THE COMPLETION OF LOCALLY REFINED SIMPLICIAL PARTITIONS CREATED BY BISECTION

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ABSTRACT. Recently, in [Ste05], we proved that an adaptive finite element method based on newest vertex bisection in two space dimensions for solving elliptic equations, which is essentially the method from [*SINUM*, 38 (2000), 466–488] by Morin, Nochetto, and Siebert, converges with the optimal rate. The number of triangles N in the output partition of such a method is generally larger than the number M of triangles that in all intermediate partitions have been marked for bisection, because additional bisections are needed to retain conforming meshes. A key ingredient to our proof was a result from [*Numer. Math.*, 97(2004), 219–268] by Binev, Dahmen and DeVore saying that $N-N_0 \leq CM$ for some absolute constant C, where N_0 is the number of triangles from the initial partition that have never been bisected. In this paper, we extend this result to bisection algorithms of n-simplices, with that generalizing the result concerning optimality of the adaptive finite element method to general space dimensions.

1. Introduction

Nowadays, adaptive finite element methods are a popular tool for the numerical solution of boundary value problems. Compared to non-adaptive finite element methods, they have the potential to achieve the optimal work-accuracy balance allowed by the polynomial degree, under much milder smoothness conditions on the solution of the boundary value problem.

The basic loop of an adaptive finite element method consists of computing the finite element solution with respect to the current partition; computing an a posteriori error estimator, being a sum of local error indicators associated to the individual elements; a marking of those elements for refinement which correspond to the largest error indicators; and finally, the construction of the next partition by refining the marked elements, generally *together with elements in some surrounding* in order to retain structural properties of the partition needed to apply the error estimator in the next iteration. We refer to this refinement of elements in the surrounding of the marked ones as the *completion* of the partition.

In this paper, we confine ourselves to partitions into n-simplices, as basic refinement step we use bisection, and as the structural property of the partition we require *conformity*, meaning that the intersection of any two different simplices in

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the partition is either empty or a common hyperface of both simplices. In this setting, in order to retain conformity, a bisection of a simplex has to be complemented by bisections of some of its neighbours, which in turn may induce bisections of their neighbours and so on. The complexity of this completion process is being studied in this paper. The advantage of the sketched approach is that highly locally refined partitions can be generated, the arising simplices are uniformly shape regular, and that finite element spaces with respect to refined partitions are nested. Alternatively, one may consider non-conforming partitions generated by other refinement strategies. In that case, a valid error estimator will require that the "amount of nonconformity" is bounded, among other things meaning that the number of hanging vertices per element has to be uniformly bounded. So also then refinements cannot be made on a purely individual element basis, and similar questions arise as with the approach studied here.

In [BDD04], considering conforming partitions into triangles generated by the so-called newest vertex bisection rule starting from some fixed initial conforming triangulation, Binev, Dahmen and DeVore showed that the total number of triangles in the partition at termination of the adaptive finite element method is bounded by some absolute multiple of the number of triangles that were marked for refinement in all iterations. In other words, all additional bisections to retain conformity of all intermediate partitions inflate the final number of triangles by not more than a constant factor. In [Ste05], we used this result to prove optimal computational complexity of an adaptive linear finite element method, essentially the method introduced in [MNS00], in the following sense: Whenever for some $s > 0$, the solution can be approximated within a tolerance $\varepsilon > 0$ in energy norm by a continuous piecewise linear function with respect to a partition generated by newest vertex bisection with $\mathcal{O}(\varepsilon^{-1/s})$ triangles, and one knows how to approximate the right-hand side in the dual norm with the same rate with piecewise constants, then this adaptive method produces approximations that converge with this rate, using a number of operations that is of the order of the number of triangles in the output partition. This result can be generalized to higher order elements and/or more than two space dimensions, for the latter generalization *assuming* that the result of Binev, Dahmen and DeVore concerning newest vertex bisection of triangles can be generalized to more space dimensions, which is the topic of this paper.

Bisection of n-simplices has been studied in [Bän91, Kos94, AMP00] for $n = 3$, and in [Mau95, Tra97] for general n, and this work has been inspired by all of these references. In order to be able to generalize the result from [BDD04], it will be important that each uniform refinement of the fixed initial conforming partition is conforming. Here with a uniform refinement, we mean a partition in which all simplices have been created by an equal number of bisections. Conformity of all uniform refinements is not guaranteed for the methods from [Bän91, AMP00]. The other methods require conditions on the initial partition in addition to conformity. In this paper, we develop a method that requires less stringent conditions on the initial partition than those from [Mau95, Tra97], although also with our method for $n > 2$ conformity alone is not sufficient. Due to the different way of formulating such conditions, it is not completely clear how our conditions for $n = 3$ compare to those from [Kos94]. For $n = 2$, our method is equal to newest vertex bisection, and thus applies to any conforming initial triangulation. We show that in any case any conforming partition of n-simplices can be refined to a valid initial partition for our bisection method. For our bisection method being applied inside an adaptive finite element method, we show that the result from [BDD04] generalizes to n-dimensions: The total number of n-simplices in the partition at termination of the method can be bounded by some absolute multiple of the number of n-simplices that were marked for refinement in all iterations.

This paper is organized as follows: In §2, we study recurrent bisections of a single n-simplex, and show that all its descendents fall into n congruency classes modulo scalings. Under a connectivity condition on the domain, in §3, we show that in order to verify conformity of a partition, we only have to check whether $(n-1)$-dimensional hyperfaces match to $(n-1)$-dimensional hyperfaces, and thus may ignore lower-dimensional hyperfaces. In §4, under some conditions on the initial partition, we show that all uniform refinements are conforming. In Appendix A, we show that these conditions can always be satisfied by some initial refinement of any given conforming subdivision into n-simplices. In §5, we demonstrate how local refinements can be made while retaining conformity. Finally, in §6, we prove that the result of Binev, Dahmen and DeVore generalizes to n-dimensions.

2. Bisection of a single simplex

Let $2 \leq n \leq m$. An n-simplex, or briefly, simplex T in $\mathbb{R}^m$ is the convex hull of $n+1$ points $x_0, \ldots, x_n \in \mathbb{R}^m$ that do not lie on a $(n-1)$-dimensional hyperplane. We will identify T with the set of its vertices $\{x_0, \ldots, x_n\}$. For $0 \leq k \leq n-1$, a simplex spanned by $k+1$ vertices of T is called a hyperface of T. For $k = n-1$, it will be called a *true hyperface*, and for $k \leq n-2$ it will called a *lower dimensional hyperface.*

Corresponding to a simplex $\{x_0, \ldots, x_n\}$, we will identify $\frac{1}{2}(n+1)!$ *tagged* simplices given by all possible *ordered* sequences $(\{x_0, x_1\}, x_2, \ldots, x_n)$. Although, for convenience, in the following we will write a tagged simplex as $(x_0, x_1, x_2, \ldots, x_n)$, the ordering of the first two coordinates is arbitrary.

Given a tagged simplex $T = (x_0, x_1, x_2, \ldots, x_n)$, its children are the tagged simplices $(x_0, x_2, \ldots, x_n, \frac{x_0+x_1}{2})$ and $(x_1, x_2, \ldots, x_n, \frac{x_0+x_1}{2})$. So these children are generated by bisecting the edge $\{x_0, x_1\}$ of T, i.e., by connecting its midpoint with the other vertices $x_2, \ldots, x_n$. See Figure 1 for an illustration. The edge $\{x_0, x_1\}$

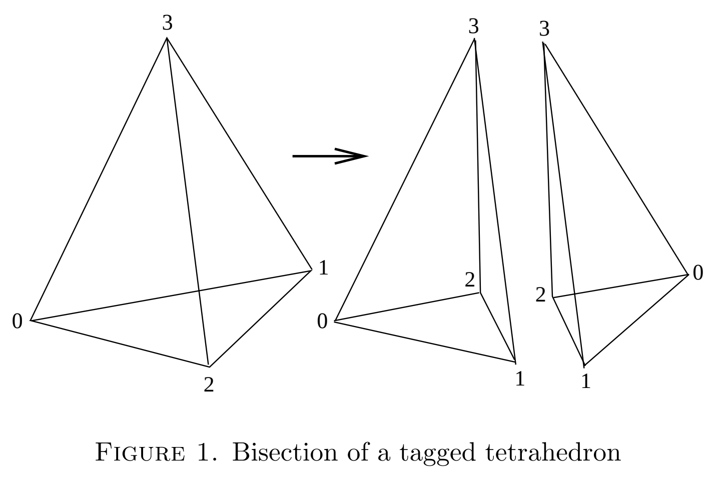

FIGURE 1. Bisection of a tagged tetrahedron

is called the *refinement edge* of T. In the $n = 2$ case, the vertex opposite to this edge is known as the *newest vertex*. A tagged simplex that is created by applying ℓ recursive bisections to T is called a level ℓ descendent of T. Every different tagging of a simplex creates a different set of descendents.

Remark 2.1. The bisection rules in [Mau95, Tra97], and for $n = 3$ in [Bän91, Kos94], are not equal to ours. If T has refinement edge $\{x_0, x_1\}$, then with our approach the children have refinement edges $\{x_0, x_2\}$ and $\{x_1, x_2\}$, whereas in these references, assuming T has level 0, the children's refinement edges are $\{x_0, x_2\}$ and $\{x_1, x_n\}$, which, for $n > 2$, are not only different from our edges, but also do depend on the ordering of x_0 and x_1. With our approach children are uniquely determined by the ordered sequence $(\{x_0, x_1\}, x_2, \dots, x_n)$ of vertices, and so there is no need to keep track of a "type" indicating the depth of refinement.

A result similar to the following theorem can be found in [Mau95, Tra97], however with more complicated proofs.

Theorem 2.2. *Let T be a tagged n-simplex in $\mathbb{R}^n$, and let $\lambda_T(x)$ denote the barycentric coordinates of $x \in \mathbb{R}^n$ with respect to T. Then, with respect to the distance function $\operatorname{dist}(x, y) = \|\lambda_T(x) - \lambda_T(y)\|$, all 2^ℓ level ℓ-descendents of T are mutually congruent. Moreover, in this metric, the level $\ell + n$ descendents are congruent to the level ℓ descendents up to a magnification factor $\frac{1}{2}$.*

Proof. The children of T are mapped into each other by permuting the first 2 barycentric coordinates. Moreover, since the descendents of the children are the images under linear maps of the children, all level ℓ descendents of T come in pairs that are mapped into each other by this reflection. With the induction hypothesis that the level $\ell - 1$ descendents of either of both children are mutually congruent, which we thus verified for $\ell - 1 = 1$, we infer that the level ℓ descendents of T are mutually congruent. The proof is completed by the observation that $(x_0, \frac{x_0+x_2}{2}, \dots, \frac{x_0+x_n}{2})$ is one of the level n-descendants of T, which is mapped onto T by the similarity transformation $\lambda \mapsto 2(\lambda - \lambda(x_0))$ magnifying distances to x_0 with a factor 2. □

3. Partitions and conformity

A locally finite collection P of mutually essentially disjoint n-simplices in $\mathbb{R}^n$ is called a *partition* of the domain $\Omega = \Omega(P)$, defined as the interior of the union of these simplices. A partition P is called *conforming* when the intersection of any two different $T, T' \in P$ is either empty, or a hyperface of both simplices. Different simplices T, T' that share a true hyperface will be called *neighbours*.

Next, we will see that under a mild condition on the domain, in order to verify whether a partition is conforming, we only have to check that there is no true hyperface of a simplex whose interior has non-empty intersection with a lower-dimensional hyperface of another simplex.

Theorem 3.1. *Let $\Omega = \Omega(P)$ be such that for any $x \in \overline{\Omega}$, for all sufficiently small open balls $B \subset \mathbb{R}^n$ that contain x, $\Omega \cap B$ is connected. Then P is conforming if and only if any two different $T, T' \in P$ for which $T \cap T'$ contains a point interior to a true hyperface of T are neighbours.*

Proof. One implication is obvious. For the other, let two different $T, T' \in P$ with $T \cap T' \neq \emptyset$ be given. For $x \in T \cap T'$, let P_x be the collection of $S \in P$ that contain

x. Let $B \ni x$ be an open ball such that $\Omega \cap B$ is connected, and contained in $\cup_{S \in P_x} S$. Let Σ be the subset of $\Omega \cap B$ created by removing any point that is on a lower dimensional hyperface of any of the $S \in P_x$. Then Σ is also connected, and it contains points $y \in T$, $y' \in T'$.

Let $T = S_0, \ldots, S_p = T'$ be the ordered sequence of simplices in P_x that is passed when traveling along a path in Σ connecting y and y'. By assumption, and the construction of Σ, for any $1 \leq i \leq p$, S_{i-1} and S_i are neighbours. We will now show that for $1 \leq q \leq p$, $\cap_{i=0}^{q} S_i$ is a hyperface of S_q. For $q = 1$ it is true, and let us assume that it is true for a $q - 1 \geq 1$. Then $\cap_{i=0}^{q} S_i$, being the intersection of the hyperfaces $\cap_{i=0}^{q-1} S_i$ and $S_{q-1} \cap S_q$ of S_{q-1}, is a hyperface of S_{q-1} that is contained in $S_{q-1} \cap S_q$, and thus is a hyperface of S_q. By applying this result for $q = p$, we conclude that x is contained in the hyperface $\cap_{i=0}^{p} S_i$ of T'. Since $x \in T \cap T'$ was arbitrary, we infer that $T \cap T'$ is the union of hyperfaces of T', and so, because $T \cap T'$ is convex, that $T \cap T'$ is a hyperface of T', and similarly of T. □

The above theorem is generally not valid without the condition on Ω as illustrated in Figure 2.

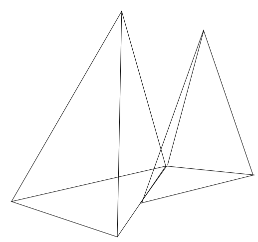

Figure 2. A nonconforming partition, although there is no face that contains a point of another tetrahedron in its interior.

4. Partitions created by refinements

In the remainder of this paper, we will exclusively consider partitions of tagged simplices that can be created by recurrent bisections, as discussed in §2, starting from some *fixed initial partition P_0 of tagged simplices.* So whenever we refer to a partition P, we mean a partition of this kind, and any $T \in P$ is a descendent, with some level $\ell(T)$, of a simplex from P_0. A partition P is a *uniform refinement* of P_0 when all its simplices have the same level.

From Theorem 2.2, we infer that all partitions are *uniformly shape regular* only dependent on P_0 and n, meaning that the ratio of the radii of the smallest circumscribed and largest inscribed balls of any T is uniformly bounded, only dependent on P_0 and n. More particular, there exist constants $d, D > 0$, only dependent on P_0 and n, such that for any T,

$$d 2^{-\ell(T)} \leq \operatorname{meas}(T), \quad \operatorname{diam}(T) \leq D 2^{-\ell(T)/n}. \tag{4.1}$$

We will *always assume that P_0 satisfies the following 3 conditions:*

(a) With $\Omega = \Omega(P_0)$, for any $x \in \overline{\Omega}$, for all sufficiently small open balls $B \subset \mathbb{R}^n$ that contain x, $\Omega \cap B$ is connected.

(b) P_0 is conforming.

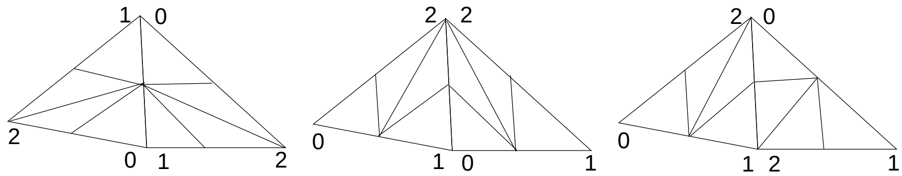

FIGURE 3. Matching neighbours for $n = 2$, and their level 2 descendents

(c) Any two neighbouring tagged simplices $T = (x_0, \dots, x_n)$, $T' = (x_0', \dots, x_n')$ from P_0 *match* in the following sense: If $\{x_0, x_1\}$ or $\{x_0', x_1'\}$ is on $T \cap T'$, then $\{x_0, x_1\} = \{x_0', x_1'\}$ and $x_\ell = x_\ell'$ for all but one $\ell \in \{2, \dots, n\}$. Otherwise, $x_\ell = x_\ell'$ for $\ell \in \{3, \dots, n\}$.

Note that because of (a) and since the domain does not change by refinements, we can always rely on Theorem 3.1 to check conformity of any P.

For $n = 2$, (c) is equivalent to the condition that, if, for any two neighbours T, T', $T \cap T'$ is the refinement edge of either T or T', then it is the refinement edge of both. See Figure 3 for an illustration. It is known, see [BDD04] and the references therein, that for any conforming partition into triangles there exists a local numbering of the vertices that satisfies (c).

Remark 4.1. Instead of (c), in [Mau95, Tra97] the stronger condition was required that $x_\ell = x_\ell'$ for all but one $\ell \in \{0, \dots, n\}$. As shown in [Tra97], for a simply connected domain, it can be satisfied if and only if each $(n-2)$-dimensional hyperface not on the boundary of the domain is shared by an even number of simplices. So for $n = 2$, it means that the valence of any interior vertex should be even.

We do not expect that for $n > 2$ Condition (c) can be satisfied for each partition. E.g., for $n = 3$, one may verify that it requires that the total number of tetrahedra, that shares a refinement edge of some tetrahedron which edge is not on the boundary of the domain, is even. Therefore, as an alternative, inspired by such a construction for $n = 3$ in [Kos94], in Appendix A we show that any conforming partition of n-simplices can be refined to a conforming partition P_0 that allows a local numbering of the vertices that satisfies (c).

We now proceed assuming P_0 satisfies (a)-(c).

Theorem 4.2. *Any uniform refinement P of P_0 is conforming.*

Proof. In addition to the statement we will show that all neighbours in P satisfy the matching condition (c). Let $\ell(T) = \ell \geq 1$ for all $T \in P$, and assume that the statement is true for $\ell - 1$. Let T, T' be two different simplices from P such that $T \cap T'$ contains a point interior to a true hyperface of T. If T, T' have the same parent $(x_0, \dots, x_n)$, then $T = (x_0, x_2, \dots, \frac{x_0+x_1}{2})$, $T' = (x_1, x_2, \dots, \frac{x_0+x_1}{2})$ are matching neighbours.

If T, T' have different parents $(x_0, \dots, x_n)$ and $(x_0', \dots, x_n')$, then these parents are matching neighbours by the induction hypothesis. So if $\{x_0, x_1\}$ is on the interface between these parents, then $\{x_0, x_1\} = \{x_0', x_1'\}$, and $x_i = x_i'$ for all but one $i \in \{2, \dots, n\}$. Without loss of generality assuming that $x_0 = x_0'$ and $x_1 = x_1'$, either $T = (x_0, x_2, \dots, \frac{x_0+x_1}{2})$ and $T' = (x_0', x_2', \dots, \frac{x_0'+x_1'}{2})$, or similar with x_0 and x_0' replaced by x_1 and x_1', respectively, showing that T, T' are matching neighbours. If $\{x_0, x_1\}$ is not on the interface between the parents, then $\{x_0', x_1'\}$ is not on this interface, and $x_i = x'_i$ for $i \in \{3, \dots, n\}$. Without loss of generality we assume that x_0, x'_0 are not on the interface, and so $\{x_1, x_2\} = \{x'_1, x'_2\}$. We conclude that $T = (x_1, x_2, \dots, \frac{x_0+x_1}{2})$ and $T' = (x'_1, x'_2, \dots, \frac{x'_0+x'_1}{2})$, and that they are matching neighbours. □

Remark 4.3. Given the rule for bisecting a tagged simplex, along the lines of above proof one may verify that the matching condition (c) on neighbours in P_0 is also *necessary* for obtaining conformity of all uniform refinements.

Remark 4.4. In [Bän91, AMP00], algorithms for bisection tetrahedra, i.e., for $n = 3$, are formulated that do not require a matching of neighbours in the initial partition. With these methods, however, Theorem 4.2 is generally not valid; only uniform refinements with levels divisible by n are guaranteed to be conforming. The result of Theorem 4.2, however, will be heavily used in the following. An interesting open question is whether the tetrahedra on level n generated by the algorithms from [Bän91, AMP00] can be re-tagged so that (c) is satisfied. For $n = 2$, starting with an arbitrary tagging of the triangles, the corresponding statement is valid (see [BDD04, p229]).

In the following, tagged neighbours will be called *compatibly divisible* when they have the same refinement edge. For a partition P, and $T \in P$, we set

$$N(P, T) := \{\text{neighbours } T' \text{ of } T \text{ in } P \text{ that contain the refinement edge of } T\}.$$

Corollary 4.5. *For any partition P, $T \in P$, and $T' \in N(P, T)$, either*

- *$\ell(T') = \ell(T)$ and T, T' are compatibly divisible, or*
- *$\ell(T') = \ell(T) - 1$ and T is compatibly divisible with one of both children of T'.*

Proof. For some $p \geq 2$, let T_1, T_2 be neighbours with $\ell(T_1) = \ell(T_2) - p$. Then there is a level p descendent of T_1 that contains a point of T_2 interior to a true hyperface. Theorem 4.2 shows that this level p descendent is a neighbour of T_2, i.e., that it has a true hyperface in common with T_2 and thus with T_1. Since a level p descendent of T_1 has $n - p < n - 1$ vertices in common with T_1, we arrive at a contradiction, and conclude that the levels of neighbours differ at most one.

Now let $T' \in N(P, T)$ with $\ell(T') = \ell(T) + 1$. Then again Theorem 4.2 shows that one of both children of T is a neighbour of T'. However, since T has its refinement edge on $T \cap T'$ this cannot be the case.

Concerning the two remaining cases, if $\ell(T') = \ell(T)$ then T, T' are indeed compatibly divisible, since otherwise the uniform refinement with simplices of level $\ell(T) + 1$ would not be conforming.

If $\ell(T') = \ell(T) - 1$, then one of both children of T' has a point of T interior to a true hyperface, so that they are neighbours by Theorem 4.2. Since the uniform refinement with simplices of level $\ell(T) + 1$ is conforming, we conclude that this child and T are compatibly divisible. □

5. Local refinements while retaining conformity

Let P be a conforming partition, and let $M \subset P$ be a subset of simplices that have been marked for bisection. After bisecting the simplices from M, a generally nonconforming partition P' arises. To restore conformity, one may apply the following completion algorithm:

complete$[P']$
`for` $T \in P'$, *for which there exists a* $T' \in P'$ *such that* $T \cap T'$ *contains a point interior to a hyperface of* T, *whereas* T *and* T' *are no neighbours*
`do` *bisect* T
`until` *such* T *do not exist*

Since the only way to cure the situation as decribed in the `for`-statement, or towards curing it, is to bisect T, **complete**$[P']$ outputs the smallest conforming refinement of P', assuming that a conforming refinement exists. This, however, holds true, since with $\ell = \max_{T' \in P'} \ell(T')$, the uniform partition with simplices of level ℓ is a conforming refinement of P'. When implementing **complete**, care has to be taken to ensure that the computational work is of the order of the number of bisections that are made.

An alternative for first bisecting all simplices in M and then restoring conformity by a call of **complete**, is, when running over all $T \in M$, for each T to replace the current partition P by its smallest conforming refinement in which T has been bisected. A call of the routine **refine**$[P,T]$ given below determines such a partition. Since it bisects generally more simplices than only T, it may happen that it bisects $T' \in M$ for which a call has not yet been made, which call thus then can be skipped. In other words, the number of calls of **refine** is never larger than the number of marked simplices. Since also with this approach, only simplices are bisected that either are marked, or whose bisection is unavoidable for obtaining a conforming partition, again we end up with the smallest conforming partition in which all marked simplices are bisected.

The following routine **refine**$[P,T]$ is a generalization to n-dimensions of such a routine by Kossaczký in [Kos94] for bisecting tetrahedra. Based on Corollary 4.5, the idea is to determine, possibly by recursive calls, a closed set of compatibly divisible neighbours that share the refinement edge with T, after which this set of simplices can be simultaneously bisected without introducing non-conformities.

refine$[P,T] \to P'$*:*
% P *is a conforming partition and* $T \in P$.
$K := \emptyset$; $F = \{T\}$
`do` $Fnew := \emptyset$
    `forall` $T' \in F$ `do`
        `forall` $T'' \in N(P,T')$ *with* $T'' \notin F \cup K$ `do`
            `if` T'' *compatibly divisible with* T'
            `then` $Fnew := Fnew \cup \{T''\}$
            `else` $P :=$ **refine**$[P,T'']$
                    add to $Fnew$ *the child of* T'' *that is a neighbour of* T'
            `endif`
        `endfor`
    `endfor`
    $K := K \cup F$
    $F := Fnew$
`until` $F = \emptyset$
create P' *from* P *by simultaneously bisecting all* $T' \in K$

Theorem 5.1. $P' :=$ **refine**$[P,T]$ *terminates, and* P' *is smallest conforming refinement of* P *in which* T *has been bisected. If* $T' \in P'$ *is newly created by the call, then* $\ell(T') \leq \ell(T) + 1$.

Proof. Let $\ell(T) = 0$. Then Corollary 4.5 shows that there will be no recursive calls of **refine**, and that just before any evaluation of the `until`-statement, all $T' \in K$ have the same refinement edge as T, and satisfy $\ell(T') = \ell(T)$ and $N(P, T') \subset K \cup F$. If $F \neq \emptyset$, then in the next iteration of the `do-until` loop, the set K will be extended. Since, on the other hand, from the uniform shape regularity we know that the cardinality of K is bounded, we conclude that this loop terminates. After termination, $F = \emptyset$, and so for all $T' \in K$, $N(P, T') \subset K$, by Theorem 3.1 meaning that by bisecting all $T' \in K$ conformity is retained. It is clear that we cannot confine bisection to a smaller set of simplices, and that $\ell(T') = \ell(T) + 1$ for any newly created T'.

Assuming that for some $\ell - 1 \geq 1$, the statement is true for T with $\ell(T) = \ell - 1$, let us consider T with $\ell(T) = \ell$. Possible recursive calls of **refine**$[P, T'']$ are unavoidable, where Corollary 4.5 shows that $\ell(T'') = \ell(T) - 1$. The induction hypothesis then shows that such a call outputs the smallest conforming partition in which T'' has been bisected, and moreover, that it does not bisect any simplex that is already in $K \cup F$, since that would create simplices with levels larger than $\ell(T) = \ell(T'') + 1$. Now the proof is completed using the same arguments as in the $\ell(T) = 0$ case. □

Assuming that the datastructures allow that the determination of $N(P, T)$ requires not more than an absolute constant number of operations, note that the number of operations needed for $P' :=$ **refine**$[P, T]$ is $\mathcal{O}(\#P' - \#P)$.

In addition to the properties of **refine** shown in Theorem 5.2, we have

Theorem 5.2. *With the constant D from* (4.1)*, any newly created T' by the call* **refine**$[P, T]$ *satisfies*

$$d(T', T) := \inf_{x' \in T',\, x \in T} |x' - x| \leq D 2^{1/n} \sum_{k=\ell(T')}^{\ell(T)} 2^{-k/n} \qquad \left(< \frac{D 2^{1/n}}{1 - 2^{-1/n}} 2^{-\ell(T')} \right).$$

Proof. For $\ell(T) = 0$, any newly created T' is a child of a $\tilde{T}$ that has its refinement edge on ∂T, so that $d(T', T) = 0$. Note that in this case the sum over k is empty since $\ell(T') = \ell(T) + 1$.

Assuming that the theorem holds for $\ell(T) = \ell - 1 \geq 0$, let us consider T with $\ell(T) = \ell$. If T' is created by bisection of any simplex from the set K, then the statement is proven as in the $\ell(T) = 0$ case. If T' is created by a recursive call **refine**$[P, T'']$, then using $T \cap T'' \neq \emptyset$, the induction hypothesis shows that

$$\begin{aligned} d(T', T) &\leq d(T', T'') + \operatorname{diam}(T'') \\ &\leq D 2^{1/n} \sum_{k=\ell(T')}^{\ell(T'')} 2^{-k/n} + D 2^{-\ell(T'')/n} = D 2^{1/n} \sum_{k=\ell(T')}^{\ell(T)} 2^{-k/n}, \end{aligned}$$

by $\ell(T'') = \ell(T) - 1$. □

The routine **refine**, that provides an alternative for the straightforward bisection of marked simplices complemented with a call of **complete**, is here discussed mainly because its properties proven in Theorems 5.1 and 5.2 will allow us, in the next section, to bound the complexity of a recurrent marking and completion process. It turns out, however, that an implementation of this process by means of calls of **refine** is particularly efficient. For this reason, this approach is followed in the adaptive finite element package ALBERTA ([SS05]).

Remark 5.3. Inside adaptive finite element methods, simplices can be marked for multiple bisections. This means that not only these simplices should be bisected, but also some of their descendents, with the obvious restriction that a descendent can only be on the list for bisection when its parent is. For example, for $n = 2$, the adaptive finite element method introduced in [MNS00] selects triangles for their bisection, and that of their children and 2 of their 4 grandchildren. The evaluation of such multiple markings can be done by scheduling them as an ordered sequence of groups of single markings, where the marking of a child is in the next group as that of its parent. After finding the smallest conforming refinement in which all simplices from a group are bisected, it may happen that bisections corresponding to markings from next groups already have taken place, so that these markings can be deleted.

6. The complexity of a recurrent marking and completion process

We study the following algorithm:

$P := P_0$
`do` *mark some set* $\bar{M} \subset P$ *for bisection*
  `for` $T \in \bar{M}$ `do`
    `if` $T \in P$ *% i.e., if it has not been yet bisected as a byproduct of a*
        % previous call of **refine** *in this* `for`*-loop*
    `then` $P :=$ **refine**$[P, T]$
    `endif`
  `endfor`
`until` *satisfied*

As we have seen, the output partition of this algorithm is the smallest conforming refinement of P_0 in which all marked simplices have been bisected. After the preparations from the previous sections, the proof of the following main theorem concerning this algorithm follows the lines of the proof of the corresponding theorem for $n = 2$ by Binev, Dahmen and DeVore. Since there are some small modifications, we include the proof for the reader's concenience.

Theorem 6.1 (generalizes [BDD04, Theorem 2.4] for $n = 2$)**.** *With M being the set of simplices for which a call of* **refine** *is made in above algorithm, which set is thus not larger than the union of all marked simplices, for the output partition P it holds that $\#(P\backslash(P \cap P_0)) \lesssim \#M$, only dependent on the constants d, D from* (4.1), *and n.*

Proof. Fixing n, let $a : I\!N_0 \cup \{-1\} \to I\!R^+$, $b : I\!N_0 \to I\!R^+$ be some sequences with $\sum_{p=-1}^{\infty} a(p) < \infty$, $\sum_{p=0}^{\infty} b(p)2^{-p/n} < \infty$, and $\inf_{p\geq 0} b(p)a(p) > 0$. Valid instances are $a(p) = (p+2)^{-1}$ and $b(p) = 2^{p/(n+1)}$. Let $A := D(\frac{2^{1/n}}{1-2^{-1/n}}+1)\sum_{p=0}^{\infty} b(p)2^{-p/n}$.

Inside this proof, P will always denote the output partition of the algorithm, whereas any intermediate partition will be denoted as $\tilde{P}$. We define $\lambda : P \times M$ by

$$\lambda(T', T) = \begin{cases} a(\ell(T) - \ell(T')) & \text{if } d(T', T) < A2^{-\ell(T')/n} \text{ and } \ell(T') \leq \ell(T) + 1, \\ 0 & \text{otherwise.} \end{cases}$$

For any fixed $T \in P$, and $\ell' \in I\!N_0$ with $\ell' \leq \ell(T) + 1$, there exists a uniformly bounded number, only dependent on d and D, of $T' \in P$ with $d(T', T) < A2^{-\ell(T')/n}$

and $\ell(T') = \ell'$. In view of the definition of λ, we thus have $\sum_{T'\in P} \lambda(T',T) \lesssim \sum_{p=-1}^{\infty} a(p) < \infty$, and so $\sum_{T\in M}\sum_{T'\in P} \lambda(T',T) \lesssim \#M$.

In the second part of this proof, we are going to show that for all $T' \in P\backslash(P\cap P_0)$,

$$\sum_{T\in M} \lambda(T',T) \gtrsim 1, \tag{6.1}$$

only dependent on d, D and n, so that

$$\#(P\backslash(P\cap P_0) \lesssim \sum_{T'\in P\backslash(P\cap P_0)} \sum_{T\in M} \lambda(T',T) \le \sum_{T\in M}\sum_{T'\in P} \lambda(T',T) \lesssim \#M,$$

as required.

Let $T_0 \in P\backslash(P\cap P_0)$. For $j \ge 0$, given that T_j has been defined and assuming that it is not in P_0, we let $T_{j+1} \in M$ be such that T_j has been created by the call **refine**$[\bar{P}, T_{j+1}]$. Let s be the smallest integer such that $\ell(T_s) = \ell(T_0)-1$. Note that such an s exists since at some point the sequence ends with a $T_{\bar{j}} \in P_0$, thus with $\ell(T_{\bar{j}}) = 0$, whereas the value $\ell(T_0)-1$ cannot be passed without being attained because $\ell(T_{j+1}) \ge \ell(T_j)-1$ by Theorem 5.1. From Theorem 5.2 and (4.1), for $1 \le j \le s$ we have

$$\begin{aligned}
d(T_0,T_j) &\le d(T_0,T_1) + \mathrm{diam}(T_1) + d(T_1,T_j)\\
&\le \sum_{k=1}^{j} d(T_{k-1},T_k) + \sum_{k=1}^{j-1} \mathrm{diam}(T_k)\\
&< \sum_{k=1}^{j} \frac{D2^{1/n}}{1-2^{-1/n}} 2^{-\ell(T_{k-1})/n} + \sum_{k=1}^{j-1} D2^{-\ell(T_k)/n}\\
&< D(1+\frac{2^{1/n}}{1-2^{-1/n}}) \sum_{k=0}^{j-1} 2^{-\ell(T_k)/n}\\
&= D(1+\frac{2^{1/n}}{1-2^{-1/n}}) \sum_{p=0}^{\infty} m(p,j) 2^{-(\ell(T_0)+p)/n},
\end{aligned}$$

where $m(p,j)$ denotes the number of $k \le j-1$ with $\ell(T_k) = \ell(T_0)+p$.

In case $m(p,s) \le b(p)$ for all p, then the definition of the constant A shows that $d(T_0,T_s) < A2^{-\ell(T_0)/n}$, and so by definition of λ, we conclude that $\lambda(T_0,T_s) = a(\ell(T_s)-\ell(T_0)) = a(-1)$, which shows (6.1).

Otherwise, there exist p with $m(p,s) > b(p)$. For each of those p, there exists a smallest $j = j(p)$ with $m(p,j(p)) > b(p)$. We denote p that gives rise to the smallest $j(p)$ as p^*, and denote $j(p^*)$ as j^*. Thus $m(p,j^*-1) \le b(p)$ for all p, and $m(p^*,j^*-1) = b(p^*)$. As in the case that $m(p,s) \le b(p)$ for all p, we find that for all $k \le j^*-1$, $d(T_0,T_k) < A2^{-\ell(T_0)/n}$ and $\lambda(T_0,T_k) = a(\ell(T_k)-\ell(T_0))$. In view of the definition of $m(\cdot,\cdot)$, we find that

$$\begin{aligned}
\sum_{\{k\le j^*-2:\ell(T_k)=\ell(T_0)+p^*\}} \lambda(T_0,T_k) &= m(p^*,j^*-1)a(p^*)\\
&= b(p^*)a(p^*) \ge \inf_{p\ge 0} b(p)a(p) > 0,
\end{aligned}$$

showing (6.1) also in this case. □

The only properties that have been used in this proof are (4.1), and that of **refine** given in Theorems 5.1 and 5.2.

Appendix A. An initial refinement to satisfy Condition (c)

Suppose we are given some conforming partition of n-simplices. Generalizing upon the construction by Kossaczký in [Kos94] for $n = 3$, in this appendix we construct a conforming refinement consisting of tagged simplices that satisfies Condition (c).

We start with constructing a conforming subdivision of any n-simplex into $\frac{1}{2}(n+1)!$ subsimplices, together with a global labeling of vertices and a marking of edges in this subdivision that satisfy the following conditions:

- A vertex on a marked edge has no label,
- the other vertices are labeled with numbers $2, \dots, n$,
- each subsimplex contains vertices with labels $2, \dots, n$ and two vertices on a marked edge
- the subdivision and labeling/marking is symmetric in the barycentric coordinates of the original (macro-) simplex.

For $n = 2$, we subdivide a triangle into tree subtriangles by connecting the vertices with the centroid. This centroid is labeled with number 2, and the edges of the original (macro-) triangle are marked. Clearly above conditions are satisfied.

For $n \geq 3$, assuming we have defined a valid subdivision and labeling and marking of any $(n-1)$-simplex, we define this for an n-simplex as follows: Create $(n+1)$ subsimplices by connecting the vertices with the centroid. Label the centroid with number n. Each of the subsimplices shares a face with the original (macro-) simplex. Use the subdivision of any $(n-1)$-simplex to subdivide these faces into $\frac{1}{2}n!$ labeled/marked $(n-1)$-simplices. Connect the vertices on the faces with the centroid to end with a subdivision into $(n+1) * \frac{1}{2}n! = \frac{1}{2}(n+1)!$ simplices with a valid labeling/marking. See Figure 4 for an illustration.

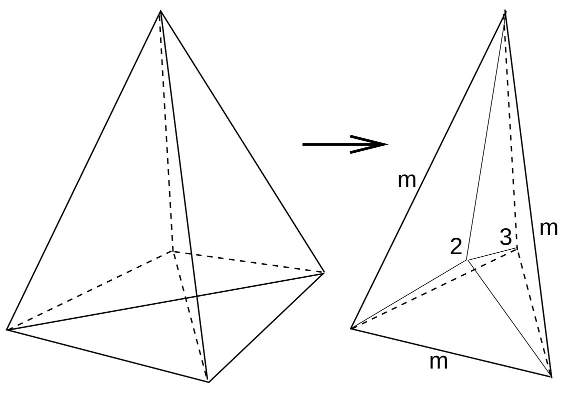

Figure 4. Subdivision of a tetrahedron into 4×3 tetrahedra with the labeling of vertices and marking of edges

Returning to the given conforming partition of n-simplices, we subdivide each of its simplices into $\frac{1}{2}(n+1)!$ subsimplices as above. Clearly this refined partition, that will serve as the initial partition P_0, is also conforming. Tagging the simplices in P_0 means specifying a local ordering of the vertices in each simplex. We simply let each simplex inherit the labeling of the vertices from the macro-simplex that contains it, with the addition that both vertices on the marked edge are numbered 0 and 1 in arbitrary order, see Figure 5 for an illustration for $n = 2$. Neighbours

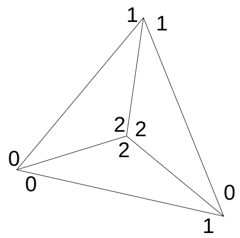

FIGURE 5. Local numbering of the vertices of the subtriangles of a macro-triangle

within one macro-simplex are obviously matching, since their numbering of the vertices on the interface between them is the same modulo permutations of 0 and 1. The same is valid for neighbours from different macro-simplices, because of the symmetry of the labeling in the barycentric coordinates. We conclude that P_0 satisfies Condition (c).

Remark A.1. For any two tagged neighbours $T = (x_0, \dots, x_n)$, $T' = (x'_0, \dots, x'_n)$ in P_0 as constructed above, possibly after permuting x'_0 and x'_1, it holds that $x_\ell = x'_\ell$ for all but one ℓ, which, in case $T \cap T'$ does not contain the refinement edges is more than required by Condition (c). It is not clear whether the extra freedom allowed by Condition (c) can be used to find a subdivision of each macro-simplex into a smaller number of subsimplices such that the resulting P_0 still satisfies (c).

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