Models of Non-Well-Founded Sets via an Indexed Final Coalgebra Theorem

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Abstract

The paper uses the formalism of indexed categories to recover the proof of a standard final coalgebra theorem, thus showing existence of final coalgebras for a special class of functors on categories with finite limits and colimits. This is then put to use in the context of a Heyting pretopos with a class of small maps, in order to build the final coalgebra for the $\mathcal{P}_s$ functor. This is then proved to provide a model for various set theories with the Anti-Foundation Axiom, depending on the chosen axiomatisation for the class of small maps.

1 Introduction

> *The explicit use of bisimulation for set theory goes back to the work on non-wellfounded sets by Aczel(1988). It would be of interest to construct sheaf models for the theory of non-wellfounded sets from our axioms for small maps.*
>
> – Joyal and Moerdijk, 1995

Since its first appearance in the book by Joyal and Moerdijk [9], algebraic set theory has always claimed the virtue of being able to describe, in a single framework, various different set theories. In fact, the paradigm of "various axioms for small maps determine axioms for the corresponding set theory" has been put to work first in the aforementioned book, and then in the work by Awodey et al. [4], thus modelling such theories as **CZF**, **IZF**, **BIST**, **CST** and so on. However, despite the suggestion in [9], it appears that up until now no one ever tried to put small maps to use in order to model a set theory with the Anti-Foundation Axiom **AFA**.

This papers provides a first step in this direction. In particular, we build a categorical model of the weak constructive theory $\mathbf{CZF}_0$ of (possibly) non-well-founded sets, studied by Aczel and Rathjen in [1] . Classically, the universe of non-well-founded sets is known to be the final coalgebra of the powerclass functor [2]. Therefore, it should come as no surprise that we build such a model from the final coalgebra for the functor $\mathcal{P}_s$ for a class of small maps.

Perhaps more surprising is the fact that such a coalgebra *always* exists. We prove this by means of a final coalgebra theorem, for a certain class of functors on a finitely complete and cocomplete category. The intuition that guided us along the argument is a standard proof of a final coalgebra theorem by Aczel and Mendler [3] for set-based functors on the category of classes. Given one such functor, they first consider the coproduct of all small coalgebras, and show that this is a weakly terminal coalgebra. Then, they quotient by the largest bisimulation on it, to obtain a final coalgebra. The argument works more generally for any functor of which we know that there is a generating family of coalgebras, for in that case we can restrict our attention to such a family, and perform the construction as above. The condition of a functor being set-based assures that we are in such a situation.

Our argument is a recasting of the given one in the internal language of a category. Unfortunately, the technicalities that arise when externalising an argument which is given in the internal language can be off-putting, at times. For instance, the internalisation of colimits forces us to work in the context of indexed categories and indexed functors. Within this context, we say that an indexed functor (which turns pullbacks into weak pullbacks) is small-based when there is a "generating family" of coalgebras. For such functors we prove an indexed final coalgebra theorem. We then apply our machinery to the case of a Heyting pretopos with a class of small maps, to show that the functor $\mathcal{P}_s$ is small-based and therefore has a final coalgebra. As a byproduct, we are able to build the M-type for any small map f (i.e. the final coalgebra for the polynomial functor P_f associated to f).

For sake of clarity, we have tried to collect as much indexed category theory as we could in a separate section. This forms the content of Section 2, and we advise the uninterested reader to skip all the details of the proofs therein. This should not affect readability of Section 3, where we prove our final coalgebra results. Finally, in Section 4 we prove that the final $\mathcal{P}_s$-coalgebra is a model of the theory $\mathbf{CZF}_0$+**AFA**.

Our choice to focus on a weak set theory such as $\mathbf{CZF}_0$ is deliberate, since stronger theories can be modelled simply by adding extra requirements for the class of small maps. For example, we can model the theory **CST** of Myhill [12] (plus **AFA**), by adding the Exponentiation Axiom, or $\mathbf{IZF}^-$+**AFA** by adding the Powerset, Separation and Collection axioms from [9, p. 65]. And we can force the theory to be classical by working in a boolean pretopos. This gives a model of $\mathbf{ZF}^-$+**AFA**, the theory presented in Aczel's book [2], aside from the Axiom of Choice. We also expect that, by working in a similar setting to the one in [11], it should be possible to build a model of the theory $\mathbf{CZF}^-$+**AFA**, which was extensively studied by M. Rathjen in [13, 14].

As a final remark, we would like to point out that the present results fit in the general picture described by the two present authors in [6]. (Incidentally, we expect that, together with the results on sheaves therein, they should yield an answer to the question by Joyal and Moerdijk which we quoted in opening this introduction.) There, we suggested that the established connection between Martin-Löf type theory, constructive set theory and the theory of ΠW-pretoposes had an analogous version in the case of non-well-founded structures. While trying to make the correspondence between the categorical and the set theoretical sides of the picture precise, it turned out that the M-types in ΠM-pretoposes are not necessary in order to obtain a model of some non-well-founded set theory. This phenomenon resembles the situation in [10], where Lindström built a model of $\mathbf{CZF}^{-}$+$\mathbf{AFA}$ out of a Martin-Löf type theory with one universe, without making any use of M-types.

2 Generating objects in indexed categories

As we mentioned before, our aim is to prove a final coalgebra theorem for a special class of functors on finitely complete and cocomplete categories. The proof of such results will be carried out by repeating in the internal language of such a category $\mathcal{C}$ a classical set-theoretic argument. This forces us to consider $\mathcal{C}$ as an indexed category, via its canonical indexing $\mathbb{C}$, whose fibre over an object X is the slice category $\mathcal{C}/X$. We shall then focus on endofunctors on $\mathcal{C}$ which are components over 1 of indexed endofunctors on $\mathbb{C}$. For such functors, we shall prove the existence of an indexed final coalgebra, under suitable assumptions. The component over 1 of this indexed final coalgebra will be the final coalgebra of the original $\mathcal{C}$-endofunctor.

Although the setting in which we shall work is rather specific, it turns out that all the basic machinery needed for the proof can be stated in a more general context. This section collects as much of the indexed category theoretic material as possible, hoping to leave the other sections easier to read for a less experienced reader.

So, for this section, $\mathcal{S}$ will be a cartesian category, which we use as a base for indexing. Our notations for indexed categories and functors follow those of [8, Chapters B1 and B2], to which we refer the reader for all the relevant definitions.

We will mostly be concerned with *$\mathcal{S}$-cocomplete categories*, i.e. $\mathcal{S}$-indexed categories in which each fibre is finitely cocomplete, finite colimits are preserved by reindexing functors, and these have left adjoints satisfying the Beck-Chevalley condition. Under these assumptions it immediately follows that:

Lemma 2.1 *If the fibre $\mathcal{C} = \mathcal{C}^1$ of an $\mathcal{S}$-cocomplete $\mathcal{S}$-indexed category $\mathbb{C}$ has a terminal object T, then this is an indexed terminal object, i.e. X^*T is terminal in $\mathcal{C}^X$ for all X in $\mathcal{S}$.*

The first step in the set-theoretic argument to build the final coalgebra is to identify a "generating family" of coalgebras, in the sense that any other coalgebra is the colimit of all coalgebras in that family that map to it. If we want to express this in the internal language, we need to introduce the concept of internal colimits in indexed categories. To this end, we first recall that an *internal category* $\mathbb{K}$ in $\mathcal{S}$ consists of a diagram

$$K_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} K_0,$$

where d_1 is the *domain* map, d_0 is the *codomain* one and they have a common left inverse i, satisfying the usual conditions. There is also a notion of internal functor between internal categories, and this gives rise to the category of internal categories in $\mathcal{S}$ (see [8, Section B2.3] for the details).

An *internal diagram* L *of shape* $\mathbb{K}$ in an $\mathcal{S}$-indexed category $\mathbb{C}$ consists of an internal $\mathcal{S}$-category $\mathbb{K}$, together with an object L in $\mathcal{C}^{K_0}$, together with a map $d_1^* L \longrightarrow d_0^* L$ in $\mathcal{C}^{K_1}$ which interacts properly with the categorical structure of $\mathbb{K}$. Moreover, one can consider the notion of *morphism of internal diagrams*, and these data define the category $\mathbb{C}^{\mathbb{K}}$ of *internal diagrams of shape* $\mathbb{K}$ *in* $\mathbb{C}$.

An indexed functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ induces an ordinary functor $F^{\mathbb{K}} : \mathbb{C}^{\mathbb{K}} \longrightarrow \mathbb{D}^{\mathbb{K}}$ between the corresponding categories of internal diagrams of shape $\mathbb{K}$. Dually, given an internal functor $F : \mathbb{K} \longrightarrow \mathbb{J}$, this (contravariantly) determines by reindexing of $\mathbb{C}$ an ordinary functor on the corresponding categories of internal diagrams: $F^* : \mathbb{C}^{\mathbb{J}} \longrightarrow \mathbb{C}^{\mathbb{K}}$. We say that $\mathbb{C}$ has *internal left Kan extensions* if these reindexing functors have left adjoints, denoted by Lan_F. In the particular case where $\mathbb{J} = 1$, the trivial internal category with one object, we write $\mathbb{K}^* : \mathbb{C} \longrightarrow \mathbb{C}^{\mathbb{K}}$ for the functor, and $\mathrm{colim}_{\mathbb{K}}$ for its left adjoint $\mathsf{Lan}_{\mathbb{K}}$, and we call $\mathrm{colim}_{\mathbb{K}} L$ the *internal colimit* of L.

Definition 2.2 Suppose $\mathbb{C}$ and $\mathbb{D}$ are $\mathcal{S}$-indexed categories with internal colimits of shape $\mathbb{K}$. Then, we say that an $\mathcal{S}$-indexed functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ *preserves colimits* if the canonical natural transformation filling the square

$$\begin{tikzcd}
\mathbb{C}^{\mathbb{K}} \arrow[r, "F^{\mathbb{K}}"] \arrow[d, "\mathrm{colim}_{\mathbb{K}}"'] & \mathbb{D}^{\mathbb{K}} \arrow[d, "\mathrm{colim}_{\mathbb{K}}"] \\
\mathbb{C} \arrow[r, "F"'] & \mathbb{D}
\end{tikzcd}$$

is an isomorphism.

It follows at once from Proposition B2.3.20 in [8] that:

Proposition 2.3 *If $\mathbb{C}$ is an $\mathcal{S}$-cocomplete $\mathcal{S}$-indexed category, then it has colimits of internal diagrams and left Kan extensions along internal functors in $\mathcal{S}$. Moreover, if an indexed functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ between $\mathcal{S}$-cocomplete categories preserves $\mathcal{S}$-indexed colimits, then it also preserves internal colimits.*

When forming the internal diagram of those coalgebras that map into a given one, say (A, α), we need to select over an object of maps to A. In order to consider such objects of arrows in the internal language, we need to introduce the following concept:

Definition 2.4 An object E in the fibre $\mathcal{C}^U$ of an $\mathcal{S}$-indexed category $\mathbb{C}$ is called *exponentiable* if for any object A in any fibre $\mathcal{C}^I$ there is an *exponential* A^E fitting in a span

$$U \xleftarrow{s} A^E \xrightarrow{t} I \tag{1}$$

in $\mathcal{S}$ and a *generic arrow* $\varepsilon : s^* E \longrightarrow t^* A$ in $\mathcal{C}^{A^E}$, with the following universal property: for any other span in $\mathcal{S}$

$$U \xleftarrow{x} J \xrightarrow{y} I$$

and any arrow $\psi\colon x^*E \longrightarrow y^*A$ in $\mathcal{C}^J$, there is a unique arrow in $\mathcal{S}$ $\chi\colon J \longrightarrow A^E$ such that $s\xi = x$, $t\chi = y$ and $\chi^*\varepsilon \cong \psi$ (via the canonical isomorphisms arising from the two previous equalities).

Remark 2.5 It follows from the definition, via a standard diagram chasing, that the reindexing along an arrow $f\colon V \longrightarrow U$ in $\mathcal{S}$ of an exponentiable object E in $\mathcal{C}^U$ is again exponentiable.

Remark 2.6 We advise the reader to check that, in case $\mathcal{C}$ is a cartesian category and $\mathbb{C}$ is its canonical indexing over itself, the notion of exponentiable object agrees with the standard one of exponentiable map.

Given an exponentiable object E in $\mathcal{C}^U$ and an object A in $\mathcal{C}^I$, the *canonical cocone from E to A* is in the internal language the cocone of those morphisms from E to A. Formally, it is described as the internal diagram $(\mathbb{K}^A, L^A)$, where the internal category $\mathbb{K}^A$ and the diagram object L^A are defined as follows. K_0^A is the object A^E, with arrows s and t as in (1), and K_1^A is the pullback

$$\begin{tikzcd} K_1^A \arrow[r, "d_0"] \arrow[d, "x"'] & K_0^A \arrow[d, "s"] \\ E^E \arrow[r, "\bar{t}"'] & U, \end{tikzcd}$$

where

$$U \xleftarrow{\;\bar{s}\;} E^E \xrightarrow{\;\bar{t}\;} U$$

is the exponential of E with itself. In the fibres over A^E and E^E we have generic maps $\varepsilon\colon s^*E \longrightarrow t^*A$ and $\bar{\varepsilon}\colon \bar{s}^*E \longrightarrow \bar{t}^*E$, respectively.

The codomain map d_0 of $\mathbb{K}^A$ is the top row of the pullback above, whereas the d_1 is induced by the composite

$$(\bar{s}x)^*E \xrightarrow{x^*\bar{\varepsilon}} (\bar{t}x)^*E \cong (sd_0)^*E \xrightarrow{d_0^*\varepsilon} (td_0)^*A$$

via the universal property of A^E and ε.

The internal diagram L^A is now the object s^*E in $\mathcal{C}^{K_0^A}$, and the arrow from $d_1^*L^A$ to $d_0^*L^A$ is (modulo the coherence isomorphisms) $x^*\bar{\varepsilon}$.

When the colimit of the canonical cocone from E to A is A itself, we can think of A as being generated by the maps from E to it. Therefore, it is natural to introduce the following terminology.

Definition 2.7 The object E is called a *generating object* if, for any A in $\mathcal{C} = \mathcal{C}^1$, $A = \mathrm{colim}_{\mathbb{K}^A} L^A$.

Later, we shall see how F-coalgebras form an indexed category. Then, a generating object for this category will provide, in the internal language, a "generating family" of coalgebras. The set-theoretical argument then goes on by taking the coproduct of all coalgebras in that family. This provides a weakly terminal coalgebra. Categorically, the argument translates to the following result.

Proposition 2.8 *Let $\mathbb{C}$ be an $\mathcal{S}$-cocomplete $\mathcal{S}$-indexed category with a generating object E in $\mathcal{C}^U$. Then, $\mathcal{C} = \mathcal{C}^1$ has a weakly terminal object.*

Proof. We build a weakly terminal object in $\mathcal{C}$ by taking the internal colimit of the diagram $(\mathbb{K}, L)$ in $\mathbb{C}$, where $K_0 = U$, $K_1 = E^E$ (with domain and codomain maps being $\overline{s}$ and $\overline{t}$, respectively), $L = E$ and the map from $d_0^* L$ to $d_1^* L$ is precisely $\overline{\varepsilon}$.

Given an object $A = \mathrm{colim}_{\mathbb{K}^A} L^A$ in $\mathcal{C}$, notice that the serially commuting diagram

$$\begin{tikzcd} K_1^A \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, "x"'] & K_0^A \arrow[d, "s"] \\ E^E \arrow[r, shift left, "\overline{s}"] \arrow[r, shift right, "\overline{t}"'] & U \end{tikzcd}$$

defines an internal functor $J\colon \mathbb{K}^A \longrightarrow \mathbb{K}$. We have a commuting triangle of internal $\mathcal{S}$-categories

$$\begin{tikzcd} \mathbb{K}^A \arrow[rr, "J"] \arrow[dr] & & \mathbb{K} \arrow[dl] \\ & 1. & \end{tikzcd}$$

Taking left adjoint along the reindexing functors which this induces on categories of internal diagrams, we get that $\mathrm{colim}_{\mathbb{K}^A} \cong \mathrm{colim}_{\mathbb{K}} \circ \mathsf{Lan}_J$. Hence, to give a map from $A = \mathrm{colim}_{\mathbb{K}^A} L^A$ to $Q = \mathrm{colim}_{\mathbb{K}} L$ it is sufficient to give a morphism of internal diagrams from $(\mathbb{K}, \mathsf{Lan}_J L^A)$ to $(\mathbb{K}, L)$, or, equivalently, from $(\mathbb{K}^A, L^A)$ to $(\mathbb{K}^A, J^* L)$, but the reader can easily check that these two diagrams are in fact the same. $\square$

Once the coproduct of coalgebras in the "generating family" is formed, the set-theoretic argument is concluded by quotienting it by its largest bisimulation. One way to build such a bisimulation constructively is to identify a generating family of bisimulations and then taking their coproduct.

This suggests that we apply Proposition 2.8 twice; first in the indexed category of coalgebras, in order to obtain a weakly terminal coalgebra (G, γ), and then in the (indexed) category of bisimulations over (G, γ). To this end, we need to prove cocompleteness and existence of a generating object for these categories. The language of inserters allows us to do that in a uniform way.

Instead of giving the general definition of an inserter in a 2-category, we describe it explicitly here for the 2-category of $\mathcal{S}$-indexed categories.

Definition 2.9 Given two $\mathcal{S}$-indexed categories $\mathbb{C}$ and $\mathbb{D}$ and two parallel $\mathcal{S}$-indexed functors $F, G\colon \mathbb{C} \longrightarrow \mathbb{D}$, the *inserter* $\mathbb{I} = \mathbb{I}\mathrm{ns}(F, G)$ *of* F *and* G has as fibre $\mathcal{I}^X$ the category whose objects are pairs (A, α) consisting of an object A in $\mathcal{C}^X$ and an arrow in $\mathcal{D}^X$ from $F^X A$ to $G^X A$, an arrow $\phi\colon (A, \alpha) \longrightarrow (B, \beta)$ being a map $\phi\colon A \longrightarrow B$ in $\mathcal{C}^X$ such that $G^X(\phi)\alpha = \beta F^X(\phi)$.

The reindexing functor for a map $f: Y \longrightarrow X$ in $\mathcal{S}$ takes an object (A, α) in $\mathcal{I}^X$ to the object $(f^*A, f^*\alpha)$, where $f^*\alpha$ has to be read modulo the coherence isomorphisms of $\mathbb{D}$, but we shall ignore these thoroughly.

There is an indexed *forgetful* functor $U: \mathbb{I}\text{ns}(F, G) \longrightarrow \mathbb{C}$ which takes a pair (A, α) to its carrier A; the maps α determine an indexed natural transformation $FU \longrightarrow GU$. The triple $(\mathbb{I}\text{ns}(F, G), U, FU \longrightarrow GU)$ has a universal property, like any good categorical construction, but we will not use it in this paper. The situation is depicted as below:

$$\mathbb{I}\text{ns}(F, G) \xrightarrow{U} \mathbb{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{D} \tag{2}$$

A tedious but otherwise straightforward computation, yields the proof of the following:

Lemma 2.10 *Given an inserter as in (2), if $\mathbb{C}$ and $\mathbb{D}$ are $\mathcal{S}$-cocomplete and F preserves indexed colimits, then $\mathbb{I}\text{ns}(F, G)$ is $\mathcal{S}$-cocomplete and U preserves colimits (in other words, U creates colimits). In particular, $\mathbb{I}\text{ns}(F, G)$ has all internal colimits, and U preserves them.*

Example 2.11 We shall be interested in two particular inserters, during our work. One is the indexed category $F - \mathbb{C}\text{oalg}$ of coalgebras for an indexed endofunctor F on $\mathbb{C}$, which can be presented as the inserter

$$\mathbb{I}\text{ns}(\mathsf{Id}, F) \xrightarrow{U} \mathbb{C} \underset{F}{\overset{\mathsf{Id}}{\rightrightarrows}} \mathbb{C}. \tag{3}$$

More concretely, $(F - \mathbb{C}\text{oalg})^I = F^I - \text{coalg}$ consists of pairs (A, α) where A is an object and $\alpha: A \longrightarrow F^I A$ a map in $\mathcal{C}^I$, and morphisms from such an (A, α) to a pair (B, β) are morphisms $\phi: A \longrightarrow B$ in $\mathcal{C}^I$ such that $F^I(\phi)\alpha = \beta\phi$. The reindexing functors are the obvious ones.

The other inserter we shall need is the indexed category $\mathbb{S}\text{pan}(M, N)$ of spans over two objects M and N in $\mathcal{C}^1$ of an indexed category. This is the inserter

$$\mathbb{I}\text{ns}(\Delta, \langle M, N\rangle) \xrightarrow{U} \mathbb{C} \underset{\langle M,N\rangle}{\overset{\Delta}{\rightrightarrows}} \mathbb{C} \times \mathbb{C} \tag{4}$$

Where $\mathbb{C} \times \mathbb{C}$ is the product of $\mathbb{C}$ with itself (which is defined fibrewise), Δ is the diagonal functor (also defined fibrewise), and $\langle M, N\rangle$ is the pairing of the two constant indexed functors determined by M and N. By this we mean that an object in $\mathcal{C}$ is mapped to the pair (M, N) and an object in $\mathcal{C}^X$ is mapped to the pair (X^*M, X^*N).

Remark 2.12 Notice that, in both cases, the forgetful functors preserve $\mathcal{S}$-indexed colimits in $\mathbb{C}$, hence both $F-\mathbb{C}\text{oalg}$ and $\mathbb{S}\text{pan}(M, N)$ are $\mathcal{S}$-cocomplete, and also internally cocomplete, if $\mathbb{C}$ is.

In order to apply Proposition 2.8 to our indexed categories, we will need to find a generating object for them. This will be achieved by means of the following two lemmas.

First of all, consider an $\mathcal{S}$-indexed inserter $\mathbb{I} = \mathbb{I}\mathrm{ns}(F, G)$ as in (2), such that F preserves exponentiable objects. Then, given an exponentiable object E in $\mathcal{C}^U$, we can define an arrow $\overline{U} \xrightarrow{r} U$ in $\mathcal{S}$ and an object $(\overline{E}, \overline{\varepsilon})$ in $\mathcal{I}^{\overline{U}}$, as follows.

We form the generic map $\varepsilon: s^* F^U E \longrightarrow t^* G^U E$ associated to the exponential of $F^U E$ and $G^U E$ (which exists because F preserves exponentiable objects), and then define $\overline{U}$ as the equaliser of the following diagram

$$\overline{U} \xrightarrow{e} (G^U E)^{F^U E} \underset{t}{\overset{s}{\rightrightarrows}} U, \tag{5}$$

the arrow $r: \overline{U} \longrightarrow U$ being one of the two equal composites $se = te$.

We then put $\overline{E} = r^* E$ and

$$\overline{\varepsilon} \;=\; F^{\overline{U}}(r^* E) \xrightarrow{\cong} e^* s^* F^U E \xrightarrow{e^* \varepsilon} e^* t^* G^U E \xrightarrow{\cong} G^{\overline{U}}(r^* E).$$

The pair $(\overline{E}, \overline{\varepsilon})$ defines an object in $\mathcal{I}^{\overline{U}}$.

Lemma 2.13 *The object $(\overline{E}, \overline{\varepsilon})$ is exponentiable in $\mathbb{I}\mathrm{ns}(F, G)$.*

Proof. Consider an object (A, α) in a fibre $\mathcal{I}^X$. Then, we define the exponential $(A, \alpha)^{(\overline{E}, \overline{\varepsilon})}$ as follows.

First, we build the exponential

$$\overline{U} \xleftarrow{s} A^{\overline{E}} \xrightarrow{t} X$$

of A and $\overline{E}$ in $\mathbb{C}$, with generic map $\chi: s^* \overline{E} \longrightarrow t^* A$. Because F preserves exponentiable objects, we can also form the exponential in $\mathbb{D}$

$$\overline{U} \xleftarrow{\overline{s}} G^X A^{F^{\overline{U}} \overline{E}} \xrightarrow{\overline{t}} X$$

with generic map $\overline{\chi}: \overline{s}^* F^{\overline{U}} \overline{E} \longrightarrow \overline{t}^* G^X A$. By the universal property of $\overline{\chi}$, the two composites in $\mathcal{D}^{A^{\overline{E}}}$

$$s^* F^{\overline{U}} \overline{E} \xrightarrow{\cong} F^{A^{\overline{E}}} s^* \overline{E} \xrightarrow{F^{A^{\overline{E}}} \chi} F^{A^{\overline{E}}}(t^* A) \xrightarrow{\cong} t^* F^X A \xrightarrow{t^* \alpha} t^* G^X A$$

and

$$s^* F^{\overline{U}} \overline{E} \xrightarrow{s^* \overline{\varepsilon}} s^* G^{\overline{U}} \overline{E} \xrightarrow{\cong} G^{A^{\overline{E}}} s^* \overline{E} \xrightarrow{G^{A^{\overline{E}}} \chi} G^{A^{\overline{E}}} t^* A \xrightarrow{\cong} t^* G^X A$$

give rise to two maps in $\mathcal{S}$ $p_1, p_2: A^{\overline{E}} \longrightarrow G^X A^{F^{\overline{U}} \overline{E}}$, whose equaliser i has as domain the exponential $(A, \alpha)^{(\overline{E}, \overline{\varepsilon})}$.

The generic map $(si)^*(\overline{E},\overline{\varepsilon})\longrightarrow(ti)^*(A,\alpha)$ in $\mathcal{I}^{(A,\alpha)^{(\overline{E},\overline{\varepsilon})}}$ associated to this exponential forms the central square of the following diagram, which commutes because the sides if its inner square are the reindexing along the maps $p_1 i = p_2 i$ of the generic map $\overline{\chi}$ above:

$$\begin{tikzcd}
(si)^* F^{\overline{U}}\overline{E} \arrow[r, "(si)^*\overline{\varepsilon}"] \arrow[d, "\cong"'] & (si)^* G^{\overline{U}}\overline{E} \arrow[d, "\cong"] \\
F^{(A,\alpha)^{(\overline{E},\overline{\varepsilon})}}(si)^*\overline{E} \arrow[r, "(si)^*(\overline{E},\overline{\varepsilon})"] \arrow[d, "F^{(A,\alpha)^{(\overline{E},\overline{\varepsilon})}} i^*\chi"'] & G^{(A,\alpha)^{(\overline{E},\overline{\varepsilon})}}(si)^*\overline{E} \arrow[d, "G^{(A,\alpha)^{(\overline{E},\overline{\varepsilon})}} i^*\chi"] \\
F^{(A,\alpha)^{(\overline{E},\overline{\varepsilon})}}(ti)^* A \arrow[r, "(ti)^*(A,\alpha)"] \arrow[d, "\cong"'] & G^{(A,\alpha)^{(\overline{E},\overline{\varepsilon})}}(ti)^* A \arrow[d, "\cong"] \\
(ti)^* F^X A \arrow[r, "(ti)^*\alpha"'] & (ti)^* G^X A.
\end{tikzcd}$$

The verification of its universal property is a lengthy but straightforward exercise. $\square$

Next, we find a criterion for the exponentiable object $(\overline{E},\overline{\varepsilon})$ to be generating.

Lemma 2.14 *Consider an inserter of $\mathcal{S}$-indexed categories as in (2), where $\mathbb{C}$ and $\mathbb{D}$ are $\mathcal{S}$-cocomplete, and F preserves $\mathcal{S}$-indexed colimits. If $(\overline{E},\overline{\varepsilon})$ is an exponentiable object in $\mathcal{I}^{\overline{U}}$ and for any (A,α) in $\mathcal{I}^1$ the equation*

$$\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} UL^{(A,\alpha)} \cong U(A,\alpha) = A$$

holds, where $(\mathbb{K}^{(A,\alpha)}, L^{(A,\alpha)})$ is the canonical cocone from $(\overline{E},\overline{\varepsilon})$ to (A,α), then $(\overline{E},\overline{\varepsilon})$ is generating in $\mathbb{I}\mathrm{ns}(F,G)$.

Proof. Recall from Lemma 2.10 that $\mathbb{I}\mathrm{ns}(F,G)$ is internally cocomplete and the forgetful functor $U\colon \mathbb{I}\mathrm{ns}(F,G)\longrightarrow\mathbb{C}$ preserves internal colimits. Therefore, given an arbitrary object (A,α) in $\mathcal{I}^1$, we can always form the colimit $(B,\beta) = \mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} L^{(A,\alpha)}$. All we need to show is that $(B,\beta)\cong(A,\alpha)$. The isomorphism between B and A exists because, by the assumption,

$$B = U(B,\beta) = U\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} L^{(A,\alpha)} \cong \mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} UL^{(A,\alpha)} \cong A.$$

Now, it is not too hard to show that the transpose of the composite

$$\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} F^{\overline{U}} U^{\overline{U}} L^{(A,\alpha)} = FU\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} L^{(A,\alpha)} \xrightarrow{\beta} GU\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} L^{(A,\alpha)}$$

is (modulo isomorphisms preserved through the adjunction $\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} \dashv \mathbb{K}^{(A,\alpha)^*}$) the transpose of α. Hence, $\beta\cong\alpha$ and we are done. $\square$

As an example, we can show the following result about the indexed category of spans:

Proposition 2.15 *Given an $\mathcal{S}$-cocomplete indexed category $\mathbb{C}$ and two objects M and N in $\mathcal{C}^1$, if $\mathbb{C}$ has a generating object, then so does the indexed category of spans $\mathbb{P} = \mathbb{S}\mathrm{pan}(M, N)$.*

Proof. Recall from Example 2.11 that the functor $U: \mathbb{S}\mathrm{pan}(M, N) \longrightarrow \mathbb{C}$ creates indexed and internal colimits. If E in $\mathcal{C}^U$ is a generating object for $\mathbb{C}$, then, by Lemma 2.13 we can build an exponentiable object

$$(\overline{E}, \overline{\varepsilon}) = M \xleftarrow{\overline{\varepsilon}_1} \overline{E} \xrightarrow{\overline{\varepsilon}_2} N$$

in $\mathcal{P}^{\overline{U}}$. We are now going to prove that $\mathbb{S}\mathrm{pan}(M, N)$ meets the requirements of Lemma 2.14 to show that $\overline{E}$ is a generating object.

To this end, consider a span

$$(A, \alpha) = M \xleftarrow{\alpha_1} A \xrightarrow{\alpha_2} N$$

in $\mathcal{P}^1$. Then, we can form the canonical cocone $(\mathbb{K}^{(A,\alpha)}, L^{(A,\alpha)})$ from $(\overline{E}, \overline{\varepsilon})$ to (A, α) in $\mathbb{S}\mathrm{pan}(M, N)$, and the canonical cocone $(\mathbb{K}^A, L^A)$ from E to A in $\mathbb{C}$. The map $r: \overline{U} \longrightarrow U$ of (5) induces an internal functor $u: \mathbb{K}^{(A,\alpha)} \longrightarrow \mathbb{K}^A$, which is an isomorphism. Therefore, the induced reindexing functor $u^*: \mathbb{C}^{\mathbb{K}^A} \longrightarrow \mathbb{C}^{\mathbb{K}^{(A,\alpha)}}$ between the categories of internal diagrams in $\mathbb{C}$ is also an isomorphism, and hence $\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} u^* \cong \mathrm{colim}_{\mathbb{K}^A}$. Moreover, it is easily checked that $u^* L^A = U L^{(A,\alpha)}$. Therefore, we have

$$\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} U L^{(A,\alpha)} \cong \mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} u^* L^A \cong \mathrm{colim}_{\mathbb{K}^A} L^A \cong A$$

and this finishes the proof. $\square$

3 Final coalgebra theorems

In this section, we are going to use the machinery of Section 2 in order to prove an indexed final coalgebra theorem. We then introduce the notion of a class of small maps for a Heyting pretopos with an (indexed) natural number object, and apply the theorem in order to derive existence of final coalgebras for various functors in this context. In more detail, we shall show that every small map has an M-type, and that the functor $\mathcal{P}_s$ has a final coalgebra.

3.1 An indexed final coalgebra theorem

In this section, $\mathcal{C}$ is a category with finite limits and stable finite colimits (that is, its canonical indexing $\mathbb{C}$ is a $\mathcal{C}$-cocomplete $\mathcal{C}$-indexed category), and F is an indexed endofunctor over it (we shall write F for F^1). Recall from Remark 2.12 that the indexed category $F\text{−}\mathbb{C}\mathrm{oalg}$ is $\mathcal{C}$-cocomplete (and the indexed forgetful functor U preserves indexed colimits).

We say that F is *small-based* whenever there is an exponentiable object (E, ε) in F^U – coalg such that, for any other F-coalgebra (A, α), the canonical cocone $(\mathbb{K}^{(A,\alpha)}, L^{(A,\alpha)})$ from (E, ε) to (A, α) has the property that

$$\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} UL^{(A,\alpha)} \cong U(A, \alpha) = A. \tag{6}$$

It is immediate from Example 2.11 and Lemma 2.14 that, whenever there is a pair (E, ε) making F small-based, this is automatically a generating object in F – $\mathbb{C}$oalg. We shall make an implicit use of this generating object in the proof of:

Theorem 3.1 *Let F be a small-based indexed endofunctor on a category $\mathcal{C}$ as above. If F^1 takes pullbacks to weak pullbacks, then F has an indexed final coalgebra.*

Before giving a proof, we need to introduce a little technical lemma:

Lemma 3.2 *If $F = F^1$ turns pullbacks into weak pullbacks, then every pair of arrows*

$$(A, \alpha) \xrightarrow{\phi} (C, \gamma) \xleftarrow{\psi} (B, \beta)$$

can be completed to a commutative square by the arrows

$$(A, \alpha) \xleftarrow{\mu} (P, \chi) \xrightarrow{\nu} (B, \beta)$$

in such a way that the underlying square in $\mathcal{C}$ is a pullback. Moreover, if ψ is a coequaliser in $\mathcal{C}$, then so is μ.

Proof. We build P as the pullback of ψ and ϕ in $\mathcal{C} = \mathcal{C}^1$. Then, since F turns pullbacks into weak pullbacks, there is a map $\chi\colon P \longrightarrow FP$, making both μ and ν into coalgebra morphisms. The second statement follows at once by the assumption that finite colimits in $\mathcal{C}$ are stable. □

Proof of Theorem 3.1. Because F – $\mathbb{C}$oalg is $\mathcal{C}$-cocomplete, it is enough, by Lemma 2.1, to show that the fibre over 1 of this indexed category admits a terminal object.

Given that (E, ε) is a generating object in F – $\mathbb{C}$oalg, Proposition 2.8 implies the existence of a weakly terminal F-coalgebra (G, γ). The classical argument now goes on taking the quotient of (G, γ) by the maximal bisimulation on it, in order to obtain a terminal coalgebra. We do that as follows. Let $\mathbb{B} = \mathbb{S}\mathrm{pan}((G, \gamma), (G, \gamma))$ be the indexed category of spans over (G, γ), i.e. bisimulations. Then, by Remark 2.12, $\mathbb{B}$ is a $\mathcal{C}$-cocomplete $\mathcal{C}$-indexed category, and by Proposition 2.15 it has a generating object. Applying again Proposition 2.8, we get a weakly terminal span (i.e. a weakly terminal bisimulation)

$$(G, \gamma) \xleftarrow{\lambda} (B, \beta) \xrightarrow{\rho} (G, \gamma).$$

We now want to prove that the coequaliser

$$
\begin{tikzcd}
(B,\beta) \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\rho"'] & (G,\gamma) \arrow[r, "q"] & (T,\tau)
\end{tikzcd}
$$

is a terminal F-coalgebra.

It is obvious that (T,τ) is weakly terminal, since (G,γ) is. On the other hand, suppose (A,α) is an F-coalgebra and $f,g\colon (A,\alpha)\longrightarrow(T,\tau)$ are two coalgebra morphisms; then, by Lemma 3.2, the pullback s (resp. t) in $\mathcal{C}$ of q along f (resp. g) is a coequaliser in $\mathcal{C}$, which carries the structure of a coalgebra morphism into (A,α). One further application of Lemma 3.2 to s and t yields a commutative square in $F-$coalg

$$
\begin{tikzcd}
(P,\pi) \arrow[r, "s'"] \arrow[d, "t'"'] & \bullet \arrow[d, "t"] \\
\bullet \arrow[r, "s"'] & (A,\alpha)
\end{tikzcd}
$$

whose underlying square in $\mathcal{C}$ is a pullback. Furthermore, the composite $d = ts' = st'$ is a regular epi in $\mathcal{C}$, hence an epimorphism in $F-$coalg.

Write $\widetilde{s}$ (resp. $\widetilde{t}$) for the composite of t' (resp. s') with the projection of the pullback of f (resp. g) and q to G. Then, the triple $((P,\pi),\widetilde{s},\widetilde{t})$ is a span over (G,γ); hence, there is a morphism of spans

$$\chi\colon((P,\pi),\widetilde{s},\widetilde{t})\longrightarrow((B,\beta),\lambda,\rho).$$

It is now easy to compute that $fd = q\lambda\chi = q\rho\chi = gd$, hence $f = g$, and the proof is complete. $\square$

3.2 Small maps

We are now going to consider on $\mathcal{C}$ a class of *small maps*. This will allow us to show that certain polynomial functors, as well as the powerclass functor, are small-based, and therefore we will be able to apply Theorem 3.1 to obtain a final coalgebra for them.

From now on, $\mathcal{C}$ will denote a Heyting pretopos with an *(indexed) natural number object*. That is, an object $\mathbb{N}$, together with maps $0\colon 1\longrightarrow\mathbb{N}$ and $s\colon\mathbb{N}\longrightarrow\mathbb{N}$ such that, for any object P and any pair of arrows $f\colon P\longrightarrow Y$ and $t\colon P\times Y\longrightarrow Y$, there is a unique arrow $\overline{f}\colon P\times\mathbb{N}\longrightarrow Y$ such that the following commutes:

$$
\begin{tikzcd}
P\times 1 \arrow[r, "\mathsf{id}\times 0"] \arrow[d, "\cong"'] & P\times\mathbb{N} \arrow[r, "\mathsf{id}\times s"] \arrow[d, "\langle p_1,\overline{f}\rangle"] & P\times\mathbb{N} \arrow[d, "\overline{f}"] \\
P \arrow[r, "\langle\mathsf{id},f\rangle"'] & P\times Y \arrow[r, "t"'] & Y.
\end{tikzcd}
$$

It then follows that each slice $\mathcal{C}/X$ has a natural number object $X\times\mathbb{N}\longrightarrow X$ in the usual sense. Notice that such categories have all finite colimits, and these are stable under pullback.

A class $\mathcal{S}$ of arrows in $\mathcal{C}$ is called a class of *small maps* if it satisfies the following axioms:

(S1) $\mathcal{S}$ is closed under composition and identities;

(S2) if in a pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & B \arrow[d, "f"] \\ C \arrow[r] & D \end{tikzcd}$$

$f \in \mathcal{S}$, then $g \in \mathcal{S}$;

(S3) for every object C in $\mathcal{C}$, the diagonal $\Delta_C\colon C \longrightarrow C \times C$ is in $\mathcal{S}$;

(S4) given an epi $e\colon C \longrightarrow D$ and a commutative triangle

$$\begin{tikzcd} C \arrow[rr, "e"] \arrow[dr, bend right, "f"'] & & D \arrow[dl, bend left, "g"] \\ & A, & \end{tikzcd}$$

if f is in $\mathcal{S}$, then so is g;

(S5) if $f\colon C \longrightarrow A$ and $g\colon D \longrightarrow A$ are in $\mathcal{S}$, then so is their copairing

$$[f, g]\colon C + D \longrightarrow A.$$

We say that an arrow in $\mathcal{S}$ is *small.* A *small relation* between objects A and B is a subobject $R \rightarrowtail A \times B$ such that its composite with the projection on B is small. We shall say that X is a *small object* if the unique map $X \longrightarrow 1$ is small. A *small subobject* R of an object A is a subobject $R \rightarrowtail A$ in which R is small.

On a class of small maps, we also require representability of small relations by means of a *powerclass* object:

(P1) for any object C in $\mathcal{C}$ there is an object $\mathcal{P}_s(C)$ and a natural correspondence between maps $I \longrightarrow \mathcal{P}_s(C)$ and small relations between I and C.

In particular, the identity on $\mathcal{P}_s(C)$ determines a small relation $\in_C \subseteq \mathcal{P}_s(C) \times C$. We think of $\mathcal{P}_s(C)$ as the object of all small subobjects of C; the relation $\in_C$ then becomes the elementhood relation between elements of C and small subobjects of C. The association $C \mapsto \mathcal{P}_s(C)$ defines a covariant functor (in fact, a monad) on $\mathcal{C}$. We further require the two following axioms:

(I) The natural number object $\mathbb{N}$ is small;

(R) There exists a *universal small map* $\pi\colon E \longrightarrow U$ in $\mathcal{C}$, such that any other small map $f\colon A \longrightarrow B$ fits in a diagram

$$\begin{tikzcd} A \arrow[d, "f"'] & \bullet \arrow[l] \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ B & \bullet \arrow[l, "q"] \arrow[r] & U \end{tikzcd}$$

where both squares are pullbacks and q is epi.

It can now be proved that a class $\mathcal{S}$ satisfying these axioms induces a class of small maps on each slice $\mathcal{C}/C$. Moreover, the reindexing functor along a small map $f\colon C \longrightarrow D$ has a right adjoint $\Pi_f\colon \mathcal{C}/C \longrightarrow \mathcal{C}/D$. In particular, it follows that all small maps are exponentiable in $\mathcal{C}$.

Remark 3.3 The axioms that we have chosen for our class of small maps subsume all of the Joyal-Moerdijk axioms in [9, pp. 6–8], except for the collection axiom (**A7**). In particular, the Descent Axiom (**A3**) can be seen to follow from axioms (**S1**) – (**S5**) and (**P**).

Conversely, the axioms of Joyal and Moerdijk imply all of our axioms except for (**S3**) and (**I**). Our results in Section 4 will imply that, by adding these axioms, a model of the weak set theory $\mathbf{CZF}_0$ can be obtained in the setting of [9].

3.3 Final coalgebras in categories with small maps

From now on, we shall consider on $\mathcal{C}$ a class of small maps $\mathcal{S}$. Using their properties, we are now going to prove the existence of the M-type for every small map $f\colon D \longrightarrow C$, as well as the existence of a final $\mathcal{P}_s$-coalgebra.

Let us recall from [6] that an exponentiable map $f\colon D \longrightarrow C$ in a cartesian category $\mathcal{C}$ induces on it a *polynomial endofunctor* P_f, defined by

$$P_f(X) = \sum_{c \in C} X^{D_c}.$$

Its final coalgebra, when it exists, is called the *M-type* associated to f. In fact, the functor P_f is the component over 1 of an *indexed polynomial endofunctor*, still denoted by P_f, which can be presented as the composite $P_f = \Sigma_C \Pi_f D^*$ of three indexed functors. It is therefore immediate that P_f preserves pullbacks. Of course, the *indexed M-type* of f is the indexed final coalgebra of P_f.

Theorem 3.4 *If $f\colon D \longrightarrow C$ is a small map in $\mathcal{C}$, then f has an (indexed) M-type.*

Proof. In order to obtain an (indexed) final P_f-coalgebra, we want to apply Theorem 3.1, and for this, what remains to be checked is that P_f is small-based. To this end, we first need to find an exponentiable coalgebra $(\overline{E}, \overline{\varepsilon})$, and then to verify condition (6).

The universal small map $\pi\colon E \longrightarrow U$ in $\mathcal{C}$ is exponentiable, as we noticed after the presentation of axiom **(R)**. Hence, unwinding the construction preceding Lemma 2.13, we obtain an exponentiable object in P_f–$\mathbb{C}$oalg. Using the internal language of $\mathcal{C}$, we can describe $(\overline{E}, \overline{\varepsilon})$ as follows.

The object $\overline{U}$ on which $\overline{E}$ lives is described as

$$\overline{U} = \{(u \in U, t\colon E_u \longrightarrow P_f(E_u))\},$$

and $\overline{E}$ is now defined as

$$\overline{E} = \{(u \in U, t\colon E_u \longrightarrow P_f(E_u), e \in E_u)\}.$$

The coalgebra structure $\overline{\varepsilon}\colon \overline{E} \longrightarrow P_f^{\overline{U}}\overline{E}$ takes a triple (u, t, e) (with $te = (c, r)$) to the pair $(c, s\colon D_c \longrightarrow \overline{E})$, where the map s takes an element $d \in D_c$ to the triple $(u, t, r(d))$.

Given a coalgebra (A, α), the canonical cocone from $(\overline{E}, \overline{\varepsilon})$ to it takes the following form. The internal category $\mathbb{K}^{(A,\alpha)}$ is given by

$$\begin{array}{rcl} K_0^{(A,\alpha)} & = & \{(u \in U, t\colon E_u \to P_f(E_u), m\colon E_u \to A) \mid P_f(m)t = \alpha m\}; \\ K_1^{(A,\alpha)} & = & \{(u, t, m, u', t', m', \phi\colon E_u \to E_{u'}) \mid (u, t, m), (u', t', m') \in K_0^{(A,\alpha)}, \\ & & \quad t'\phi = P_f(\phi)t \text{ and } m'\phi = m\}. \end{array}$$

(Notice that, in writing the formulae above, we have used the functor P_f in the internal language of $\mathcal{C}$; we can safely do that because the functor is indexed. We shall implicitly follow the same reasoning in the proof of Theorem 4.4 below, in order to build an (indexed) final $\mathcal{P}_s$-coalgebra.)

The diagram $L^{(A,\alpha)}$ is specified by a coalgebra over $K_0^{(A,\alpha)}$, but for our purposes we only need to consider its carrier, which is

$$UL^{(A,\alpha)} = \{(u, t, m, e) \mid (u, t, m) \in K_0^{(A,\alpha)} \text{ and } e \in E_u\}.$$

Condition (6) says that the colimit of this internal diagram in $\mathcal{C}$ is A, but this is implied by the conjunction of the two following statements, which we are now going to prove:

1. For all $a \in A$ there exists $(u, t, m, e) \in UL^{(A,\alpha)}$ such that $me = a$;
2. If (u_0, t_0, m_0, e_0) and (u_1, t_1, m_1, e_1) are elements of $UL^{(A,\alpha)}$ such that $m_0e_0 = m_1e_1$, then there exist $(u, t, m, e) \in UL^{(A,\alpha)}$ and coalgebra morphisms $\phi_i\colon E_u \longrightarrow E_{u_i}$ $(i = 0, 1)$ such that $m_i\phi_i = m$ and $\phi_i e = e_i$.

Condition 2) is trivial: given (u_0, t_0, m_0, e_0) and (u_1, t_1, m_1, e_1), Lemma 3.2 allows us to fill a square

$$\begin{tikzcd} (P, \gamma) \arrow[r] \arrow[d] & (E_{u_0}, t_0) \arrow[d, "m_0"] \\ (E_{u_1}, t_1) \arrow[r, "m_1"'] & (A, \alpha), \end{tikzcd}$$

in such a way that the underlying square in $\mathcal{C}$ is a pullback (hence, P is a small object). Therefore, (P, γ) is isomorphic to a coalgebra (E_u, t), and, under this isomorphism, the span

$$(E_{u_0}, t_0) \longleftarrow (P, \gamma) \longrightarrow (E_{u_1}, t_1)$$

takes the form

$$(E_{u_0}, t_0) \xleftarrow{\phi_0} (E_u, t) \xrightarrow{\phi_1} (E_{u_1}, t_1).$$

Moreover, since $m_0 e_0 = m_1 e_1$, there is an $e \in E_u$ such that $\phi_i e = e_i$. Then, defining m as any of the two composites $m_i \phi_i$, the element (u, t, m, e) in $UL^{(A,\alpha)}$ satisfies the desired conditions.

As for condition 1), fix an element $a \in A$. We build a subobject $\langle a \rangle$ of A inductively, as follows:

$$\begin{array}{rcl} \langle a \rangle_0 & = & \{a\}; \\ \langle a \rangle_{n+1} & = & \displaystyle\bigcup_{a' \in \langle a \rangle_n} t(D_c) \text{ where } \alpha a' = (c, t\colon D_c \longrightarrow A). \end{array}$$

Then, each $\langle a \rangle_n$ is a small object, because it is a small-indexed union of small objects. For the same reason (since, by axiom **(I)**, $\mathbb{N}$ is a small object) their union $\langle a \rangle = \bigcup_{n \in \mathbb{N}} \langle a \rangle_n$ is small, and it is a subobject of A. It is not hard to see that the coalgebra structure α induces a coalgebra α' on $\langle a \rangle$ (in fact, $\langle a \rangle$ is the smallest subcoalgebra of (A, α) containing a, i.e. the subcoalgebra *generated* by a), and, up to isomorphism, this is a coalgebra $t\colon E_u \longrightarrow P_f E_u$, with embedding $m\colon E_u \longrightarrow A$. Via the isomorphism $E_u \cong \langle a \rangle$, the element a becomes an element $e \in E_u$ such that $me = a$. Hence, we get the desired 4-tuple (u, t, m, e) in $UL^{(A,\alpha)}$.

This concludes the proof of the theorem. $\square$

Theorem 3.5 *The powerclass functor $\mathcal{P}_s$ has an (indexed) final coalgebra.*

Proof. It is easy to check that $\mathcal{P}_s$ is the component on 1 of an indexed functor, and that it maps pullbacks to weak pullbacks.

Therefore, once again, we just need to verify that $\mathcal{P}_s$ is small-based. We proceed exactly like in the proof of Theorem 3.4 above, except for the construction of the coalgebra $(\langle a \rangle, \alpha')$ generated by an element $a \in A$ in 1). Given a $\mathcal{P}_s$-coalgebra (A, α), we construct the subcoalgebra of (A, α) generated by a as follows. First, we define inductively the subobjects

$$\begin{array}{rcl} \langle a \rangle_0 & = & \{a\}; \\ \langle a \rangle_{n+1} & = & \displaystyle\bigcup_{a' \in \langle a \rangle_n} \alpha(a'). \end{array}$$

Each $\langle a \rangle_n$ is a small object, and so is their union $\langle a \rangle = \bigcup_{n \in \mathbb{N}} \langle a \rangle_n$. The coalgebra structure α' is again induced by restriction of α on $\langle a \rangle$. $\square$

4 The final $\mathcal{P}_s$-coalgebra as a model of AFA

Our standing assumption in this section is that $\mathcal{C}$ is a Heyting pretopos with an (indexed) natural number object and a class $\mathcal{S}$ of small maps. In the last section, we proved that in this case the $\mathcal{P}_s$-functor has a final coalgebra in $\mathcal{C}$.

Now we will explain how this final coalgebra can be used to model various set theories with the Anti-Foundation Axiom. First we work out the case for the weak constructive theory $\mathbf{CZF}_0$, and then we indicate how the same method can be applied to obtain models for stronger, better known or classical set theories.

Our presentation of $\mathbf{CZF}_0$ follows that of Aczel and Rathjen in [1]; the same theory appears under the name of **BCST*** in the work of Awodey and Warren in [5]. It is a first-order theory whose underlying logic is intuitionistic; its non-logical symbols are a binary relation symbol ϵ and a constant ω, to be thought of as membership and the set of (von Neumann) natural numbers, respectively. Two more symbols will be added for sake of readability, as we proceed to state the axioms.

The axioms for $\mathbf{CZF}_0$ are (the universal closures) of the following statements:

(Extensionality) $\forall x\, (x \,\epsilon\, a \leftrightarrow x \,\epsilon\, b) \rightarrow a = b$

(Pairing) $\exists t\, (z \,\epsilon\, t \leftrightarrow (z = x \vee z = y))$

(Union) $\exists t\, (z \,\epsilon\, t \leftrightarrow \exists y\, (z \,\epsilon\, y \wedge y \,\epsilon\, x))$

(Emptyset) $\exists x\, (z \,\epsilon\, x \leftrightarrow \bot)$

(Intersection) $\exists t\, (z \,\epsilon\, t \leftrightarrow (z \,\epsilon\, a \wedge z \,\epsilon\, b))$

(Replacement) $\forall x \,\epsilon\, a\, \exists! y\, \phi \rightarrow \exists z\, (y \,\epsilon\, z \leftrightarrow \exists x \,\epsilon\, a\, \phi)$

Two more axioms will be added, but before we do so, we want to point out that all instances of Δ_0-separation follow from these axioms, i.e. we can deduce all instances of

(Δ_0-Separation) $\exists t\, (x \,\epsilon\, t \leftrightarrow (x \,\epsilon\, a \wedge \phi))$

where ϕ is a formula in which t does not occur and all quantifiers are bounded. Furthermore, in view of the above axioms, we can introduce a new constant $\emptyset$ to denote the empty set, and a function symbol s which maps a set x to its "successor" $x \cup \{x\}$. This allows us to formulate concisely our last axioms:

(Infinity-1) $\emptyset \,\epsilon\, \omega \wedge \forall x \,\epsilon\, \omega\, (sx \,\epsilon\, \omega)$

(Infinity-2) $\psi(\emptyset) \wedge \forall x \,\epsilon\, \omega\, (\psi(x) \rightarrow \psi(sx)) \rightarrow \forall x \,\epsilon\, \omega\; \psi(x)$.

It is an old observation by Rieger that models for set theory can be obtained as fixpoints for the powerclass functor (see [15]). In the context of algebraic set theory, a similar result holds (see, for example, [7]), and the same is true here.

Theorem 4.1 *Every $\mathcal{P}_s$-fixpoint in $\mathcal{C}$ provides a model of* $\mathbf{CZF}_0$.

Proof. Suppose we have a fixpoint $E\colon V \longrightarrow \mathcal{P}_s V$, with inverse I. We call y the *name* of a small subobject $A \subseteq V$, when $E(y)$ is its corresponding element in $\mathcal{P}_s(V)$. We interpret the predicate $x \epsilon y$ as an abbreviation of the sentence

$x \in E(y)$ in the internal language of $\mathcal{C}$. Then, the validation of the axioms for $\mathbf{CZF}_0$ goes as follows.

Extensionality holds because two small subobjects $E(x)$ and $E(y)$ of V are equal if and only if, in the internal language of $\mathcal{C}$, $z \in E(x) \leftrightarrow z \in E(y)$. The pairing of two elements x and y represented by two arrows $1 \longrightarrow V$, is given by $I(l)$, where l is the name of the (small) image of their copairing $[x, y]: 1+1 \longrightarrow V$. The union of the sets contained in a set x is interpreted by applying the multiplication of the monad $\mathcal{P}_s$ to $(\mathcal{P}_s E)(E(x))$. The intersection of two elements x and y in V is given by $I(E(x) \cap E(y))$, where the intersection is taken in $\mathcal{P}_s(V)$. The empty set is interpreted by $I(m)$, where m is the name of the (small) subobject $0 \rightarrowtail V$.

For the Replacement axiom, consider a, and suppose that for every $x \in a$ there exists a unique y such that ϕ. Then, the subobject $\{y \mid \exists x \,\epsilon\, a\ \phi\}$ of V is covered by $E(a)$, hence small. Applying I to its name, we get the image of ϕ.

Finally, the Infinity axioms follow from the axiom **(I)**. In fact, the empty set models an arrow $1 \longrightarrow V$, and this, together with the map $s: V \longrightarrow V$ which takes an element x to $x \cup \{x\}$, allows us to apply the definition of an (indexed) natural number object to obtain a morphism $\alpha: \mathbb{N} \longrightarrow V$. Since $\mathbb{N}$ is small, so is the image of α, as a subobject of V, and applying I to its name we get an N in V which validates the axioms Infinity-1 and Infinity-2. $\square$

A final $\mathcal{P}_s$-coalgebra further satisfies the Anti-Foundation Axiom. To formulate this axiom, we define the following notions. A (directed) graph consists of a pair of sets (n, e) such that $n \subseteq e \times e$. A colouring of such a graph is a function c assigning to every node $x \epsilon n$ a set $c(x)$ such that

$$c(x) = \{c(y) \mid (x, y) \,\epsilon\, e\}.$$

This can be formulated solely in terms of ϵ using the standard encoding of pairs and functions. In ordinary set theory (with classical logic and the Foundation Axiom), the only graphs that have a colouring are well-founded trees and these colourings are then necessarily unique.

The Anti-Foundation Axiom says:

(AFA) Every graph has a unique colouring.

Proposition 4.2 *If $\mathcal{C}$ has an (indexed) final $\mathcal{P}_s$-coalgebra, then this is a model for the theory* $\mathbf{CZF}_0$+**AFA**.

Proof. We clearly have to check just **AFA**, since any final coalgebra is a fixpoint. To this end, note first of all that, because (V, E) is an indexed final coalgebra, we can think of it as a final $\mathcal{P}_s$-coalgebra in the internal logic of $\mathcal{C}$.

So, suppose we have a graph (n, e) in V. Then, n (internally) has the structure of a $\mathcal{P}_s$-coalgebra $\nu: n \longrightarrow \mathcal{P}_s n$, by sending a node $x \,\epsilon\, n$ to the (small) set of nodes $y \,\epsilon\, n$ such that $(x, y) \,\epsilon\, e$. The colouring of n is now given by the unique $\mathcal{P}_s$-coalgebra map $\gamma: n \longrightarrow V$. $\square$

By Theorem 3.5, it then follows at once:

Corollary 4.3 *Every Heyting pretopos with a natural number object and class of small maps contains a model of* $\mathbf{CZF}_0$+**AFA**.

We end this paper by explaining how this result can be extended to theories stronger than $\mathbf{CZF}_0$. For example, to the set theory **CST** introduced by Myhill in [12]. This theory is closely related to (in fact, intertranslatable with) $\mathbf{CZF}_0$+**Exp**, where **Exp** is (the universal closure of) the following axiom.

(Exponentiation) $\exists t\, (f \,\epsilon\, t \leftrightarrow \mathrm{Fun}(f, x, y))$

Here, the predicate $\mathrm{Fun}(f, x, y)$ expresses the fact that f is a function from x to y, and it can be formally written as the conjunction of $\forall a \,\epsilon\, x\, \exists! b \,\epsilon\, y\ (a, b) \,\epsilon\, f$ and $\forall z \,\epsilon\, f\, \exists a \,\epsilon\, x, b \,\epsilon\, y\ (z = (a, b))$.

Theorem 4.4 *Assume the class of small maps also satisfies*

(E) *The functor* Π_f *preserves small maps for any* f *in* $\mathcal{S}$.

Then, $\mathcal{C}$ *contains a model of* **CST**+**AFA**.

Proof. We already saw how the final $\mathcal{P}_s$-coalgebra (V, E) models $\mathbf{CZF}_0$+**AFA**. Now, **(E)** implies that A^B is small, if A and B are, so, in $E(y)^{E(x)}$ is always small. This gives rise to a small subobject of V, by considering the image of the morphism that sends a function $f \in E(y)^{E(x)}$ to the element in V representing its graph. The image under I of the name of this small object is the desired exponential t. $\Box$

Another example of a stronger theory which can be obtained by imposing further axioms for small maps is provided by $\mathbf{IZF}^-$, which is intuitionistic **ZF** without the Foundation Axiom. It is obtained by adding to $\mathbf{CZF}_0$ the following axioms:

(Powerset) $\exists y\, \forall z\, (z \,\epsilon\, y \leftrightarrow \forall w \,\epsilon\, z\, (w \,\epsilon\, x))$

(Full Separation) $\exists y\, \forall z\, (z \,\epsilon\, y \leftrightarrow z \,\epsilon\, x \wedge \phi)$

(Collection) $\forall y \,\epsilon\, x\, \exists w\ \phi \rightarrow \exists z\, \forall y \,\epsilon\, x\, \exists w \,\epsilon\, z\ \phi$

(In Full Separation, y is not allowed to occur in ϕ.)

The proof of the following theorem should now be routine (if not, the reader should consult [7]):

Theorem 4.5 *Assume the class of small maps* $\mathcal{S}$ *also satisfies*

(P2) *if* $X \longrightarrow B$ *belongs to* $\mathcal{S}$, *then so does* $\mathcal{P}_s(X \longrightarrow B)$;

(F) *every monomorphism is small;*

(C) *for any two arrows $p: Y \longrightarrow X$ and $f: X \longrightarrow A$ where p is epi and f belongs to $\mathcal{S}$, there exists a quasi-pullback square of the form*

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "g"'] & Y \arrow[r, "p"] & X \arrow[d, "f"] \\ B \arrow[rr, "h"'] & & A \end{tikzcd}$$

where h is epi and g belongs to $\mathcal{S}$.

Then, $\mathcal{C}$ contains a model of $\mathbf{IZF}^{-}+\mathbf{AFA}$.

Corollary 4.6 *If the pretopos $\mathcal{C}$ is Boolean, then classical logic is also true in the model, which will therefore validate* $\mathbf{ZF}^{-}+\mathbf{AFA}$*, Zermelo-Fraenkel set theory with Anti-Foundation instead of Foundation.*

We believe that a similar result should be derivable for Aczel's set theory **CZF** in the setting of [11].

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