An optimal adaptive wavelet method for nonsymmetric and indefinite elliptic problems*

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Abstract

In this paper, we modify the adaptive wavelet algorithm from [Technical Report 1325, Department of Mathematics, Utrecht University, March 2005] so that it applies directly, i.e., without forming the normal equation, not only to self-adjoint elliptic operators but also to such operators to which generally nonsymmetric lower order terms are added, assuming that the resulting operator equation is well-posed. We show that the algorithm has optimal computational complexity.

Keywords: Adaptive methods, strongly elliptic operators, wavelets, optimal computational complexity, best N-term approximation

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1 Introduction

Let $\mathcal{H}$ be a real Hilbert space and let $\mathcal{H}'$ denote its dual. Given a boundedly invertible linear operator $L : \mathcal{H} \to \mathcal{H}'$ and a linear functional $f \in \mathcal{H}'$, we consider the problem of finding $u \in \mathcal{H}$ such that

$$Lu = f.$$

As an example of $\mathcal{H}$ one can think of the Sobolev space H^t on a domain or manifold, possibly incorporating essential boundary conditions. Then the weak formulation of (scalar) linear differential or integral equations of order $2t$ leads to the above type of equations.

Let $\Psi = \{\psi_\lambda \in \mathcal{H} : \lambda \in \nabla\}$ be a *Riesz basis* for $\mathcal{H}$ with a countable index set ∇. We consider Ψ formally as a column vector whose entries are elements of $\mathcal{H}$. Let $u = \mathbf{u}^T\Psi$ where $\mathbf{u}$ is a column vector in $\ell_2 := \ell_2(\nabla)$. Then the above problem is *equivalent* to finding $\mathbf{u} \in \ell_2$ satisfying the infinite matrix-vector system

$$\mathbf{L}\mathbf{u} = \mathbf{f}, \tag{1.1}$$

where $\mathbf{L} := \langle \psi_\lambda, L\psi_\mu \rangle_{\lambda,\mu\in\nabla} : \ell_2 \to \ell_2$ is boundedly invertible and $\mathbf{f} := \langle f, \psi_\lambda \rangle_{\lambda\in\nabla} \in \ell_2$. Here $\langle \cdot, \cdot \rangle$ denotes the duality product on $\mathcal{H} \times \mathcal{H}'$. In the following, we will also use $\langle \cdot, \cdot \rangle$ to denote $\langle \cdot, \cdot \rangle_{\ell_2}$ if there is no risk of confusion, and use $\|\cdot\|$ to denote $\|\cdot\|_{\ell_2}$ as well as $\|\cdot\|_{\ell_2\to\ell_2}$. Note that $\langle \mathbf{v}, \mathbf{w} \rangle_{\ell_2}$ can also be written as $\mathbf{w}^T\mathbf{v}$.

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Our goal is now to approximate the solution $\mathbf{u}$ of (1.1) by a finitely supported vector. In order to assess the quality of this approximation, we consider the following. Let $\gamma_n(\mathbf{u})$ denote the nth largest coefficient in modulus of $\mathbf{u}$. For $0 < \tau < 2$, the space $\ell_\tau^w := \ell_\tau^w(\nabla)$ is defined by

$$\ell_\tau^w = \left\{ \mathbf{u} \in \ell_2 : |\mathbf{u}|_{\ell_\tau^w} := \sup_n n^{1/\tau} |\gamma_n(\mathbf{u})| < \infty \right\}.$$

It is easily verified that $\ell_\tau \hookrightarrow \ell_\tau^w \hookrightarrow \ell_{\tau+\delta}$ for any $\delta \in (0, 2-\tau]$, justifying why ℓ_τ^w is called *weak* ℓ_τ. The expression $|\mathbf{u}|_{\ell_\tau^w}$ defines only a quasi-norm since in general it satisfies only a generalized triangle inequality. Now let us denote by $\mathbf{u}_N$ a *best N-term approximation* for $\mathbf{u}$, i.e., a vector with at most N nonzero coefficients that has the smallest possible ℓ_2-distance to $\mathbf{u}$. Then for each $\tau \in (0, 2)$, setting $s = \frac{1}{\tau} - \frac{1}{2}$, the membership $\mathbf{u} \in \ell_\tau^w$ is equivalent to

$$\sup_N N^s \|\mathbf{u} - \mathbf{u}_N\| \eqsim |\mathbf{u}|_{\ell_\tau^w}, \tag{1.2}$$

see, e.g., [CDD01, Proposition 3.2]. Note that $\|u - \mathbf{u}_N^T \Psi\|_{\mathcal{H}} \eqsim \|\mathbf{u} - \mathbf{u}_N\|$. Here and in the following, in order to avoid the repeated use of generic but unspecified constants, by $D \lesssim E$ we mean that D can be bounded by a multiple of E, independently of parameters which D and E may depend on. Obviously, $D \gtrsim E$ is defined as $E \lesssim D$, and $D \eqsim E$ as $D \lesssim E$ and $D \gtrsim E$.

Considering the Sobolev space H^t and a basis Ψ of sufficiently smooth *wavelet* type, the theory of nonlinear approximation ([DeV98, Coh00]) tells us that if both

$$0 < s < \tfrac{d-t}{n}, \tag{1.3}$$

where d is the *order* of the wavelets and n is the space dimension, and u is in the *Besov space* $B_\tau^{sn+t}(L_\tau)$, with $\tau = (\frac{1}{2} + s)^{-1}$, then $\mathbf{u} \in \ell_\tau^w$. The condition here involving Besov regularity is much milder than the corresponding condition $u \in H^{sn+t}$ involving Sobolev regularity that would be needed to guarantee the same rate of convergence with linear approximation in the span of N wavelets corresponding to the "coarsest levels." Indeed, assuming a sufficiently smooth right-hand side, for several boundary value problems it was proven that the solution has a much higher Besov than Sobolev regularity [Dah99, DD97]. Note that with wavelets of order d, the maximum rate that can be expected by only imposing appropriate smoothness conditions on the solution is $\frac{d-t}{n}$. On general domains or manifolds, suitable wavelet bases for H^t have been constructed in [DS99a, CTU99, CM00, DS99b, Ste04a, HS04].

The aforementioned convergence rates under the mild Besov regularity assumption concern best N-term approximations, whose computation, however, requires full knowledge of the solution $\mathbf{u}$, which is only implicitly given. In [CDD01, CDD02], iterative methods for solving $\mathbf{L}\mathbf{u} = \mathbf{f}$ were developed that produce a sequence of approximations that converges with the same rate as that of the best N-term approximations, taking a number of operations that is equivalent to their support sizes. Both properties show that these methods are of *optimal computational complexity*. The methods have been generalized or quantitatively improved in e.g. [Ste03, DFR04, GHS05].

The methods apply under the condition that $\mathbf{L}$ is *symmetric, positive definite* (SPD), which is equivalent to $\langle Lv, w\rangle = \langle v, Lw\rangle$, $v, w \in \mathcal{H}$, and $\langle Lv, v\rangle \gtrsim \|v\|_{\mathcal{H}}^2$, $v \in \mathcal{H}$, i.e., that L is self-adjoint and $\mathcal{H}$-elliptic. For the case that L does not have both properties, in [CDD02] alternatives were sketched to reformulate $\mathbf{L}\mathbf{u} = \mathbf{f}$ as an equivalent well-posed infinite matrix-vector problem with a symmetric, positive definite system matrix, as via the normal equations, or, in case the equation represents a saddle point problem, by using the reformulation as a positive definite system introduced in [BP88].

Throughout this paper, we will consider the operators of type $L = A + B$ where A is self-adjoint and $\mathcal{H}$-elliptic, and B is compact. Now in general $\mathbf{L}$ is no longer SPD, hence the above mentioned adaptive wavelet methods cannot be applied directly. Following [CDD02], one can consider the normal equation $\mathbf{L}^T\mathbf{L}\mathbf{u} = \mathbf{L}^T\mathbf{f}$; however, the main disadvantage of this approach is that the condition number of the system is squared, while the quantitative properties of the methods depend sensitively on the conditioning of the system. In this paper, we will modify the adaptive wavelet algorithm from [GHS05] so that it applies directly to the system $\mathbf{L}\mathbf{u} = \mathbf{f}$, avoiding the normal equations. The analysis in [GHS05] extensively uses the Galerkin orthogonality, which in the present case has to be replaced by only a *quasi-orthogonality* property. It should be mentioned that this quasi-orthogonality property has been used in [MN04] in a convergence proof of an adaptive finite element method. By proving the quasi-orthogonality property for the present general setting and extending the complexity analysis in [GHS05], we will show that our algorithm has optimal computational complexity.

This paper is organized as follows. In the following section, we derive results on Ritz-Galerkin approximations to the exact solution, and in the last section, the adaptive wavelet algorithm is constructed and analyzed.

2 Ritz-Galerkin approximations

Let $\mathcal{H} \hookrightarrow \mathcal{Y}$ be separable real Hilbert spaces with compact embedding, and let $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and $b : \mathcal{Y} \times \mathcal{H} \to \mathbb{R}$ be bounded bilinear forms. We assume that the bilinear form a is symmetric and elliptic, which implies that $|\!|\!| \cdot |\!|\!| := a(\cdot, \cdot)^{\frac{1}{2}}$ is an equivalent norm on $\mathcal{H}$, i.e.,

$$|\!|\!| v |\!|\!| \eqsim \|v\|_{\mathcal{H}} \qquad v \in \mathcal{H}. \tag{2.1}$$

In particular, the operator $A : \mathcal{H} \to \mathcal{H}'$ defined by $\langle Av, w \rangle = a(v, w)$ for $v, w \in \mathcal{H}$, is boundedly invertible. Moreover, since $B : \mathcal{H} \to \mathcal{H}'$ defined by $\langle Bv, w \rangle = b(v, w)$ for $v, w \in \mathcal{H}$, is compact, the linear operator $L := A + B$ is a Fredholm operator of index zero. Therefore, assuming that L is injective, $L : \mathcal{H} \to \mathcal{H}'$ is boundedly invertible, in particular meaning that the linear operator equation

$$Lu = f, \tag{2.2}$$

has a unique solution for $f \in \mathcal{H}'$.

For our analysis we will need the following mild *regularity assumption* on the adjoint L' of L: There is a Hilbert space $\mathcal{X} \hookrightarrow \mathcal{H}$ with compact embedding, such that $(L')^{-1} : \mathcal{Y}' \to \mathcal{X}$ is bounded. The following lemma gives a means to check this assumption.

Lemma 2.1. *Let either $A^{-1} : \mathcal{Y}' \to \mathcal{X}$ or $L^{-1} : \mathcal{Y}' \to \mathcal{X}$ be bounded. Then $(L')^{-1} : \mathcal{Y}' \to \mathcal{X}$ is bounded.*

Proof. We treat the first case only. The other case is analogous. The operator B extends to a bounded mapping from $\mathcal{Y}$ to $\mathcal{H}'$. So $L' = A + B' : \mathcal{X} \to \mathcal{Y}'$ is bounded. Now consider the equation $L'u = f$. We know that there exists a unique solution $u \in \mathcal{H}$ with $\|u\|_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}'}$ and thus $\|B'u\|_{\mathcal{Y}'} \lesssim \|u\|_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}'} \lesssim \|f\|_{\mathcal{Y}'}$. From $Au = f - B'u$, we now infer that $\|u\|_{\mathcal{X}} \lesssim \|f\|_{\mathcal{Y}'}$. $\square$

Example 2.2. For some Lipschitz domain $\Omega \subset \mathbb{R}^n$, with $\mathcal{H} := H_0^1(\Omega)$ let $L : \mathcal{H} \to \mathcal{H}'$ be defined by

$$\langle Lv, w\rangle = -\sum_{j,k=1}^n \langle a_{jk}\partial_k v, \partial_j w\rangle_{L_2} + \sum_{k=1}^n \langle b_k \partial_k v, w\rangle_{L_2} + \langle cv, w\rangle_{L_2} \quad v, w \in \mathcal{H}.$$

If the coefficients satisfy $a_{jk}, b_k, c \in L_\infty$ then $L : \mathcal{H} \to \mathcal{H}'$ is bounded. Moreover, if the matrix $[a_{jk}]$ is symmetric and uniformly positive definite a.e. in Ω, then the bilinear form $a(\cdot,\cdot) := -\sum_{j,k=1}^n \langle a_{jk}\partial_k\cdot, \partial_j\cdot\rangle_{L_2}$ is symmetric and satisfies (2.1). If either $b_k = 0$, $1 \le k \le n$ and $c \ge 0$ a.e. or $c \ge \beta > 0$ a.e., then the generalized maximum principle implies that L is injective, cf. [Sta65]. Also if $L = A - \eta^2$ for a constant $\eta \in \mathbb{R}$, then the injectivity is guaranteed as long as η^2 is not an eigenvalue of A. With $\mathcal{Y}_\sigma := (L_2(\Omega), H_0^1(\Omega))_{1-\sigma,2}$ for some $\sigma \in (0,1]$, where $(X,Y)_{\theta,p}$ denotes the intermediate space between X and Y obtained by the real interpolation method, the bilinear form $b(\cdot,\cdot) := \sum_{k=1}^n \langle b_k\partial_k\cdot,\cdot\rangle_{L_2} + \langle c\cdot,\cdot\rangle_{L_2} : \mathcal{Y}_\sigma \times \mathcal{H} \to \mathbb{R}$ is bounded for any $\sigma \in (0,1]$. If the coefficients a_{jk}, $1 \le j,k \le n$, are Lipschitz continuous, then with $\mathcal{X}_\sigma := (H_0^1(\Omega), H^2(\Omega) \cap H_0^1(\Omega))_{\sigma,2}$ it is known that $A^{-1} : \mathcal{Y}'_\sigma \to \mathcal{X}_\sigma$ is bounded for any $\sigma \in (0, \frac{1}{2})$, cf. [Sav98]. Furthermore, the embeddings $\mathcal{X}_\sigma \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{Y}_\sigma$ are compact. From Lemma 2.1 we conclude that all aforementioned conditions are satisfied.

Example 2.3. Let L be the operator considered in the above example. We assume that the domain Ω is Lipschitz, the coefficients a_{jk}, b_k, c are constant and that the matrix $[a_{jk}]$ is symmetric and positive definite. Then the single layer and hypersingular boundary integral operators corresponding to the differential operator L can be written as the sum of a bounded $\mathcal{H}$-elliptic operator $A : \mathcal{H} \to \mathcal{H}'$ and a compact operator $B : \mathcal{H} \to \mathcal{H}'$, see [Cos88]. With Γ being the boundary of the underlying domain Ω, here the energy space is $\mathcal{H} = H^t(\Gamma)$ with $t = -\frac{1}{2}$ for the single layer operator and $t = \frac{1}{2}$ for the hypersingular integral operator. A close inspection of the proofs of [CW86, Theorem 3.9] and [Cos88, Theorem 2] reveals that in both cases, the operator A is self-adjoint and that with $\mathcal{Y}_\sigma := H^{t-\sigma}(\Gamma)$ where t has the appropriate value depending on the case, the operator B can be extended to a bounded operator $\mathcal{Y}_\sigma \to \mathcal{H}'$ for any $\sigma \in (0, \frac{1}{2}]$. Assuming the injectivity of $L : \mathcal{H} \to \mathcal{H}'$, in [Cos88] it is shown that with $\mathcal{X}_\sigma := H^{t+\sigma}(\Gamma)$, $L^{-1} : \mathcal{Y}'_\sigma \to \mathcal{X}_\sigma$ is bounded for any $\sigma \in [0, \frac{1}{2}]$. The injectivity depends on the particular case at hand, see [McL00] for some important cases.

We consider a sequence of finite dimensional closed subspaces $V_0 \subset V_1 \subset \ldots \subset \mathcal{H}$ satisfying

$$\inf_{v_j \in V_j} \|v - v_j\|_{\mathcal{H}} \le \alpha_j \|v\|_{\mathcal{X}} \qquad v \in \mathcal{X}, \tag{2.3}$$

with $\lim_{j\to\infty} \alpha_j = 0$.

Remark 2.4. Such a sequence exists since the embedding $\mathcal{X} \hookrightarrow \mathcal{H}$ is compact, cf. [Sch01].

Example 2.5. Let $\mathcal{H} = H^t$ and $\mathcal{X} = H^{t+\sigma}$. Then for standard finite element or spline spaces V_j subordinate to dyadic subdivisions of an initial mesh, the approximation property (2.3) is satisfied with $\alpha_j \eqsim 2^{-j\sigma}$, for any $t < \gamma$ and $\sigma \le d - t$, where d is the polynomial order of the spaces V_j, and $\gamma = \sup_j\{s : V_j \subset H^s\}$, see e.g. [Ngu05].

For a finite dimensional closed subspace $S \subset \mathcal{H}$ such that $V_j \subseteq S$ for some j, we consider the Ritz-Galerkin problem

$$\langle Lu_S, v_S\rangle = \langle f, v_S\rangle \qquad \text{for all } v_S \in S. \tag{2.4}$$

It is well known that for j being sufficiently large, a unique solution u_S to the above problem exists, and that u_S is a near best approximation to u in the energy norm $|\!|\!|\cdot|\!|\!|$. In the weaker norm $\|\cdot\|_{\mathcal{Y}}$, convergence of higher order than (2.3) can be obtained via an Aubin-Nitsche duality argument, cf. [Sch74]. These results are recalled in the following lemma, where for convenience we also include a proof.

Lemma 2.6. *There is an absolute constant $j_0 \in \mathbb{N}_0$ (not depending on S) such that for all $j \geq j_0$, (2.4) has a unique solution with*

$$|||u - u_S||| \leq [1 + O(\alpha_j)] \inf_{v \in S} |||u - v|||. \tag{2.5}$$

Moreover, for $j \geq j_0$ we have

$$\|u - u_S\|_{\mathcal{Y}} \leq O(\alpha_j) |||u - u_S|||. \tag{2.6}$$

Proof. Suppose that a solution u_S to (2.4) exists. Then we trivially have

$$\langle L(u - u_S), v_S \rangle = 0 \qquad \forall v_S \in S. \tag{2.7}$$

Using this and the boundedness of $b : \mathcal{Y} \times \mathcal{H} \to \mathbb{R}$, for arbitrary $v_S \in S$ we get

$$\begin{aligned} |||u - u_S|||^2 &= \langle L(u - u_S), u - u_S \rangle - b(u - u_S, u - u_S) \\ &= \langle L(u - u_S), u - v_S \rangle - b(u - u_S, u - u_S) \\ &= a(u - u_S, u - v_S) + b(u - u_S, u_S - v_S) \\ &\leq |||u - u_S||| |||u - v_S||| + O(1) \|u - u_S\|_{\mathcal{Y}} \|u_S - v_S\|_{\mathcal{H}}. \end{aligned} \tag{2.8}$$

We estimate $\|u - u_S\|_{\mathcal{Y}}$ by an Aubin-Nitsche duality argument. For $w \in \mathcal{Y}'$ we infer that

$$\begin{aligned} \langle u - u_S, w \rangle = \langle L(u - u_S), (L')^{-1} w - w_S \rangle &\leq \|L\|_{\mathcal{H} \to \mathcal{H}'} \|u - u_S\|_{\mathcal{H}} \|(L')^{-1} w - w_S\|_{\mathcal{H}} \\ &\leq \|L\|_{\mathcal{H} \to \mathcal{H}'} \|u - u_S\|_{\mathcal{H}} \alpha_j \|(L')^{-1} w\|_{\mathcal{X}} \\ &\leq \|L\|_{\mathcal{H} \to \mathcal{H}'} \|u - u_S\|_{\mathcal{H}} \alpha_j \|(L')^{-1}\|_{\mathcal{Y}' \to \mathcal{X}} \|w\|_{\mathcal{Y}'}, \end{aligned}$$

where we used (2.7), (2.3) and the boundedness of $(L')^{-1} : \mathcal{Y}' \to \mathcal{X}$. We have

$$\|u - u_S\|_{\mathcal{Y}} = \sup_{w \in \mathcal{Y}'} \frac{\langle u - u_S, w \rangle}{\|w\|_{\mathcal{Y}'}},$$

and subsequently using (2.1) we arrive at (2.6). Substituting (2.6) into (2.8), we get

$$|||u - u_S||| \leq |||u - v_S||| + O(\alpha_j) \|u_S - v_S\|_{\mathcal{H}}.$$

For the last term, from the triangle inequality and (2.1), we have

$$\|v - u_S\|_{\mathcal{H}} \lesssim |||u - u_S||| + |||u - v_S|||.$$

Now choosing j_0 sufficiently large, we finally obtain (2.5).

Since (2.4) is a finite dimensional system, existence and uniqueness are equivalent. To see the uniqueness, it is sufficient to prove that $f = 0$ implies $u_S = 0$. By linearity and invertibility of L, we have $u = 0$ if $f = 0$, and so (2.5) implies that $u_S = 0$. The proof is completed. $\square$

The following observation concerning quasi-orthogonality is an easy generalization of [MN04, Lemma 2.1].

Lemma 2.7. *For some $j \geq j_0$ with j_0 being the absolute constant from Lemma 2.6, let $S_0 \subset S_1 \subset \mathcal{H}$ be finite dimensional subspaces satisfying $V_j \subseteq S_0$. Let $u_0 \in S_0$ and $u_1 \in S_1$ be the solutions to the Galerkin problems $\langle Lu_0, v\rangle = \langle f, v\rangle$ $\forall v \in S_0$ and $\langle Lu_1, v\rangle = \langle f, v\rangle$ $\forall v \in S_1$, respectively. Then we have*

$$\left| \|u-u_0\|^2 - \|u-u_1\|^2 - \|u_1-u_0\|^2 \right| \leq O(\alpha_j)\left(\|u-u_0\|^2 + \|u-u_1\|^2\right). \tag{2.9}$$

Proof. We have $\|u-u_0\|^2 = \|u-u_1\|^2 + \|u_1-u_0\|^2 + 2a(u-u_1, u_1-u_0)$. Using (2.7), boundedness of $b : \mathcal{Y} \times \mathcal{H} \to \mathbb{R}$, and the triangle inequality, we estimate the absolute value of the last term as

$$\begin{aligned} |2a(u-u_1, u_1-u_0)| &= |2b(u-u_1, u_1-u_0)| \\ &\lesssim \|u-u_1\|_{\mathcal{Y}} \|u_1-u_0\|_{\mathcal{H}} \\ &\leq \|u-u_1\|_{\mathcal{Y}} \left(\|u-u_1\|_{\mathcal{H}} + \|u-u_0\|_{\mathcal{H}}\right) \end{aligned}$$

Now using (2.6), and applying the inequality $2ab \leq a^2 + b^2$, $a, b \in \mathbb{R}$, we conclude the proof by

$$\begin{aligned} |2a(u-u_1, u_1-u_0)| &\leq O(\alpha_j)\left(\|u-u_1\|^2 + \|u-u_1\|\|u-u_0\|\right) \\ &\leq O(\alpha_j)\left(\|u-u_1\|^2 + \|u-u_0\|^2\right). \end{aligned}$$

$\square$

Using a *Riesz basis* for $\mathcal{H}$, we will now transform (2.2) into an equivalent infinite matrix-vector system in ℓ_2. Let $\Psi = \{\psi_\lambda : \lambda \in \nabla\}$ be a Riesz basis for $\mathcal{H}$ of wavelet type. We assume that for some $\nabla_0 \subset \nabla_1 \subset \ldots \subset \nabla$, the subspaces defined by $V_j = \operatorname{span}\{\psi_\lambda : \lambda \in \nabla_j\}$, $j \in \mathbb{N}_0$, satisfies (2.3) with $\lim_{j\to\infty} \alpha_j = 0$.

Example 2.8. With the spaces V_j described in Example 2.5, wavelet bases satisfying the above condition have been constructed e.g. in [DS99a, CTU99, CM00, DS99b, Ste04a, HS04].

Writing $u = \mathbf{u}^T\Psi$ for some $\mathbf{u} \in \ell_2$, $\mathbf{u}$ satisfies

$$\mathbf{L}\mathbf{u} = \mathbf{f}, \tag{2.10}$$

where $\mathbf{L} := \langle \psi_\lambda, L\psi_\mu\rangle_{\lambda,\mu\in\nabla} : \ell_2 \to \ell_2$ is boundedly invertible and $\mathbf{f} := \langle f, \psi_\lambda\rangle_{\lambda\in\nabla} \in \ell_2$. Similarly to $\mathbf{L}$, we define also the matrices $\mathbf{A} := \langle \psi_\lambda, A\psi_\mu\rangle_{\lambda,\mu\in\nabla} = a(\psi_\mu, \psi_\lambda)_{\lambda,\mu\in\nabla}$ and $\mathbf{B} := \langle \psi_\lambda, B\psi_\mu\rangle_{\lambda,\mu\in\nabla} = b(\psi_\mu, \psi_\lambda)_{\lambda,\mu\in\nabla}$, so that $\mathbf{L} = \mathbf{A} + \mathbf{B}$. The matrix $\mathbf{A}$ is symmetric positive definite, so $\langle \mathbf{A}\cdot, \cdot\rangle$ is an inner product on ℓ_2, and the induced norm $\|\cdot\|$ satisfies

$$\|\mathbf{v}\|^2 := \langle \mathbf{A}\mathbf{v}, \mathbf{v}\rangle = a(\mathbf{v}^T\Psi, \mathbf{v}^T\Psi) = \|\mathbf{v}^T\Psi\|^2 \qquad \mathbf{v} \in \ell_2.$$

Furthermore, one can verify that for any $\mathbf{v} \in \ell_2$, $\Lambda \subseteq \nabla$, $\mathbf{v}_\Lambda \in \ell_2(\Lambda)$,

$$\|\mathbf{A}\mathbf{v}\| \leq \|\mathbf{A}\|^{\frac{1}{2}} \|\mathbf{v}\| \leq \|\mathbf{A}\|\|\mathbf{v}\|, \qquad \|\mathbf{v}_\Lambda\| \leq \|\mathbf{A}^{-1}\|^{\frac{1}{2}} \|\mathbf{P}_\Lambda \mathbf{A}\mathbf{v}_\Lambda\|. \tag{2.11}$$

For any $\mathbf{v}, \mathbf{w} \in \ell_2$, we have $\langle \mathbf{B}\mathbf{v}, \mathbf{w}\rangle = b(\mathbf{v}^T\Psi, \mathbf{w}^T\Psi) \lesssim \|\mathbf{v}^T\Psi\|_{\mathcal{Y}} \|\mathbf{w}^T\Psi\|_{\mathcal{H}} \lesssim \|\mathbf{v}^T\Psi\|_{\mathcal{Y}} \|\mathbf{w}\|$, implying the following estimate which will be used often in the rest of this section.

$$\|\mathbf{B}\mathbf{v}\| = \sup_{0\neq\mathbf{w}\in\ell_2} \frac{\langle \mathbf{B}\mathbf{v}, \mathbf{w}\rangle}{\|\mathbf{w}\|} \lesssim \|\mathbf{v}^T\Psi\|_{\mathcal{Y}} \qquad \mathbf{v} \in \ell_2. \tag{2.12}$$

For some $\Lambda \subset \nabla$, let $S = \operatorname{span}\{\psi_\lambda : \lambda \in \Lambda\}$. Then $u_S = \mathbf{u}_\Lambda^T \Psi \in S$ is the solution to the Galerkin problem (2.4) if and only if $\mathbf{u}_\Lambda \in \ell_2(\Lambda)$ satisfies

$$\mathbf{P}_\Lambda \mathbf{L} \mathbf{I}_\Lambda \mathbf{u}_\Lambda = \mathbf{P}_\Lambda \mathbf{f}, \tag{2.13}$$

where $\mathbf{P}_\Lambda : \ell_2 \to \ell_2(\Lambda)$ is the orthogonal projector onto $\ell_2(\Lambda)$, and $\mathbf{I}_\Lambda$ denotes the trivial inclusion $\ell_2(\Lambda) \to \ell_2$. In the following, we will refer to $\mathbf{u}_\Lambda$ as the *Galerkin solution* with respect to the index set Λ. From Lemma 2.6 we know that this solution exists and is unique when $\nabla_j \subseteq \Lambda$ for some $j \geq j_0$.

Lemma 2.9. *Let $\mathbf{P}_\Lambda$ and $\mathbf{I}_\Lambda$ be as above. Then for any $\Lambda \supseteq \nabla_j$ for some $j \geq j_0$ we have*

$$\|(\mathbf{P}_\Lambda \mathbf{L} \mathbf{I}_\Lambda)^{-1}\| \leq \|\mathbf{A}^{-1}\| \left[1 + \|\mathbf{B}\mathbf{L}^{-1}\| + O(\alpha_j)\right].$$

Proof. Recalling that $\mathbf{L}(\mathbf{u} - \mathbf{u}_\Lambda) \perp \ell_2(\Lambda)$ and that $\mathbf{A} = \mathbf{L} - \mathbf{B}$, we have

$$\begin{aligned}\|\mathbf{u}_\Lambda\|^2 &\leq \|\mathbf{A}^{-1}\| \|\!|\mathbf{u}_\Lambda|\!\|^2 = \|\mathbf{A}^{-1}\| \left[\langle \mathbf{L}\mathbf{u}_\Lambda, \mathbf{u}_\Lambda\rangle - \langle \mathbf{B}\mathbf{u}_\Lambda, \mathbf{u}_\Lambda\rangle\right] \\ &= \|\mathbf{A}^{-1}\| \left[\langle \mathbf{L}\mathbf{u}, \mathbf{u}_\Lambda\rangle - \langle \mathbf{B}\mathbf{u}, \mathbf{u}_\Lambda\rangle + \langle \mathbf{B}(\mathbf{u} - \mathbf{u}_\Lambda), \mathbf{u}_\Lambda\rangle\right].\end{aligned}$$

Now applying the Cauchy-Schwarz inequality gives

$$\|\mathbf{u}_\Lambda\| \leq \|\mathbf{A}^{-1}\| \left[\|\mathbf{L}\mathbf{u}\| + \|\mathbf{B}\mathbf{u}\| + \|\mathbf{B}(\mathbf{u} - \mathbf{u}_\Lambda)\|\right]. \tag{2.14}$$

For the last term in the brackets, using the estimates (2.12), (2.6) and (2.5), we have

$$\|\mathbf{B}(\mathbf{u} - \mathbf{u}_\Lambda)\| \lesssim \|u - \mathbf{u}_\Lambda^T \Psi\|_{\mathcal{Y}} \leq O(\alpha_j) \|\!|u - \mathbf{u}_\Lambda^T \Psi|\!\| \leq O(\alpha_j) \inf_{\mathbf{v} \in \ell_2(\Lambda)} \|\!|\mathbf{u} - \mathbf{v}|\!\| \leq O(\alpha_j) \|\mathbf{u}\|.$$

We substitute it into (2.14) to get

$$\|\mathbf{u}_\Lambda\| \leq \|\mathbf{A}^{-1}\| \left[\|\mathbf{L}\mathbf{u}\| + \|\mathbf{B}\mathbf{u}\| + O(\alpha_j)\|\mathbf{u}\|\right] \leq \|\mathbf{A}^{-1}\| \left[1 + \|\mathbf{B}\mathbf{L}^{-1}\| + O(\alpha_j)\|\mathbf{L}^{-1}\|\right] \|\mathbf{f}\|.$$

Since this estimate holds in particular for arbitrary $\mathbf{f} = \mathbf{P}_\Lambda \mathbf{f}$, taking into account that $\mathbf{u}_\Lambda = (\mathbf{P}_\Lambda \mathbf{L} \mathbf{I}_\Lambda)^{-1} \mathbf{P}_\Lambda \mathbf{f}$ the proof is completed. □

The following lemma generalizes [GHS05, Lemma 1.2] to the present case of nonsymmetric and indefinite operators, and provides a way to extend a given set $\Lambda_0 \subset \nabla$ such that the error of the Galerkin solution with respect to the extended set is reduced by a constant factor.

Lemma 2.10. *Suppose that $\mathbf{u}_0 \in \ell_2(\Lambda_0)$ is the solution to $\mathbf{P}_{\Lambda_0} \mathbf{L} \mathbf{I}_{\Lambda_0} \mathbf{u}_0 = \mathbf{P}_{\Lambda_0} \mathbf{f}$ with $\Lambda_0 \supseteq \nabla_j$ for j sufficiently large. For a constant $\mu \in (0, 1)$, let $\nabla \supset \Lambda_1 \supset \Lambda_0$ be such that*

$$\|\mathbf{P}_{\Lambda_1}(\mathbf{f} - \mathbf{L}\mathbf{u}_0)\| \geq \mu \|\mathbf{f} - \mathbf{L}\mathbf{u}_0\|. \tag{2.15}$$

Then, for $\mathbf{u}_1 \in \ell_2(\Lambda_1)$ being the solution to $\mathbf{P}_{\Lambda_1} \mathbf{L} \mathbf{I}_{\Lambda_1} \mathbf{u}_1 = \mathbf{P}_{\Lambda_1} \mathbf{f}$, it holds that

$$\|\!|\mathbf{u} - \mathbf{u}_1|\!\| \leq \left[1 - \kappa(\mathbf{A})^{-1} \mu^2 + O(\alpha_j)\right]^{\frac{1}{2}} \|\!|\mathbf{u} - \mathbf{u}_0|\!\|.$$

Proof. In this proof, we use the notations $u_0 = \mathbf{u}_0^T\Psi$ and $u_1 = \mathbf{u}_1^T\Psi$. We have

$$\|\mathbf{L}(\mathbf{u}_1 - \mathbf{u}_0)\|^2 = \|\mathbf{A}(\mathbf{u}_1 - \mathbf{u}_0)\|^2 + 2\langle \mathbf{A}(\mathbf{u}_1 - \mathbf{u}_0), \mathbf{B}(\mathbf{u}_1 - \mathbf{u}_0)\rangle + \|\mathbf{B}(\mathbf{u}_1 - \mathbf{u}_0)\|^2.$$

The first term on the right hand side is bounded from above by using the first inequality from (2.11). We estimate the second term by using (2.12) as

$$|2\langle \mathbf{A}(\mathbf{u}_1 - \mathbf{u}_0), \mathbf{B}(\mathbf{u}_1 - \mathbf{u}_0)\rangle| \leq 2\|\mathbf{A}(\mathbf{u}_1 - \mathbf{u}_0)\|\|\mathbf{B}(\mathbf{u}_1 - \mathbf{u}_0)\| \lesssim |\!|\!| u_1 - u_0 |\!|\!| \|u_1 - u_0\|_{\mathcal{Y}}.$$

For the third term we have $\|\mathbf{B}(\mathbf{u}_1 - \mathbf{u}_0)\|^2 \lesssim \|u_1 - u_0\|_{\mathcal{Y}}^2$. Combining these estimates, and taking into account (2.6), we conclude that

$$\begin{aligned}\|\mathbf{L}(\mathbf{u}_1 - \mathbf{u}_0)\|^2 &\leq \|\mathbf{A}\| |\!|\!| u_1 - u_0 |\!|\!|^2 + O(1) |\!|\!| u_1 - u_0 |\!|\!| \|u_1 - u_0\|_{\mathcal{Y}} \\ &\leq \|\mathbf{A}\| |\!|\!| u_1 - u_0 |\!|\!|^2 + O(\alpha_j)\left(|\!|\!| u - u_0 |\!|\!|^2 + |\!|\!| u - u_1 |\!|\!|^2\right).\end{aligned} \tag{2.16}$$

On the other hand, we have

$$\|\mathbf{L}(\mathbf{u} - \mathbf{u}_0)\|^2 = \|\mathbf{A}(\mathbf{u} - \mathbf{u}_0)\|^2 + 2\langle \mathbf{A}(\mathbf{u} - \mathbf{u}_0), \mathbf{B}(\mathbf{u} - \mathbf{u}_0)\rangle + \|\mathbf{B}(\mathbf{u} - \mathbf{u}_0)\|^2.$$

The first term can be bounded from below by using the last inequality in (2.11) with $\Lambda = \nabla$. By using (2.12) and (2.6), we bound the second term as

$$|2\langle \mathbf{A}(\mathbf{u} - \mathbf{u}_0), \mathbf{B}(\mathbf{u} - \mathbf{u}_0)\rangle| \lesssim |\!|\!| u - u_0 |\!|\!| \|u - u_0\|_{\mathcal{Y}} \leq O(\alpha_j) |\!|\!| u - u_0 |\!|\!|^2. \tag{2.17}$$

Estimating the third term by zero, we infer

$$\|\mathbf{L}(\mathbf{u} - \mathbf{u}_0)\|^2 \geq \|\mathbf{A}^{-1}\|^{-1} |\!|\!| u - u_0 |\!|\!|^2 - O(\alpha_j) |\!|\!| u - u_0 |\!|\!|^2. \tag{2.18}$$

By hypothesis we have $\|\mathbf{L}(\mathbf{u}_1 - \mathbf{u}_0)\| \geq \|\mathbf{P}_{\Lambda_1}\mathbf{L}(\mathbf{u}_1 - \mathbf{u}_0)\| = \|\mathbf{P}_{\Lambda_1}(\mathbf{f} - \mathbf{L}\mathbf{u}_0)\| \geq \mu\|\mathbf{L}(\mathbf{u} - \mathbf{u}_0)\|$. Combining this with (2.16) and (2.18), we get

$$\|\mathbf{A}\| |\!|\!| u_1 - u_0 |\!|\!|^2 + O(\alpha_j) |\!|\!| u - u_1 |\!|\!|^2 \geq \mu^2 \|\mathbf{A}^{-1}\|^{-1} |\!|\!| u - u_0 |\!|\!|^2 - O(\alpha_j) |\!|\!| u - u_0 |\!|\!|^2.$$

Now by using that $|\!|\!| u_1 - u_0 |\!|\!| \leq |\!|\!| u - u_0 |\!|\!|^2 - |\!|\!| u - u_1 |\!|\!|^2 + O(\alpha_j)(|\!|\!| u - u_0 |\!|\!|^2 + |\!|\!| u - u_1 |\!|\!|^2)$ by (2.9), and choosing j sufficiently large we finish the proof. $\square$

In the following lemma it is showed that for sufficiently small μ and $\mathbf{u} \in \ell_\tau^w$, for a set Λ_1 as in Lemma 2.10 that has minimal cardinality, $\#(\Lambda_1 \backslash \Lambda_0)$ can be bounded in terms of $\|\mathbf{f} - \mathbf{L}\mathbf{u}_0\|$ and $|\mathbf{u}|_{\ell_\tau^w}$ only, cf. [GHS05, Lemma 2.1].

Lemma 2.11. *For some $s > 0$ and $\tau = (\frac{1}{2} + s)^{-1}$ let $\mathbf{u} \in \ell_\tau^w$, and let $\mu \in (0, \kappa(\mathbf{A})^{-\frac{1}{2}})$. Assume that $\mathbf{u}_0 \in \ell_2(\Lambda_0)$ is the solution to $\mathbf{P}_{\Lambda_0}\mathbf{L}\mathbf{I}_{\Lambda_0}\mathbf{u}_0 = \mathbf{P}_{\Lambda_0}\mathbf{f}$ with $\Lambda_0 \supseteq \nabla_j$ for a sufficiently large j. Then, the* smallest *set $\Lambda_1 \supset \Lambda_0$ with*

$$\|\mathbf{P}_{\Lambda_1}(\mathbf{f} - \mathbf{L}\mathbf{u}_0)\| \geq \mu\|\mathbf{f} - \mathbf{L}\mathbf{u}_0\| \tag{2.19}$$

satisfies

$$\#(\Lambda_1 \backslash \Lambda_0) \lesssim \|\mathbf{f} - \mathbf{L}\mathbf{u}_0\|^{-1/s} |\mathbf{u}|_{\ell_\tau^w}^{1/s}.$$

Proof. With a constant $\lambda > 0$ to be chosen later, let N be such that a best N-term approximation $\mathbf{u}_N$ for $\mathbf{u}$ satisfies $|\!|\!|\mathbf{u}-\mathbf{u}_N|\!|\!| \leq \lambda |\!|\!|\mathbf{u}-\mathbf{u}_0|\!|\!|$. Since $\mathbf{L}$ is boundedly invertible we have $|\!|\!|\mathbf{u}-\mathbf{u}_0|\!|\!| \gtrsim \|\mathbf{f}-\mathbf{L}\mathbf{u}_0\|$ and thus, in view of (1.2), $N \lesssim \|\mathbf{f}-\mathbf{L}\mathbf{u}_0\|^{-1/s}|\mathbf{u}|_{\ell_\tau^w}^{1/s}$. Let $\Lambda := \Lambda_0 \cup \operatorname{supp} \mathbf{u}_N \supset \Lambda_0$. We are going to show that for a suitable λ, and j sufficiently large, $\|\mathbf{P}_\Lambda(\mathbf{f}-\mathbf{L}\mathbf{u}_0)\| \geq \mu\|\mathbf{f}-\mathbf{L}\mathbf{u}_0\|$. Then by definition of Λ_1 we may conclude that

$$\#(\Lambda_1 \backslash \Lambda_0) \lesssim \#(\Lambda \backslash \Lambda_0) \leq N \lesssim \|\mathbf{f}-\mathbf{L}\mathbf{u}_0\|^{-1/s}|\mathbf{u}|_{\ell_\tau^w}^{1/s}.$$

Now we will show that the above claim is valid. The solution to $\mathbf{P}_\Lambda\mathbf{L}\mathbf{I}_\Lambda\mathbf{u}_\Lambda = \mathbf{P}_\Lambda\mathbf{f}$ satisfies

$$\begin{aligned} |\!|\!|\mathbf{u}-\mathbf{u}_\Lambda|\!|\!| &\leq [1+O(\alpha_j)]|\!|\!|\mathbf{u}-\mathbf{u}_N|\!|\!| \leq [1+O(\alpha_j)]\|\mathbf{A}\|^{\frac{1}{2}}\|\mathbf{u}-\mathbf{u}_N\| \\ &\leq \lambda[1+O(\alpha_j)]\|\mathbf{A}\|^{\frac{1}{2}}|\!|\!|\mathbf{u}-\mathbf{u}_0|\!|\!|, \end{aligned} \tag{2.20}$$

where we have used (2.5) and the second inequality from (2.11). We have

$$\|\mathbf{P}_\Lambda\mathbf{L}(\mathbf{u}_\Lambda-\mathbf{u}_0)\|^2 \geq \|\mathbf{P}_\Lambda\mathbf{A}(\mathbf{u}_\Lambda-\mathbf{u}_0)\|^2 + 2\langle\mathbf{P}_\Lambda\mathbf{A}(\mathbf{u}_\Lambda-\mathbf{u}_0), \mathbf{B}(\mathbf{u}_\Lambda-\mathbf{u}_0)\rangle.$$

The first term in the right hand side can be bounded from below by using the last inequality from (2.11). Estimating the second term as

$$|2\langle\mathbf{P}_\Lambda\mathbf{A}(\mathbf{u}_\Lambda-\mathbf{u}_0), \mathbf{B}(\mathbf{u}_\Lambda-\mathbf{u}_0)\rangle| \lesssim |\!|\!|\mathbf{u}_\Lambda-\mathbf{u}_0|\!|\!|\|\mathbf{u}_\Lambda-\mathbf{u}_0\| \leq O(\alpha_j)\left(|\!|\!|\mathbf{u}-\mathbf{u}_\Lambda|\!|\!|^2 + |\!|\!|\mathbf{u}-\mathbf{u}_0|\!|\!|^2\right),$$

we get

$$\|\mathbf{P}_\Lambda\mathbf{L}(\mathbf{u}_\Lambda-\mathbf{u}_0)\|^2 \geq \|\mathbf{A}^{-1}\|^{-1}|\!|\!|\mathbf{u}_\Lambda-\mathbf{u}_0|\!|\!|^2 - O(\alpha_j)\left(|\!|\!|\mathbf{u}-\mathbf{u}_\Lambda|\!|\!|^2 + |\!|\!|\mathbf{u}-\mathbf{u}_0|\!|\!|^2\right).$$

Now by using that $|\!|\!|\mathbf{u}_\Lambda-\mathbf{u}_0|\!|\!| \geq |\!|\!|\mathbf{u}-\mathbf{u}_0|\!|\!|^2 - |\!|\!|\mathbf{u}-\mathbf{u}_\Lambda|\!|\!|^2 - O(\alpha_j)(|\!|\!|\mathbf{u}-\mathbf{u}_0|\!|\!|^2 + |\!|\!|\mathbf{u}-\mathbf{u}_\Lambda|\!|\!|^2)$ by (2.9), and applying (2.20), we have

$$\begin{aligned} \|\mathbf{P}_\Lambda\mathbf{L}(\mathbf{u}_\Lambda-\mathbf{u}_0)\|^2 &\geq [1-O(\alpha_j)]\|\mathbf{A}^{-1}\|^{-1}|\!|\!|\mathbf{u}-\mathbf{u}_0|\!|\!|^2 - [1+O(\alpha_j)]\|\mathbf{A}^{-1}\|^{-1}|\!|\!|\mathbf{u}-\mathbf{u}_\Lambda|\!|\!|^2 \\ &\quad - O(\alpha_j)\left[|\!|\!|\mathbf{u}-\mathbf{u}_\Lambda|\!|\!|^2 + |\!|\!|\mathbf{u}-\mathbf{u}_0|\!|\!|^2\right] \\ &\geq [1-O(\alpha_j)]\,\|\mathbf{A}^{-1}\|^{-1}|\!|\!|\mathbf{u}-\mathbf{u}_0|\!|\!|^2 - [1+O(\alpha_j)]\,\|\mathbf{A}^{-1}\|^{-1}|\!|\!|\mathbf{u}-\mathbf{u}_\Lambda|\!|\!|^2 \\ &\geq \left[1-\lambda^2\|\mathbf{A}\| - O(\alpha_j)\right]\|\mathbf{A}^{-1}\|^{-1}|\!|\!|\mathbf{u}-\mathbf{u}_0|\!|\!|^2. \end{aligned}$$

On the other hand, we have

$$\|\mathbf{L}(\mathbf{u}-\mathbf{u}_0)\|^2 \leq [1+O(\alpha_j)]\,\|\mathbf{A}\||\!|\!|\mathbf{u}-\mathbf{u}_0|\!|\!|^2.$$

Combining the last two estimates we infer

$$\|\mathbf{P}_\Lambda(\mathbf{f}-\mathbf{L}\mathbf{u}_0)\|^2 \geq \kappa(\mathbf{A})^{-1}\left[1-\lambda^2\|\mathbf{A}\| - O(\alpha_j)\right]\|\mathbf{f}-\mathbf{L}\mathbf{u}_0\|^2.$$

Choose a value of the constant $\lambda > 0$ such that $\kappa(\mathbf{A})^{-\frac{1}{2}}(1-\lambda^2\|\mathbf{A}\|)^{\frac{1}{2}} > \mu$. Then for j sufficiently large, we have $\|\mathbf{P}_\Lambda(\mathbf{f}-\mathbf{L}\mathbf{u}_0)\| \geq \mu\|\mathbf{f}-\mathbf{L}\mathbf{u}_0\|$, thus completing the proof. □

3 The adaptive algorithm

In this section, we will formulate an adaptive wavelet algorithm for solving (1.1) and analyse its convergence behaviour. To give a rough idea before going through the rigorous treatment, the algorithm starts with an initial index set Λ and computes an approximate residual of the exact Galerkin solution with respect to the index set Λ. Having computed the approximate residual, we use Lemma 2.10 and Lemma 2.11 to extend the set Λ such that the error in the new Galerkin solution is a constant factor smaller where the cardinality of the extention is up to a constant factor minimal, and this process is repeated until the computed residual is satisfactorily small.

Ideally, our algorithm should produce approximations that converge with the same rate as that of the best N-term approximations, taking a number of operations that is equivalent to their support sizes, cf. (1.2). However, usually this is achieved for a limited range of convergence rates. Whether this range is reasonably large can be answered by looking at the connection between the smoothness of a function in the continuous space and the best N-term approximation rate of the corresponding vector in the discrete space. For some $s > 0$, we say that the algorithm has *optimal computational complexity* for the *convergence rate* s, if, whenever $\mathbf{u} \in \ell^w_\tau$ with $\frac{1}{\tau} = s + \frac{1}{2}$, for any given tolerance $\varepsilon > 0$, it computes an approximation $\mathbf{w}$ to the exact solution $\mathbf{u}$ such that $\|\mathbf{u} - \mathbf{w}\| \leq \varepsilon$ and $\#\operatorname{supp}\mathbf{w} \lesssim \varepsilon^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau}$, and the cost of determining $\mathbf{w}$ is bounded by an absolute multiple of the same expression. In view of (1.3), since, by imposing whatever smoothness conditions on the solution u generally the convergence rate of best N-term approximations cannot be higher than $\frac{d-t}{n}$, it is fully satisfactory if the algorithm has optimal computational complexity for $s \in (0, \frac{d-t}{n})$.

In order to use Lemma 2.10, we need to compute the residual $\mathbf{r} := \mathbf{f} - \mathbf{L}\mathbf{u}_0$ for a Galerkin solution $\mathbf{u}_0$. Generally, this residual has infinitely many nonzero coefficients, and has to be approximated by a finitely supported vector. We will approximate it by approximating the vector $\mathbf{f}$ and the matrix-vector product $\mathbf{L}\mathbf{u}_0$ separately. In connection with that, we assume that for some constant $s^* > 0$, $\mathbf{L}$ is s^*-*computable*, meaning that for any $s < s^*$, for all $N \in \mathbb{N}$, there is an infinite matrix $\mathbf{L}_N$, having in each column $\mathcal{O}(N)$ non-zero entries, whose computations require $\mathcal{O}(N)$ operations, such that

$$\|\mathbf{L} - \mathbf{L}_N\| \lesssim N^{-s}. \tag{3.1}$$

Under this assumption, the adaptive approximate matrix-vector product **APPLY** from [CDD01, Ste03] can be shown to have the following properties:

APPLY$[\mathbf{w}, \varepsilon] \to \mathbf{z}$. *The input satisfies $\varepsilon > 0$, and $\mathbf{w}$ is finitely supported. The output satisfies $\|\mathbf{L}\mathbf{w} - \mathbf{z}\| \leq \varepsilon$, with for any $s < s^*$, $\#\operatorname{supp}\mathbf{z} \lesssim \varepsilon^{-1/s}|\mathbf{w}|^{1/s}_{\ell^w_\tau}$, where, as always, $\tau = (\frac{1}{2} + s)^{-1}$, and the number of arithmetic operations and storage locations required by this call being bounded by some absolute multiple of $\varepsilon^{-1/s}|\mathbf{w}|^{1/s}_{\ell^w_\tau} + \#\operatorname{supp}\mathbf{w} + 1$.*

Remark 3.1. In the sequel we will construct an adaptive wavelet algorithm and prove that it has optimal computational complexity for the convergence rates less than s^*. For sufficiently smooth wavelets, that have sufficiently many vanishing moments, and for both differential operators with piecewise sufficiently smooth coefficients, or singular integral operators on sufficiently smooth manifolds, the results from [Ste04b, GS04, GS06] show that for some $s^* \geq \frac{d-t}{n}$, $\mathbf{L}$ is s^*-computable. This result is satisfactory as explained earlier.

The construction of a sequence of approximations for $\mathbf{u}$ that converges with a certain rate requires the availability of a sequence of approximations for $\mathbf{f}$ that convergence with at least that rate. It can be shown that for $s < s^*$, with $\tau = (\frac{1}{2} + s)^{-1}$, if $\mathbf{u} \in \ell^w_\tau$, then $\mathbf{f} \in \ell^w_\tau$, with $|\mathbf{f}|_{\ell^w_\tau} \lesssim |\mathbf{u}|_{\ell^w_\tau}$, and so $\sup_N N^s \|\mathbf{f} - \mathbf{f}_N\| \lesssim |\mathbf{u}|_{\ell^w_\tau}$, which, however does not tell how to *construct* an approximation $\mathbf{g}$ which is qualitatively as good as $\mathbf{f}_N$ with a comparable support size. We will assume the availability of the following routine, whose realization depends on the right-hand side at hand.

$\mathbf{RHS}[\varepsilon] \to \mathbf{g}$ *with* $\|\mathbf{f} - \mathbf{g}\| \leq \varepsilon$, *such that if* $\mathbf{u} \in \ell^w_\tau$, *and* $s < s^*$, *then* $\#\operatorname{supp} \mathbf{g} \lesssim \varepsilon^{-1/s} |\mathbf{u}|^{1/s}_{\ell^w_\tau}$, *and the number of arithmetic operations and storage locations required by this call is bounded by some absolute multiple of* $\varepsilon^{-1/s} |\mathbf{u}|^{1/s}_{\ell^w_\tau} + 1$.

Assuming that we use the above subroutines to compute the residual and extend the current set Λ_0 to a set Λ as in Lemma 2.10, we now need to choose a way to compute the Galerkin solution $\mathbf{u}_\Lambda$ on the extended set Λ. Computing the Galerkin solution requires inverting the system (2.13). In view of obtaining a method of optimal complexity, we will solve the system approximately using an iterative method. Here we formulate a subroutine to solve the Galerkin system (2.13) approximately.

$\mathbf{GALSOLVE}[\Lambda, \mathbf{w}_0, \delta, \varepsilon] \to \mathbf{w}_\Lambda$
% The input should satisfy $\delta \geq \|\mathbf{P}_\Lambda(\mathbf{f} - \mathbf{L}\mathbf{w}_0)\|$.
% Let N *be such that, with* $\mathbf{L}_N$ *from* (3.1),
% $\varrho := \|\mathbf{L} - \mathbf{L}_N\| \|\mathbf{A}^{-1}\| \left[2 + \|\mathbf{B}\mathbf{L}^{-1}\|\right] \leq \frac{\varepsilon}{4\varepsilon + 4\delta}$. *Set* $\tilde{\mathbf{L}}_\Lambda := \mathbf{P}_\Lambda \mathbf{L}_N \mathbf{I}_\Lambda$.
`if` $\delta \leq \varepsilon$ `then` *set* $\mathbf{w}_\Lambda := \mathbf{w}_0$ *and terminate the subroutine* `end if`
$\tilde{\mathbf{r}}_\Lambda := \mathbf{P}_\Lambda(\mathbf{RHS}[\frac{\varepsilon}{4}] - \mathbf{APPLY}[\mathbf{w}_0, \frac{\varepsilon}{4}])$
Apply a suitable iterative method for solving $\tilde{\mathbf{L}}_\Lambda \mathbf{x} = \tilde{\mathbf{r}}_\Lambda$, *e.g., Conjugate Gradients to the Normal Equations, to find an* $\tilde{\mathbf{x}}$ *with* $\|\tilde{\mathbf{r}}_\Lambda - \tilde{\mathbf{L}}_\Lambda \tilde{\mathbf{x}}\| \leq \frac{\varepsilon}{4}$
$\mathbf{w}_\Lambda := \mathbf{w}_0 + \tilde{\mathbf{x}}$

Theorem 3.2. *If* $\Lambda \supseteq \nabla_j$ *with* j *sufficiently large, the output of* $\mathbf{w}_\Lambda := \mathbf{GALSOLVE}[\Lambda, \mathbf{w}_0, \delta, \varepsilon]$ *satisfies* $\|\mathbf{P}_\Lambda(\mathbf{f} - \mathbf{L}\mathbf{w}_\Lambda)\| \leq \varepsilon$, *and for any* $s < s^*$, *the number of arithmetic operations and storage locations required by the call is bounded by some absolute multiple of* $\varepsilon^{-1/s}(|\mathbf{w}_0|^{1/s}_{\ell^w_\tau} + |\mathbf{u}|^{1/s}_{\ell^w_\tau}) + c(\delta/\varepsilon)\#\Lambda$, *with* $c : \mathbb{R}_+ \to \mathbb{R}_+$ *being some non-decreasing function.*

Proof. In this proof, j is assumed to be sufficiently large whenever needed. With the short-hand notation $\mathbf{L}_\Lambda = \mathbf{P}_\Lambda \mathbf{L} \mathbf{I}_\Lambda$, using Lemma 2.9 and estimating $1 + O(\alpha_j) \leq 2$, we have

$$\|\mathbf{L}_\Lambda^{-1}(\tilde{\mathbf{L}}_\Lambda - \mathbf{L}_\Lambda)\| \leq \|\mathbf{L}_\Lambda^{-1}\| \|\mathbf{L}_N - \mathbf{L}\| \leq \|\mathbf{A}^{-1}\| \left[1 + \|\mathbf{B}\mathbf{L}^{-1}\| + O(\alpha_j)\right] \|\mathbf{L}_N - \mathbf{L}\| \leq \varrho < 1.$$

This implies that $\mathbf{I} + \mathbf{L}_\Lambda^{-1}(\tilde{\mathbf{L}}_\Lambda - \mathbf{L}_\Lambda)$ is invertible with $\|(\mathbf{I} + \mathbf{L}_\Lambda^{-1}(\tilde{\mathbf{L}}_\Lambda - \mathbf{L}_\Lambda))^{-1}\| \leq \frac{1}{1-\varrho}$. Now writing $\tilde{\mathbf{L}}_\Lambda = \mathbf{L}_\Lambda(\mathbf{I} + \mathbf{L}_\Lambda^{-1}(\tilde{\mathbf{L}}_\Lambda - \mathbf{L}_\Lambda))$ and using Lemma 2.9 again, we conclude that $\tilde{\mathbf{L}}_\Lambda$ is invertible with

$$\|\tilde{\mathbf{L}}_\Lambda^{-1}\| \leq \frac{1}{1-\varrho} \|\mathbf{L}_\Lambda^{-1}\| \leq \frac{1}{1-\varrho} \|\mathbf{A}^{-1}\| \left[2 + \|\mathbf{B}\mathbf{L}^{-1}\|\right]. \tag{3.2}$$

We have

$$\|\tilde{\mathbf{L}}_\Lambda - \mathbf{L}_\Lambda\| \|\tilde{\mathbf{L}}_\Lambda^{-1}\| \leq \|\mathbf{L}_N - \mathbf{L}\| \frac{1}{1-\varrho} \|\mathbf{A}^{-1}\| \left[2 + \|\mathbf{B}\mathbf{L}^{-1}\|\right] \leq \frac{\varrho}{1-\varrho},$$

and $\|\tilde{\mathbf{r}}_\Lambda\| \leq \|\mathbf{P}_\Lambda(\mathbf{f} - \mathbf{L}\mathbf{w}_0)\| + \|\mathbf{P}_\Lambda(\mathbf{f} - \mathbf{L}\mathbf{w}_0) - \tilde{\mathbf{r}}_\Lambda\| \leq \delta + \frac{\varepsilon}{2}$. Setting $\mathbf{r}_\Lambda := \mathbf{P}_\Lambda(\mathbf{f} - \mathbf{L}\mathbf{w}_0)$ and writing

$$\mathbf{P}_\Lambda(\mathbf{f} - \mathbf{L}\mathbf{w}_\Lambda) = \mathbf{r}_\Lambda - \mathbf{P}_\Lambda \mathbf{L}\tilde{\mathbf{x}} = (\mathbf{r}_\Lambda - \tilde{\mathbf{r}}_\Lambda) + (\tilde{\mathbf{r}}_\Lambda - \tilde{\mathbf{L}}_\Lambda \tilde{\mathbf{x}}) + (\tilde{\mathbf{L}}_\Lambda - \mathbf{P}_\Lambda \mathbf{L})\tilde{\mathbf{L}}_\Lambda^{-1}(\tilde{\mathbf{r}}_\Lambda + \tilde{\mathbf{L}}_\Lambda \tilde{\mathbf{x}} - \tilde{\mathbf{r}}_\Lambda),$$

we find

$$\|\mathbf{P}_\Lambda(\mathbf{f}-\mathbf{L}\mathbf{w}_\Lambda)\| \leq \tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{4}+\tfrac{\varrho}{1-\varrho}(\delta+\tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{4}) \leq \varepsilon.$$

The properties of **APPLY** and **RHS** show that the cost of the computation of $\tilde{\mathbf{r}}_\Lambda$ is bounded by some multiple of $\varepsilon^{-1/s}(|\mathbf{w}_0|_{\ell^w_\tau}^{1/s}+|\mathbf{u}|_{\ell^w_\tau}^{1/s})$. We know that $\|\tilde{\mathbf{L}}_\Lambda\| \lesssim 1$ uniformly in ε and δ. So taking into account (3.2) we have $\kappa(\tilde{\mathbf{L}}_\Lambda) \lesssim 1$ uniformly in ε and δ. Since by (3.1), $\tilde{\mathbf{L}}_\Lambda$ is sparse and can be constructed in $\mathcal{O}(\#\Lambda)$ operations, where the proportionality coefficient is only dependent on an upper bound for δ/ε, and the required number of iterations of the iterative method is bounded, the proof is completed. □

Remark 3.3. If the symmetric part of $\mathbf{L}$ is positive definite, then the spectrum of $\tilde{\mathbf{L}}_\Lambda$ lies in the open right half of the complex plane, and so one can use the GMRES method for the solution of the linear system in **GALSOLVE**, cf. [EES83, SS86]. In this case, the proof of the preceding theorem works verbatim.

Next, we combine the above subroutines into an algorithm which approximately computes the residual $\mathbf{f}-\mathbf{L}\mathbf{u}_\Lambda$ for a given set $\Lambda \subset \nabla$. We get an approximate Galerkin solution as a byproduct because we use **GALSOLVE** to approximate the Galerkin solution $\mathbf{u}_\Lambda$.

$\mathbf{GALRES}[\Lambda,\mathbf{w}_0,\rho_0,\varepsilon] \to [\mathbf{r}_k,\mathbf{w}_k,\rho_k]$:
% The input should satisfy $\rho_0 \geq \|\mathbf{f}-\mathbf{L}\mathbf{w}_0\|$.
% Let $\omega,\gamma \in (0,1)$ *and* $\theta > 0$ *be constants.*

$k := 0$, $\zeta_0 := \theta\rho_0$, $\delta_0 := \rho_0$
`do` $k := k+1$
  $\zeta_k := \zeta_{k-1}/2$
  $\delta_k := \gamma\zeta_k\left(\|\mathbf{L}\|\|\mathbf{A}^{-1}\|\left[2+\|\mathbf{B}\mathbf{L}^{-1}\|\right]\right)^{-1}$
  $\mathbf{w}_k := \mathbf{GALSOLVE}[\Lambda,\mathbf{w}_{k-1},\delta_{k-1},\delta_k]$
  $\mathbf{r}_k := \mathbf{RHS}[(1-\gamma)\zeta_k/2] - \mathbf{APPLY}[\mathbf{w}_k,(1-\gamma)\zeta_k/2]$
  $\delta_k := \min\{\delta_{k-1},\delta_k\}$
`until` $\rho_k := \|\mathbf{r}_k\|+(1-\gamma)\zeta_k \leq \varepsilon$ `or` $\zeta_k \leq \omega\|\mathbf{r}_k\|$

Remark 3.4. In the above algorithm, as opposed to the algorithm in [GHS05], we are forced to place the Galerkin solver inside the loop that computes the current residual with a sufficient accuracy. The reason is that in Lemma 2.10 and Lemma 2.11 the vector $\mathbf{u}_0$ must be the Galerkin solution on its support, whereas in the corresponding Lemma 1.2 and Lemma 2.1 from [GHS05] this vector could be arbitrary.

Remark 3.5. In view of [GHS05, Remark 2.2 and Remark 2.6], taking into account that ρ_0 is an upper bound on the residual of $\mathbf{w}_0$, a reasonable choice for the value of θ is $\theta \approx \frac{2\omega}{(1+\omega)(1-\gamma)}$.

Theorem 3.6. *If* $\Lambda \supseteq \nabla_j$ *for some sufficiently large* j*, then* $[\mathbf{r},\mathbf{w},\rho] := \mathbf{GALRES}[\Lambda,\mathbf{w}_0,\rho_0,\varepsilon]$ *terminates with* $\|\mathbf{f}-\mathbf{L}\mathbf{w}\| \leq \rho$*, and either* $\rho \leq \varepsilon$ *or* $\|\mathbf{r}-(\mathbf{f}-\mathbf{L}\mathbf{u}_\Lambda)\| \leq \omega\|\mathbf{r}\|$*. Furthermore, under the same condition we have* $\rho \gtrsim \min\{\rho_0,\varepsilon\}$*. In addition, if for some* $s < s^*$ *and* $\tau = (\frac{1}{2}+s)^{-1}$*,* $\mathbf{u} \in \ell^w_\tau$*, then* $\#\operatorname{supp}\mathbf{r} \lesssim \rho^{-1/s}|\mathbf{u}|_{\ell^w_\tau}^{1/s}+(\rho_0/\rho)^{1/s}\#\Lambda$ *and the number of arithmetic operations and storage locations required by the call is bounded by some absolute multiple of* $\rho^{-1/s}|\mathbf{u}|_{\ell^w_\tau}^{1/s}+(\rho_0/\rho)^{1/s}(\#\Lambda+1)$*.*

Proof. If at evaluation of the until-clause for the k-th iteration, $\zeta_k > \omega\|\mathbf{r}_k\|$, then $\rho_k = \|\mathbf{r}_k\| + (1-\gamma)\zeta_k < (\omega^{-1}+1-\gamma)\zeta_k$. Since ζ_k is halved in each iteration, we infer that, if not by $\zeta_k \le \omega\|\mathbf{r}_k\|$, the inner loop will terminate by $\rho_k \le \varepsilon$.

Let K be the value of k at the termination of the loop. First we will show $\rho \gtrsim \min\{\rho_0, \varepsilon\}$. When the loop terminates in the first iteration, i.e., when $K=1$, we have $\rho_1 = \|\mathbf{r}_1\| + (1-\gamma)\zeta_1 \gtrsim \rho_0$. In the case the loop terminates with $\rho_K \le \varepsilon$ we have $\|\mathbf{r}_{K-1}\| + 2(1-\gamma)\zeta_K > \varepsilon$ and $2\zeta_K > \omega\|\mathbf{r}_{K-1}\|$, so we conclude

$$\rho_K \ge (1-\gamma)\zeta_K > \frac{(1-\gamma)\omega(\|\mathbf{r}_{K-1}\| + 2(1-\gamma)\zeta_K)}{2+2\omega(1-\gamma)} > \frac{(1-\gamma)\omega\varepsilon}{2+2\omega(1-\gamma)}.$$

Since after any evaluation of $\mathbf{r}_k$ inside the algorithm, $\|\mathbf{r}_k - (\mathbf{f} - \mathbf{L}\mathbf{w}_k)\| \le (1-\gamma)\zeta_k$, for any $1 \le k \le K$, ρ_k is an upper bound on $\|\mathbf{f} - \mathbf{L}\mathbf{w}_k\|$. Together with the condition on ρ_0 this guarantees that the subroutine **GALSOLVE** is called with a valid parameter δ_{k-1}. By applying Lemma 2.9 for sufficiently large j, we have

$$\begin{aligned}\|\mathbf{r}_k - (\mathbf{f} - \mathbf{L}\mathbf{u}_\Lambda)\| &\le \|\mathbf{r}_k - (\mathbf{f} - \mathbf{L}\mathbf{w}_k)\| + \|\mathbf{L}(\mathbf{u}_\Lambda - \mathbf{w}_k)\| \\ &\le (1-\gamma)\zeta_k + \|\mathbf{L}\|\|(\mathbf{P}_\Lambda\mathbf{L}\mathbf{I}_\Lambda)^{-1}\|\|\mathbf{P}_\Lambda(\mathbf{f} - \mathbf{L}\mathbf{w}_k)\| \\ &\le (1-\gamma)\zeta_k + \|\mathbf{L}\|\cdot\|\mathbf{A}^{-1}\|[1 + \|\mathbf{B}\mathbf{L}^{-1}\| + O(\alpha_j)]\cdot\delta_k \le \zeta_k,\end{aligned}$$

and therefore the condition $\zeta_k \le \omega\|\mathbf{r}_k\|$ implies $\|\mathbf{r}_k - (\mathbf{f} - \mathbf{L}\mathbf{u}_\Lambda)\| \le \omega\|\mathbf{r}_k\|$. This proves the first part of the theorem.

The properties of **RHS**, **APPLY** and **GALSOLVE** imply that the cost of k-th iteration can be bounded by some multiple of $\zeta_k^{-1/s}(|\mathbf{w}_{k-1}|_{\ell_\tau^w}^{1/s} + |\mathbf{u}|_{\ell_\tau^w}^{1/s} + |\mathbf{w}_k|_{\ell_\tau^w}^{1/s}) + c(\frac{\delta_{k-1}}{\delta_k})\#\Lambda + \#\Lambda + 1$, where $c(\cdot)$ is the non-decreasing function from Theorem 3.2. Since any vector $\mathbf{w}_k$ determined inside the algorithm satisfies $\|\mathbf{u} - \mathbf{w}_k\| \lesssim \rho_0$, from $|\mathbf{w}_k|_{\ell_\tau^w} \lesssim |\mathbf{u}|_{\ell_\tau^w} + (\#\mathrm{supp}\,\mathbf{w}_k)^s\|\mathbf{w}_k - \mathbf{u}\|$ ([CDD01, Lemma 4.11]), we infer that $|\mathbf{w}_k|_{\ell_\tau^w} \lesssim |\mathbf{u}|_{\ell_\tau^w} + (\#\Lambda)^s\rho_0$. At any iteration the ratio $\frac{\delta_{k-1}}{\delta_k}$ can be bounded by a multiple of $\max\{\frac{\delta_0}{\delta_1}, 2\} \lesssim \frac{\rho_0}{\zeta_1} + 1 \lesssim 1$. By the geometric decrease of ζ_k inside the loop, the above considerations imply that the total cost of the algorithm can be bounded by some multiple of $\zeta_K^{-1/s}(|\mathbf{u}|_{\ell_\tau^w}^{1/s} + \rho_0^{1/s}\#\Lambda) + K(\#\Lambda + 1)$. Taking into account the value of ζ_0, and the geometric decrease of ζ_i inside the loop, we have $K(\#\Lambda+1) = K\rho_0^{-1/s}\rho_0^{1/s}(\#\Lambda+1) \lesssim \zeta_K^{-1/s}\rho_0^{1/s}(\#\Lambda+1)$. The number of nonzero coefficients in $\mathbf{r}_K$ is bounded by an absolute multiple of $\zeta_K^{-1/s}(|\mathbf{u}|_{\ell_\tau^w}^{1/s} + \rho_0^{1/s}\#\Lambda)$ so the theorem is proven upon showing that $\zeta_K \gtrsim \rho_K$. When the loop terminates in the first iteration, i.e., when $K=1$, we have $\rho_1 = \|\mathbf{r}_1\| + (1-\gamma)\zeta_1 \le \|\mathbf{f} - \mathbf{L}\mathbf{w}_0\| + 2(1-\gamma)\zeta_1 \lesssim \rho_0 + \zeta_1 \lesssim \zeta_1$, and when the loop terminates with $\zeta_K \ge \omega\|\mathbf{r}_K\|$, we have $\rho_K = \|\mathbf{r}_K\| + (1-\gamma)\zeta_K \le (\frac{1}{\omega} + 1 - \gamma)\zeta_K$. In the other case, we have $\omega\|\mathbf{r}_{K-1}\| \le 2\zeta_K$, and so from $\|\mathbf{r}_K - \mathbf{r}_{K-1}\| \le \zeta_K + 2\zeta_K$, we infer $\|\mathbf{r}_K\| \le \|\mathbf{r}_{K-1}\| + 3\zeta_K \le (\frac{2}{\omega} + 3)\zeta_K$, so that $\rho_K \le (\frac{2}{\omega} + 4 - \gamma)\zeta_K$. □

We now define our adaptive wavelet solver.

SOLVE$[\varepsilon] \to \mathbf{w}_k$:
% Let j be a sufficiently large fixed integer,
% $\rho_0 \ge \|\mathbf{f}\|$, and $\alpha \in (0,1)$ be constants.

$k := 0$, $\mathbf{w}_0 := 0$, $\Lambda_1 := \nabla_j$
`do` $k := k+1$
    $[\mathbf{r}_k, \mathbf{w}_k, \rho_k] :=$**GALRES**$[\Lambda_k, \mathbf{w}_{k-1}, \rho_{k-1}, \varepsilon]$

```
    if ρ_k > ε
    then determine a set ∇ ⊃ Λ_{k+1} ⊃ Λ_k, with, up to some absolute constant factor,
             minimal cardinality, such that ‖P_{Λ_{k+1}} r_k‖ ≥ α‖r_k‖
    else terminate the subroutine
enddo
```

Remark 3.7. Employing an efficient exact sorting algorithm, one can determine the set Λ_{k+1} with true minimal cardinality, in $O\left(\#\mathrm{supp}\,\mathbf{r}\cdot\log(\#\mathrm{supp}\,\mathbf{r})\right)$ operations. If one allows "minimal cardinality up to some absolute constant factor", as in the above algorithm, the log-factor can be removed via binary bins, see [GHS05, Remark 2.3] and [Ste03] for details.

Theorem 3.8. $\mathbf{w} := \mathbf{SOLVE}[\varepsilon]$ *terminates with* $\|\mathbf{f}-\mathbf{L}\mathbf{w}\| \leq \varepsilon$. *In addition, let the parameters* α *and* ρ_0 *in* **SOLVE**, *and* ω *in* **GALRES**, *be selected such that* $\frac{\alpha+\omega}{1-\omega} < \kappa(\mathbf{A})^{-\frac{1}{2}}$ *and* $\rho_0 \lesssim \|\mathbf{f}\|$, *and let* $\varepsilon \lesssim \|\mathbf{f}\|$. *Then, if for some* $s<s^*$, *and* $\tau = (\frac{1}{2}+s)^{-1}$, $\mathbf{u}\in\ell^w_\tau$, *we have* $\#\mathrm{supp}\,\mathbf{w} \lesssim \varepsilon^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau}$ *and the number of arithmetic operations and storage locations required by the call is bounded by some absolute multiple of the same expression.*

Proof. Before we come to the actual proof, first we indicate the need for the conditions involving ρ_0, $\|\mathbf{f}\|$ and ε. If $\rho_0 \not\lesssim \|\mathbf{f}\|$ we cannot bound the cost of the first call of **GALRES**. If $\varepsilon \not\lesssim \|\mathbf{f}\|$, then $\varepsilon^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau}$ might be arbitrarily small, whereas **SOLVE** takes in any case some arithmetic operations.

Abbreviating $\mathbf{P}_{\Lambda_k}$ as $\mathbf{P}_k$, let $\mathbf{u}_k\in\ell_2(\Lambda_k)$ be the solution of the Galerkin system $\mathbf{P}_k\mathbf{L}\mathbf{u}_k = \mathbf{P}_k\mathbf{f}$. Assume that the k-th call of **GALRES** terminates with $\rho_k>\varepsilon$ and thus with $\|\mathbf{r}_k - (\mathbf{f}-\mathbf{L}\mathbf{u}_k)\| \leq \omega\|\mathbf{r}_k\|$. Then we have

$$\alpha\|\mathbf{r}_k\| \leq \|\mathbf{P}_{k+1}\mathbf{r}_k\| = \|\mathbf{P}_{k+1}[\mathbf{r}_k-(\mathbf{f}-\mathbf{L}\mathbf{u}_k)+(\mathbf{f}-\mathbf{L}\mathbf{u}_k)]\| \leq \omega\|\mathbf{r}_k\| + \|\mathbf{P}_{k+1}(\mathbf{f}-\mathbf{L}\mathbf{u}_k)\|,$$

giving $\|\mathbf{P}_{k+1}(\mathbf{f}-\mathbf{L}\mathbf{u}_k)\| \geq (\alpha-\omega)\|\mathbf{r}_k\|$. Defining $\nu_k := \|\mathbf{r}_k\| + \|\mathbf{r}_k-(\mathbf{f}-\mathbf{L}\mathbf{u}_k)\|$ we have $\|\mathbf{f}-\mathbf{L}\mathbf{u}_k\| \leq \nu_k \leq (1+\omega)\|\mathbf{r}_k\|$, and using this we obtain

$$\|\mathbf{P}_{k+1}(\mathbf{f}-\mathbf{L}\mathbf{u}_k)\| \geq \tfrac{\alpha-\omega}{1+\omega}\nu_k \geq \tfrac{\alpha-\omega}{1+\omega}\|\mathbf{f}-\mathbf{L}\mathbf{u}_k\|,$$

so that Lemma 2.10 shows that $|\!|\!|\mathbf{u}-\mathbf{u}_{k+1}|\!|\!| \leq \left[1-\kappa(\mathbf{A})^{-1}(\frac{\alpha-\omega}{1+\omega})^2+O(\alpha_j)\right]^{\frac{1}{2}}|\!|\!|\mathbf{u}-\mathbf{u}_k|\!|\!|$. Taking into account that $\nu_k \leq (1+\omega)\|\mathbf{r}_k\| < (1+\omega)\rho_k$ and that $\|\mathbf{f}-\mathbf{L}\mathbf{u}_k\| \geq \|\mathbf{P}_{k+1}(\mathbf{f}-\mathbf{L}\mathbf{u}_k)\| \gtrsim \nu_k$, we have $\rho_k \eqsim \nu_k \eqsim \|\mathbf{f}-\mathbf{L}\mathbf{u}_k\| \eqsim |\!|\!|\mathbf{u}-\mathbf{u}_k|\!|\!|$ as long as $\rho_k>\varepsilon$. By the conditions that $\alpha>\omega$ and that j is sufficiently large, it holds that $\rho_k \lesssim \xi^{k-1}\rho_1$ for certain $\xi<1$, so that **SOLVE** terminates, say directly after the K-th iteration. This proves the first part of the theorem.

With $\mu = \frac{\alpha+\omega}{1-\omega}$, for $1\leq k<K$ let $\nabla\supset\Lambda\supset\Lambda_k$ be the *smallest* set with

$$\|\mathbf{P}_\Lambda(\mathbf{f}-\mathbf{L}\mathbf{u}_k)\| \geq \mu\|\mathbf{f}-\mathbf{L}\mathbf{u}_k\|.$$

Since $\mu < \kappa(\mathbf{A})^{\frac{1}{2}}$ by the condition on ω and α, and $\|\mathbf{f}-\mathbf{L}\mathbf{u}_k\| \leq \nu_k$, an application of Lemma 2.11 shows that $\#(\Lambda\backslash\Lambda_k) \lesssim \nu_k^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau}$. On the other hand, using Theorem 3.6 twice we have $\mu\|\mathbf{r}_k\| \leq \mu\|\mathbf{f}-\mathbf{L}\mathbf{u}_k\| + \mu\omega\|\mathbf{r}_k\| \leq \|\mathbf{P}_\Lambda(\mathbf{f}-\mathbf{L}\mathbf{u}_k)\| + \mu\omega\|\mathbf{r}_k\| \leq \|\mathbf{P}_\Lambda\mathbf{r}_k\| + (1+\mu)\omega\|\mathbf{r}_k\|$ or $\|\mathbf{P}_\Lambda\mathbf{r}_k\| \geq \alpha\|\mathbf{r}_k\|$. Thus by construction of Λ_{k+1} we conclude that

$$\#(\Lambda_{k+1}\backslash\Lambda_k) \lesssim \#(\Lambda\backslash\Lambda_k) \lesssim \nu_k^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau} \lesssim \rho_k^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau} \quad \text{for } 1\leq k<K.$$

Since $\Lambda_1 \lesssim 1 \lesssim \rho_0^{-1/s} |\mathbf{u}|_{\ell_\tau^w}^{1/s}$ by $\rho_0 \lesssim \|\mathbf{f}\| \lesssim |\mathbf{u}|_{\ell_\tau^w}$, with $\Lambda_0 := \emptyset$ we have

$$\#\Lambda_k = \sum_{i=0}^{k-1} \#(\Lambda_{i+1} \backslash \Lambda_i) \lesssim (\sum_{i=0}^{k-1} \rho_i^{-1/s}) |\mathbf{u}|_{\ell_\tau^w}^{1/s} \lesssim \rho_{k-1}^{-1/s} |\mathbf{u}|_{\ell_\tau^w}^{1/s} \quad \text{for } 1 \leq k \leq K. \tag{3.3}$$

In view of Remark 3.7, we infer that the cost of determining the set Λ_{k+1} is of order $\#\mathrm{supp}\, \mathbf{r}_k$. From Theorem 3.6, we have $\#\mathrm{supp}\, \mathbf{r}_k \lesssim \rho_k^{-1/s} |\mathbf{u}|_{\ell_\tau^w}^{1/s} + (\rho_{k-1}/\rho_k)^{1/s} \#\Lambda_k$ and that the cost of the k-th call of **GALRES** is of order $\rho_k^{-1/s} |\mathbf{u}|_{\ell_\tau^w}^{1/s} + (\rho_{k-1}/\rho_k)^{1/s}(\#\Lambda_k + 1)$, implying that the cost of the k-th iteration of **SOLVE** can be bounded by an absolute multiple of the latter expression. Now by using (3.3) and $1 \lesssim \rho_0^{-1/s} |\mathbf{u}|_{\ell_\tau^w}^{1/s}$, and taking into account the geometric decrease of ρ_k we conclude that the total cost of the algorithm can be bounded by an absolute multiple of $\rho_K^{-1/s} |\mathbf{u}|_{\ell_\tau^w}^{1/s}$. From Theorem 3.6 we have $\rho_K \gtrsim \min\{\rho_{K-1}, \varepsilon\} \gtrsim \varepsilon$, where the second inequality follows from $\rho_{K-1} > \varepsilon$ when $K > 1$ and by assumption when $K = 1$. This completes the proof. □

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