INVARIANT DENSITIES FOR RANDOM β-EXPANSIONS

KARMA DAJANI AND MARTIJN DE VRIES

ABSTRACT. Let $\beta > 1$ be a non-integer. We consider expansions of the form $\sum_{i=1}^{\infty} \frac{d_i}{\beta^i}$, where the digits $(d_i)_{i\geq 1}$ are generated by means of a Borel map K_β defined on $\{0,1\}^{\mathbb{N}} \times [0, \lfloor\beta\rfloor/(\beta-1)]$. We show existence and uniqueness of an absolutely continuous K_β-invariant probability measure w.r.t. $m_p \otimes \lambda$, where m_p is the Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ with parameter p $(0 < p < 1)$ and λ is the normalized Lebesgue measure on $[0, \lfloor\beta\rfloor/(\beta-1)]$. Furthermore, this measure is of the form $m_p \otimes \mu_{\beta,p}$, where $\mu_{\beta,p}$ is equivalent with λ. We establish the fact that the measure of maximal entropy and $m_p \otimes \lambda$ are mutually singular. In case the number 1 has a finite greedy expansion with positive coefficients, the measure $m_p \otimes \mu_{\beta,p}$ is Markov. In the last section we answer a question concerning the number of universal expansions, a notion introduced in [EK].

1. INTRODUCTION

Let $\beta > 1$ be a non-integer. In this paper we consider expansions of numbers x in $J_\beta := [0, \lfloor\beta\rfloor/(\beta-1)]$ of the form

$$x = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i}$$

with $a_i \in \{0, 1, \ldots, \lfloor\beta\rfloor\}$. We shall refer to expansions of this form as $(\beta-)$*expansions* or *expansions in base* β. The largest expansion in lexicographical order is the *greedy expansion*; [P], [R1], [R2], and the smallest is the *lazy expansion*; [JS], [EJK], [DK1]. The greedy expansion is obtained by iterating the greedy transformation $T_\beta : J_\beta \to J_\beta$, defined by

$$T_\beta(x) = \beta x - d \quad \text{for} \quad x \in C(d),$$

where

$$C(j) = \left[\frac{j}{\beta}, \frac{j+1}{\beta}\right), \quad j \in \{0, \ldots, \lfloor\beta\rfloor - 1\},$$

and

$$C(\lfloor\beta\rfloor) = \left[\frac{\lfloor\beta\rfloor}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}\right].$$

1991 *Mathematics Subject Classification.* Primary:28D05, Secondary:11K16, 28D20, 37A35, 37A45.

Key words and phrases. greedy expansions, lazy expansions, absolutely continuous invariant measures, measures of maximal entropy, Markov chains, universal expansions.

The greedy expansion of $x \in J_\beta$ is given by $x = \sum_{i=1}^{\infty} d_i(x)/\beta^i$, where $d_i(x) = d$ if and only if $T_\beta^{i-1}(x) \in C(d)$. Let $\ell : J_\beta \to J_\beta$ be given by

$$\ell(x) = \frac{\lfloor \beta \rfloor}{\beta - 1} - x,$$

then the lazy transformation $L_\beta : J_\beta \to J_\beta$ is defined by

$$L_\beta(x) = \beta x - d \quad \text{for} \quad x \in \Delta(d) = \ell\left(C(\lfloor \beta \rfloor - d)\right),\ d \in \{0, \dots, \lfloor \beta \rfloor\}.$$

The lazy expansion of $x \in J_\beta$ is given by $x = \sum_{i=1}^{\infty} \tilde{d}_i(x)/\beta^i$, where $\tilde{d}_i(x) = d$ if and only if $L_\beta^{i-1}(x) \in \Delta(d)$.

We denote by μ_β the extended T_β-invariant *Parry* measure (see [P],[G]) on J_β which is absolutely continuous with respect to Lebesgue measure, and with density

$$h_\beta(x) \;=\; \begin{cases} \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \frac{1}{\beta^n}\, 1_{[0, T_\beta^n(1))}(x) & 0 \le x < 1, \\ 0 & 1 \le x \le \lfloor \beta \rfloor/(\beta - 1), \end{cases}$$

where $F(\beta)$ is the normalizing constant. Define the *lazy* measure ρ_β on J_β by $\rho_\beta = \mu_\beta \circ \ell^{-1}$. It is easy to see ([DK1]) that ℓ is a continuous isomorphism between $(J_\beta, \mu_\beta, T_\beta)$ and $(J_\beta, \rho_\beta, L_\beta)$.

In order to produce other expansions in a dynamical way, a new transformation K_β was introduced in [DK2]. The expansions generated by iterating this map are random mixtures of greedy and lazy expansions. This is done by superimposing the greedy map and the corresponding lazy map on J_β. In this way one obtains $\lfloor \beta \rfloor$ intervals on which the greedy map and the lazy map differ. These intervals are given by

$$S_k \;=\; \left[\frac{k}{\beta}, \frac{\lfloor \beta \rfloor}{\beta(\beta - 1)} + \frac{k-1}{\beta}\right], \quad k = 1, \dots, \lfloor \beta \rfloor,$$

which one refers to as *switch regions*. On S_k, the greedy map assigns the digit k, while the lazy map assigns the digit $k - 1$. Outside these switch regions both maps are identical, and hence they assign the same digits. Now define other expansions in base β by randomizing the choice of the map used in the switch regions. So, whenever x belongs to a switch region, flip a coin to decide which map will be applied to x, and hence which digit will be assigned. To be more precise, partition the interval J_β into switch regions S_k and *equality regions* E_k, where

$$E_k \;=\; \left(\frac{\lfloor \beta \rfloor}{\beta(\beta - 1)} + \frac{k-1}{\beta}, \frac{k+1}{\beta}\right), \quad k = 1, \dots, \lfloor \beta \rfloor - 1,$$

$$E_0 \;=\; \left[0, \frac{1}{\beta}\right) \quad \text{and} \quad E_{\lfloor \beta \rfloor} \;=\; \left(\frac{\lfloor \beta \rfloor}{\beta(\beta - 1)} + \frac{\lfloor \beta \rfloor - 1}{\beta}, \frac{\lfloor \beta \rfloor}{\beta - 1}\right].$$

Let

$$S \;=\; \bigcup_{k=1}^{\lfloor \beta \rfloor} S_k, \quad \text{and} \quad E \;=\; \bigcup_{k=0}^{\lfloor \beta \rfloor} E_k,$$

and consider $\Omega = \{0,1\}^{\mathbb{N}}$ with product σ-algebra $\mathcal{A}$. Let $\sigma : \Omega \to \Omega$ be the left shift, and define $K_\beta : \Omega \times J_\beta \to \Omega \times J_\beta$ by

$$K_\beta(\omega, x) \;=\; \begin{cases} (\omega, \beta x - k) & x \in E_k,\ k = 0, 1, \ldots, \lfloor \beta \rfloor, \\ \\ (\sigma(\omega), \beta x - k) & x \in S_k \ \text{ and } \ \omega_1 = 1,\ k = 1, \ldots, \lfloor \beta \rfloor, \\ \\ (\sigma(\omega), \beta x - k + 1) & x \in S_k \ \text{ and } \ \omega_1 = 0,\ k = 1, \ldots, \lfloor \beta \rfloor. \end{cases}$$

The elements of Ω represent the coin tosses ('heads'=1 and 'tails'=0) used every time the orbit $\{K_\beta^n(\omega, x) : n \ge 0\}$ hits $\Omega \times S$. Let

$$d_1 \;=\; d_1(\omega, x) \;=\; \begin{cases} k & \text{if } \ x \in E_k,\ k = 0, 1, \ldots, \lfloor \beta \rfloor, \\ & \text{or } \ (\omega, x) \in \{\omega_1 = 1\} \times S_k,\ k = 1, \ldots, \lfloor \beta \rfloor, \\ \\ k - 1 & \text{if } \ (\omega, x) \in \{\omega_1 = 0\} \times S_k,\ k = 1, \ldots, \lfloor \beta \rfloor, \end{cases}$$

then

$$K_\beta(\omega, x) \;=\; \begin{cases} (\omega, \beta x - d_1) & \text{if } \ x \in E, \\ \\ (\sigma(\omega), \beta x - d_1) & \text{if } \ x \in S. \end{cases}$$

Set $d_n \;=\; d_n(\omega, x) \;=\; d_1\left(K_\beta^{n-1}(\omega, x)\right)$, and let $\pi_2 : \Omega \times J_\beta \to J_\beta$ be the canonical projection onto the second coordinate. Then

$$\pi_2\left(K_\beta^n(\omega, x)\right) \;=\; \beta^n x - \beta^{n-1} d_1 - \cdots - \beta d_{n-1} - d_n,$$

and rewriting yields

$$x \;=\; \frac{d_1}{\beta} + \cdots + \frac{d_n}{\beta^n} + \frac{\pi_2\left(K_\beta^n(\omega, x)\right)}{\beta^n}.$$

This shows that for all $\omega \in \Omega$ and for all $x \in J_\beta$ one has that

$$x \;=\; \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \;=\; \sum_{i=1}^{\infty} \frac{d_i(\omega, x)}{\beta^i}.$$

The random procedure just described shows that with each $\omega \in \Omega$ corresponds an algorithm that produces an expansion in base β. Furthermore, if we identify the point (ω, x) with $(\omega, (d_1(\omega, x), d_2(\omega, x), \ldots))$, then the action of K_β on the second coordinate corresponds to the left shift.

Let $<_{lex}$ and $\le_{lex}$ denote the lexicographical ordering on both Ω and $\{0, \ldots, \lfloor \beta \rfloor\}^{\mathbb{N}}$. We recall from [DdV] the following basic properties of random β-expansions.

Theorem 1. *Suppose $\omega, \omega' \in \Omega$ are such that $\omega <_{lex} \omega'$, then*

$$(d_1(\omega, x), d_2(\omega, x), \ldots) \le_{lex} (d_1(\omega', x), d_2(\omega', x), \ldots).$$

Theorem 2. *Let $x \in J_\beta$ and let $x = \sum_{i=1}^{\infty} a_i/\beta^i$ with $a_i \in \{0, 1, \ldots, \lfloor \beta \rfloor\}$ be an expansion of x in base β. Then there exists an $\omega \in \Omega$ such that for all $i \ge 1$, $a_i = d_i(\omega, x)$.*

In [DdV] it is shown that there exists a unique measure of maximal entropy ν_β for the map K_β. It is the main goal of this paper to investigate the relationship between this measure and the measure $m_p \otimes \lambda$, where λ is the normalized Lebesgue measure on J_β and m_p is the Bernoulli measure on $\Omega = \{0,1\}^{\mathbb{N}}$ with parameter p $(0 < p < 1)$:

$$m_p(\{\omega_1 = i_1, \ldots, \omega_n = i_n\}) = p^{\sum_{j=1}^n i_j}(1-p)^{n-\sum_{j=1}^n i_j}.$$

In order to prove that ν_β and $m_p \otimes \lambda$ are mutually singular, we introduce in the next section another K_β-invariant probability measure. This measure is a product measure $m_p \otimes \mu_{\beta,p}$ and we show in Section 3 that K_β is ergodic w.r.t. this measure. Furthermore, the measures $m_p \otimes \lambda$ and $m_p \otimes \mu_{\beta,p}$ are shown to be equivalent. These facts enable us to conclude that the measures ν_β and $m_p \otimes \lambda$ are mutually singular. Moreover, it follows that $m_p \otimes \mu_{\beta,p}$ is the unique absolutely continuous K_β-invariant probability measure w.r.t. $m_p \otimes \lambda$. The measure $\mu_{\beta,p}$ satisfies the important relationship

$$\mu_{\beta,p} = p \cdot \mu_{\beta,p} \circ T_\beta^{-1} + (1-p) \cdot \mu_{\beta,p} \circ L_\beta^{-1}.$$

In Section 4 we show that if 1 has a finite greedy expansion with positive coefficients, then the measure $m_p \otimes \mu_{\beta,p}$ is Markov, and we determine the measure $\mu_{\beta,p}$ explicitly. In Section 5 we discuss some open problems. As an application of some of the results in this paper, we also show that for λ-a.e. $x \in J_\beta$, there exist $2^{\aleph_0}$ so called universal expansions of x in base β.

2. The skew product transformation R_β

Define the *skew product* transformation R_β on $\Omega \times J_\beta$ as follows.

$$R_\beta(\omega, x) \; = \; \begin{cases} (\sigma(\omega), T_\beta x) & \text{if } \; \omega_1 = 1, \\ (\sigma(\omega), L_\beta x) & \text{if } \; \omega_1 = 0. \end{cases}$$

On the set $\Omega \times J_\beta$, we consider the σ-algebra $\mathcal{A} \otimes \mathcal{B}$, where $\mathcal{A}$ is the product σ-algebra on Ω and $\mathcal{B}$ is the Borel σ-algebra on J_β. Let μ be an arbitrary probability measure on J_β. It is easy to see that $m_p \otimes \mu$ is R_β-invariant if and only if $\mu = p \cdot \mu \circ T_\beta^{-1} + (1-p) \cdot \mu \circ L_\beta^{-1}$. The following result shows that a product measure of the form $m_p \otimes \mu$ is K_β-invariant if and only if it is R_β-invariant.

Lemma 1. *$m_p \otimes \mu \circ K_\beta^{-1} = m_p \otimes \mu \circ R_\beta^{-1} = m_p \otimes \nu$, where*

$$\nu = p \cdot \mu \circ T_\beta^{-1} + (1-p) \cdot \mu \circ L_\beta^{-1}.$$

Proof. Denote by C an arbitrary cylinder in Ω and let (a,b) be an interval in J_β. It suffices to verify that the measures coincide on sets of the form $C \times (a,b)$, since the collection of these sets forms a generating π-system. Furthermore, let $[i, C] = \{\omega_1 = i\} \cap \sigma^{-1}(C)$ for $i = 0, 1$. Note that $E \cap T_\beta^{-1}(a,b) = E \cap L_\beta^{-1}(a,b)$, and that

$$\begin{aligned} K_\beta^{-1}(C \times (a,b)) \;&=\; C \times (E \cap T_\beta^{-1}(a,b)) \\ &\quad \cup [0, C] \times (S \cap L_\beta^{-1}(a,b)) \\ &\quad \cup [1, C] \times (S \cap T_\beta^{-1}(a,b)). \end{aligned}$$

Hence,

$$\begin{aligned} m_p \otimes \mu \circ K_\beta^{-1}(C \times (a,b)) &= p \cdot m_p(C) \cdot \mu(T_\beta^{-1}(a,b)) \\ &\quad +(1-p) \cdot m_p(C) \cdot \mu(L_\beta^{-1}(a,b)) \\ &= m_p \otimes \nu(C \times (a,b)). \end{aligned}$$

On the other hand,

$$R_\beta^{-1}(C \times (a,b)) = [0,C] \times L_\beta^{-1}(a,b) \cup [1,C] \times T_\beta^{-1}(a,b),$$

and the result follows. □

Let $\mathfrak{D} = \mathfrak{D}(J_\beta, \mathcal{B}, \lambda)$ denote the space of probability density functions on J_β with respect to λ. A measurable transformation $T : J_\beta \to J_\beta$ is called nonsingular if $\lambda(T^{-1}B) = 0$ whenever $\lambda(B) = 0$.

If μ is absolutely continuous w.r.t. λ with probability density $f = d\mu/d\lambda$ and if T is a nonsingular transformation, then $\mu \circ T^{-1}$ is absolutely continuous w.r.t. λ with probability density $P_T f$ (say). Equivalently, the Frobenius-Perron operator $P_T : \mathfrak{D} \to \mathfrak{D}$ is defined as a linear operator such that for $f \in \mathfrak{D}$, $P_T f$ is the function for which

$$\int_B P_T f d\lambda = \int_{T^{-1}B} f d\lambda \quad \text{for all} \quad B \in \mathcal{B}.$$

Existence and uniqueness (λ-a.e.) follow from the Radon-Nikodým Theorem. A nonsingular transformation $T : J_\beta \to J_\beta$ is said to be a Lasota-Yorke type map (L-Y map) if T is piecewise monotone and C^2. Piecewise monotone and C^2 means that there exists a partition $\mathcal{P} = \{[a_{i-1}, a_i] : i = 1, \ldots, k\}$, such that for each $i = 1, \ldots, k$, the restriction of T to (a_{i-1}, a_i) is monotone and extends to a C^2 map on $[a_{i-1}, a_i]$. For such a transformation the Frobenius-Perron operator can be computed explicitly (see [BG, page 86]) by the formula

$$P_T f(x) = \sum_{T(y)=x} \frac{f(y)}{|T'(y)|}. \tag{1}$$

If, in addition, $|T'(x)| \geq \alpha > 1$ for each $x \in (a_{i-1}, a_i)$, $i = 1, \ldots, k$, then we say that T is a piecewise expanding L-Y map. Let $T_1, \ldots, T_n$ be L-Y maps on J_β. Define for $f \in \mathfrak{D}$, $Pf = \sum_{i=1}^n p_i \cdot P_{T_i} f$, where $(p_1, \ldots, p_n)$ is a probability vector. We recall the following important theorem, due to Pelikan; [Pel]. For more results concerning invariant densities of L-Y maps see [LY], [LiY], [Pel].

Theorem 3. *Suppose that for all $x \in J_\beta$, $\sum_{i=1}^n \frac{p_i}{|T_i'(x)|} \leq \gamma < 1$, where $T_i'(x)$ is the appropriate one sided derivative at the endpoints of $\mathcal{P}$. Then for all $f \in \mathfrak{D}$, the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} P^j f = f^*$$

exists in $L_1(J_\beta, \lambda)$. Furthermore, $Pf^* = f^*$ and one can choose f^* to be of bounded variation.

Since T_β and L_β are both piecewise expanding L-Y maps, it follows at once from Theorem 3 that for all $f \in \mathfrak{D}$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} P^j f = f^*$$

exists in $L_1(J_\beta, \lambda)$, where

$$Pf = p \cdot P_{T_\beta} f + (1-p) \cdot P_{L_\beta} f.$$

Define for $f \in \mathfrak{D}$ the probability measure μ_f by

$$\mu_f(B) = \int_B f d\lambda \qquad [B \in \mathcal{B}].$$

Observe that $Pf = f$ if and only if

$$\mu_f = p \cdot \mu_f \circ T_\beta^{-1} + (1-p) \cdot \mu_f \circ L_\beta^{-1},$$

i.e., if and only if $m_p \otimes \mu_f$ is R_β-invariant (cf. Lemma 1).

Let $\mathbf{1}$ denote the constant function equal to 1 on J_β and consider the function $\mathbf{1}^*$, given by

$$\mathbf{1}^* = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} P^j \mathbf{1} \quad \text{in } L_1(J_\beta, \lambda).$$

We shall assume that the function $\mathbf{1}^*$ is of bounded variation. Note that this is possible by Theorem 3. It follows easily from the definition of bounded variation that the left- and right hand limits of $\mathbf{1}^*$ at every point $x \in J_\beta$ exist and that the function $\mathbf{1}^*$ is continuous up to countably many points. Now we modify the function $\mathbf{1}^*$ in such a way that it becomes lower semicontinuous. Replace $\mathbf{1}^*(x)$ at every discontinuity point x in the interior of J_β, by setting

$$\mathbf{1}^*(x) = \min\{\mathbf{1}^*(x^-), \mathbf{1}^*(x^+)\}$$

and replace $\mathbf{1}^*(x)$ by its left- or right hand limit if x is an endpoint of J_β. From now on we work with this modified version of $\mathbf{1}^*$ which we denote again by $\mathbf{1}^*$. In the next theorem, we show that this function is bounded below by a positive constant $d > 0$, everywhere on J_β.

Theorem 4. *The skew product transformation R_β is ergodic w.r.t. the measure $m_p \otimes \mu_{\mathbf{1}^*}$. Furthermore, the measures $m_p \otimes \mu_{\mathbf{1}^*}$ and $m_p \otimes \lambda$ are equivalent and the density $\mathbf{1}^*$ is bounded below by a positive constant d, everywhere on J_β.*

Proof. Since $P\mathbf{1}^* = \mathbf{1}^*$, it follows from Lemma 1 that the measure $m_p \otimes \mu_{\mathbf{1}^*}$ is R_β-invariant. It is well-known that the greedy transformation T_β is ergodic w.r.t. its unique absolutely continuous invariant measure, which is the Parry measure μ_β (see Section 1). Similarly, the lazy transformation is ergodic w.r.t. its unique absolutely continuous invariant measure. This implies [Pel, Corollary 7] that the skew product transformation R_β is ergodic w.r.t. $m_p \otimes \mu_{\mathbf{1}^*}$. Since the random Frobenius-Perron operator P is integral preserving w.r.t. λ, we have that

$$\int_{J_\beta} \mathbf{1}^* d\lambda = 1.$$

In particular, there exists a point x in the interior of J_β, for which $\mathbf{1}^*(x) > 0$. By lower semicontinuity of $\mathbf{1}^*$, there exist an open interval $(a,b) \subset J_\beta$ and a constant $c > 0$ such that $\mathbf{1}^*(x) > c$ for each $x \in (a,b)$. Rewriting (1) one gets

$$P_{T_\beta} f(x) = \frac{1}{\beta} \sum_{T_\beta y = x} f(y)\ ,\ P_{L_\beta} f(x) = \frac{1}{\beta} \sum_{L_\beta y = x} f(y), \tag{2}$$

see also [P, Theorem 1]. It follows that $\mathbf{1}^*$ is the unique probability density function (λ-a.e.) satisfying

$$\mathbf{1}^*(x) = \frac{p}{\beta} \sum_{T_\beta y = x} \mathbf{1}^*(y) + \frac{1-p}{\beta} \sum_{L_\beta y = x} \mathbf{1}^*(y).$$

Hence, for λ-a.e. $x \in T_\beta(a,b)$, we have that

$$\mathbf{1}^*(x) > \frac{pc}{\beta}.$$

By induction, we have that for each n and for λ-a.e. $x \in T_\beta^n(a,b)$,

$$\mathbf{1}^*(x) > \frac{p^n c}{\beta^n}.$$

It is easy to verify that there exist a number $\delta > 0$ and a positive integer n, such that

$$T_\beta^n(a,b) \supset [x, x+\delta),$$

where x is a discontinuity point of T_β. Hence,

$$T_\beta^{n+1}(a,b) \supset [0, \beta\delta).$$

Moreover, there exists a positive integer m, such that

$$L_\beta^m([0,\beta\delta)) = J_\beta.$$

Using the same argument as before, we conclude that for λ-a.e. $x \in J_\beta$,

$$\mathbf{1}^*(x) > d := \frac{p^{n+1}(1-p)^m c}{\beta^{n+m+1}}.$$

Hence, the function $\mathbf{1}^*$ is larger or equal than d at every continuity point of $\mathbf{1}^*$. Due to our modification of $\mathbf{1}^*$ at discontinuity points, the function $\mathbf{1}^*$ is everywhere larger or equal than d. The equivalence of $m_p \otimes \mu_{\mathbf{1}^*}$ and $m_p \otimes \lambda$ is an immediate consequence. □

Since any invariant probability measure absolutely continuous w.r.t. an ergodic invariant probability measure coincides with this measure, we have from Theorem 3 and Theorem 4 that for all $f \in \mathfrak{D}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} P^j f = \mathbf{1}^* \quad \text{in } L_1(J_\beta, \lambda).$$

Remarks 1. (1) From now on we write $\mu_{\beta,p}$ instead of $\mu_{\mathbf{1}^*}$, since the measure depends on both β and p. It is the unique probability measure, absolutely continuous w.r.t. λ, satisfying the relationship

$$\mu_{\beta,p} = p \cdot \mu_{\beta,p} \circ T_\beta^{-1} + (1-p) \cdot \mu_{\beta,p} \circ L_\beta^{-1}. \tag{3}$$

(2) Recall that $\ell : J_\beta \to J_\beta$, given by $\ell(x) = \lfloor \beta \rfloor/(\beta - 1) - x$, satisfies $T_\beta \circ \ell = \ell \circ L_\beta$. It follows from the previous remark that $\mu_{\beta,p} \circ \ell^{-1} = \mu_{\beta,1-p}$. In particular, we see that the invariant density $\mathbf{1}^*$ is symmetric on J_β if $p = 1/2$.

(3) Let $T_1, \ldots, T_n$ be piecewise expanding L-Y maps on J_β and let $(p_1, \ldots, p_n)$ be a probability vector. Recently it has been shown by Boyarsky, Góra and Islam (see [BGI]) that functions $f \in \mathfrak{D}$ satisfying $f = Pf = \sum_{i=1}^n p_i \cdot P_{T_i} f$, are bounded below by a positive constant on their support (λ-a.e.). Hence, the fact that $\mathbf{1}^*$ is bounded below by a positive constant on J_β can also be deduced from their result combined with the equivalence of $m_p \otimes \lambda$ and $m_p \otimes \mu_{\beta,p}$.

(4) It is well-known that the Parry measure μ_β is the unique probability measure, absolutely continuous w.r.t. λ and satisfying equation (3) with $p = 1$. Note however that μ_β and λ are *not* equivalent on J_β. Similarly, the lazy measure ρ_β and λ are not equivalent. For this reason, we restrict ourselves in this paper to values of the parameter p in the open interval $(0, 1)$.

3. Main Theorem

It is the object of this section to show that the measure of maximal entropy ν_β for the map K_β and the measure $m_p \otimes \lambda$ are mutually singular.

Let $D = \{0, 1, \ldots, \lfloor \beta \rfloor\}^{\mathbb{N}}$ be equipped with the product σ-algebra $\mathcal{D}$ and let σ' be the left shift on D. Define the function $\varphi : \Omega \times J_\beta \to D$ by

$$\varphi(\omega, x) = (d_1(\omega, x), d_2(\omega, x), \ldots).$$

Clearly, φ is measurable and $\varphi \circ K_\beta = \sigma' \circ \varphi$. Furthermore, Theorem 2 implies that φ is surjective. Let

$$Z = \{(\omega, x) \in \Omega \times J_\beta : K_\beta^n(\omega, x) \in \Omega \times S \text{ for infinitely many } n \geq 0\},$$

and

$$D' = \{(a_1, a_2, \ldots) \in D : \sum_{i=1}^{\infty} \frac{a_{j+i-1}}{\beta^i} \in S \text{ for infinitely many } j \geq 1\}.$$

Observe that $K_\beta^{-1}(Z) = Z$, $(\sigma')^{-1}(D') = D'$ and that the restriction $\varphi' : Z \to D'$ of the map φ to Z is a bimeasurable bijection. Let $\mathbb{P}$ denote the uniform product measure on D. We recall from [DdV] that the measure ν_β defined on $\mathcal{A} \otimes \mathcal{B}$ by $\nu_\beta(A) = \mathbb{P}(\varphi(Z \cap A))$ is the unique K_β-invariant measure of maximal entropy $\log(1 + \lfloor \beta \rfloor)$. It was also shown that the projection of ν_β on the second coordinate is an infinite convolution of Bernoulli measures (see [E1], [E2]). More precisely, consider the purely discrete probability measures $\{\delta_i\}_{i \geq 1}$ defined on J_β and determined by:

$$\delta_i(\{k\beta^{-i}\}) = \frac{1}{\lfloor \beta \rfloor + 1} \quad \text{for} \quad k = 0, 1, \ldots, \lfloor \beta \rfloor.$$

Let δ_β be the corresponding infinite Bernoulli convolution,

$$\delta_\beta = \lim_{n\to\infty} \delta_1 * \cdots * \delta_n,$$

then $\nu_\beta \circ \pi_2^{-1} = \delta_\beta$.

For $\omega \in \Omega$, let $\overline{\omega}$ be given by

$$\overline{\omega} = (\overline{\omega_1}, \overline{\omega_2}, \ldots) = (1-\omega_1, 1-\omega_2, \ldots).$$

Concerning the projection $\pi_1 : \Omega \times J_\beta \to \Omega$ of the measure ν_β on the first coordinate, we have the following lemma.

Lemma 2. *For $n \geq 1$ and $i_1, \ldots, i_n \in \{0,1\}$, we have*

$$\nu_\beta \circ \pi_1^{-1}(\{\omega_1 = i_1, \ldots, \omega_n = i_n\}) = \nu_\beta \circ \pi_1^{-1}(\{\overline{\omega_1} = i_1, \ldots, \overline{\omega_n} = i_n\}).$$

Proof. Define the map $r : D \to D$ by

$$r(a_1, a_2, \ldots) = (\lfloor\beta\rfloor - a_1, \lfloor\beta\rfloor - a_2, \ldots).$$

It follows easily by induction that for $i \geq 1$ and $(\omega, x) \in \Omega \times J_\beta$,

$$d_i(\omega, x) = \lfloor\beta\rfloor - d_i(\overline{\omega}, \ell(x)).$$

Hence,

$$\varphi(\omega, x) = r \circ \varphi(\overline{\omega}, \ell(x)).$$

Since the map r is clearly invariant w.r.t. $\mathbb{P}$, the assertion follows. $\square$

In particular, it follows from Lemma 2 that $\nu_\beta \circ \pi_1^{-1}(\{\omega_i = 1\}) = \frac{1}{2}$, for all $i \geq 1$. However, in general, the measure $\nu_\beta \circ \pi_1^{-1}$ is *not* the uniform Bernoulli measure on $\{0,1\}^{\mathbb{N}}$. For instance, using the techniques in [DdV, Section 4], one easily shows that if the greedy expansion of 1 in base β satisfies $1 = \frac{1}{\beta} + \frac{1}{\beta^3}$, then $\nu_\beta \circ \pi_1^{-1}$ provides a counter example. In the case that 1 has a finite greedy expansion with positive coefficients, it has been shown in [DdV, Theorem 8] that $\nu_\beta \circ \pi_1^{-1}$ *is* the uniform Bernoulli measure. The next lemma shows that the K_β-invariant measures ν_β and $m_p \otimes \mu_{\beta,p}$ are different, for all non-integer $\beta > 1$ and $0 < p < 1$.

Lemma 3. $\nu_\beta \neq m_p \otimes \mu_{\beta,p}$.

Proof. According to Theorem 4, there exists a constant $c > 0$, such that $\mathbf{1}^*(x) \geq c$ for all $x \in J_\beta$. Choose $n \in \mathbb{N}$ such that $\frac{1}{\beta} + \frac{1}{\beta^n} \in S_1$. Now, suppose the converse is true, *i.e.*, suppose that the measures ν_β and $m_p \otimes \mu_{\beta,p}$ coincide. In particular, we assume that ν_β is a product measure and that $\delta_\beta = \mu_{\beta,p}$.

On the one hand we infer from Lemma 2 that

$$\nu_\beta(\{\omega_1 = 1\} \times J_\beta \,|\, \Omega \times [\frac{1}{\beta}, \frac{1}{\beta} + \frac{1}{\beta^n})) = \frac{1}{2}.$$

On the other hand, since the digits $(d_i)_{i\geq 1}$ form a uniform Bernoulli process under ν_β,

$$\begin{aligned}
& \nu_\beta(\{\omega_1=1\}\times J_\beta \,\big|\, \Omega\times[\frac{1}{\beta},\frac{1}{\beta}+\frac{1}{\beta^n})) \\
= \;& \nu_\beta(\{d_1=1\}|\Omega\times[\frac{1}{\beta},\frac{1}{\beta}+\frac{1}{\beta^n})) \\
= \;& \frac{\nu_\beta(\{d_1=1,d_2=0,\ldots,d_n=0,\sum_{i=1}^\infty \frac{d_{n+i}}{\beta^i}\in[0,1)\})}{\mu_{\beta,p}([\frac{1}{\beta},\frac{1}{\beta}+\frac{1}{\beta^n}))} \\
\leq \;& \frac{1}{c}\left(\frac{\beta}{\lfloor\beta\rfloor+1}\right)^n \delta_\beta([0,1)).
\end{aligned}$$

Passing to the limit, we get a contradiction. □

Define the map $F:\Omega\times J_\beta\to D$ by

$$F(\omega,x)=(d_1(\omega,x),d_1(R_\beta(\omega,x)),d_1(R_\beta^2(\omega,x)),\ldots).$$

We have that $\sum_{i=1}^\infty d_1(R_\beta^{i-1}(\omega,x))/\beta^i=x$ for all $(\omega,x)\in\Omega\times J_\beta$. Moreover, the map F is surjective and $\sigma'\circ F=F\circ R_\beta$. Hence F is a factor map and σ' is ergodic w.r.t. the measure $\rho=m_p\otimes\mu_{\beta,p}\circ F^{-1}$. Note however, that the map F is not injective, even if we restrict F to the set for which R_β hits $\Omega\times S$ infinitely many times; this is due to the fact that in equality regions only one digit can be assigned. It follows from Theorem 4 and Birkhoff's Ergodic Theorem that ρ is concentrated on D'. Therefore, the measure ρ' defined on $\mathcal{A}\otimes\mathcal{B}$ by $\rho'(A)=\rho(\varphi(A\cap Z))$ is a K_β-invariant probability measure and K_β is ergodic w.r.t. ρ'.

Lemma 4. $\rho'=m_p\otimes\mu_{\beta,p}$.

Proof. Let

$$\begin{aligned}
&A_{00}=\{\omega_1=0\}\times S_1 \qquad && A_{\lfloor\beta\rfloor 1}=\{\omega_1=1\}\times S_{\lfloor\beta\rfloor}\\
&A_{02}=\Omega\times E_0 && A_{\lfloor\beta\rfloor 2}=\Omega\times E_{\lfloor\beta\rfloor}
\end{aligned}$$

and

$$\begin{aligned}
A_{i0} &= \{\omega_1=0\}\times S_{i+1}\\
A_{i1} &= \{\omega_1=1\}\times S_i\\
A_{i2} &= \Omega\times E_i,
\end{aligned}$$

for $1\leq i\leq\lfloor\beta\rfloor-1$. Note that for all i, $\varphi^{-1}(\{d_1=i\})$ is the union of the sets A_{ij}. It is enough to show that $\rho'=m_p\otimes\mu_{\beta,p}$ on sets of the form

$$\varphi^{-1}(\{d_1=i_1,\ldots,d_n=i_n\}).$$

Now,

$$\varphi^{-1}(\{d_1=i_1,\ldots,d_n=i_n\})=\bigcup_{j_1,\ldots,j_n} A_{i_1j_1}\cap\cdots\cap K_\beta^{-n+1}A_{i_nj_n},$$

where the union is taken over all $j_1,\ldots,j_n$ for which $A_{i_1j_1},\ldots,A_{i_nj_n}$ are defined. Hence, it is enough to show that

$$\rho'(A_{i_1j_1}\cap\cdots\cap K_\beta^{-n+1}A_{i_nj_n})=m_p\otimes\mu_{\beta,p}(A_{i_1j_1}\cap\cdots\cap K_\beta^{-n+1}A_{i_nj_n}).$$

It is easy to see that the set $A_{i_1j_1}\cap\cdots\cap K_\beta^{-n+1}A_{i_nj_n}$ is a product set. Denote the projection on the second coordinate by $V_{i_1j_1\ldots i_nj_n}$.
Define

$$\mathcal{U}=\{(0,0),(\lfloor\beta\rfloor,1)\}\cup\{(i,j):1\le i\le\lfloor\beta\rfloor-1,j\in\{0,1\}\}$$

and

$$\{\ell_1,\ldots,\ell_L\}=\{\ell:(i_\ell,j_\ell)\in\mathcal{U}\}\subset\{1,\ldots,n\},\quad \ell_1<\cdots<\ell_L.$$

Then,

$$A_{i_1j_1}\cap\cdots\cap K_\beta^{-n+1}A_{i_nj_n}=\{\omega_1=j_{\ell_1},\ldots,\omega_L=j_{\ell_L}\}\times V_{i_1j_1\ldots i_nj_n}. \tag{4}$$

Note that for all $x\in V_{i_1j_1\ldots i_nj_n}$,

$$F^{-1}\circ\varphi(\{\omega_1=j_{\ell_1},\ldots,\omega_L=j_{\ell_L}\}\times\{x\})=\{\omega_{\ell_1}=j_{\ell_1},\ldots,\omega_{\ell_L}=j_{\ell_L}\}\times\{x\}.$$

Therefore,

$$F^{-1}\circ\varphi(A_{i_1j_1}\cap\cdots\cap K_\beta^{-n+1}A_{i_nj_n})=\{\omega_{\ell_1}=j_{\ell_1},\ldots,\omega_{\ell_L}=j_{\ell_L}\}\times V_{i_1j_1\ldots i_nj_n}. \tag{5}$$

The assertion follows immediately from (4) and (5). □

From Theorem 4, Lemma 3, Lemma 4 and the ergodicity of K_β w.r.t. ρ' and ν_β, we arrive at the following theorem.

Theorem 5. *The measures ν_β and $m_p\otimes\lambda$ are mutually singular.*

Remark 2. If $\beta\in(1,2)$ is a Pisot number, the mutual singularity of ν_β and $m_p\otimes\lambda$ is a simple consequence of the fact that in this case δ_β and λ are mutually singular (see [E1],[E2]).

4. Finite greedy expansion of 1 with positive coefficients, and the Markov property of the random β-expansion

In this section we assume that the greedy expansion of 1 in base β satisfies $1=b_1/\beta+b_2/\beta^2+\cdots+b_n/\beta^n$ with $b_i\ge1$ for $i=1,\ldots,n$ and $n\ge2$ (note that $\lfloor\beta\rfloor=b_1$). It has been shown in [DdV] that in this case the dynamics of K_β can be identified with a subshift of finite type with an irreducible adjacency matrix.

We exhibit the measure $m_p\otimes\mu_{\beta,p}$ obtained in the previous section explicitly. Moreover, it turns out that K_β is exact w.r.t. $m_p\otimes\mu_{\beta,p}$. The mutual singularity of ν_β and $m_p\otimes\lambda$, *i.e.*, Theorem 5, will be derived by elementary means, independent of the results established in the previous sections.

The analysis of the case $\beta^2=b_1\beta+1$ needs some adjustments. For this reason, we assume here that $\beta^2\ne b_1\beta+1$, and refer the reader to [DdV, Remarks 6(2)] for the appropriate modifications needed for the case $\beta^2=b_1\beta+1$. We first recall some results obtained in [DdV] briefly, without proof.

We begin by a proposition which plays a crucial role in finding the Markov partition describing the dynamics of K_β.

Proposition 1. *Suppose* 1 *has a finite greedy expansion of the form* $1 = b_1/\beta + b_2/\beta^2 + \cdots + b_n/\beta^n$. *If* $b_j \geq 1$ *for* $1 \leq j \leq n$, *then*

(i) $T_\beta^i 1 = L_\beta^i 1 \in E_{b_{i+1}}$, $0 \leq i \leq n-2$.

(ii) $T_\beta^{n-1} 1 = L_\beta^{n-1} 1 = \frac{b_n}{\beta} \in S_{b_n}$, $T_\beta^n 1 = 0$, *and* $L_\beta^n 1 = 1$.

(iii) $T_\beta^i(\frac{b_1}{\beta-1} - 1) = L_\beta^i(\frac{b_1}{\beta-1} - 1) \in E_{b_1 - b_{i+1}}$, $0 \leq i \leq n-2$.

(iv) $T_\beta^{n-1}(\frac{b_1}{\beta-1} - 1) = L_\beta^{n-1}(\frac{b_1}{\beta-1} - 1) = \frac{b_1}{\beta(\beta-1)} + \frac{b_1 - b_n}{\beta} \in S_{b_1 - b_n + 1}$, $T_\beta^n(\frac{b_1}{\beta-1} - 1) = \frac{b_1}{\beta-1} - 1$, *and* $L_\beta^n(\frac{b_1}{\beta-1} - 1) = \frac{b_1}{\beta-1}$.

To find the Markov chain behind the map K_β, one starts by refining the partition

$$\mathcal{E} = \{E_0, S_1, E_1, \ldots, S_{b_1}, E_{b_1}\}$$

of $\left[0, \frac{b_1}{\beta-1}\right]$, using the orbits of 1 and $\frac{b_1}{\beta-1} - 1$ under the transformation T_β. We place the endpoints of $\mathcal{E}$ together with $T_\beta^i 1$, $T_\beta^i(\frac{b_1}{\beta-1} - 1)$, $i = 0, \ldots, n-2$, in increasing order. We use these points to form a new partition $\mathcal{C}$ which is a refinement of $\mathcal{E}$, consisting of intervals. We write $\mathcal{C}$ as

$$\mathcal{C} = \{C_0, C_1, \ldots, C_L\}.$$

We choose $\mathcal{C}$ to satisfy the following. For $0 \leq i \leq n-2$,

- $T_\beta^i 1 \in C_j$ if and only if $T_\beta^i 1$ is a left endpoint of C_j,
- $T_\beta^i(\frac{b_1}{\beta-1} - 1) \in C_j$ if and only if $T_\beta^i(\frac{b_1}{\beta-1} - 1)$ is a right endpoint of C_j.

Note that this choice is possible, since the points $T_\beta^i 1, T_\beta^i(\frac{b_1}{\beta-1} - 1)$ for $0 \leq i \leq n-2$, are all different. From the dynamics of K_β on this refinement, one reads the following properties of $\mathcal{C}$.

p1. $C_0 = \left[0, \frac{b_1}{\beta-1} - 1\right]$ and $C_L = \left[1, \frac{b_1}{\beta-1}\right]$.

p2. For $i = 0, 1, \ldots, b_1$, E_i can be written as a finite disjoint union of the form $E_i = \cup_{j \in M_i} C_j$ with $M_0, M_1, \ldots, M_{b_1}$ disjoint subsets of $\{0, 1, \ldots, L\}$. Further, the number of elements in M_i equals the number of elements in $M_{b_1 - i}$.

p3. For each S_i there corresponds exactly one $j \in \{0, 1, \ldots, L\} \setminus \cup_{k=0}^{b_1} M_k$ such that $S_i = C_j$.

p4. If $C_j \subset E_i$, then $T_\beta(C_j) = L_\beta(C_j)$ is a finite disjoint union of elements of $\mathcal{C}$, say $T_\beta(C_j) = C_{i_1} \cup \cdots \cup C_{i_l}$. Since $\ell(C_j) = C_{L-j} \subset E_{b_1 - i}$, it follows that $T_\beta(C_{L-j}) = C_{L-i_1} \cup \cdots \cup C_{L-i_l}$.

p5. If $C_j = S_i$, then $T_\beta(C_j) = C_0$ and $L_\beta(C_j) = C_L$.

To define the underlying subshift of finite type associated with the map K_β, we consider the $(L+1) \times (L+1)$ matrix $A = (a_{i,j})$ with entries in $\{0, 1\}$ defined by

$$a_{i,j} = \begin{cases} 1 & \text{if } i \in \cup_{k=0}^{b_1} M_k \text{ and } \lambda(C_j \cap T_\beta(C_i)) = \lambda(C_j), \\ 0 & \text{if } i \in \cup_{k=0}^{b_1} M_k \text{ and } C_i \cap T_\beta^{-1} C_j = \varnothing, \\ 1 & \text{if } i \in \{0, \ldots, L\} \setminus \cup_{k=0}^{b_1} M_k \text{ and } j = 0, L, \\ 0 & \text{if } i \in \{0, \ldots, L\} \setminus \cup_{k=0}^{b_1} M_k \text{ and } j \neq 0, L. \end{cases}$$

Let Y denote the topological Markov chain (or the subshift of finite type) determined by the matrix A. That is, $Y = \{y = (y_i) \in \{0, 1, \ldots, L\}^{\mathbb{N}} : a_{y_i, y_{i+1}} = 1\}$. We let σ_Y be the left shift on Y. For ease of notation, we denote by $s_1, \ldots, s_{b_1}$ the states $j \in \{0, \ldots, L\} \setminus \cup_{k=0}^{b_1} M_k$ corresponding to the switch regions $S_1, \ldots, S_{b_1}$ respectively.

For each $y \in Y$, we associate a sequence $(e_i) \in \{0, 1, \ldots, b_1\}^{\mathbb{N}}$ and a point $x \in \left[0, \frac{b_1}{\beta - 1}\right]$ as follows. Let

$$e_j = \begin{cases} i & \text{if } y_j \in M_i, \\ i & \text{if } y_j = s_i \text{ and } y_{j+1} = 0, \\ i - 1 & \text{if } y_j = s_i \text{ and } y_{j+1} = L. \end{cases} \tag{6}$$

Now set

$$x = \sum_{j=1}^{\infty} \frac{e_j}{\beta^j}. \tag{7}$$

Our aim is to define a map $\psi : Y \to \Omega \times \left[0, \frac{b_1}{\beta - 1}\right]$ that commutes the actions of K_β and σ_Y. Given $y \in Y$, equations (6) and (7) describe what the second coordinate of ψ should be. In order to be able to associate an $\omega \in \Omega$, one needs that $y_i \in \{s_1, \ldots, s_{b_1}\}$ infinitely often. For this reason it is not possible to define ψ on all of Y, but only on an invariant subset. To be more precise, let

$$Y' = \{y = (y_1, y_2, \ldots) \in Y : \, y_i \in \{s_1, \ldots, s_{b_1}\} \text{ for infinitely many } \, i\text{'s}\}.$$

Define $\psi : Y' \to \Omega \times \left[0, \frac{b_1}{\beta - 1}\right]$ as follows. Let $y = (y_1, y_2, \ldots) \in Y'$, and define x as given in (7). To define a point $\omega \in \Omega$ corresponding to y, we first locate the indices $n_i = n_i(y)$ where the realization y of the Markov chain is in state s_r for some $r \in \{1, \ldots, b_1\}$. That is, let $n_1 < n_2 < \cdots$ be the indices such that $y_{n_i} = s_r$ for some $r = 1, \ldots, b_1$. Define

$$\omega_j = \begin{cases} 1 & \text{if } y_{n_j + 1} = 0, \\ 0 & \text{if } y_{n_j + 1} = L. \end{cases}$$

Now set $\psi(y) = (\omega, x)$.

The following two lemmas reflect the fact that the dynamics of K_β is essentially the same as that of the Markov chain Y.

Lemma 5. *Let $y \in Y'$ be such that $\psi(y) = (\omega, x)$. Then,*

(i) $y_1 = k$ *for some* $k \in \bigcup_{i=0}^{b_1} M_i \Rightarrow \; x \in C_k$.
(ii) $y_1 = s_i$, $y_2 = 0 \;\Rightarrow\; x \in S_i$ *and* $\omega_1 = 1$ *for* $i = 1, \ldots, b_1$.
(iii) $y_1 = s_i$, $y_2 = L \;\Rightarrow\; x \in S_i$ *and* $\omega_1 = 0$ *for* $i = 1, \ldots, b_1$.

Lemma 6. *For $y \in Y'$, we have*

$$\psi \circ \sigma_Y(y) \;=\; K_\beta \circ \psi(y).$$

We now consider on Y the Markov measure $Q_{\beta,p}$ with transition matrix $P=(p_{i,j})$, given by

$$p_{i,j} = \begin{cases} \lambda(C_i \cap T_\beta^{-1}C_j)/\lambda(C_i) & \text{if } i \in \cup_{k=0}^{b_1} M_k, \\ p & \text{if } i \in \{0,\dots,L\} \setminus \cup_{k=0}^{b_1} M_k \text{ and } j=0, \\ 1-p & \text{if } i \in \{0,\dots,L\} \setminus \cup_{k=0}^{b_1} M_k \text{ and } j=L, \\ 0 & \text{if } i \in \{0,\dots,L\} \setminus \cup_{k=0}^{b_1} M_k \text{ and } j \neq 0, L, \end{cases}$$

and initial distribution the corresponding stationary distribution π.

Theorem 6. *$Q_{\beta,p} \circ \psi^{-1}$ is a product measure of the form $m_p \otimes \mu$.*

Proof. Define the measure μ on $\left[0, \frac{b_1}{\beta-1}\right]$ by

$$\mu(B) = \sum_{j=0}^{L} \frac{\lambda(B \cap C_j)}{\lambda(C_j)} \cdot \pi(j) \qquad [B \in \mathcal{B}].$$

Define the Markov partition $\mathcal{P}_0$ of $\Omega \times \left[0, \frac{b_1}{\beta-1}\right]$ by

$$\mathcal{P}_0 = \{\Omega \times C_j : j \in \cup_{k=0}^{b_1} M_k\} \cup \{\{\omega_1 = i\} \times S_j : i=0,1,\ j=1,\dots,b_1\}.$$

and let $\mathcal{P}_n = \mathcal{P}_0 \vee K_\beta^{-1}\mathcal{P}_0 \vee \cdots \vee K_\beta^{-n}\mathcal{P}_0$. It is straightforward to see that the inverse images of elements in $\mathcal{P}_n$ under ψ are cylinders in Y and that for each element $P \in \mathcal{P}_n$, $m_p \otimes \mu(P) = Q_{\beta,p} \circ \psi^{-1}(P)$. It follows that $Q_{\beta,p} \circ \psi^{-1} = m_p \otimes \mu$. □

Since P is an irreducible transition matrix, σ_Y is ergodic w.r.t. $Q_{\beta,p}$ and $\pi(i) > 0$ for all $i \in \{0,\dots,L\}$. It follows from Lemma 6 that K_β is ergodic w.r.t. $m_p \otimes \mu$. Furthermore, it is immediately seen from the definition that μ is equivalent with λ. Hence, the measure $Q_{\beta,p} \circ \psi^{-1}$ is equivalent with $m_p \otimes \lambda$.

Proposition 2. *The map K_β is exact w.r.t. $m_p \otimes \mu_{\beta,p}$. Moreover, $\mu = \mu_{\beta,p}$.*

Proof. It follows from Lemma 1 and Remarks 1(1) that $\mu = \mu_{\beta,p}$. Since the transition matrix P is also aperiodic, σ_Y is exact w.r.t. $Q_{\beta,p}$. It follows from Lemma 6 that K_β is exact w.r.t. $m_p \otimes \mu_{\beta,p}$. □

It also follows from the above proposition that the density $\mathbf{1}^*$ assumes the constant value $\pi(j)/\lambda(C_j)$ on the interval C_j, $j \in \{0,\dots,L\}$.

Example 1. Let $\beta = G = \frac{1}{2}(1+\sqrt{5})$ and let $g = G-1 = \frac{1}{2}(\sqrt{5}-1)$. Note that $1 = 1/\beta + 1/\beta^2$. In this case, we let $\mathcal{C} = \mathcal{E}$, since 1 and $1/(\beta-1)-1$ are already endpoints of intervals in $\mathcal{E}$. Using the techniques in this section it is easily verified that the dynamical system $(\Omega \times J_\beta, \mathcal{A} \otimes \mathcal{B}, m_p \otimes \mu_{\beta,p}, K_\beta)$ is measurably isomorphic to the Markov chain with transition matrix P, given by

$$P = \begin{pmatrix} g & g^2 & 0 \\ p & 0 & 1-p \\ 0 & g^2 & g \end{pmatrix},$$

and stationary distribution π determined by $\pi P = \pi$.

It remains to prove that $Q_{\beta,p} \circ \psi^{-1}$ and ν_β are mutually singular. Since K_β is ergodic w.r.t. both measures, it suffices to show that the measures do not coincide.

Lemma 7. $\nu_\beta \neq Q_{\beta,p} \circ \psi^{-1}$.

Proof. We distinguish between the cases $p = 1/2$ and $p \neq 1/2$.
Suppose $p = 1/2$. On the one hand we have that for all $i \in \{1, \dots, \lfloor\beta\rfloor\}$

$$\frac{i}{\beta} + \sum_{i=2}^{\infty} \frac{d_i}{\beta^i} \in S_i \Longleftrightarrow \sum_{i=1}^{\infty} \frac{d_{i+1}}{\beta^i} \in C_0,$$

$$\frac{i-1}{\beta} + \sum_{i=2}^{\infty} \frac{d_i}{\beta^i} \in S_i \Longleftrightarrow \sum_{i=1}^{\infty} \frac{d_{i+1}}{\beta^i} \in C_L.$$

Using the fact that the digits $(d_i)_{i\geq 1}$ form a uniform Bernoulli process under ν_β, a simple calculation yields that

$$\nu_\beta(\Omega \times S) = \frac{\lfloor\beta\rfloor}{\lfloor\beta\rfloor + 1} \cdot \nu_\beta(\Omega \times C_0) + \frac{\lfloor\beta\rfloor}{\lfloor\beta\rfloor + 1} \cdot \nu_\beta(\Omega \times C_L).$$

Since $\nu_\beta(\Omega \times C_0) = \nu_\beta(\Omega \times C_L)$, it follows that

$$\frac{\nu_\beta(\Omega \times S)}{\nu_\beta(\Omega \times C_0)} = \frac{2\lfloor\beta\rfloor}{\lfloor\beta\rfloor + 1}.$$

On the other hand, it follows from $\pi P = \pi$ that

$$\pi(0) = \frac{1}{\beta}\pi(0) + \frac{1}{2}(\pi(s_1) + \cdots + \pi(s_{b_1})).$$

Rewriting one gets

$$\frac{\pi(s_1) + \cdots + \pi(s_{b_1})}{\pi(0)} = \frac{Q_{\beta,p} \circ \psi^{-1}(\Omega \times S)}{Q_{\beta,p} \circ \psi^{-1}(\Omega \times C_0)} = \frac{2(\beta - 1)}{\beta}.$$

However,

$$\frac{2(\beta-1)}{\beta} \neq \frac{2\lfloor\beta\rfloor}{\lfloor\beta\rfloor + 1}$$

for all non-integer β, in particular for the β's under consideration.
Suppose $p \neq 1/2$. In this case, the assertion follows from the fact that the projection of ν_β on the first coordinate is the uniform Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ [DdV, Theorem 8]. Note that this result is applicable since 1 has a finite greedy expansion with positive coefficients. □

The mutual singularity of ν_β and $m_p \otimes \lambda$ follows as before.

5. Open problems and final remarks

1. We have not been able to find an explicit formula for $\mathbf{1}^*$. Recall that the Parry density $h_\beta = P_{T_\beta} h_\beta$ is given by

$$h_\beta(x) = \frac{1}{F(\beta)} \sum_{x < T_\beta^n(1)} \frac{1}{\beta^n}.$$

(see Section 1). We expect that the density $\mathbf{1}^*$ can be expressed in a similar way, but now the random orbits of 1 as well as the random orbits of the complementary point $\frac{\lfloor\beta\rfloor}{\beta-1} - 1$ are involved. Let us consider an example.

Example 2. Let $p = 1/2$ and $\beta = 3/2$. Note that in this case $\frac{\lfloor\beta\rfloor}{\beta-1} - 1 = 1$. Rewriting (2) one gets

$$P_{T_\beta} f(x) = \frac{1}{\beta} \sum_{i=0}^{1} f(\frac{x+i}{\beta}) \cdot 1_{[0,1)}(x) + \frac{1}{\beta} f(\frac{x+1}{\beta}) \cdot 1_{[1,2]}(x)$$

and

$$P_{L_\beta} f(x) = \frac{1}{\beta} f(\frac{x}{\beta}) \cdot 1_{[0,1]}(x) + \frac{1}{\beta} \sum_{i=0}^{1} f(\frac{x+i}{\beta}) \cdot 1_{(1,2]}(x).$$

It is easy to verify that $\mathbf{1} \in \mathfrak{D}$ satisfies $P\mathbf{1} = \mathbf{1}$, hence $\mathbf{1}^* = \mathbf{1}$. It follows that $m_{1/2} \otimes \lambda$ is $K_{3/2}$-invariant.

2. We have not been able to give an explicit formula for $h_{m_p \otimes \mu_{\beta,p}}(K_\beta)$. However, in the special case that $\beta^2 = b_1\beta + 1$, the entropy is already calculated in [DK2]:

$$h_{m_p \otimes \mu_{\beta,p}}(K_\beta) = \log\beta - \frac{b_1}{1+\beta^2}\left(p\log p + (1-p)\log(1-p)\right).$$

Since in this case $\pi(s_i) = \frac{1}{1+\beta^2}$, $i = 1, \ldots, b_1$, it follows that

$$h_{m_p \otimes \mu_{\beta,p}}(K_\beta) = \log\beta - \mu_{\beta,p}(S)\left(p\log p + (1-p)\log(1-p)\right).$$

One might conjecture that this formula holds in general.

3. Fix $p \in (0,1)$. It is a direct consequence of Birkhoff's Ergodic Theorem, Theorem 4 and the ergodicity of K_β w.r.t. $m_p \otimes \mu_{\beta,p}$, that for $m_p \otimes \lambda$-a.e. $(\omega, x) \in \Omega \times J_\beta$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_{\Omega\times S}(K_\beta^i(\omega, x)) = \mu_{\beta,p}(S) > 0. \tag{8}$$

In particular, we infer from (8), that the set

$$G = \{x \in J_\beta : x \text{ has a unique expansion in base } \beta\}$$

has Lebesgue measure zero, since for all $(\omega, x) \in \Omega \times G$, $K_\beta^n(\omega, x) \in \Omega \times E$, for all $n \ge 0$. Let $T_0 = L_\beta$, $T_1 = T_\beta$, and let

$$N = \bigcup_{n=1}^{\infty} \{x \in J_\beta : T_{u_1} \circ \cdots \circ T_{u_n} x \in G, \text{ for some } u_1, \ldots, u_n \in \{0,1\}\}.$$

Since the greedy map and the lazy map are nonsingular, $\lambda(N) = 0$. Note that $\Omega \times J_\beta \setminus N \subset Z$ and that for $x \in J_\beta \setminus N$, different elements of Ω give rise to different expansions of x in base β. We conclude that for λ-a.e. $x \in J_\beta$, there exist $2^{\aleph_0}$ expansions of x in base β. For a more elementary proof of this fact in case $\beta \in (1, 2)$, we refer to [S1].

4. Erdős and Komornik introduced in [EK] the notion of universal expansions. They called an expansion $(d_1, d_2, \ldots)$ in base β of some $x \in J_\beta$ universal if for each (finite) block $b_1 \ldots b_n$ consisting of digits in the set $\{0, \ldots, \lfloor\beta\rfloor\}$, there exists an index $k \geq 1$, such that $d_k \ldots d_{k+n-1} = b_1 \ldots b_n$. They proved that there exists a number $\beta_0 \in (1, 2)$, such that for each $\beta \in (1, \beta_0)$, *every* $x \in (0, 1/(\beta - 1))$ has a universal expansion in base β. Subsequently, Sidorov proved in [S2] that for a given $\beta \in (1, 2)$ and for λ-a.e. $x \in J_\beta$, there exists a universal expansion of x in base β. We now strengthen his result and the preceding remark by the following theorem.

Theorem 7. *For any non-integer $\beta > 1$, and for λ-a.e. $x \in J_\beta$, there exist $2^{\aleph_0}$ universal expansions of x in base β.*

In order to prove Theorem 7 we need the following lemma.

Lemma 8. *Let $\beta > 1$ be a non-integer and let $p \in (0, 1)$. Then, for $n \geq 1$ and $i_1, \ldots, i_n \in \{0, \ldots, \lfloor\beta\rfloor\}$, we have that*

$$m_p \otimes \mu_{\beta,p}(\{d_1 = i_1, \ldots, d_n = i_n\}) > 0.$$

Proof. By Theorem 4, it suffices to show that

$$m_p \otimes \lambda(\{d_1 = i_1, \ldots, d_n = i_n\}) > 0.$$

It is easy to verify that there exists a sequence $(j_1, j_2, \ldots) \in D$, starting with $i_1 \ldots i_n$, such that the numbers $x_1, \ldots, x_n$, given by

$$x_r = \sum_{i=1}^{\infty} \frac{j_{i+r-1}}{\beta^i}, \quad r = 1, \ldots, n,$$

are elements of $J_\beta \setminus \partial(S)$, where $\partial(S)$ denotes the boundary of S. Consider for $m \geq 1$, the set

$$I_m = \left[\sum_{i=1}^{n+m} \frac{j_i}{\beta^i}, \sum_{i=1}^{n+m} \frac{j_i}{\beta^i} + \sum_{i=n+m+1}^{\infty} \frac{\lfloor\beta\rfloor}{\beta^i}\right].$$

Let $y \in I_m$ and let $(a_1, a_2, \ldots)$ be an expansion y, starting with $j_1 \ldots j_{n+m}$. Define

$$y_r = \sum_{i=1}^{\infty} \frac{a_{i+r-1}}{\beta^i}, \quad r = 1, \ldots, n.$$

Choose m large enough, so that for each $r = 1, \ldots, n$, x_r and y_r are elements of the same equal or switch region, regardless of the values of the digits a_ℓ, $\ell > n + m$, and hence regardless of the chosen element $y \in I_m$. Note that this is possible because $x_r \notin \partial(S)$ for $r = 1, \ldots, n$. Denote the set of indices $r \in \{1, \ldots, n\}$ for which $x_r \in S$ by $\{\ell_1, \ldots, \ell_L\}$. Then, for suitably chosen $u_1, \ldots, u_L \in \{0, 1\}$, we have that

$$\{\omega_1 = u_1, \ldots, \omega_L = u_L\} \times I_m \subset \{d_1 = i_1, \ldots, d_n = i_n\}$$

and the conclusion follows. □

Proof of Theorem 7. Fix $p \in (0,1)$ and let $b_1 \dots b_n$ be an arbitrary block. Using Birkhoff's Ergodic Theorem, Theorem 4, Lemma 8 and the ergodicity of K_β w.r.t. $m_p \otimes \mu_{\beta,p}$, we may conclude that for $m_p \otimes \lambda$-a.e. $(\omega, x) \in \Omega \times J_\beta$, the block $b_1 \dots b_n$ occurs in

$$(d_1(\omega,x), d_2(\omega,x), \dots). \tag{9}$$

with positive limiting frequency $m_p \otimes \mu_{\beta,p}(\{d_1 = b_1, \dots, d_n = b_n\})$. In particular, we have that for $m_p \otimes \lambda$-a.e. $(\omega, x) \in \Omega \times J_\beta$, the block $b_1 \dots b_n$ occurs in (9). Since there are only countably many blocks, we have that for $m_p \otimes \lambda$-a.e. $(\omega, x) \in \Omega \times J_\beta$, the expansion (9) is universal in base β. An application of Fubini's Theorem yields that there exists a Borel set $B \subset J_\beta \setminus N$ of full Lebesgue measure and there exist sets $A_x \in \mathcal{A}$ with $m_p(A_x) = 1\,(x \in B)$, such that for all $x \in B$ and $(\omega, x) \in A_x \times \{x\}$, the expansion (9) is universal in base β. Since the sets A_x have necessarily the cardinality of the continuum and since different elements of Ω give rise to different expansions of x in base β for any $x \in J_\beta \setminus N$, the assertion follows. □

5. An expansion $(a_1, a_2, \dots)$ in base β of some number $x \in J_\beta$ is called *normal* if each block $i_1 \dots i_n$ with digits in $\{0, \dots, \lfloor\beta\rfloor\}$ occurs in $(a_1, a_2, \dots)$ with limiting frequency $(\lfloor\beta\rfloor + 1)^{-n}$. Note that a normal expansion is in particular universal.

Fix $p \in (0,1)$. Since $\nu_\beta \neq m_p \otimes \mu_{\beta,p}$ and since both measures ν_β and $m_p \otimes \mu_{\beta,p}$ are concentrated on Z, there exists a block $i_1 \dots i_n$ such that

$$m_p \otimes \mu_{\beta,p}(\{d_1 = i_1, \dots, d_n = i_n\}) \neq (\lfloor\beta\rfloor + 1)^{-n}.$$

Hence, for $m_p \otimes \lambda$-a.e. $(\omega, x) \in \Omega \times J_\beta$, the expansion (9) is universal but *not* normal. On the other hand, Sidorov proved in [S2], that there exists a Borel set $V \subset (1,2)$ of full Lebesgue measure, such that for each $\beta \in V$ and for λ-a.e. $x \in J_\beta$, there exists a normal expansion of x in base β.

Acknowledgements. We are very grateful to the referee of [DdV] for suggesting the problem discussed in this paper. We also thank Pawel Góra, Pierre Liardet and Boris Solomyak for very fruitful discussions concerning the proof of Theorem 4. Finally, we would like to thank the referee of this paper for many helpful suggestions concerning the presentation of this paper.

References

[BG] Boyarsky, A., Góra, P. – *Laws of chaos*, Probability and Its Applications, Birkhäuser, Boston, (1997).

[BGI] Boyarsky, A., Góra, P., Islam, M.S. – *Invariant densities of random maps have lower bounds on their supports*, `http://www.mathstat.concordia.ca/pg/Support_Random6.pdf`

[DK1] Dajani, K., Kraaikamp, C. – *From Greedy to Lazy Expansions, and their driving dynamics*, Expo. Math. **20** (2002), no.4, 315-327.

[DK2] Dajani, K., Kraaikamp, C. – *Random β-expansions*, Ergodic Theory Dynam. Systems. **23** (2003), no.2, 461-479.

[DdV] Dajani, K., de Vries, M. – *Measures of maximal entropy for random β-expansions*, J. Eur. Math. Soc. (JEMS) **7** (2005), no.1, 51-68.

[E1] Erdős, P. – *On a family of symmetric Bernoulli convolutions*, Amer. J. Math. **61** (1939), 974-976.

[E2] Erdős, P. – *On the smoothness properties of Bernoulli convolutions*, Amer. J. Math. **62** (1940), 180-186.

[EJK] Erdős, P., Joó, I., Komornik, V. – *Characterization of the unique expansions* $1 = \sum_{i=1}^{\infty} q^{-n_i}$ *and related problems*, Bull. Soc. Math. France **118** (1990), no.3, 377–390.

[EK] Erdős, P., Komornik, V. – *Developments in non-integer bases*, Acta Math. Hung. **79** (1998), 57-83.

[JS] Joó, I., Schnitzer, F.J. – *Expansions with respect to non-integer bases*, Grazer Mathematische Berichte, **329**. Karl-Franzens-Universität Graz, Graz, (1996).

[G] Gel'fond, A.O. – *A common property of number systems*, Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 809–814.

[LY] Lasota A., Yorke J.A. – *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1973), 481-488.

[LiY] Li T.-Y., Yorke J.A. – *Ergodic transformations from an interval into itself*, Trans. Amer. Math. Soc. **235** (1978), 183-192.

[P] Parry, W. – *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416.

[Pel] Pelikan, S. – *Invariant densities for random maps on the interval*, Trans. Amer. Math. Soc. **281** (1984), no.2, 813-825.

[R1] Rényi, A. – *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hung. **8** (1957), 477-493.

[R2] Rényi, A. – *On algorithms for the generation of real numbers*, Magyar Tud. Akad. Mat. Fiz. Oszt. Kzl. **7** (1957), 265–293.

[S1] Sidorov, N. – *Almost every number has a continuum of β-expansions*, Amer. Math. Monthly **110** (2003), no.9, 838-842.

[S2] Sidorov, N. – *Universal β-expansions*, Period. Math. Hungar. **47** (2003), no.1-2, 221-231.

Universiteit Utrecht, Fac. Wiskunde en Informatica and MRI, Budapestlaan 6, P.O. Box 80.000, 3508 TA Utrecht, the Netherlands
E-mail address: dajani@math.uu.nl

Vrije Universiteit Amsterdam, Faculty of Exact Sciences, de Boelelaan 1081, 1081 HV Amsterdam, the Netherlands
E-mail address: mdvries@cs.vu.nl