend the original two-strategy set to a continuous one. This can be done by allowing for strategies that play one of the pure strategies with a certain probability. We compare our results with those of Vaughan [15], who analyses the replicator dynamics for the pure strategies, to which a stochastic perturbation term is added, leading to a Fokker-Planck equation. There are similarities between the two models, but also striking differences. In the stochastic mutation version, there is always only one attracting, stationary, distribution, which for small mutations converges to a point-distribution (Dirac-delta function). In other words, all players eventually use the same strategy. In the deterministic mutation version, however, we find the possibility of two attracting stationary distributions existing simultaneously. Also, in certain cases the limiting distribution for small mutations is not concentrated on a point, but has the whole strategy space as support.

A more complicated example, namely the Ultimatum Game, is treated in section 4. In this game, a sum of money is to be split by two players. The first player proposes the split and the second player then has the choice to accept the split or refuse, in which case both players get nothing. The solution offered by standard game theory is for the second player to accept any amount (something is always better than nothing) and for the first player, therefore, to offer the lowest possible amount. In our model we find that the offers converge to a Gauss-like distribution around a mean that is not equal to zero. The position of the mean and the width of the distribution seem to converge to zero when the rate of mutation vanishes. However, this convergence is slow, so that even for very small values of the mutation rate, the average offer is well above zero. Also, the dynamics shows two time-scales. A random starting distribution is initially attracted to a member of a certain set of distributions, which can have an average offer much larger than zero. Then, on a time-scale inversely proportional to the rate of mutation, the solution converges to the, unique, stationary solution.

In section 5 we discuss the results and suggest some topics for further research.

## 2 From the discrete to the continuous equation

The equation for the replicator dynamics with continuous strategy space will be derived by a limiting process, starting from the equations for the discrete case.

Following Hofbauer and Sigmund [4], we consider an infinite population of players and a set of strategies  $S_1, \dots, S_N$ . When a player  $A$  opts for strategy  $S_i$  against  $B$ , who uses  $S_j$ , the payoff to  $A$  is taken to be  $M_{ij}$ , and the payoff to  $B$  is  $M_{ji}$ . Consequently, if  $p_j(t)$  is the fraction of the whole population that at time  $t$  plays the strategy  $S_j$ , the average payoff to each player using  $S_i$  is equal to

$$\Pi_i(\mathbf{p}) \equiv \sum_{j=1}^N M_{ij} p_j. \quad (1)$$

The average payoff for the whole population therefore is

$$\bar{\Pi}(\mathbf{p}) = \sum_{i=1}^N \Pi_i(\mathbf{p}) p_i = \sum_{i,j=1}^N p_i M_{ij} p_j. \quad (2)$$

Now in the course of time, the fraction  $p_i$  changes at a rate which is proportional to the difference between the payoff to  $p_i$  and the average payoff for the whole population. In addition we assume that for each player of  $S_j$  there is a transition probability per unit time to make a spontaneous transfer to strategy  $S_i$  at a rate given by  $W_{ij}$ . In this way the *discrete replicator equation with mutation* is derived:

$$\dot{p}_i = (\Pi_i(\mathbf{p}) - \bar{\Pi}(\mathbf{p})) p_i + \sum_{j=1}^N (W_{ij} p_j - W_{ji} p_i). \quad (3)$$

It is easy to show that  $S(t) \equiv \sum_{i=1}^N p_i(t) = 1$  for all  $t$  if  $S(0) = 1$ .

In transforming to a continuous strategy space, we replace the discrete index  $i$  by a continuous variable  $s \in D \subset R^n$  and the variables  $p_i(t)$  by a probability distribution  $P(s, t)$ . The payoff matrix  $M_{ij}$  must now be replaced by a payoff function  $M(s, s')$ , which gives the payoff to strategy  $s$  when playing strategy  $s'$ .

Eq.(3) now takes the form

$$\frac{\partial P(s, t)}{\partial t} = (\Pi(s, P) - \bar{\Pi}(P)) P(s, t) + \mathcal{M}(P, s) \quad (4)$$

in which the mutation term is equal to

$$\mathcal{M}(P, s) = \int [W(s|s') P(s', t) - W(s'|s) P(s, t)] ds'. \quad (5)$$

The average payoff for strategy  $s$  and the total average payoff are given by

$$\Pi(s, P) = \int M(s, s') P(s', t) ds' \quad (6)$$

and

$$\bar{\Pi}(P) = \int \Pi(s, P) P(s, t) ds. \quad (7)$$

Let us now restrict ourselves to mutations in which only small changes in the strategies occur and apply the method which is used to derive the Fokker-Planck equation from a master equation [11]. For simplicity of presentation, we restrict ourselves to the case of a one-dimensional strategy space. Define  $\xi$  by  $s' = s - \xi$  and write the mutation rate as a function of  $s$  and  $\xi$

$$W(s'|s) = \widetilde{W}(s, s - s') = \widetilde{W}(s, \xi) \quad (8)$$

and so

$$W(s|s') = \widetilde{W}(s', s' - s) = \widetilde{W}(s - \xi, -\xi). \quad (9)$$

Assume now that  $\widetilde{W}(s, \xi)$  varies slowly in the first variable  $s$  and that due to the mutations only small variations in the strategies will occur. Then  $\widetilde{W}(s, \xi)$  is only nonvanishing when  $\xi$  is small. In the mutation term (5), which can be written as

$$\mathcal{M}(P, s) = \int [\widetilde{W}(s - \xi, -\xi) P(s - \xi, t) - \widetilde{W}(s, \xi) P(s, t)] d\xi, \quad (10)$$

we can now expand the dependence on the first variable in powers of  $\xi$  and obtain

$$\begin{aligned} \mathcal{M}(P, s) &= \int [\widetilde{W}(s, -\xi) P(s, t) - \xi \frac{\partial}{\partial s} \{\widetilde{W}(s, -\xi) P(s, t)\} + \\ &\quad + \frac{1}{2} \xi^2 \frac{\partial^2}{\partial s^2} \{\widetilde{W}(s, -\xi) P(s, t)\} + \dots - \widetilde{W}(s, \xi) P(s, t)] d\xi. \end{aligned} \quad (11)$$

Because  $\int [\widetilde{W}(s, -\xi) - \widetilde{W}(s, \xi)] P(s, t) d\xi = 0$ , the first and last term cancel, so that to second order we are left with

$$\mathcal{M}(P, s) = \frac{\partial}{\partial s} \{\alpha_1(s) P(s, t)\} + \frac{1}{2} \frac{\partial^2}{\partial s^2} \{\alpha_2(s) P(s, t)\}, \quad (12)$$

in which

$$\alpha_1(s) = - \int \xi \widetilde{W}(s, \xi) d\xi \quad \text{and} \quad \alpha_2(s) = \int \xi^2 \widetilde{W}(s, \xi) d\xi. \quad (13)$$

We will further simplify the equations by assuming that the average change in strategy due to mutations is equal to zero, so  $\alpha_1(s) = 0$ , and that the average of the square of this change is constant, so  $\alpha_2(s) = 2\sigma$ .

The final form of the *continuous replicator equation* (4) then becomes

$$\frac{\partial P(s, t)}{\partial t} = (\Pi(s, P(t)) - \bar{\Pi}(P(t))) P(s, t) + \sigma \Delta P(s, t), \quad (14)$$

where we have restored the correct dimensionality of the strategy space by replacing  $\frac{\partial^2}{\partial s^2}$  by the  $n$ -dimensional Laplace operator.  $P(s, t)$  should satisfy

$P(s, t) \geq 0$  and  $S(t) \equiv \int_D P(s, t) ds$ , should be equal to unity at all times. This last condition is fulfilled when we choose Neumann, or reflecting, boundary conditions:

$$\mathbf{n} \cdot \nabla P(s, t)|_{\partial D} = 0, \quad (15)$$

where  $\mathbf{n}$  is the normal to the boundary  $\partial D$  of the domain  $D$ .

Indeed, integrating (14) over  $D$  and using (15) we find

$$\frac{dS(t)}{dt} = \bar{\Pi}(P)(1 - S(t)), \quad (16)$$

showing that if  $S(0) = 1$  then  $S(t) = 1$  for all times. In the case that  $\sigma = 0$ , condition (15) is not required to ensure that  $S(t)$  remain constant in time.

The rate at which the total average fitness  $\bar{\Pi}(P)$  changes is given by:

$$\frac{d\bar{\Pi}(P)}{dt} = \int (\Pi(s, P) - \bar{\Pi}(P))^2 P(s, t) ds + \sigma \int \Pi(s, P) \Delta P(s, t) ds. \quad (17)$$

Therefore, when the diffusion term is absent,  $\sigma = 0$ , the average fitness cannot decrease.

Equation (14) is a nonlinear reaction-diffusion equation, where the reaction term  $\Pi(s, P(t)) - \bar{\Pi}(P(t))$  is nonlocal. On the function space of twice continuous space-differentiable and once time-differentiable functions, we can show that the solution of (14) exists for all times. This follows from the assumption that  $M(s, s')$  is bounded on  $D$ , so that  $|\Pi(s, P(t)) - \bar{\Pi}(P(t))| \leq \int_{D \times D} |M(s, s')| P(s, t) P(s', t) ds ds' + \int_D |M(s, s')| P(s', t) ds' \leq \max |M(s, s')| \left\{ \int_{D \times D} P(s, t) P(s', t) ds ds' + \int_D P(s', t) ds' \right\} = 2 \max |M(s, s')|$ .

Standard comparison theorems for parabolic equations (Pao [12]) complete the proof. Also, by standard positivity results for parabolic equations, it can be shown that the when the initial distribution  $P(s, 0) \geq 0$ , then  $P(s, t) \geq 0$  for all times  $t$ .

Numerical simulations suggest that even stronger results hold. In particular, we suspect that the solution of Eqs.(14), (15) is uniformly (in space and time) bounded in terms of the sup-norm of the initial distribution, and that it converges to a stationary solution.

For  $\sigma = 0$ , Eq.(14) has been studied by Cressman [13] and Oechsler & Riedel [14]. They show that Eq.(14) has a unique solution, for all times, on a large space of distributions, containing amongst others the Dirac-delta distributions.

### 3 Symmetric 2x2 games

In symmetric  $2 \times 2$  games there are two possible strategies, denoted by  $I$  and  $II$ . The payoff to player  $A$  is given by the payoff matrix

$$M = \begin{array}{|c||c|c|} \hline A \setminus B & I & II \\ \hline I & a & b \\ \hline II & c & d \\ \hline \end{array}.$$

#### 3.1 Discrete replicator dynamics

The discrete replicator dynamics associated with this game consists of an infinite population where a fraction  $x_1(t)$  plays the pure strategy  $I$  and a fraction  $x_2(t)$  plays the pure strategy  $II$ . The payoffs to strategies  $I$  and  $II$  are given by:

$$\Pi_I(x_1, x_2) = ax_1 + bx_2 \quad \text{and} \quad \Pi_{II}(x_1, x_2) = cx_1 + dx_2. \quad (18)$$

The average payoff to the total population is then

$$\bar{\Pi}(x_1, x_2) = \Pi_I(x_1, x_2)x_1 + \Pi_{II}(x_1, x_2)x_2 = ax_1^2 + (b+c)x_1x_2 + dx_2^2. \quad (19)$$

With a mutation rate matrix of the form  $W = \sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\sigma > 0$  and using  $x_1(t) + x_2(t) = 1$ , this leads to the following equation for  $x_1(t)$ :

$$\dot{x}_1 = x_1(1 - x_1)(B + (A - B)x_1) + \sigma(1 - 2x_1), \quad (20)$$

where  $A = a - c$  and  $B = b - d$ . The solutions of Eq.(20) for  $\sigma = 0$  are summarised in figure 1.

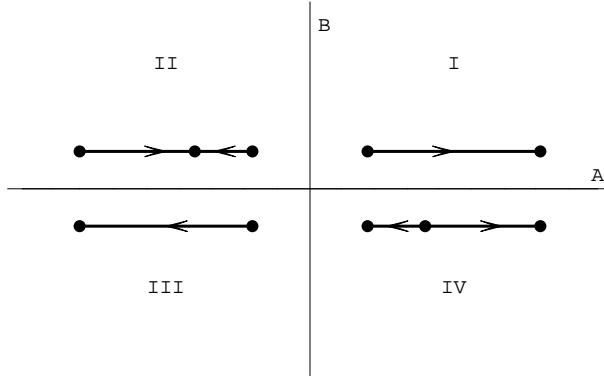


Figure 1: The four quadrants of the parameter space

A typical example of a game in the quadrant  $A > 0, B > 0$  is the classic Prisoners Dilemma, where  $a = 1, b = 5, c = 0, d = 3$ . The strategy  $I$  corresponds to 'defect' and strategy  $II$  to 'cooperate'. Figure 1 shows that for  $\sigma = 0$  the

discrete replicator dynamics Eq.(20) predicts a final outcome of  $x_1 = 1$ ,  $x_2 = 0$ , or 'All Defect'. The effect of the mutation is to shift the stable solution to a slightly lower value of  $x_1$ .

The Hawk-Dove game (also known as Chicken) has  $a = (G - C)/2$ ,  $b = G$ ,  $c = 0$ ,  $d = G/2$ , where now strategy  $I$  corresponds to 'hawk' (or 'never back down') and  $II$  to 'dove' ('allways back down'). When the cost  $C$  to the loser of a hawk-hawk fight is larger than the gain  $G$  a hawk makes when confronting a dove, we have  $A < 0$ ,  $B > 0$ . It follows from Eq.(20) that for  $\sigma = 0$  the solution tends to the stable equilibrium  $x_1 = B/(B-A) = G/C$ , describing a population where the strategies co-exist. Also in this case, the effect of the mutation term is restricted to a small shift in the location of the stable equilibrium.

The quadrant with  $A > 0$ ,  $B < 0$  is the domain of the Coordination Games, exemplified by the situation  $a = 2$ ,  $b = 0$ ,  $c = 0$ ,  $d = 1$ . In a Coordination Game it is advantageous for both players to play the same strategy. In a  $2 \times 2$  game, this leads to two equilibria. Note, however, that in the example mentioned here the situation where both play strategy  $I$  is superior to the one where both play  $II$ . For  $\sigma = 0$  there are two stable solutions and one unstable one, and the final outcome depends on the initial situation. When  $x_1(0) < \frac{1}{3}$  the solution will ultimately tend to  $x_1 = 0$ , otherwise to  $x_1 = 1$ . In other words, when the initial fraction of strategy  $I$  players is too small, the final population will consist exclusively of strategy  $II$  players, even though this is the less attractive of the two equilibria.

In the case of Coordination Games, the effect of mutation can qualitatively change this picture. With the above given values of  $a$ ,  $b$ ,  $c$  and  $d$ , we have plotted the right-hand side of Eq.(20) for several values of  $\sigma$ . (figure 2).

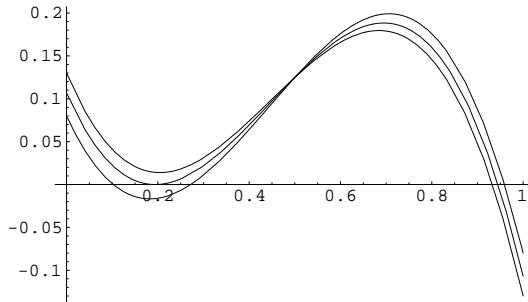


Figure 2: The bifurcation property of Eq.(20)

From this picture it follows that for values of  $\sigma$  larger than a critical value  $\sigma_c$  (in this example  $\sigma_c \approx 0.11$ ) the only equilibrium left is the optimal one near  $x_1 = 1$ , which is also the globally attracting solution.

### 3.2 Discrete replicator dynamics with a stochastic term

In [15], Eq.(20) is studied, where instead of a deterministic mutation term, a small noise term is added. This leads to the stochastic equation:

$$dx = G(x)dt + 2\sigma dW, \quad (21)$$

where  $G(x) = x(1-x)(B + (A-B)x)$  and  $W(t)$  denotes a Wiener process with zero mean and unit variance.

In this model, the mutation from one strategy to another does not happen at a fixed rate, as in the model described by Eq.(20), but rather the fraction  $x_1(t)$  is changed by a small random amount per time-step. To describe the outcome of Eq.(21), we consider the evolution of the probability density  $f(x, t)$ . The probability that at time  $t$  the fraction of strategy  $I$  players lies in the interval  $[x, x + \Delta x]$  is given by  $f(x, t)\Delta x$ . The equation for  $f(x, t)$  is the Fokker-Planck equation [11]:

$$\frac{\partial f(x, t)}{\partial t} = -\frac{\partial}{\partial x}(G(x)f(x, t)) + \sigma \frac{\partial^2 f(x, t)}{\partial x^2}. \quad (22)$$

We assume reflecting boundaries, which yield as boundary conditions:

$$\frac{\partial f(x=0, t)}{\partial x} = \frac{\partial f(x=1, t)}{\partial x} = 0. \quad (23)$$

Eq.(22) with conditions (23) has a unique, attracting, stationary distribution  $f^*(x)$  [15].

It is easy to see that this equilibrium distribution is given by

$$f^*(x) = C \exp\left(\frac{1}{\sigma} \int_0^x G(x') dx'\right), \quad (24)$$

where the constant  $C$  is determined by the condition  $\int_0^1 f^*(x) dx = 1$ .

Differentiating  $f^*(x)$  once yields

$$\frac{df^*(x)}{dx} = \frac{C}{\sigma} G(x) \exp\left(\frac{1}{\sigma} \int_0^x G(x') dx'\right), \quad (25)$$

from which it follows that the extrema of  $f^*(x)$  are the zeroes of  $G(x)$ , which are the equilibria of the discrete replicator equation (20) with  $\sigma = 0$ . Differentiating once more gives

$$\frac{d^2 f^*(x)}{dx^2} = \frac{C}{\sigma} \left(G'(x) + \frac{G^2(x)}{\sigma}\right) \exp\left(\frac{1}{\sigma} \int_0^x G(x') dx'\right), \quad (26)$$

so at an equilibrium point  $x_e$  we have that  $\frac{d^2 f^*(x=x_e)}{dx^2} = K G'(x_e)$  with  $K > 0$ . Therefore, a stable equilibrium of Eq.(20) with  $\sigma = 0$  corresponds to a maximum of  $f^*(x)$ , and an unstable equilibrium to a minimum.

In the limit  $\sigma \rightarrow 0$ , the stationary distribution  $f^*(x)$  will tend to a point-distribution, where the total probability is concentrated on one point. In the case of the Hawk-Dove Game and the Prisoners Dilemma it is clear that this point-distribution is concentrated on the unique stable equilibrium of the corresponding discrete replicator equation. For Coordination Games, there are two

stable equilibria in the discrete case. Correspondingly, the stationary distribution has two local maximuma, namely at  $x = 0$  and at  $x = 1$ . In [15] it is proved that for  $\sigma \rightarrow 0$  one of these two maxima will eventually dominate. The equilibrium that finally emerges is the one which has the largest basin of attraction in the discrete case, and is known in the game-theory literature as the risk-dominant equilibrium.

### 3.3 Continuous replicator dynamics

The two strategy set of  $2 \times 2$  games can be extended to a continuum of strategies, each of which indicated by a real number  $x \in [0, 1]$ . For the payoff function  $M(x, x')$  we choose a simple interpolation between the four payoff values of the symmetric  $2 \times 2$  game:

$$M(x, x') = xx'a + x(1 - x')b + (1 - x)x'c + (1 - x)(1 - x')d. \quad (27)$$

This game can be considered as the underlying discrete  $2 \times 2$  game where now mixed strategies are allowed, in the following sense. A strategy  $x \in [0, 1]$  means that the player will use pure strategy  $I$  with probability  $x$ . We now assume that two players, using strategies  $x$  and  $x'$  respectively, at one encounter play each other a large number of times. Eq.(27) then gives the expectation value of the payoff to the first player. Because our players will soon become aware of the Law of Large Numbers, they won't bother with playing against each other, but at an encounter simply settle for the payoff given by Eq.(27), making this a deterministic game.

The expressions for  $\Pi(x, P)$  and  $\bar{\Pi}(P)$  are easy to calculate:

$$\begin{aligned} \Pi(x, P) &= \int_0^1 M(x, x')P(x', t) dx \\ &= \int_0^1 (xx'a + x(1 - x')b + (1 - x)x'c + (1 - x)(1 - x')d)P(x', t) dx' \\ &= d + (b - d)x + (c - d)\bar{x}(t) + (a - b - c + d)x\bar{x}(t) \end{aligned}$$

in which

$$\bar{x}(t) = \int_0^1 xP(x, t) dx \quad (28)$$

is the average strategy. Then

$$\begin{aligned} \bar{\Pi}(P) &= \int_0^1 \Pi(x, P)P(x, t) dx = \\ &= d + (b + c - 2d)\bar{x}(t) + (a - b - c + d)\bar{x}^2(t), \end{aligned}$$

and

$$\begin{aligned} \Pi(x, P) - \bar{\Pi}(P) &= (b - d)x - (b - d)\bar{x}(t) + (a - b - c + d)(x\bar{x}(t) - \bar{x}^2(t)) \\ &= (B + (A - B)\bar{x}(t))(x - \bar{x}(t)), \end{aligned} \quad (29)$$

where  $A$  and  $B$  are the same parameters as defined in section 3.1.

Eq.(14) for  $P(x, t)$  now becomes:

$$\frac{\partial P(x, t)}{\partial t} = (B + (A - B)\bar{x}(t))(x - \bar{x}(t))P(x, t) + \sigma \frac{\partial^2 P(x, t)}{\partial x^2}, \quad (30)$$

with boundary conditions:

$$\left( \frac{\partial P(x, t)}{\partial x} \right)_{x=0} = \left( \frac{\partial P(x, t)}{\partial x} \right)_{x=1} = 0, \quad (31)$$

and an initial distribution  $P(x, 0)$ .

### 3.3.1 The equation without mutation

As was noted before, when considering Eq.(30) without mutation, i.e. with  $\sigma = 0$ , it is not necessary to impose the boundary conditions (31) in order to ensure that  $\int_0^1 P(x, t) dx$  remain constant. In [13] it is shown that

$$\frac{\partial P(x, t)}{\partial t} = (B + (A - B)\bar{x}(t))(x - \bar{x}(t))P(x, t), \quad P(x, 0) = P_0(x) \quad (32)$$

has a unique solution for all  $t > 0$ . In this subsection we will analyse the asymptotic behaviour of the solution of Eq.(32) as  $t \rightarrow \infty$ .

Firstly, the equation for the average  $\bar{x}(t)$  is given by:

$$\frac{d\bar{x}}{dt} = (B + (A - B)\bar{x}(t))(\bar{x}^2(t) - (\bar{x}(t))^2), \quad x(0) = \int_0^1 xP_0(x)dx, \quad (33)$$

where  $\bar{x}^2(t) = \int_0^1 x^2 P(x, t) dx$ . The factor  $\bar{x}^2(t) - (\bar{x}(t))^2$  is always positive. For the four different regions of the  $(A, B)$  parameter plane, as illustrated in figure 1, Eq.(33) implies the following.

$(A, B) \in I$ : in this case  $(B + (A - B)\bar{x}(t)) > 0$  for all  $t$ , so  $\bar{x}(t)$  is an increasing function. Because  $P(x, t) = 0$  for  $x \notin [0, 1]$ , the distribution will accumulate at  $x = 1$  and  $\lim_{t \rightarrow \infty} \bar{x}(t) = 1$ .

$(A, B) \in II$ : Eq.(33) now has an attractive fixed point at  $\bar{x} = B/(B - A)$ . Therefore,  $\lim_{t \rightarrow \infty} \bar{x}(t) = B/(B - A)$

$(A, B) \in III$ : similar to  $I$ , but now  $\lim_{t \rightarrow \infty} \bar{x}(t) = 0$ .

$(A, B) \in IV$ : Eq.(33) has a repelling fixed point at  $\bar{x} = B/(B - A)$ . It follows that  $\lim_{t \rightarrow \infty} \bar{x}(t) = 0$  if  $\bar{x}(0) < B/(B - A)$  and  $\lim_{t \rightarrow \infty} \bar{x}(t) = 1$  if  $\bar{x}(0) > B/(B - A)$ .

In the cases  $I$ ,  $III$  and  $IV$  the limiting distributions  $P_\infty(x) = \lim_{t \rightarrow \infty} P(x, t)$  are point-distributions concentrated on one of the endpoints of the domain  $[0, 1]$ .

This is comparable with the results in the model described in section 3.2 when  $\sigma \rightarrow 0$ : eventually all players will play either one or the other of the pure strategies. The situation is quite different for case *II*, i.e. for Hawk-Dove type games. We will show that in this case,  $P_\infty(x)$  depends on the initial condition  $P_0(x)$ , but is in general non-zero on all of  $[0, 1]$ . This is a dramatic difference with the above mentioned model. There, the population is divided into a fraction  $B/(B - A)$  of the population who play pure strategy *I* and the rest who play strategy *II*. In our model, where the players have access to a continuum of mixed strategies, we do not find that everybody plays the mixed strategy  $x = B/(B - A)$  (play strategy *I* with probability  $B/(B - A)$ ), as might be expected. Rather, the final outcome is a population who's members play a broad range of strategies, although the average value of the strategies played is  $\bar{x} = B/(B - A)$ . The details are as follows.

Every distribution  $P(x)$  with  $\int_0^1 xP(x) dx = \bar{x} = B/(B - A)$  is a solution of Eq.(32). We will now show that this set of distributions is attractive.

Let  $\lambda(t) = (B + (A - B)\bar{x}(t))$ . The solution of Eq.(32) then is

$$P(x, t) = P_0(x) \exp\left(\int_0^t \lambda(t') (x - \bar{x}(t')) dt'\right). \quad (34)$$

Using  $\int_0^1 P(x, t) dx = 1$  and writing  $g(t) = \int_0^t \lambda(t') dt$ , it follows that

$$\exp\left(\int_0^t \lambda(t') \bar{x}(t') dt\right) = \int_0^1 P_0(x) e^{xg(t)} dx. \quad (35)$$

Therefore

$$P(x, t) = \frac{P_0(x) e^{xg(t)}}{\int_0^1 P_0(x) e^{xg(t)} dx}. \quad (36)$$

Differentiating relation (35) with respect to  $t$  yields

$$\lambda(t) \bar{x}(t) \exp\left(\int_0^t \lambda(t') \bar{x}(t') dt\right) = g'(t) \int_0^1 x P_0(x) e^{xg(t)} dx.$$

Using  $g'(t) = \lambda(t)$ , and Eq.(35), this reduces to

$$\bar{x}(t) = \frac{\int_0^1 x P_0(x) e^{xg(t)} dx}{\int_0^1 P_0(x) e^{xg(t)} dx}. \quad (37)$$

Since  $\lim_{t \rightarrow \infty} \bar{x}(t) = B/(B - A)$ , then  $g(t) = \int_0^t (B + (A - B)\bar{x}(t')) dt'$  either tends to a finite limit or diverges. Using the change of variable  $u = x g(t)$ , we can write:

$$\bar{x}(t) = \frac{1}{g(t)} \frac{\int_0^{g(t)} u P_0\left(\frac{u}{g(t)}\right) e^u du}{\int_0^{g(t)} P_0\left(\frac{u}{g(t)}\right) e^u du}.$$

Therefore, when  $g(t) \rightarrow \infty$ , then

$$\bar{x}(t) = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon \int_0^{1/\varepsilon} u P_0(\varepsilon u) e^u du}{\int_0^{1/\varepsilon} P_0(\varepsilon u) e^u du} = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon \int_0^{1/\varepsilon} u e^u du}{\int_0^{1/\varepsilon} e^u du} = 1.$$

By a similar argument, when  $g(t) \rightarrow -\infty$  then  $\bar{x}(t) \rightarrow 0$ . Since  $\lim_{t \rightarrow \infty} \bar{x}(t) = B/(B-A)$  is neither 0 nor 1, it follows that  $\lim_{t \rightarrow \infty} g(t) = g_0$  is finite. Therefore, as  $t \rightarrow \infty$  the solution Eq.(36) tends to

$$P_\infty(x) = \frac{P_0(x) e^{xg_0}}{\int_0^1 P_0(x) e^{xg_0} dx},$$

which is clearly not a point-distribution.

The value of  $g_0$  can be found from Eq.(37), which in the limit  $t \rightarrow \infty$  reads:

$$\frac{B}{B-A} = \frac{\int_0^1 x P_0(x) e^{xg_0} dx}{\int_0^1 P_0(x) e^{xg_0} dx}.$$

This equation for  $g_0$  can be solved numerically for given values of  $A$  and  $B$  and a given initial distribution  $P_0(x)$ . In figure 3, two examples, both with  $A = -3$  and  $B = 7$ , are given for different initial distributions, shown in the left column. In the right column the final distributions are plotted.

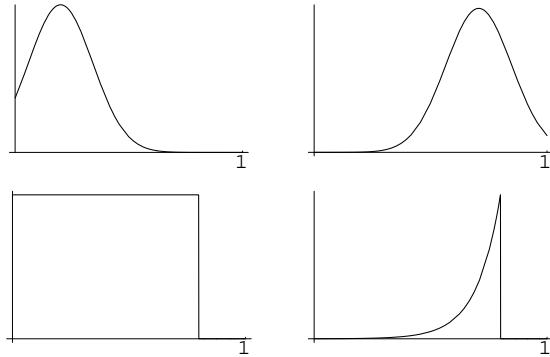


Figure 3: Initial and final distributions without mutations

### 3.3.2 Stationary solutions of the full equation

Numerical experiments show that all solutions of the full equation (30) converge to a time-independent solution. The equation for these stationary solutions is given by:

$$\sigma \frac{d^2 P(x)}{dx^2} + (B + (A - B)\bar{x})(x - \bar{x})P(x) = 0, \quad (38)$$

$$P'(0) = P'(1) = 0, \quad P(x) \geq 0, \quad \int_0^1 P(x) dx = 1. \quad (39)$$

Rather than giving a definition of  $\bar{x}$  in terms of  $P(x)$ , we take  $\bar{x}$  to be a free parameter and impose the conditions (39). Integrating Eq. (38) over  $x$  from 0 to 1 then yields:

$$(B + (A - B)\bar{x})(\int_0^1 xP(x) dx - \bar{x}) = 0. \quad (40)$$

Assuming that  $B + (A - B)\bar{x} \neq 0$ , the equality

$$\bar{x} = \int_0^1 xP(x) dx. \quad (41)$$

then follows automatically.

Let  $s(\bar{x}) = \text{sign}(B + (A - B)\bar{x})$  and  $\kappa(\bar{x}) = |\frac{B + (A - B)\bar{x}}{\sigma}|^{1/3}$ . Then the solution of Eq. (38) is given by:

$$P(x) = a \text{Ai}[-s(\bar{x})\kappa(\bar{x})(x - \bar{x})] + b \text{Bi}[-s(\bar{x})\kappa(\bar{x})(x - \bar{x})], \quad (42)$$

where the Airy functions  $\text{Ai}(z)$  and  $\text{Bi}(z)$  are the standard linearly independent solutions of  $y''(z) - zy(z) = 0$ . These functions are plotted in figure 4.

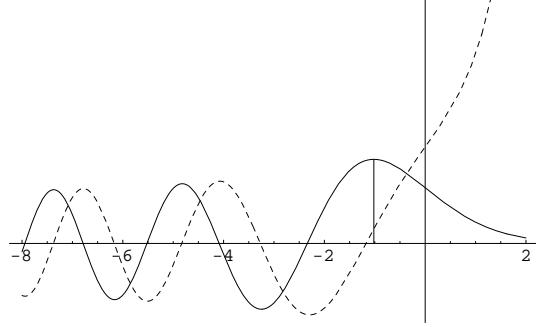


Figure 4: Airy functions  $\text{Ai}(z)$  and  $\text{Bi}(z)$

The position  $-\eta_0$  of the first maximum of  $\text{Ai}(z)$  is indicated by a vertical line segment. The curve for  $\text{Bi}(z)$  is dashed. Imposing the boundary conditions  $P'(0) = P'(1) = 0$  gives

$$\begin{aligned} a \text{Ai}'[s(\bar{x})\kappa(\bar{x})\bar{x}] + b \text{Bi}'[s(\bar{x})\kappa(\bar{x})\bar{x}] &= 0 \\ a \text{Ai}'[s(\bar{x})\kappa(\bar{x})(\bar{x} - 1)] + b \text{Bi}'[s(\bar{x})\kappa(\bar{x})(\bar{x} - 1)] &= 0, \end{aligned}$$

so that a non-trivial solution only exists if  $\bar{x}$  is a solution to the "eigenvalue equation":

$$\text{Ai}'[s(\bar{x})\kappa(\bar{x})\bar{x}] \text{Bi}'[s(\bar{x})\kappa(\bar{x})(\bar{x} - 1)] - \text{Bi}'[s(\bar{x})\kappa(\bar{x})\bar{x}] \text{Ai}'[s(\bar{x})\kappa(\bar{x})(\bar{x} - 1)] = 0. \quad (43)$$

Corresponding to a solution  $\bar{x}$  of Eq. (43), we find that the solution (42) can be written as:

$$P(x) = \quad (44)$$

$$= c(Bi'[s(\bar{x})\kappa(\bar{x})\bar{x}]Ai[s(\bar{x})\kappa(\bar{x})(\bar{x}-x)] - Ai'[s(\bar{x})\kappa(\bar{x})\bar{x}]Bi[s(\bar{x})\kappa(\bar{x})(\bar{x}-x)]),$$

in which  $c$  is determined by the normalisation condition. We found that, although the eigenvalue equation (43) can have many solutions, there will be at most three that correspond to a distribution (44) with the property that  $P(x) \geq 0$  for all  $x \in [0, 1]$ . For values of A and B in the regions I and II of figure 1, typified by PD- and HD-games, there is only one solution. The distributions calculated for  $\sigma = 0.001$ , are shown in figure 5.

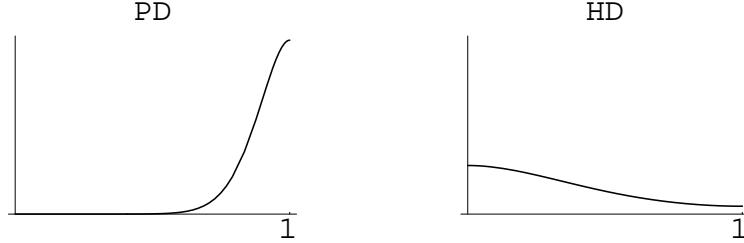


Figure 5: Stationary distributions for PD- and HD-games

For the PD-case the parameters are  $A = 1$ ,  $B = 2$  and  $\bar{x} = 0.901$ . For the HD-case they are  $A = -2$ ,  $B = 1$  and  $\bar{x} = 0.342$ . For  $(A,B) \in IV$ , i.e., for Coordination Games, there exist three solutions for this value of  $\sigma$ . The corresponding distributions are plotted in one picture (figure 6).

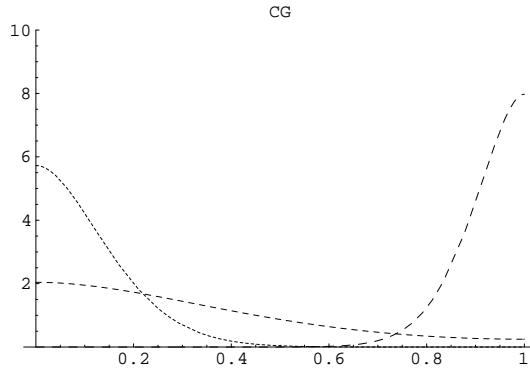


Figure 6: Stationary distributions for CG-game

With  $A = 2$  and  $B = -1$  we find for the average strategies  $\bar{x} = 0.118$ ,  $\bar{x} = 0.323$  and  $\bar{x} = 0.915$ .

We now study the behaviour of the solution of (38) and (39) for  $\sigma \rightarrow 0$ . First consider the case that  $|B + (A - B)\bar{x}| > \text{const}$ , independent of  $\sigma$ . Then  $\kappa(\bar{x}) = |\frac{B+(A-B)\bar{x}}{\sigma}|^{1/3} \rightarrow \infty$  as  $\sigma \rightarrow 0$ . From figure 4 it is clear that if  $s(\bar{x}) = \text{sign}(B + (A - B)\bar{x}) = 1$ , then  $Bi'[s(\bar{x})\kappa(\bar{x})\bar{x}] \rightarrow \infty$ ,  $Bi'[s(\bar{x})\kappa(\bar{x})(\bar{x}-1)]$  remains bounded and  $Ai'[s(\bar{x})\kappa(\bar{x})\bar{x}] \rightarrow 0$ , as  $\sigma \rightarrow 0$ . This implies that  $\bar{x}$  can only be a solution of Eq.(43) if  $Ai'[s(\bar{x})\kappa(\bar{x})(\bar{x}-1)] \rightarrow 0$ . From figure 4, it then follows that  $\kappa(\bar{x})(\bar{x}-1) \rightarrow -\eta_0$ , where  $\eta_0 = 1.01879\dots$  is the smallest positive solution of  $Ai'(-\eta_0) = 0$ , so that  $\bar{x} \rightarrow 1$  and  $\kappa(\bar{x}) \rightarrow |\frac{A}{\sigma}|^{1/3}$ . More precisely, we have:

$$\bar{x} = 1 - \left| \frac{\sigma}{A} \right|^{1/3} \eta_0 \quad \text{as } \sigma \rightarrow 0$$

This solution is consistent with the assumption  $s(\bar{x}) = 1$  if and only if  $A > 0$ , as in the Prisoners Dilemma and the Cooperation Game. The corresponding asymptotic expression for  $P(x)$  becomes:

$$P(x) = cAi[-\kappa(\bar{x})(x - \bar{x})] = cAi\left[\left(\frac{A}{\sigma}\right)^{1/3}(1 - x) - \eta_0\right].$$

Note that for  $\sigma \rightarrow 0$  this distribution becomes sharply peaked at  $x = 1$ , with the width of the peak proportional to  $\sigma^{1/3}$ .

By a similar reasoning, it is found that for  $B < 0$  (as in the Cooperation Game) there exists a solution such that

$$\bar{x} = \left| \frac{\sigma}{B} \right|^{1/3} \eta_0 \quad \text{and} \quad P(x) = cAi\left[\left|\frac{B}{\sigma}\right|^{1/3}x - \eta_0\right]$$

as  $\sigma \rightarrow 0$ .

In the previous section we found that for  $\sigma = 0$  the Hawk-Dove Game and the Cooperation Game have solutions with  $\bar{x} = \frac{B}{B-A}$ , which we call a central solution. This motivates us to look for solutions for which  $\kappa(\bar{x}) = |\frac{B+(A-B)\bar{x}}{\sigma}|^{1/3}$  remains bounded as  $\sigma \rightarrow 0$ . We therefore assume that

$$\bar{x} = \frac{B}{B-A} + \frac{\alpha}{B-A}\sigma + \dots \quad \text{as } \sigma \rightarrow 0, \quad (45)$$

where  $\alpha \in R$  is as yet unknown. Substituting Eq.(45) into Eq.(43) and taking the limit  $\sigma \rightarrow 0$  yields:

$$Ai'\left[\frac{B}{B-A}\alpha^{1/3}\right]Bi'\left[\frac{A}{B-A}\alpha^{1/3}\right] - Bi'\left[\frac{B}{B-A}\alpha^{1/3}\right]Ai'\left[\frac{A}{B-A}\alpha^{1/3}\right] = 0, \quad (46)$$

where  $\alpha^{1/3}$  is understood to mean  $\text{sign}(\alpha)|\alpha|^{1/3}$ . This is the equation from which  $\alpha$  must be solved. Although Eq.(46) has many zeroes, only one corresponds to a positive distribution given by:

$$P(x) = c(Bi'\left[\beta\alpha^{1/3}\right]Ai[\alpha^{1/3}(x - \beta)] - Ai'\left[\beta\alpha^{1/3}\right]Bi[\alpha^{1/3}(x - \beta)]), \quad (47)$$

with  $\beta = \frac{B}{B-A}$ . For  $A = -2$  and  $B = 1$ , (a Hawk-Dove game), it is plotted in figure 7.

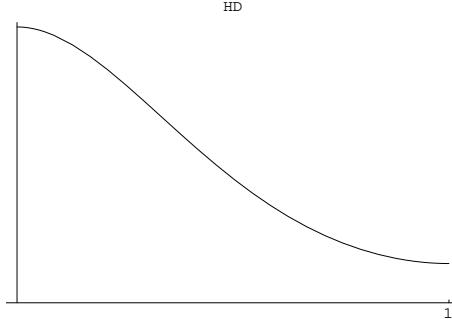


Figure 7: Asymptotic HD-distribution for  $\sigma \rightarrow 0$

We note that Eqs. (46) and (47) are invariant under  $A \rightarrow -A$ ,  $B \rightarrow -B$ . Therefore, as  $\sigma \rightarrow 0$ , we find the same central solution for the Coordination Game with  $A = 2$  and  $B = -1$ . From numerical simulations we find, however, that the central solution is stable in the Hawk-Dove Game, but unstable in the Coordination Game.

Away from the limit  $\sigma \rightarrow 0$ , we can track the fate of the central solution as  $\sigma$  grows. Since the full eigenvalue-equation (43) is not invariant under  $A \rightarrow -A$ ,  $B \rightarrow -B$ , the situation is different for Hawk-Dove Games as opposed to Coordination Games. For Hawk-Dove Games, we find that the central solution persists for all values of the diffusion coefficient  $\sigma$ . However, for Coordination Games we find that above a critical value  $\sigma_c$  of  $\sigma$ , the unstable central solution disappears, together with one of the two stable solutions, leaving only one attracting, stationary solution. When  $A < |B|$ , the solution at  $x = 1$  remains, and when  $A > |B|$ , the solution near  $x = 0$  survives. This can be summarised by saying that for large enough values of  $\sigma$ , the only attracting solution is a stationary solution near the risk-dominated solution of the discrete equation. The bifurcation process is illustrated in figure 8, where for two values of  $\sigma$  the left-hand-side of Eq.(43) is plotted as a function of  $\bar{x}(\sigma)$ . For  $\sigma = 0.0035$  this function has three zeroes, whereas for  $\sigma = 0.0039$  there is only one. This closely resembles the situation in the discrete case as described in section 2. However, the critical values for  $\sigma$  in the discrete case and in the continuous case are not comparable (0.11 vs. 0.0037).

### 3.3.3 Summary

Here we summarise and compare the results of the three types of games, using the discrete model, the discrete model with a stochastic term and the continuous model with deterministic mutation.

The Prisoners Dilemma is straightforward. In the discrete case the population will eventually play All Defect. This does not change when the strategy-space is made continuous and adding deterministic mutation simply changes the

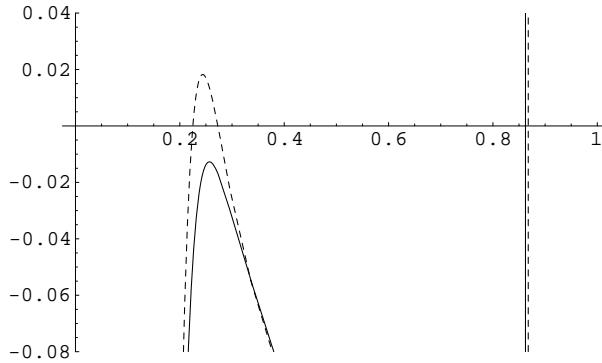


Figure 8: Bifurcation of CG-games

limiting delta-distribution to a finite peak with width proportional to  $\sigma^{1/3}$ . The contrast with the corresponding stochastic equation is that there the width of the peak is narrower and proportional to  $\sigma^{1/2}$ .

In the discrete version of the Coordination Game, the population will eventually play either  $x_1 = 0$  or  $x_1 = 1$ , where the final outcome depends on the initial condition. If the mutation rate is larger than a certain threshold, only the stable solution around the risk-dominant solution remains. This result remains the same in the continuous case. Depending on the initial distribution of strategies, the final distribution will be sharply peaked (width of peak proportional to  $\sigma^{1/3}$ ) around either  $x = 0$  or  $x = 1$ . When the mutation rate becomes larger than a critical value, only the distribution around the risk-dominant solution remains. We note that this critical value is much smaller in the continuous case than in the discrete case. The stochastic equation has only one, attracting, stationary solution. When the mutation rate  $\sigma$  is small, this distribution is almost completely concentrated around the risk dominant solution, where again the width of the distribution is proportional to  $\sigma^{1/2}$ .

The Hawk-Dove Game shows the following behaviour. In the discrete case, there is an asymptotically stable solution for all values of  $\sigma > 0$ , which for  $\sigma = 0$  has the value  $x_1 = B/(B - A)$ . The stochastic equation has a unique attracting stationary solution, peaked around  $x = B/(B - A)$  and with a width proportional to  $\sigma^{1/2}$ . Without mutation,  $\sigma = 0$ , the continuous equation has an asymptotically stable invariant set of solutions, consisting of all distributions  $P(x)$  with average  $\bar{x} = B/(B - A)$ . Depending on the initial distribution, the solution converges to a member of this set. When the mutation term is added,  $\sigma > 0$ , only one, attracting, solution remains. In the limit  $\sigma \rightarrow 0$ , this solution converges to (47). This solution is not concentrated on a single point, but has the whole interval  $[0, 1]$  as support. Such a solution is sometimes referred to as polymorphic. For small  $\sigma$ , numerical experiments show that a starting distribution initially converges to a distribution close to the solution it would reach if  $\sigma = 0$ , but then slowly (on a time-scale of  $1/\sigma$ ) evolves to the unique limiting solution. This behaviour resembles that of a singularly perturbed

ordinary differential equation, which in the unperturbed case has an attracting set of fixed points, and where after adding the perturbation, the attracting invariant set survives. In this invariant set we would have, in this analogy, one attracting fixed point left.

## 4 The Ultimatum Game

This game has attracted a great deal of interest, mainly as a model to explain the occurrence of strong reciprocity in populations of supposedly selfish individuals.(Binmore [16], Fehr & Gachter [17]). Here "strong reciprocity" means the willingness to share, but also to punish egotistical behaviour in others, even at a cost to oneself. See Bowles & Gintis [18].

The game is played by two players. The first player is given a certain amount of money and proposes a split of this money with the second player. This second player has the choice between accepting the offer, or rejecting it, in which case neither of the two players will receive anything. An obvious strategy for the second player is to accept every offer, since something is better than nothing. Realising this, the first player will maximise his share by offering the lowest possible amount to the second player. This (combined) strategy is sometimes referred to as "the rational solution" (Page & Nowak [19]), "the subgame-perfect equilibrium" (Seymour [20]) or the strategy of "Homo Economicus" (Bowles & Gintis [21]).

An evolutionary version of this game, taking into account mutations, was studied in [19] and by Nowak et al. [22], using adaptive dynamics. In adaptive dynamics models, the population is assumed to always be monomorphic, *i.e.*, everybody plays the same strategy. Every now and then, a mutant is introduced. If the strategy of the mutant is more successful than the resident strategy, it will quickly spread in the population, thus becoming the new resident strategy. For the Ultimatum Game it was found that in the absence of any restrictions, the solution of the adaptive dynamics model indeed converged to the "rational solution".

The Ultimatum Game was also studied by Seymour [20], using replicator dynamics. He included mutations as a given, exogeneous term and found that other solutions can emerge, far from the "subgame-perfect solution", depending on the form and intensity of the mutation function.

We model the Ultimatum Game as follows. The strategy space is  $S = [0, 1] \times [0, 1]$ , where a strategy  $s = (x, y) \in S$  means that the player, in the role of nr. 1, will offer a fraction  $x$ , while in the role of nr. 2 he will reject any offer lower than  $y$ .

In one round, the players will play the role of nr. 1 and nr. 2 alternately. This leads to a payoff function giving the payoff to strategy  $s$  when playing strategy  $s'$  (the factor  $1/2$  has been omitted):

$$M(s, s') = M(x, y|x', y') = (1 - x)\Theta(x - y') + x'\Theta(x' - y), \quad (48)$$

in which the Heaviside function is defined by

$$\Theta(z) = \begin{cases} 1 & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases} \quad (49)$$

Before writing down the full replicator equation (14) for this case we first introduce a number of abbreviations:

$$H(x, t) = \int_0^1 P(x, y, t) dy \quad \text{and} \quad V(y, t) = \int_0^1 P(x, y, t) dx, \quad (50)$$

which are normalised as

$$\int_0^1 H(x, t) dx = \int_0^1 V(y, t) dy = 1. \quad (51)$$

$Q(x, t)$  and  $R(y, t)$  are defined by

$$Q(x, t) = \int_0^x V(y, t) dy \quad \text{and} \quad R(y, t) = \int_y^1 xH(x, t) dx, \quad (52)$$

so that

$$Q(0, t) = 0 \quad \text{and} \quad Q(1, t) = 1 \quad \text{and} \quad R(1, t) = 0. \quad (53)$$

In terms of these functions the local and global averages take the form

$$\Pi(x, y, P) = (1 - x)Q(x, t) + R(y, t) \quad (54)$$

and

$$\bar{\Pi}(P) = c_1(t) + c_2(t), \quad (55)$$

with

$$c_1(t) = \int_0^1 (1 - x)Q(x, t)H(x, t) dx \quad c_2(t) = \int_0^1 R(y, t)V(y, t) dy. \quad (56)$$

At last the replicator equation becomes

$$\frac{\partial P(x, y, t)}{\partial t} = [(1 - x)Q(x, t) + R(y, t) - c_1(t) - c_2(t)]P(x, y, t) + \sigma\Delta P(x, y, t), \quad (57)$$

in which  $\Delta$  is the two-dimensional Laplace operator. The boundary condition is

$$\nabla P \cdot \mathbf{n} = 0 \quad \text{on the boundary.} \quad (58)$$

In what follows we will restrict ourselves to solutions which can be written as the product of two normalised functions of  $(x, t)$  and of  $(y, t)$  respectively. It then necessarily follows that

$$P(x, y, t) = H(x, t)V(y, t). \quad (59)$$

With this restriction we easily show, by integrating Eq.(57) over  $y$ , that

$$\frac{\partial H(x, t)}{\partial t} = [(1 - x)Q(x, t) - c_1(t)]H(x, t) + \sigma \frac{\partial^2 H(x, t)}{\partial x^2}. \quad (60)$$

The boundary conditions are

$$\frac{\partial H(x = 0, t)}{\partial x} = \frac{\partial H(x = 1, t)}{\partial x} = 0 \quad \text{for all } t. \quad (61)$$

Integration over  $x$  of Eq.(57) leads to

$$\frac{\partial V(y, t)}{\partial t} = [R(y, t) - c_2(t)]V(y, t) + \sigma \frac{\partial^2 V(y, t)}{\partial y^2}, \quad (62)$$

with boundary conditions

$$\frac{\partial V(y = 0, t)}{\partial y} = \frac{\partial V(y = 1, t)}{\partial y} = 0 \quad \text{for all } t. \quad (63)$$

We note that for  $\sigma = 0$  these equations are the same as those studied in [20], where it is assumed that there are two separate populations of players, one where all members always play the role of nr. 1 and the other with nr. 2 players.

#### 4.1 The equation without mutation

For  $\sigma = 0$  the equations become:

$$\begin{aligned} \frac{\partial H(x, t)}{\partial t} &= [(1 - x)Q(x, t) - c_1(t)]H(x, t) \\ \frac{\partial V(y, t)}{\partial t} &= [R(y, t) - c_2(t)]V(y, t). \end{aligned} \quad (64)$$

Similar to the Hawk-Dove Game, we identify a set of stationary solutions:

$$\begin{aligned} H_0(x) &= \delta(x - \bar{x}) \\ V_0(y) &= \begin{cases} v(y) & \text{if } y < \bar{x} \\ 0 & \text{if } y > \bar{x} \end{cases} \\ c_1 &= 1 - \bar{x}, \quad c_2 = \bar{x} \end{aligned} \quad (65)$$

where  $v(y)$  is an arbitrary function with  $\int_0^{\bar{x}} v(y) dy = 1$  and  $\delta(z)$  is the Dirac- $\delta$  distribution. The solution (65) is easily checked, by noting that  $R(y) = \bar{x}\Theta(\bar{x} - y)$  and that  $Q(\bar{x}) = 1$ . We have also used  $z\delta(z) \equiv 0$ . The interpretation of this solution is clear: player nr. 1 always offers  $\bar{x}$ , so player nr. 2 will always receive this amount, as long as his acceptance threshold is below  $\bar{x}$ . The average payoff is therefore  $\bar{\Pi} = c_1 + c_2 = 1$  and any distribution of the  $y$ -values below  $\bar{x}$  is stationary, given this distribution of  $x$ . The limit  $\bar{x} \rightarrow 0$  corresponds to the subgame-perfect solution.

To show that (65) represents an attracting set of solutions, we use the same reasoning as in section 3.3.1, and find

$$\exp\left[\int_0^t c_1(t') dt'\right] = \int_0^1 H_0(x) \exp[(1-x) \int_0^t Q(x, t') dt'] dx \equiv h(t) \quad (66)$$

$$H(x, t) = \frac{H_0(x)}{h(t)} \exp[(1-x) \int_0^t Q(x, t') dt'] \quad (67)$$

$$\exp\left[\int_0^t c_2(t') dt'\right] = \int_0^1 V_0(y) \exp\left[\int_0^t R(y, t') dt'\right] dy \equiv v(t) \quad (68)$$

$$V(y, t) = \frac{V_0(y)}{v(t)} \exp\left[\int_0^t R(y, t') dt'\right]. \quad (69)$$

By differentiating Eq. (66) and (68), respectively, we obtain:

$$c_1(t) = \frac{1}{h(t)} \int_0^1 (1-x)Q(x, t)H_0(x) \exp[(1-x) \int_0^t Q(x, t') dt'] dx \quad (70)$$

$$c_2(t) = \frac{1}{v(t)} \int_0^1 R(y, t)V_0(y) \exp\left[\int_0^t R(y, t') dt'\right] dy. \quad (71)$$

Assuming that  $V(y, t)$  converges to a stationary distribution (as all numerical results show), then  $(1-x) \int_0^t Q(x, t') dt'$  converges to a function with a finite number of isolated local maxima. One of these, say  $x = \bar{x}$ , is the absolute maximum, and it follows from Eq.(67) that  $H(x, t)$  converges to  $\delta(x - \bar{x})$ . From Eq.(52) it follows that  $R(y, t)$  converges to  $R(y) = \bar{x} \Theta(\bar{x} - y)$  and  $V(y, t)$  converges to  $c V_0(y) \Theta(\bar{x} - y)$ , with  $c$  a normalization constant. From Eq.(56) it follows that  $c_1(t)$  converges to  $1 - \bar{x}$  and  $c_2(t)$  to  $\bar{x}$ .

The above considerations show that if the solution of Eq.(64) converges to a stationary solution, it must be a member of the invariant set (65). However, the value of  $\bar{x}$  cannot be predicted from the above formula's. Numerical solution of Eq.(64) shows that for random initial distributions of  $H(x, 0)$  and  $V(y, 0)$  on the whole interval  $[0, 1]$ , the functions  $H(x, t)$  and  $V(y, t)$  indeed approach the form of Eq.(65) for  $t \rightarrow \infty$ . The average strategy  $\bar{x}$ , based on 100 simulations, takes values between 0.12 and 0.30, with a mean value equal to 0.22 and a standard deviation of 0.04. We note that a uniform distribution of both  $H(x, 0)$  and  $V(y, 0)$  also leads to a value of  $\bar{x} = 0.22$ .

## 4.2 The equation with mutation

We have found numerically that when  $\sigma > 0$ , all solutions of the full equations (60) and (62) tend to a unique solution of the stationary equations:

$$\sigma \frac{d^2 H(x)}{dx^2} + [(1-x)Q(x) - c_1]H(x) = 0 \quad (72)$$

and

$$\sigma \frac{d^2V(y)}{dy^2} + [R(y) - c_2]V(y) = 0. \quad (73)$$

The boundary values are those of Eqs.(61) and(63).

Unfortunately, we have not been able to find closed form expressions for the solutions of these equations. There are two ways to approximate the stationary solution, which lead to the same result. First, we numerically solved the full equations(60) and (62), by discretising space, solving the resulting coupled set of ordinary differential equations and considering the solution as  $t \rightarrow \infty$ . In the second method, we define the following seven functions

$$\begin{aligned} F_1(x) &= H(x), & F_2(x) &= \frac{dH(x)}{dx}, & F_3(x) &= V(x), & F_4(x) &= \frac{dV(x)}{dx}, \\ F_5(x) &= Q(x), & F_6(x) &= R(x), & F_7(x) &= \int_x^1 H(x') dx'. \end{aligned} \quad (74)$$

In terms of these functions and with  $k = 1/\sigma$ , the stationary equations can now be written as

$$\begin{aligned} \frac{dF_1(x)}{dx} &= F_2(x) \\ \frac{dF_2(x)}{dx} &= -k[(1-x)F_5(x) - c_1]F_1(x) \\ \frac{dF_3(x)}{dx} &= F_4(x) \\ \frac{dF_4(x)}{dx} &= -k[F_6(x) - c_2]F_3(x) \\ \frac{dF_5(x)}{dx} &= F_3(x) \\ \frac{dF_6(x)}{dx} &= -xF_1(x) \\ \frac{dF_7(x)}{dx} &= -F_1(x) \end{aligned} \quad (75)$$

These equation can be solved numerically by starting the integration from the following values at  $x = 1$

$$\begin{aligned} F_1(1) &= a, & F_2(1) &= 0, & F_3(1) &= b, & F_4(1) &= 0, \\ F_5(1) &= 1, & F_6(1) &= 0, & F_7(1) &= 0, \end{aligned} \quad (76)$$

and using a standard routine to arrive at the values of these functions in  $x = 0$ . The numbers  $(a,b,c_1,c_2)$  are as yet unknown. They should be chosen in such a way that the boundary conditions at  $x = 0$  be satisfied, *i.e.*,

$$(F_2(0), F_4(0), F_5(0), F_7(0)) = (0, 0, 0, 1). \quad (77)$$

This matching of four numbers by varying four other numbers should be possible in many ways. It turns out, however, that the requirement of positivity of  $F_1(x)$

and  $F_3(x)$  in the whole interval  $[0, 1]$  makes the solution unique. A root finding routine of Mathematica does the job. For  $\sigma = 0.001$  the stationary solutions  $H(x)$  and  $V(y)$  are shown in figure 9.

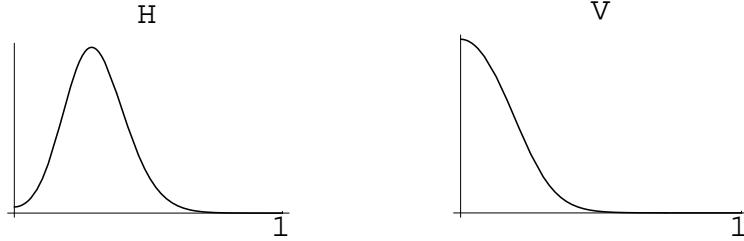


Figure 9: Stationary solution of Eqs.(72) and (73) for  $\sigma = 0.001$

We note that  $H(x)$  has a Gauss-like distribution around a mean value  $\bar{x} = 0.3172$ , while  $V(y)$  is approximated by the right half of a Gaussian, with its maximum at  $y = 0$ . For smaller values of  $\sigma$ , the value of  $\bar{x}(\sigma)$  and the width of the peak of the  $H(x)$ -distribution decrease, but the shape of the distributions is otherwise unchanged.

For values of  $\sigma$  down to  $10^{-9}$  we have calculated the  $\sigma$ -dependence of  $\bar{x}$ . In figure 10

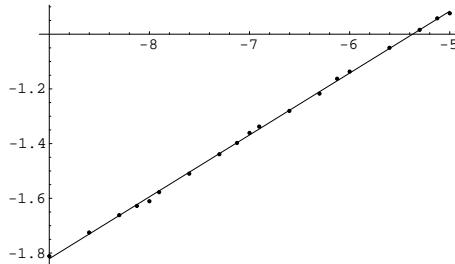


Figure 10: Log-log plot of  $\bar{x}$  as function of  $\sigma$

a log-log plot of this dependence is shown. A good fit of the data points is given by  $\bar{x}(\sigma) \approx 1.64 \times \sigma^{0.23}$ .

The numerical solution of the full time dependent equations reveals a dynamical pattern similar to the Hawk-Dove Game of section 2. Initially a distribution approaches the attracting set (65), after which it slowly converges to the unique stationary solution, on a time-scale of  $1/\sigma$ .

## 5 Conclusions

In this paper we have generalised the replicator dynamics of games with deterministic mutations, as described in [4], to the situation where players have

access to a continuous strategy space. The resulting equation (14) has a well defined solution, which, however, is not easy to analyse in general.

Our first example, the continuous version of  $2 \times 2$  symmetric games, already illustrates a number of interesting and perhaps unexpected features of this equation. Although there is no a priori reason to believe that the continuous and the discrete strategy version of the same  $2 \times 2$  game have anything to do with each other, the similarities between some results warrant our surprise at the differences in others. The continuous Prisoners Dilemma and the Coordination Game behave similar to their discrete case counterparts: the final state is a monomorphic distribution, where every player in the population plays one of the two pure strategies. The extension to a continuous strategy space and the inclusion of mutation (which has the form of a diffusion term) only leads to the existence of small variations around the single peak of the final distribution. Also in the continuous Coordination Game we encounter, as in the discrete case, a threshold value for the mutation term separating a regime with two attracting solutions from one where only a single attractor exists.

The difference occurs in the Hawk-Dove game. In the discrete version there is a stable equilibrium with Hawks and Doves coexisting. In the continuous version this does not translate into a monomorphic distribution around the mixed strategy corresponding to this equilibrium. Rather, in the unique limiting distribution the whole range of mixed strategies from pure Hawk to pure Dove is represented. The attraction to this stationary solution occurs on two time scales. On a fast time scale, the solution is attracted to the set of distributions with average corresponding to the equilibrium mixed strategy of the discrete case. Then, on a slow time scale proportional to the inverse of the mutation rate, the solution converges to the unique attractor. This two timescales phenomenon was observed in numerical simulations, and is currently awaiting a more thorough analysis. Also, there are many interesting games with three or more strategies (for instance Rock, Scissors, Paper) of which the continuous version can hold more surprises.

The results of the second example, the Ultimatum Game, are of great interest to the debate around strong reciprocity and how it could have evolved. Our model shows that replicator dynamics and a small mutation term can lead to a final outcome far from the subgame-perfect solution. Take, for instance, a mutation rate of  $10^{-3}$ . So in every time-interval, all players vary their strategies according to the cold rules of self-interest, after which a small fraction of 0.1% of the population, change their strategy just a little bit. Then we find that an initial population consisting almost entirely of cynical misers (accept everything and offer nothing), eventually turns into a world where the average offer is more than 30%!

This surprising result can be explained in the following way. Consider a situation where all proposers offer only a small share to their opponent and these opponents all have an acceptance threshold lower than this offer. Now, due to mutation, some acceptors will demand a share that is slightly larger than what is being offered. Normally, this would be a suicidal strategy. However,

also due to mutation, there will be amongst the proposers a small set who are willing to offer slightly more than their colleagues. On the one hand, these fairer-minded proposers earn slightly less from the bulk of the acceptors, but on the other hand they are the only ones to profit from the small group of high-minded mutants on the acceptor-side. The net result can be that the second effect dominates and that there will be a tendency towards higher offers.

For sufficiently small mutation rates the dynamics of the Ultimatum Game show the same structure with two timescales as the continuous Hawk-Dove game. An initial distribution is quickly attracted to a distribution where the offers are sharply peaked, and then slowly converges to the unique stationary solution.

In this case too, a more rigorous mathematical analysis is required for a better understanding of the model. In particular it would be nice to be able to calculate the value of the exponent in the formula relating the mutation rate and the average value of the offers, which in this paper we derived from numerical simulations. For this purpose singular perturbation theory seems to be an appropriate tool. Furthermore, we have only considered mutation rates that are the same for the proposers as for the acceptors. Differentiating between these may also lead to a fuller understanding.

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