# ADAPTIVE FINITE ELEMENT ALGORITHMS FOR THE STOKES PROBLEM: CONVERGENCE RATES AND OPTIMAL COMPUTATIONAL COMPLEXITY

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Abstract. Although adaptive finite element methods (FEMs) are recognized as powerful techniques for solving mixed variational problems of fluid mechanics, usually they are not even proven to converge. Only recently, in [SINUM, 40 (2002), pp.1207-1229] Bänsch, Morin and Nochetto introduced an adaptive Uzawa FEM for solving the Stokes problem, and showed its convergence. In their paper, numerical experiments indicate (quasi-) optimal triangulations for some values of the parameters, where, a theoretical explanation of these results is still open.

In this paper, we present a similar adaptive Uzawa finite element algorithm that uses a generalization of the optimal adaptive FEM of Stevenson [SINUM, 42 (2005), pp.2188-2217] as an inner solver. By adding a derefinement step to the resulting adaptive Uzawa algorithm, in order to optimize the underlying triangulation after each fixed number of iterations, we show that the overall method converges with optimal rates with linear computational complexity.

## 1. Introduction

Nowadays adaptive finite element algorithms are being used to solve efficiently partial differential equations (PDEs) arising in science and engineering. The general structure of the loop of an adaptive algorithm is

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Refine, Derefine}$$

Only recently, however, in the works of Dörfler ([Dör96]) and Morin, Nochetto, Siebert ([MNS00]), adaptive FEMs for elliptic problems were shown to converge. Later, in ([BDD04]), Binev, Dahmen and DeVore, and in ([Ste05a]), Stevenson showed that adaptive FEMs of this type converge with optimal rates and with linear computational complexity.

Typically, problems in fluid mechanics naturally lead to mixed variational problems. Concerning the adaptive solution of mixed variational problems, the situation is more complicated, and we are not aware of any proof of optimality of adaptive finite element algorithms.

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In ([BMN02]), Bänch, Morin and Nochetto introduced an adaptive FEM for the Stokes problem, and they proved convergence of the method. Since convergence alone does not imply that the adaptive method is more efficient than its non-adaptive counterpart, an analysis of the convergence rates of an adaptive approximation is important. The numerical experiments in ([BMN02]) show (quasi-) optimal triangulations for some values of the parameters, where, however, a theoretical explanation of these results is still an open question.

In this paper, we present a detailed design of adaptive FEM algorithm for the Stokes problem, and analyze its computational complexity. As in [BMN02], we apply a fixed point iteration to an infinite dimensional Schur complement operator, where to approximate the inverse of the elliptic operator we use a generalization of the optimal adaptive finite element method of Stevenson for elliptic equations. By adding a derefinement step to the resulting adaptive fixed point iteration, in order to optimize the underlying triangulation after each fixed number of iterations, we show that the resulting method converges with the optimal rate and that it has optimal computational complexity.

## 2. Mixed Variational Problems. Well-Posedness and Discretization

Let $X$ and $Q$ be Hilbert spaces, and let the bilinear forms

$$a : X \times X \to \mathbb{R}, \quad b : X \times Q \to \mathbb{R} \tag{2.1}$$

be continuous. We consider the following mixed variational problem

$$\begin{cases} \text{Given } f \in X', \; g \in Q' \\ \text{find } (u,p) \in X \times Q \text{ such that} \\ a(u,v) + b(v,p) \quad = f(v) \text{ for all } v \in X. \\ b(u,q) \qquad\qquad\quad = g(q) \text{ for all } q \in Q. \end{cases} \tag{2.2}$$

By the Riesz representation theorem, the bilinear forms $a : X \times X \to \mathbb{R}$ and $b : X \times Q \to \mathbb{R}$ induce corresponding operators $\mathscr{A} \in \mathcal{L}(X, X')$, $\mathscr{B} \in \mathcal{L}(X, Q')$ and their adjoints such that

$$\begin{aligned} a(u,v) &= (\mathscr{A}u)(v) = (\mathscr{A}^*v)(u) \qquad && \text{for all } u, v \in X \\ b(v,q) &= (\mathscr{B}v)(q) = (\mathscr{B}^*q)(v) && \text{for all } v \in X, \; q \in Q \end{aligned} \tag{2.3}$$

Now we can rewrite (2.2) as the following operator problem

$$\begin{cases} \text{Given } f \in X', \; g \in Q' \\ \text{find } (u,p) \in X \times Q \text{ such that} \\ \mathscr{A}u + \mathscr{B}^*p \quad = f \\ \mathscr{B}u \qquad\qquad = g, \end{cases} \tag{2.4}$$

or with

$$\mathscr{L} := \begin{pmatrix} \mathscr{A} & \mathscr{B}^* \\ \mathscr{B} & 0 \end{pmatrix} : X \times Q \to X' \times Q', \tag{2.5}$$

as

$$\begin{cases} \text{Given } F = (f, g) \in X' \times Q' \\ \text{find } U = (u, p) \in X \times Q \text{ such that} \\ \mathscr{L} u = F. \end{cases}$$

**Theorem 2.1.** [GR86] *The mapping (2.5) is an homeomorphism, i.e., there exist constants* $c_{\mathscr{L}}, C_{\mathscr{L}} > 0$ *such that*

$$c_{\mathscr{L}}(\|v\|_X^2 + \|q\|_Q^2)^{1/2} \leq \|\mathscr{L}(v,q)\|_{X' \times Q'} \leq C_{\mathscr{L}}(\|v\|_X^2 + \|q\|_Q^2)^{1/2} \tag{2.6}$$

*if and only if the following conditions are satisfied*

*(i) There exists a constant* $\bar{\alpha} > 0$ *such that*

$$a(v, v) \geq \bar{\alpha} \|v\|_X^2 \text{ for all } v \in Ker\, \mathscr{B} \tag{2.7}$$

*(ii) there exists a constant* $\bar{\beta} > 0$ *such that*

$$\inf_{q \in Q} \sup_{v \in X} \frac{b(v,q)}{\|v\|_X \|q\|_Q} \geq \bar{\beta} \tag{2.8}$$

## 3. Stokes Problem

The motion of an incompressible viscous fluid enclosed in a domain $\Omega \subset \mathbb{R}^2$ is described by the Stokes equations

$$-\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \sigma_{ij}(\mathbf{u}) = \tilde{f}_i \quad \text{in } \Omega \tag{3.1}$$

$$\sigma_{ij}(\mathbf{u}) = -\tilde{p}\delta_{ij} + 2\mu e_{ij}(\mathbf{u}) \quad \text{in } \Omega \tag{3.2}$$

$$e_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{in } \Omega \tag{3.3}$$

$$\mathrm{div}\mathbf{u} = 0 \quad \text{in } \Omega \tag{3.4}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega \tag{3.5}$$

where $\mathbf{u}$ is a velocity field, $\tilde{p}$ is the pressure, $\mu$ is the viscosity of the fluid, $\mathbf{f}$ defines the body forces per unit mass, and $\{\sigma_{ij}\}$, $\{e_{ij}\}$ are the stress and deformation rate tensors, respectively. Here, for simplicity of the presentation we assumed that $\Omega \subset \mathbb{R}^2$. However, we expect that the results of this paper can be generalized to the 3-D case. Elimination of $\{\sigma_{ij}\}$ leads to the equivalent formulation

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla\tilde{p} = \tilde{\mathbf{f}} & \text{in } \Omega \\ \text{div}\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.6}$$

Finally, introducing the scaled variables $p = \mu^{-1}\tilde{p}$, $\mathbf{f} = \mu^{-1}\tilde{\mathbf{f}}$, we obtain the parameter-independent Stokes problem

$$\begin{cases} -\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \text{div}\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.7}$$

Next we formulate the variational form of the Stokes problem, which is the following mixed variational problem

$$\begin{cases} \text{With } \mathbf{X} = [H_0^1(\Omega)]^2,\ Q = L_{2,0}(\Omega) = \{w \in L_2(\Omega) : \displaystyle\int_\Omega w = 0\},\ \text{and} \\ \text{for given } \mathbf{f} \in \mathbf{X}', \\ \text{find } (\mathbf{u}, p) \in \mathbf{X} \times Q \text{ such that} \\ \begin{array}{lll} a(\mathbf{u}, v) + b(\mathbf{v}, p) & = \mathbf{f}(\mathbf{v}) & \text{for all } \mathbf{v} \in \mathbf{X}. \\ b(\mathbf{u}, q) & = 0 & \text{for all } q \in Q. \end{array} \end{cases} \tag{3.8}$$

where

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} = \int_\Omega \sum_{i=1}^{2} \nabla u_i \nabla v_i \\ b(\mathbf{v}, q) &= -\int_\Omega q\,\text{div}\mathbf{v} \end{aligned} \tag{3.9}$$

We will equip the space of vector-fields $\mathbf{X}$ with norm

$$\|\mathbf{v}\|_{\mathbf{X}} := a(\mathbf{v}, \mathbf{v})^{1/2} \text{ for all } \mathbf{v} \in \mathbf{X}, \tag{3.10}$$

which is, in fact, the $[H^1(\Omega)]^2$-seminorm. By Poincaré's inequality, on $[H_0^1(\Omega)]^2$ it is a norm that is equivalent to the standard $[H^1(\Omega)]^2$-norm. We equip $\mathbf{X}'$ with dual norm

$$\|\mathbf{f}\|_{\mathbf{X}'} := \sup_{0 \neq \mathbf{v} \in \mathbf{X}} \frac{|\mathbf{f}(\mathbf{v})|}{\|\mathbf{v}\|_{\mathbf{X}}}. \tag{3.11}$$

Equipped with these norms, $\mathscr{A} : \mathbf{X} \to \mathbf{X}'$ is an isomorphism. We equip the space Q with the $L_2(\Omega)$ norm, i.e.,

$$\|q\|_Q := \|q\|_{L_2(\Omega)} \text{ for all } q \in Q. \tag{3.12}$$

Again, as we have done in (2.3), we will introduce the operators induced by the bilinear forms of the Stokes problem, where now $\mathscr{B} = -\mathrm{div}$ and $\mathscr{B}^* = \nabla$, and write the problem as the equivalent operator problem (2.4). To convince ourselves that the mapping $\mathscr{L}$ : $\mathbf{X} \times Q \to \mathbf{X}' \times Q'$ is an homeomorphism, so that the Stokes problem is well-posed, we will check the conditions of Theorem 2.1. The ellipticity (2.7) is valid with $\alpha = 1$.

To verify the inf-sup condition (2.8) we recall the following theorem from ([GR86]).

**Theorem 3.1.** *Let $\Omega$ be a bounded connected domain with Lipschitz continuous boundary. Then there exists a $c$, which depends on $\Omega$, such that*

$$\|p\|_{L_2(\Omega)} \leq c\|\nabla p\|_{[H^{-1}(\Omega)]^2} \quad \textit{for all } p \in L_{2,0}(\Omega) \tag{3.13}$$

Using Theorem 3.1, the inf-sup condition (2.8) easily follows. Indeed, given $q \in Q$, we can find $\mathbf{v} \in \mathbf{X}$, such that $\|\mathbf{v}\|_{\mathbf{X}} = 1$ and

$$\frac{1}{c}\|q\|_Q \leq \|\nabla q\|_{\mathbf{X}'} =< \mathbf{v}, \nabla q >_{\mathbf{X}\times\mathbf{X}'}= (\mathscr{B}^* q)(\mathbf{v}) = (\mathscr{B}\mathbf{v})(q) = b(\mathbf{v}, q) \tag{3.14}$$

Given some pair $(\mathbf{X}_i, Q_i) \subset \mathbf{X} \times Q$ of FEM spaces, the discrete problem reads as follows

$$\begin{cases} \text{Given } \mathbf{f} \in \mathbf{X}' \\ \text{find } (\mathbf{u}_i, p_i) \in \mathbf{X}_i \times Q_i \text{ such that} \\ a(\mathbf{u}_i, \mathbf{v}) + b(\mathbf{v}, p_i) \quad = \mathbf{f}(\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{X}_i. \\ b(\mathbf{u}_i, q) \qquad\qquad\quad = 0 \qquad\;\; \text{for all } q \in Q_i. \end{cases} \tag{3.15}$$

**Theorem 3.2.** [BF91] *Let $(\mathbf{u}, p) \in \mathbf{X} \times Q$ be the solution of Stokes problem, and assume that for some constant $\bar{\beta} > 0$ the pair $(\mathbf{X}_i, Q_i)$ satisfies*

$$\inf_{q_i \in Q_i} \sup_{\mathbf{v}_i \in \mathbf{X}_i} \frac{b(\mathbf{v}_i, q_i)}{\|\mathbf{v}_i\|_{\mathbf{X}}\|q_i\|_Q} \geq \bar{\beta}. \tag{3.16}$$

*Then the following error estimate holds for the discrete solution $(\mathbf{u}_i, p_i) \in \mathbf{X}_i \times Q_i$*

$$\|\mathbf{u} - \mathbf{u}_i\|_{\mathbf{X}} + \|p - q_i\|_Q \leq C(\inf_{\mathbf{v}_i \in \mathbf{X}_i} \|\mathbf{u} - \mathbf{v}_i\|_{\mathbf{X}} + \inf_{q_i \in Q_i} \|p - q_i\|_Q), \tag{3.17}$$

*where $C > 0$ is a constant only dependent on the bilinear forms $a$ and $b$ and the constant $\bar{\beta}$.*

Although the above error estimate indicates the importance of the discrete inf-sup condition (3.16), in this paper we develop an adaptive algorithm where there is no need to pose such a requirement on the approximation spaces.

## 4. Nonconforming triangulations and adaptive hierarchical tree-structures

Let us consider finite element approximations with respect to partitions into triangles (triangulations). In this section, we describe the type of triangulations that we shall use throughout the paper. Let $\mathcal{T}$ be a triangulation of $\Omega$. If for any pair of distinct triangles

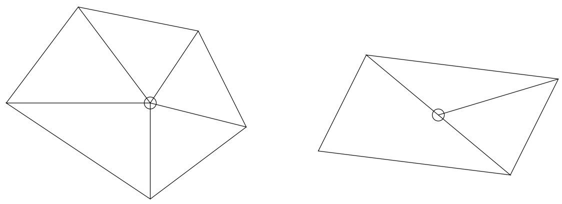


FIGURE 1. Non-hanging and hanging vertices.

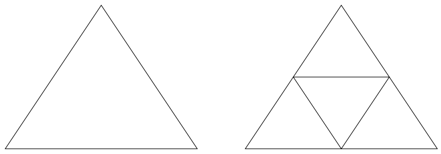


FIGURE 2. A triangle and its red-refinement.

$K_1, K_2 \in \mathcal{T}$ with $K_1 \cap K_2 \neq \emptyset$, their intersection is a common lower dimensional face of both $K_1$ and $K_2$, then $\mathcal{T}$ is called *conforming* and otherwise we shall call it a *nonconforming* triangulation. All vertices of triangles $K \in \mathcal{T}$ will be called vertices of the triangulation $\mathcal{T}$. A vertex of the triangulation $\mathcal{T}$ is called a *non-hanging* vertex if it is a vertex for all $K \in \mathcal{T}$ that contain it, otherwise it is called a *hanging* vertex (see Fig. 1). We shall say that an edge $\ell$ of a triangle $K \in \mathcal{T}$ is an edge of the triangulation $\mathcal{T}$ if it doesn't contain a hanging vertex in its interior. The set of all edges of $\mathcal{T}$ will be denoted by $\bar{\mathcal{E}}_{\mathcal{T}}$, and the set of all non-hanging vertices of $\mathcal{T}$ by $\bar{\mathcal{V}}_{\mathcal{T}}$. The set of all interior edges of $\mathcal{T}$ and the set of all interior non-hanging vertices of $\mathcal{T}$ will be denoted by $\mathcal{E}_{\mathcal{T}}$ and $\mathcal{V}_{\mathcal{T}}$ respectively. If two triangles $K_1, K_2 \in \mathcal{T}$ share some edge $\ell \in \mathcal{E}_{\mathcal{T}}$, they will be called edge neighbors, and if they share a vertex we will call them vertex neighbors.

A subdivision of a triangle $K$ into 4 subtriangles by connecting the midpoints of the edges is called a *red-refinement* of $K$ (see Fig. 2). In this paper we restrict ourselves to triangulations that can be created by (recursive), generally non-uniform red-refinements starting from some fixed initial conforming triangulation $\mathcal{T}_0$. Generally these triangulations are nonconforming. For reasons that will become clear later, we will have to limit the 'amount of nonconformity' by restricting ourselves to admissible triangulations, a concept that is defined as follows:

**Definition 4.1.** A triangulation $\mathcal{T}$ is called an *admissible* triangulation if for every edge of a $K \in \mathcal{T}$ that contains a hanging node in its interior, the endpoints of this edge are nonhanging vertices.

Figure 3 depicts an example of an admissible triangulation. Apparently local refinement produces triangulations which are generally not admissible. In the following we discuss how to repair this. To any triangle $K$ from the initial triangulation $\mathcal{T}_0$ we assign the generation $\mathrm{gen}(K) = 0$. Then the generation $\mathrm{gen}(K)$ for $K \in \mathcal{T}$ is defined as the number of subdivision steps needed to produce $K$ starting from the initial triangulation $\mathcal{T}_0$. A triangulation is called *uniform* when all triangles have the same generation. A vertex of

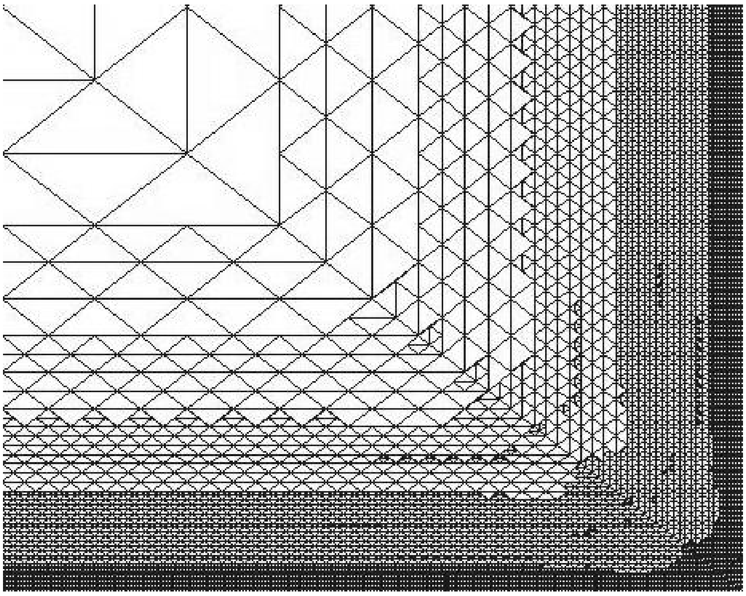


FIGURE 3. An example of admissible triangulation.

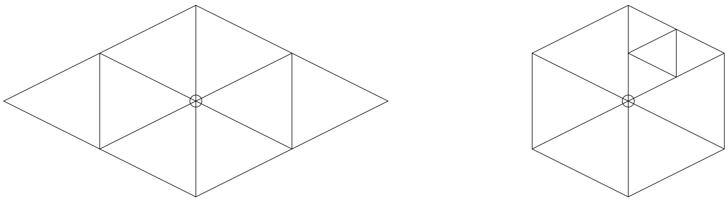


FIGURE 4. Regular and non-regular vertices.

the triangulation will be called a *regular* vertex if all triangles that contain this vertex have the same generation (see Fig. 4). Obviously, a regular vertex is non-hanging.

Corresponding to the triangulation $\mathcal{T}$, we build also a tree $T(\mathcal{T})$ that contains as nodes all the triangles which were created to construct $\mathcal{T}$ from $\mathcal{T}_0$. The roots of this tree are the triangles of the initial triangulation $\mathcal{T}_0$. When a triangle $K$ is subdivided, four new triangles appear which are called children of $K$, and $K$ is called their parent. Similarly, we introduce also grandparent/grandchildren relations. In case a triangle $K \in T(\mathcal{T})$ has no children in the tree $T(\mathcal{T})$, it is called a leaf of this tree. The set of all leaves of the given tree $T(\mathcal{T})$ will be denoted by $\mathcal{L}(T(\mathcal{T}))$. Apparently, the set of leaves $\mathcal{L}(T(\mathcal{T}))$ forms the final triangulation $\mathcal{T}$. We will call $\tilde{T}$ a subtree of $T$, denoted as $\tilde{T} \subset T$, if it contains all roots of $T$ and if for any $K \in \tilde{T}$ all its siblings and ancestors are also in $\tilde{T}$.

**Proposition 4.2.** [Ste05a] *For any triangulation $\mathcal{T}$ created by red-refinements starting from $\mathcal{T}_0$, there exists a unique sequence of triangulations $\mathcal{T}_0, \mathcal{T}_1 \ldots, \mathcal{T}_n$ with $\max_{K\in\mathcal{T}^i} gen(K) = i$, $\mathcal{T}_n = \mathcal{T}$ and where $\mathcal{T}_{i+1}$ is created from $\mathcal{T}_i$ by refining some $K \in \mathcal{T}_i$ with $gen(K) = i$. Let $\mathcal{T}_{-1} = \emptyset$ and $\mathcal{V}_{\mathcal{T}_{-1}} = \emptyset$. The following properties are valid:*

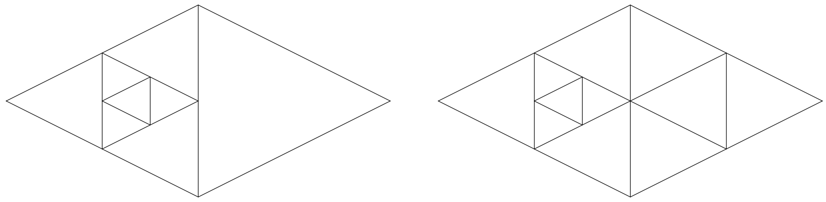


Figure 5. An example of a non-admissible triangulation (left) and its smallest admissible refinement (right).

- $\sum_{i=0}^{n} \sharp \mathcal{T}_i \leq \frac{4}{3} \sharp \mathcal{T}$,
- $\mathcal{V}_{\mathcal{T}_{i-1}} \subset \mathcal{V}_{\mathcal{T}_i}$ *and so* $\mathcal{V}_{\mathcal{T}} = \cup_{i=0}^{n} \mathcal{V}_{\mathcal{T}_i} \setminus \mathcal{V}_{\mathcal{T}_{i-1}}$ *with empty mutual intersections,*
- *a* $v \in \mathcal{V}_{\mathcal{T}_i} \setminus \mathcal{V}_{\mathcal{T}_{i-1}}$ *is not a vertex of* $\mathcal{T}_{i-1}$*, and so it is a regular vertex of* $\mathcal{T}_i$

As we noted before, in case we refine some admissible triangulation only locally, the admissibility of the triangulation might be destroyed. If a given triangulation is not admissible (see Fig. 5), it can be completed to an admissible triangulation using the following algorithm

**Algorithm 4.3.** Make Admissible

(I) $\mathcal{T}_0^a := \mathcal{T}_0$
(II) Let $i := 0$
(III) Define $\mathcal{T}_{i+1}^a$ as the union of $\mathcal{T}_{i+1}$ and, when $i \leq n-2$, the collection of children of those $K \in \mathcal{T}_i^a$ that have a vertex neighbor in $\mathcal{T}_i$ with grandchildren in $\mathcal{T}_{i+2}$
(IV) if $i \leq n-1$ then $i++$ and go to step (III)

**Proposition 4.4.** [Ste05a] *The triangulation* $\mathcal{T}^a$ *constructed by Algorithm (8.9) is an admissible refinement of* $\mathcal{T}$*. Moreover,*

$$\sharp \mathcal{T}^a \lesssim \sharp \mathcal{T} \tag{4.1}$$

In finite element analysis we need a so called 'shape regularity' requirement to avoid very small angles in the triangulation since typical FEM estimates depend on the minimal angle of the triangulation and deteriorate when it is small.

**Definition 4.5.** Let $(\mathcal{T}_i)_{i\in\mathbb{N}}$ be some family of triangulations. If, for each $K \in \mathcal{T}_i$, the ratio of the radii of the smallest circumscribed and the largest inscribed ball of $K$ is bounded uniformly in $K \in \mathcal{T}_i$ and $i \in \mathbb{N}$, then the family of triangulations $(\mathcal{T}_i)_{i\in\mathbb{N}}$ is called *shape-regular.*

Obviously, the family of triangulations created by the red-refinements is shape-regular, depending only on an initial triangulation.

## 5. Methods of Optimal Complexity for Solving the Stokes Problem

For $(\mathcal{T}_i)_i$ being the family of all triangulations that can be constructed from a fixed initial triangulation $\mathcal{T}_0$ of $\Omega$ by red-refinements, let $(\mathbf{X}_{\mathcal{T}_i}, Q_{\mathcal{T}_i}, \mathbf{Y}_{\mathcal{T}_i})_i$ be the family of finite element approximation spaces defined by

$$\mathbf{X}_{\mathcal{T}_i} := \mathbf{X} \cap [C(\Omega)]^2 \cap \prod_{K \in \mathcal{T}_i} [P_1(K)]^2, \tag{5.1}$$

$$Q_{\mathcal{T}_i} := Q \cap \prod_{K \in \mathcal{T}_i} P_0(K), \tag{5.2}$$

$$\mathbf{Y}_{\mathcal{T}_i} := \prod_{K \in \mathcal{T}_i} [P_0(K)]^2. \tag{5.3}$$

As is known, for a shape regular family of triangulations, we have the following *inverse* or *Bernstein* estimate: With $\underline{h}_i := \min_{K \in \mathcal{T}_i} \operatorname{diam}(K)$ and $s \leq t < 1 + \frac{1}{p}$,

$$\|\mathbf{v}_i\|_{[W_p^t(\Omega)]^2} \lesssim \underline{h}_i^{s-t} \|\mathbf{v}_i\|_{[W_p^s(\Omega)]^2} \quad \text{for all } \mathbf{v}_i \in \mathbf{X}_{\mathcal{T}_i}. \tag{5.4}$$

Defining for any $\mathbf{V} \subset \mathbf{X}$ and $\mathbf{w} \in \mathbf{X}$

$$e(\mathbf{w}, \mathbf{V})_{\mathbf{X}} := \inf_{\tilde{\mathbf{w}} \in \mathbf{V}} \|\mathbf{w} - \tilde{\mathbf{w}}\|_{\mathbf{X}}, \tag{5.5}$$

the error of the best approximation from the best space $\mathbf{X}_{\mathcal{T}_i}$ with underlying triangulation consisting of $n - \sharp\mathcal{T}_0$ triangles is given by

$$\sigma_n^{\mathbf{X}}(\mathbf{w}) = \inf_{\mathcal{T}_i \in (\mathcal{T}_j)_j,\ \sharp\mathcal{T}_i - \sharp\mathcal{T}_0 \leq n} e(\mathbf{w}, \mathbf{X}_{\mathcal{T}_i})_{\mathbf{X}} \tag{5.6}$$

We now define classes of vector fields for which the errors of the best approximations from the best spaces decay with certain rates.

**Definition 5.1.** For any $s > 0$, let $\mathcal{A}^s(\mathbf{X}, (\mathbf{X}_{\mathcal{T}_i})_i)$ be the class of vector fields $\mathbf{w} \in \mathbf{X}$ such that for some $M > 0$

$$\sigma_n^{\mathbf{X}}(\mathbf{w}) \leq M n^{-s} \quad n = 1, 2, \ldots. \tag{5.7}$$

We equip $\mathcal{A}^s(\mathbf{X})$ with a semi-norm defined as the smallest $M$ for which (5.7) holds:

$$|\mathbf{w}|_{\mathcal{A}^s(\mathbf{X})} := \sup_{n \geq 1} n^s \sigma_n^{\mathbf{X}}(\mathbf{w}), \tag{5.8}$$

and with norm

$$\|\mathbf{w}\|_{\mathcal{A}^s(\mathbf{X})} := |\mathbf{w}|_{\mathcal{A}^s(\mathbf{X})} + \|\mathbf{w}\|_{\mathbf{X}}. \tag{5.9}$$

In a similar way we define also classes $\mathcal{A}^s(Q) = \mathcal{A}^s(Q, (Q_{\mathcal{T}_i})_i)$ and $\mathcal{A}^s(\mathbf{X}') = \mathcal{A}^s(\mathbf{X}', (\mathbf{Y}_{\mathcal{T}_i})_i)$ using approximation spaces $(Q_{\mathcal{T}_i})_i$ and $(\mathbf{Y}_{\mathcal{T}_i})_i$ respectively. The goal of adaptive approximation is to realize the rates of the best approximation from the best spaces. Below we sketch why we expect better results with adaptive approximation than with non-adaptive approximation based on uniform refinements.

Instead of red-refinements one could also use so-called newest vertex bisections ([BDD04]). Here, a triangle is not refined into 4 but into 2 subtriangles by connecting its 'newest vertex' to the midpoint of the opposite edge. This midpoint is then the newest vertex of both subtriangles. In the initial triangulation, one of the vertices of each triangle is labeled as the newest vertex in some proper way. Let $(\acute{\mathcal{T}}_i)_i$ be the family of all triangulations created by newest vertex bisections from an initial triangulation $\mathcal{T}_0$ and let

$$X_{\acute{\mathcal{T}}_i} := H^1(\Omega) \cap C(\Omega) \cap \prod_{K \in \acute{\mathcal{T}}_i} P_1(K). \tag{5.10}$$

For simplicity, to illustrate the ideas, here we switched from vector-fields to the case of scalar functions and we dropped also the Dirichlet boundary conditions.

Recently, in [BDD04] it was shown that the approximation classes $\mathcal{A}^s(H^1(\Omega), (X_{\acute{\mathcal{T}}_i})_i)$ are (nearly) characterised by membership of certain Besov spaces:

**Theorem 5.2.** *(i) If $u \in B^{2s+1}_\tau(L_\tau(\Omega))$ with $0 \le s \le 1/2$ and $1/\tau < s + 1/2$ then*

$$\inf_{\acute{\mathcal{T}}_i \in (\acute{\mathcal{T}}_i)_i, \sharp \acute{\mathcal{T}}_i \le n} \; \inf_{w_{\acute{\mathcal{T}}_i} \in X_{\acute{\mathcal{T}}_i}} \|u - w_{\acute{\mathcal{T}}_i}\|_{H^1(\Omega)} \lesssim n^{-s} \|u\|_{B^{2s+1}_\tau(L_\tau(\Omega))} \tag{5.11}$$

*(ii) if for $u \in H^1(\Omega)$*

$$\inf_{\acute{\mathcal{T}}_i \in (\acute{\mathcal{T}}_i)_i, \sharp \acute{\mathcal{T}}_i \le n} \; \inf_{w_{\acute{\mathcal{T}}_i} \in X_{\acute{\mathcal{T}}_i}} \|u - w_{\acute{\mathcal{T}}_i}\|_{H^1(\Omega)} \lesssim n^{-s} \tag{5.12}$$

*then $u \in B^{2s+1}_\tau(L_\tau(\Omega))$, where $1/\tau = s + 1/2$.*

As is well-known, in two space dimensions and with piecewise linear approximation a rate $s \le 1/2$ is attained by approximations on uniform triangulations if and only if the approximated function belongs to $H^{2s+1}(\Omega)$. A rate $s > 1/2$ only occurs when the approximated function is exceptionally close to a finite element function, and such a rate cannot be enforced by imposing whatever smoothness condition. Above result shows that a rate $s \le 1/2$ is achieved by 'the best approximation from the best spaces' for any function from the much larger spaces $B^{2s+1}_\tau(L_\tau(\Omega))$ for any $\tau > (s + 1/2)^{-1}$. Although these results were stated for the triangulations created by newest vertex bisections, we expect the same results to be valid for the family of triangulations constructed by red-refinements. The difference between Sobolev spaces and Besov spaces is well illustrated by the DeVore diagram (Fig. 6). In this diagram, the point $(1/\tau, r)$ represents the Besov space $B^r_\tau(L_\tau(\Omega))$. Since $B^r_2(L_2(\Omega)) = H^r(\Omega)$, the Sobolev space $H^r(\Omega)$ corresponds to the point $(1/2, r)$. From this diagram we can observe that the larger $s$ is, the larger is the space $B^{2s+1}_\tau(L_\tau(\Omega))$ with $\tau = (s + 1/2)^{-1}$ compared to $H^{2s+1}(\Omega)$.

For the Stokes problem on an L-shaped domain, even with a smooth right-hand side, one generally has

$$\mathbf{u} \in [H^{1+r}(\Omega)]^2, \; p \in H^r(\Omega), \; \text{only if } r < 0.54448373678246. \tag{5.13}$$

In contrast, in [Dah99] it was shown that the solutions of the Stokes problem have arbitrarily high Besov regularity independent of the shape of the domain, depending only on the smoothnes of the right-hand side, revealing the potential of adaptive methods.

To solve the Stokes problem numerically, we need to be able to approximate the right-hand side within any given tolerance. We assume the availability of the following routine:

**Algorithm 5.3.** $\mathbf{RHS}[\mathcal{T},\mathbf{f},\varepsilon] \to [\tilde{\mathcal{T}},\mathbf{f}_{\tilde{\mathcal{T}}}]$
/* Input of the routine:

- triangulation $\mathcal{T}$
- $\mathbf{f} \in \mathbf{X}'$
- $\varepsilon > 0$

Output: admissible triangulation $\tilde{\mathcal{T}} \supset \mathcal{T}$ and an $\mathbf{f}_{\tilde{\mathcal{T}}} \in \mathbf{Y}_{\tilde{\mathcal{T}}}$ such that $\|\mathbf{f} - \mathbf{f}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}'} \leq \varepsilon$ */

As in [Ste05a], we will call the pair $(\mathbf{f}, \mathbf{RHS})$ to be $s$-optimal if there exists an absolute constant $c_f > 0$ such that for any $\varepsilon > 0$ and triangulation $\mathcal{T}$, the call of the algorithm **RHS** performs such that both

$$\sharp\tilde{\mathcal{T}} \text{ and the number of flops required by the call are } \lesssim \sharp\mathcal{T} + c_f^{1/s}\varepsilon^{-1/s} \tag{5.14}$$

Apparently, for a given $s$, such a pair can only exist if $f \in \mathscr{A}^s(\mathbf{X}')$. A realisation of the routine **RHS** depends on the right-hand side at our hands. As it was shown in [Ste05a], for $\mathbf{f} \in [L_2(\Omega)]^2$, **RHS** can be based on uniform refinements.

We will say that an adaptive method is optimal if it produces a sequence of approximations with respect to triangulations with cardinalities that are at most a constant multiple larger then the optimal ones. More precisely, we define

**Definition 5.4.** Suppose that the solution of the Stokes problem is such that $(\mathbf{u}, p) \in (\mathcal{A}^s(\mathbf{X}), \mathcal{A}^s(Q))$ and there exists a routine **RHS** such that the pair $(\mathbf{f}, \mathbf{RHS})$ is $s$-optimal. Then we shall say that an adaptive method is optimal if for any $\varepsilon > 0$ it produces a triangulation $\mathcal{T}_i$ from the family $(\mathcal{T}_j)_j$ with $\sharp\mathcal{T}_i \lesssim \varepsilon^{-1/s}(|\mathbf{u}|_{\mathcal{A}^s(\mathbf{X})}^{1/s} + \|p\|_{\mathcal{A}^s(Q)}^{1/s} + c_f^{1/s})$ and an approximation $(\mathbf{w}_{\mathcal{T}_i}, q_{\mathcal{T}_i}) \in (\mathbf{X}_{\mathcal{T}_i}, Q_{\mathcal{T}_i})$ with $\|\mathbf{u} - \mathbf{w}_{\mathcal{T}_i}\|_{\mathbf{X}} + \|p - q_{\mathcal{T}_i}\|_Q \leq \varepsilon$ taking only $\lesssim \varepsilon^{-1/s}(|\mathbf{u}|_{\mathcal{A}^s(\mathbf{X})}^{1/s} + \|p\|_{\mathcal{A}^s(Q)}^{1/s} + c_f^{1/s})$ flops.

## 6. Derefinement/optimization of finite element approximation

In this section we deal with the task how to optimize a finite element approximation, i.e., how to derefine those elements in the current triangulation that hardly contribute to the quality of the approximation, but, because of their number, may spoil the complexity of the algorithm.

6.1. **Derefinement of pressure approximation.** We first consider the approximations for the pressure. Suppose we are given $p_{\mathcal{T}} \in Q_{\mathcal{T}}$. We consider the tree $T(\mathcal{T})$ that corresponds to the triangulation $\mathcal{T}$ and for any $K \in T(\mathcal{T})$ we define an error functional as

$$e(K) := \inf_{\tilde{p}_K \in P_0(K)} \|\tilde{p}_K - (p_{\mathcal{T}})_{|K}\|_{L_2(K)}^2. \tag{6.1}$$

Then, for any subtree $\tilde{T} \subset T(\mathcal{T})$ we define

$$E(\tilde{T}) := \sum_{K \in \mathcal{L}(\tilde{T})} e(K) = \inf_{p_{\mathcal{T}(\tilde{T})} \in Q_{\mathcal{T}(\tilde{T})}} \|p_{\mathcal{T}} - p_{\mathcal{T}(\tilde{T})}\|_{L_2(\Omega)}^2 \tag{6.2}$$

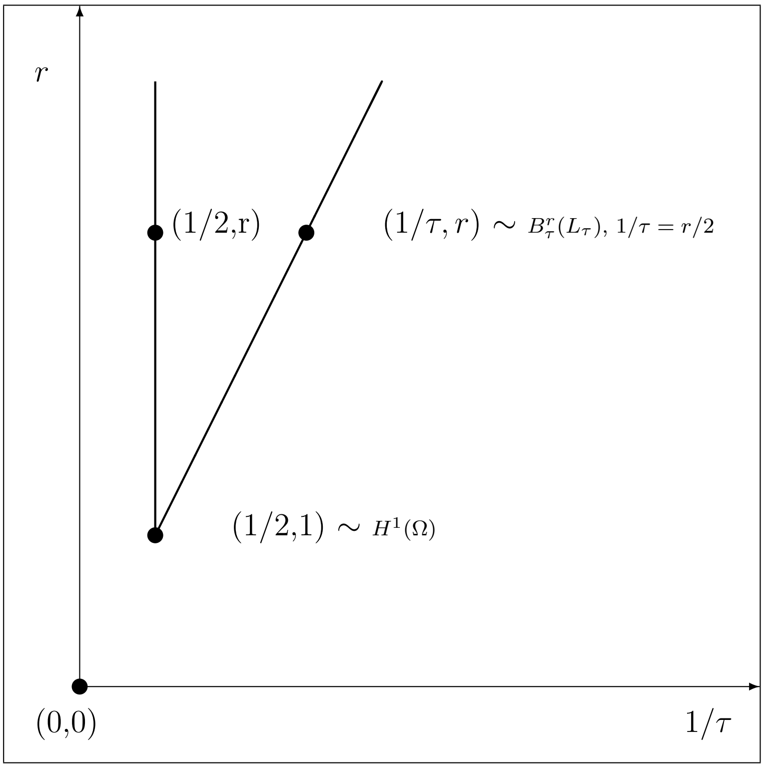



FIGURE 6. DeVore diagram. On the vertical line are the spaces $H^r(\Omega)$, and on the skew line are the spaces $B^r_\tau(L_\tau(\Omega))$ with $r = 2s+1$ and $\tau = (1/2+s)^{-1}$, i.e., $r = 2/\tau$

which is just the error in the best approximation for $p_{\mathcal{T}}$ from $Q_{\mathcal{T}(\tilde{T})}$. When implemented in a leaves-to-roots recursive way, all $e(K)$ for $K \in T(\mathcal{T})$ can be computed in $\mathcal{O}(\sharp T(\mathcal{T})) = \mathcal{O}(\sharp\mathcal{T})$ operations. Further we define a modified error functional by $\tilde{e}(K) := e(K)$ for all root nodes, and whenever $\tilde{e}(K)$ has been defined, for each child $K_i$ of $K$ by

$$\tilde{e}(K_i) := \frac{\sum_{j=1}^4 e(K_j)}{e(K) + \tilde{e}(K)} \tilde{e}(K). \tag{6.3}$$

Clearly, knowing $e(K)$ for all $K \in T(\mathcal{T})$, the computation of $\tilde{e}(K)$ for each $K \in T(\mathcal{T})$ requires not more than $\mathcal{O}(1)$ operations. Now in order to find a (quasi-) optimal piecewise constant finite element approximation $p_{\tilde{\mathcal{T}}}$ with respect to $\tilde{\mathcal{T}} \subset \mathcal{T}$ we use the following derefinement algorithm.

**Algorithm 6.1. DEREFINE-Q** $[\mathcal{T},p_{\mathcal{T}},\varepsilon] \rightarrow [\tilde{\mathcal{T}},p_{\tilde{\mathcal{T}}}]$
$\tilde{T} := T(\mathcal{T}_0)$
`while` $E(\tilde{T}) > \varepsilon^2$ `do`
compute $\rho = \max_{K\in\mathcal{L}(\tilde{T})} \tilde{e}(K)$
for all $K \in \mathcal{L}(\tilde{T})$ with $\tilde{e}(K) = \rho$ add all children of $K$ to $\tilde{T}$
`endwhile`
$\tilde{\mathcal{T}} := \mathcal{T}(\tilde{T})$

Actually, in order to avoid a log-factor in the operation count due to sorting of the $\tilde{e}(K)$ needed to find $\max_{K\in\mathcal{L}(\tilde{T})} \tilde{e}(K)$, we shall use an approximate sorting based on binary binning. With this small modification the following can be proven.

**Theorem 6.2.** *The algorithm* **DEREFINE-Q** $[\mathcal{T},p_{\mathcal{T}},\varepsilon] \rightarrow [\tilde{\mathcal{T}},p_{\tilde{\mathcal{T}}}]$ *produces a triangulation* $\tilde{\mathcal{T}}$*, and an approximation* $p_{\tilde{\mathcal{T}}}$ *such that* $\|p_{\mathcal{T}} - p_{\tilde{\mathcal{T}}}\|_Q \leq \varepsilon$. *There exist absolute constants* $t_1, T_2 > 0$, *necessarily with* $t_1 \leq 1 \leq T_2$, *such that for any triangulation* $\hat{\mathcal{T}}$ *for which there exists a* $p_{\hat{\mathcal{T}}} \in Q_{\hat{\mathcal{T}}}$ *with* $\|p_{\mathcal{T}} - p_{\hat{\mathcal{T}}}\|_Q \leq \sqrt{t_1}\varepsilon$ *we have* $\sharp\tilde{\mathcal{T}} \leq T_2\sharp\hat{\mathcal{T}}$.

*Moreover, the call of the algorithm requires* $\lesssim \sharp\mathcal{T} + \max\{0, \log(\varepsilon^{-1}\|p_{\mathcal{T}}\|_Q)\}$ *flops.*

*Proof.* See [BD04] with a small modification because of binary binning from Proposition 5.3 in [Ste05a]. $\diamondsuit$

The following corollary motivates the use of the described above derefinement routine. It states that by a call of this derefinement routine any approximation for the pressure can be turned into a (quasi-) optimal one, at the expense of making the error at most a constant factor larger.

**Corollary 6.3.** *Let* $\rho > t_1^{-1/2}$. *Then, for any* $\varepsilon > 0$, $p \in L_2(\Omega)$, *a triangulation* $\mathcal{T}$, $p_{\mathcal{T}} \in Q_{\mathcal{T}}$ *with* $\|p - p_{\mathcal{T}}\|_Q \leq \varepsilon$, *for* $[\tilde{\mathcal{T}},p_{\tilde{\mathcal{T}}}]$:=**DEREFINE-Q**$[\mathcal{T},p_{\mathcal{T}},\rho\varepsilon]$ *we have that* $\|p - p_{\tilde{\mathcal{T}}}\|_Q \leq (1+\rho)\varepsilon$ *and* $\sharp\tilde{\mathcal{T}} \leq T_2\sharp\hat{\mathcal{T}}$ *for any triangulation* $\hat{\mathcal{T}}$ *with* $\inf_{p_{\hat{\mathcal{T}}}\in Q_{\hat{\mathcal{T}}}} \|p - p_{\hat{\mathcal{T}}}\|_Q \leq (\sqrt{t_1}\rho - 1)\varepsilon$.

*Proof.* With $[\tilde{\mathcal{T}},p_{\tilde{\mathcal{T}}}]$:= **DEREFINE-Q**$[\mathcal{T},p_{\mathcal{T}},\rho\varepsilon]$, the properties of **DEREFINE-Q** ensure that $\|p_{\tilde{\mathcal{T}}} - p_{\mathcal{T}}\|_Q \leq \rho\varepsilon$ and so

$$\|p - p_{\tilde{\mathcal{T}}}\|_Q \leq \|p - p_{\mathcal{T}}\|_Q + \|p_{\mathcal{T}} - p_{\tilde{\mathcal{T}}}\|_Q \leq \varepsilon + \rho\varepsilon \leq (1+\rho)\varepsilon. \tag{6.4}$$

Further, let $\hat{\mathcal{T}}$ be a triangulation with $\inf_{p_{\hat{\mathcal{T}}}\in Q_{\hat{\mathcal{T}}}} \|p - p_{\hat{\mathcal{T}}}\|_Q \leq (\sqrt{t_1}\rho - 1)\varepsilon$, then

$$\inf_{p_{\hat{\mathcal{T}}}\in Q_{\hat{\mathcal{T}}}} \|p_{\hat{\mathcal{T}}} - p_{\mathcal{T}}\|_Q \leq \inf_{p_{\hat{\mathcal{T}}}\in Q_{\hat{\mathcal{T}}}} \|p_{\hat{\mathcal{T}}} - p\|_Q + \|p - p_{\mathcal{T}}\|_Q \leq (\sqrt{t_1}\rho - 1)\varepsilon + \varepsilon \leq \sqrt{t_1}\rho\varepsilon, \tag{6.5}$$

and so, from Corollary 6.2 we conclude that $\sharp\tilde{\mathcal{T}} \leq T_2\sharp\hat{\mathcal{T}}$. $\diamondsuit$

6.2. **Derefinement of velocity approximation.** Let $\mathcal{V}_*$ be the set of all vertices of all triangles that can be created by recursive uniform red-refinements of $\mathcal{T}_0$. $\mathcal{V}_*$ can be organized as a tree as follows. The set of the interior vertices $\mathcal{V}_{\mathcal{T}_0}$ are the roots. Assuming that the vertices after $k$ uniform refinement steps have been organized in a tree, each of the newly created vertices by the next uniform refinement step is assigned to be a child of an already existing vertex in the following way. The newly created vertex is the midpoint of an edge between two triangles. One of the 4 vertices of these triangles, not being on the boundary of the domain, is assigned to be the parent of the newly created vertex. It is not difficult to give a deterministic rule for the assignments. In [Ste05a], a wavelet basis $\tilde{\Psi} = \{\psi_v : v \in \mathcal{V}_*\}$ was constructed for $H_0^1(\Omega)$. Using this wavelet basis we define the wavelet basis for $\mathbf{X}$ as $\Psi = \{\psi_v \mathbf{e}_i : v \in \mathcal{V}_*, i \in \{1,2\}\}$. Each $\mathbf{u} \in \mathbf{X}$ has a unique representation $\mathbf{u} = \sum_{v\in\mathcal{V}_*}\sum_{i=1}^2 w_{v,i}\psi_v\mathbf{e}_i$ with respect to the wavelet basis. With $\|\mathbf{u}\|_\Psi := (\sum_{v\in\mathcal{V}_*}\sum_{i=1}^2 w_{v,i}^2)^{1/2}$, we define $\lambda_\Psi, \Lambda_\Psi > 0$ to be largest and smallest constants, respectively, such that

$$\lambda_\Psi\|\mathbf{u}\|_\Psi^2 \leq \|\mathbf{u}\|_{\mathbf{X}}^2 \leq \Lambda_\Psi\|\mathbf{u}\|_\Psi^2 \quad (\mathbf{u} \in \mathbf{X}) \tag{6.6}$$

The condition number of the wavelet basis $\Psi$ is then given by $\kappa_\Psi := \frac{\Lambda_\Psi}{\lambda_\Psi}$. Further, given an admissable triangulation $\mathcal{T}$, a wavelet basis for $\mathbf{X}_\mathcal{T}$ is given by $\Psi_\mathcal{T} = \{\psi_v\mathbf{e}_i : v \in \mathcal{V}_\mathcal{T}, i \in \{1,2\}\}$. At this point our restriction to admissible triangulations becomes essential, since the third property of Proposition 4.2 guarantees that for any $\psi_v\mathbf{e}_i \in \Psi_\mathcal{T}$ it holds that $\psi_v\mathbf{e}_i \in \mathbf{X}_\mathcal{T}$ so that this collection is indeed a basis for $\mathbf{X}_\mathcal{T}$.

In [Ste05a] a coarsening routine **COARSE** was developed to derefine a scalar continuous piecewise linear approximation. This routine is based on the transformation to the wavelet basis $\tilde{\Psi}$ and a call of a routine similar to **DEREFINE-Q** on the vertex tree corresponding to the current admissible triangulation. Having the wavelet basis $\Psi$ in our hands we can easily generalize the coarsening routine **COARSE** from [Ste05a] to an algorithm for derefinement of the vector-field approximation $\mathbf{u}_\mathcal{T} \in \mathbf{X}_\mathcal{T}$. Since this derefinement routine works precisely as **COARSE** with the wavelet basis $\Psi$ instead of $\tilde{\Psi}$ and the error functional $e(v) := \sum_{\bar{v} \text{ a descendent of v in the vertex tree}} |w_{\bar{v},1}|^2 + |w_{\bar{v},2}|^2$, we present only the prototype and properties of this algorithm.

**Algorithm 6.4. DEREFINE-X**$[\mathcal{T}, \mathbf{u}_\mathcal{T}, \varepsilon] \to [\tilde{\mathcal{T}}, \mathbf{u}_{\tilde{\mathcal{T}}}]$

/* Input:

- triangulation $\mathcal{T}$
- $\mathbf{u}_\mathcal{T} \in \mathbf{X}_\mathcal{T}$
- $\varepsilon > 0$

Output: A triangulation $\tilde{\mathcal{T}}$, and an approximation $\mathbf{u}_{\tilde{\mathcal{T}}} \in \mathbf{X}_{\tilde{\mathcal{T}}}$ such that $\|\mathbf{u}_\mathcal{T} - \mathbf{u}_{\tilde{\mathcal{T}}}\|_\Psi \leq \varepsilon$ and such that for any triangulation $\hat{\mathcal{T}}$ for which there exists a $\mathbf{u}_{\hat{\mathcal{T}}} \in \mathbf{X}_{\hat{\mathcal{T}}}$ with $\|\mathbf{u}_\mathcal{T} - \mathbf{u}_{\hat{\mathcal{T}}}\|_\Psi \leq \sqrt{t_1}\varepsilon$ we have $\sharp\tilde{\mathcal{T}} \leq C_C\sharp\hat{\mathcal{T}}$, with $C_C > 0$ being some absolute constant.

Moreover, the call of the algorithm requires $\lesssim \sharp\mathcal{T} + \max\{0, \log(\varepsilon^{-1}\|\mathbf{u}_\mathcal{T}\|_\Psi)\}$ flops. */

Similar to Corollary 6.3 we have

**Corollary 6.5.** *Let $\gamma > t_1^{-1/2}$ with $t_1$ from the Theorem 6.2. Then, for any $\varepsilon > 0$, $\mathbf{u} \in \mathbf{X}$, a triangulation $\mathcal{T}$, $\mathbf{u}_\mathcal{T} \in \mathbf{X}_\mathcal{T}$ with $\|\mathbf{u} - \mathbf{u}_\mathcal{T}\|_\Psi \leq \varepsilon$, for $[\tilde{\mathcal{T}}, \mathbf{u}_{\tilde{\mathcal{T}}}]$:=***DEREFINE-X**$[\mathcal{T},\mathbf{u}_\mathcal{T},\gamma\varepsilon]$ *we have that $\|\mathbf{u} - \mathbf{u}_{\tilde{\mathcal{T}}}\|_\Psi \leq (1+\gamma)\varepsilon$ and $\sharp\tilde{\mathcal{T}} \leq C_C\sharp\hat{\mathcal{T}}$ for any triangulation $\hat{\mathcal{T}}$ with $\inf_{\mathbf{u}_{\hat{\mathcal{T}}}\in\mathbf{X}_{\hat{\mathcal{T}}}} \|\mathbf{u} - \mathbf{u}_{\hat{\mathcal{T}}}\|_\Psi \leq (\sqrt{t_1}\gamma - 1)\varepsilon$*

Finally in this section, we define a routine **SCR**$[\mathcal{T}^1, \mathcal{T}^2] \to [\mathcal{T}]$, that constructs the smallest common refinement of the input triangulations $\mathcal{T}^1$ and $\mathcal{T}^2$.

## 7. Adaptive Fixed Point Algorithms for Stokes Problem: Analysis and Convergence

For convenience let us recall again the Stokes problem in operator notations.

$$\begin{cases} \text{Given } \mathbf{f} \in \mathbf{X}' \\ \text{find } (\mathbf{u}, p) \in \mathbf{X} \times Q \text{ such that} \\ \mathscr{A}\mathbf{u} + \mathscr{B}^* p \quad = \mathbf{f} \\ \mathscr{B}\mathbf{u} \qquad\qquad = 0 \end{cases} \tag{7.1}$$

Since $\mathscr{A}$ is an isomorphism between $\mathbf{X}$ and $\mathbf{X}'$ we can solve the first equation in (7.1) for $\mathbf{u}$

$$\mathbf{u} = \mathscr{A}^{-1}(\mathbf{f} - \mathscr{B}^* p) \tag{7.2}$$

Substituting this into the second equation in (7.1) gives

$$\mathscr{B}\mathscr{A}^{-1}\mathscr{B}^* p = \mathscr{B}\mathscr{A}^{-1}\mathbf{f} \tag{7.3}$$

The operator $\mathscr{S} := \mathscr{B}\mathscr{A}^{-1}\mathscr{B}^* : Q \to Q$, is the Schur complement operator and it is known to be symmetric positive definite with $\|\mathscr{S}\|_{Q\to Q} \leq 1$ [JH02]. The problem (7.3) can be reformulated as a fixed point problem for the operator

$$\mathscr{N}_\alpha : Q \to Q,\ q \to q - \alpha(\mathscr{S}q - \mathscr{B}\mathscr{A}^{-1}\mathbf{f}) \tag{7.4}$$

$$\begin{cases} \text{Find } \ p \in Q \text{ such that} \\ \mathscr{N}_\alpha p = p \end{cases} \tag{7.5}$$

From

$$\begin{aligned} \|\mathscr{N}_\alpha q_1 - \mathscr{N}_\alpha q_2\|_Q &= \|(I - \alpha\mathscr{S})(q_1 - q_2)\|_Q \\ &\leq \|I - \alpha\mathscr{S}\|_{Q\to Q}\|q_1 - q_2\|_Q \text{ for all } q_1, q_2 \in Q \end{aligned} \tag{7.6}$$

we see that $\mathscr{N}_\alpha$ defines a contractive mapping if $\|I - \alpha\mathscr{S}\|_{Q\to Q} < 1$. Since $\|I - \alpha\mathscr{S}\|_{Q\to Q} < 1$ if $0 < \alpha < 2/\|\mathscr{S}\|_{Q\to Q}$, the iterative process

$$p_j = (I - \alpha\mathscr{S})p_{j-1} + \alpha\mathscr{F}, \quad \mathscr{F} := \mathscr{B}\mathscr{A}^{-1}\mathbf{f} \tag{7.7}$$

converges to the solution of (7.3), due to Banach's theorem for contractive mappings. This iteration can be also viewed as a generalization of the damped Richardson method to the operator equation (7.3). The optimal choice of the parameter $\alpha$ is

$$\alpha_{opt} = \frac{2}{\|\mathscr{S}\|_{Q\to Q} + \|\mathscr{S}^{-1}\|^{-1}_{Q\to Q}}, \text{ giving } \|I - \alpha_{opt}\mathscr{S}\|_{Q\to Q} = \frac{\kappa(\mathscr{S}) - 1}{\kappa(\mathscr{S}) + 1}, \tag{7.8}$$

with $\kappa(\mathscr{S}) := \|\mathscr{S}\|_{Q\to Q}\|\mathscr{S}^{-1}\|_{Q\to Q}$. Rewritten in the following form, this iterative method is known as the Uzawa algorithm.

$$\boxed{\begin{aligned} \mathbf{u}_j &= \mathscr{A}^{-1}(\mathbf{f} - \mathscr{B}^* p_{j-1}) \\ p_j &= p_{j-1} + \alpha \mathscr{B} \mathbf{u}_j \end{aligned}} \tag{7.9}$$

Of course (7.9) is an idealized iteration, and we will analyse an inexact version of it where the application of the operator $\mathscr{A}^{-1}$ is approximated by a convergent adaptive finite element algorithm that we will develop in Section 10. At the moment it is enough to present a prototype of this routine

**Algorithm 7.1. INNERELLIPTIC**$[\bar{\mathcal{T}}, \mathbf{f}, p_{\bar{\mathcal{T}}}, \mathbf{u}_{\bar{\mathcal{T}}}, \varepsilon_0, \varepsilon] \to [\mathcal{T}, \mathbf{u}_{\mathcal{T}}]$

/* Input of the routine:

- input triangulation $\bar{\mathcal{T}}$
- $\mathbf{f} \in \mathbf{X}'$
- $p_{\bar{\mathcal{T}}} \in Q_{\bar{\mathcal{T}}}$, $\mathbf{u}_{\bar{\mathcal{T}}} \in \mathbf{X}_{\bar{\mathcal{T}}}$
- $\varepsilon_0$, such that with

$$\mathbf{u}^{p_{\bar{\mathcal{T}}}} := \mathscr{A}^{-1}(\mathbf{f} - \mathscr{B}^* p_{\bar{\mathcal{T}}}), \tag{7.10}$$

  $\|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathbf{u}_{\bar{\mathcal{T}}}\|_{\mathbf{X}} \leq \varepsilon_0$.
- $\varepsilon > 0$

Output: a triangulation $\mathcal{T} \supset \bar{\mathcal{T}}$, $\mathbf{u}_{\mathcal{T}} \in \mathbf{X}_{\mathcal{T}}$, with $\|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \leq \varepsilon$. Moreover, if for some $s > 0$ the pair $(\mathbf{f}, \mathbf{RHS})$ is $s$-optimal, and the input satisfies $\varepsilon_0 \lesssim \varepsilon$, then both the cardinality of the output triangulation $\mathcal{T}$ and the number of flops required by the routine are $\lesssim \sharp\bar{\mathcal{T}} + \varepsilon^{-1/s} c_f^{1/s}$ with $c_f$ as in the definition of $s$-optimality.
*/

Using **INNERELLIPTIC** we are ready to define our adaptive routine **AFEMSTOKESSOLVER** (Algorithm 7.3).

**Theorem 7.2.** *The algorithm* **AFEMSTOKESSOLVER**$[\mathbf{f}, \varepsilon] \to [\mathcal{T}, \mathbf{u}_{\mathcal{T}}, p_{\mathcal{T}}]$ *terminates with*

$$\|p - p_{\mathcal{T}}\|_Q \leq \varepsilon, \ \|\mathbf{u} - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \leq \lambda_{\Psi}^{-1/2}(\eta^{-1} + 1)\varepsilon \tag{7.11}$$

*Moreover, assuming $\varepsilon_0 \lesssim \|p\|_Q$, if $\mathbf{u} \in \mathcal{A}^s(\mathbf{X})$, $p \in \mathcal{A}^s(Q)$ and the pair $(\mathbf{f}, \mathbf{RHS})$ is $s$ -optimal, then both*

$$\text{the number of flops and } \sharp\mathcal{T} \lesssim \varepsilon^{-1/s}(\|\mathbf{u}\|_{\mathcal{A}^s(\mathbf{X})}^{1/s} + \|p\|_{\mathcal{A}^s(Q)}^{1/s} + c_f^{1/s}), \tag{7.12}$$

*i.e., the algorithm has optimal computational complexity.*

**Algorithm 7.3.** Adaptive Stokes Solver
**AFEMSTOKESSOLVER**$[\mathbf{f}, \varepsilon] \to [\mathcal{T}, \mathbf{u}_{\mathcal{T}}, p_{\mathcal{T}}]$
/*
Input parameters of the algorithm are: $\mathbf{f} \in \mathbf{X}'$, $\varepsilon > 0$
Output: a triangulation $\mathcal{T}$, $\mathbf{u}_{\mathcal{T}} \in \mathbf{X}_{\mathcal{T}}$, $p_{\mathcal{T}} \in Q_{\mathcal{T}}$ such that
$\|\mathbf{u} - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} + \|p - p_{\mathcal{T}}\|_Q \leq \varepsilon$
*/

Select an initial triangulation $\mathcal{T}_0^0$, approximations $\mathbf{u}_{\mathcal{T}_0^0} \in \mathbf{X}_{\mathcal{T}_0}$, $p_{\mathcal{T}_0^0} \in Q_{\mathcal{T}_0}$, and constants $0 < \gamma < 1$, $\delta < 1/2$, $\rho > t_1^{-1/2}$, with $t_1$ from Theorem 6.2, $\eta \in (\max\{\gamma, \beta\}, 1)$, where $\beta := \|I - \alpha\mathscr{S}\|_{Q\to Q}$, and $\alpha$ such that $\delta(\rho+1) < 1$, $0 < \alpha < 2/\|\mathscr{S}\|_{Q\to Q}$, and $\varepsilon_0$ such that $\|\mathbf{u} - \mathbf{u}_{\mathcal{T}_0^0}\|_{\mathbf{X}} + \|p - p_{\mathcal{T}_0^0}\|_Q \leq \varepsilon_0$.

```
M := min{j ∈ ℕ : η^j(αj + 1) ≤ δ}
N := min{i ∈ ℕ : δ((ρ + 1)δ)^(i−1) ε_0 ≤ ε}
for i = 0 ... N do
    if i ≠ 0 then
    [T^p, p_{T^p}] := DEREFINE-Q[T_{i−1}^M, p_{T_{i−1}^M}, ρδε_{i−1}]
    [T^u, u_{T^u}] := DEREFINE-X[T_{i−1}^M, u_{T_{i−1}^M}, ρλ_Ψ^{−1/2}(1/η + 1)δε_{i−1}]
    [T_i^0] := SCR[T^p, T^u], u_{T_i^0} := u_{T^u}, p_{T_i^0} := p_{T^p}
    ε_i := (ρ + 1)δε_{i−1}
    endif
    ε̄_0^0 := (κ_Ψ^{1/2}(η^{−1} + 1) + 1)ε_i
    for j = 1, ..., M do
    [T_i^j, u_{T_i^j}] := INNERELLIPTIC[T_i^{j−1}, f, p_{T_i^{j−1}}, u_{T_i^{j−1}}, ε̄_0^{j−1}, γ^j ε_i]
     p_{T_i^j} := p_{T_i^{j−1}} + α𝓑u_{T_i^j}
     ε̄_0^j := (η^j(αj + 1) + η^{j−1}(α(j − 1) + 1) + γ^j)ε_i
    endfor
endfor
T := T_N^M, p_T := p_{T_N^M}, u_T := u_{T_N^M}
```

*Proof.* By construction, we have $\|p - p_{\mathcal{T}_0^0}\|_Q \leq \varepsilon_0$. Considering the inner loop of the algorithm, it is easy to see that

$$p - p_{\mathcal{T}_i^j} = (I - \alpha\mathscr{S})(p - p_{\mathcal{T}_i^{j-1}}) + \alpha\mathscr{B}(\mathbf{u}_{\mathcal{T}_i^j} - \mathbf{u}^{p_{\mathcal{T}_i^{j-1}}}) \tag{7.13}$$

where $\mathbf{u}^{p_{\mathcal{T}_i^{j-1}}}$ corresponds to $p_{\mathcal{T}_i^{j-1}}$ as in (7.10). Since, see [JH02],

$$\|\mathscr{B}\mathbf{u}\|_Q \leq \|\mathbf{u}\|_{\mathbf{X}}, \tag{7.14}$$

with $\eta := \max\{\beta, \gamma\}$ we have

$$\begin{aligned} \|p - p_{\mathcal{T}_i^j}\|_Q &\leq \|I - \alpha\mathscr{S}\|_{Q\to Q}\|p - p_{\mathcal{T}_i^{j-1}}\|_Q + \alpha\|\mathbf{u}_{\mathcal{T}_i^j} - \mathbf{u}^{p_{\mathcal{T}_i^{j-1}}}\|_{\mathbf{X}} \\ &\leq \beta^j\|p - p_{\mathcal{T}_i^0}\|_Q + \alpha\varepsilon_i\sum_{l=0}^{j-1}\beta^l\gamma^{j-l} \\ &\leq \eta^j\{\|p - p_{\mathcal{T}_i^0}\|_Q + \alpha\varepsilon_i j\} \\ &\leq \eta^j(\alpha j + 1)\varepsilon_i \end{aligned} \tag{7.15}$$

where we used as induction hypothesis that $\|p - p_{\mathcal{T}_i^0}\|_Q \leq \varepsilon_i$. Now with $M$ as in the algorithm, we conclude that

$$\|p - p_{\mathcal{T}_i^M}\|_Q \leq \delta\varepsilon_i. \tag{7.16}$$

A call of the **DEREFINE-Q** in the next iteration of the outer loop results in

$$\|p - p_{\mathcal{T}_{i+1}^0}\|_Q \leq (\rho + 1)\delta\varepsilon_i = \varepsilon_{i+1}. \tag{7.17}$$

We can show now that a similar error reduction also holds for the velocities. Indeed,

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}^{p_{\mathcal{T}_i^{j-1}}}\|_{\mathbf{X}} &= \sup_{\mathbf{v}\in\mathbf{X}}\frac{(\mathscr{A}^{-1}\mathscr{B}^*(p_{\mathcal{T}_i^{j-1}} - p), \mathbf{v})_{\mathbf{X}}}{\|\mathbf{v}\|_{\mathbf{X}}} \\ &= \sup_{\mathbf{v}\in\mathbf{X}}\frac{(p_{\mathcal{T}_i^{j-1}} - p, \mathscr{B}\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{X}}} \\ &\leq \|p_{\mathcal{T}_i^{j-1}} - p\|_Q. \end{aligned} \tag{7.18}$$

Further,

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_{\mathcal{T}_i^j}\|_{\mathbf{X}} &\leq \|\mathbf{u} - \mathbf{u}^{p_{\mathcal{T}_i^{j-1}}}\|_{\mathbf{X}} + \|\mathbf{u}^{p_{\mathcal{T}_i^{j-1}}} - \mathbf{u}_{\mathcal{T}_i^j}\|_{\mathbf{X}} \\ &\leq \|p - p_{\mathcal{T}_i^{j-1}}\|_Q + \gamma^j\varepsilon_i, \\ &\leq (\eta^{j-1}(\alpha(j-1) + 1) + \gamma^j)\varepsilon_i. \end{aligned} \tag{7.19}$$

In particular, using definition of $M$ and $\eta$, we have

$$\|\mathbf{u} - \mathbf{u}_{\mathcal{T}_i^M}\|_{\mathbf{X}} \leq (\eta^{-1} + 1)\delta\varepsilon_i. \tag{7.20}$$

So before the call of the **DEREFINE-X** in the next iteration of the outer loop it holds

$$\|\mathbf{u} - \mathbf{u}_{\mathcal{T}_i^M}\|_\Psi \leq \lambda_\Psi^{-1/2}\|\mathbf{u} - \mathbf{u}_{\mathcal{T}_i^M}\|_{\mathbf{X}} \leq \lambda_\Psi^{-1/2}(\eta^{-1} + 1)\delta\varepsilon_i. \tag{7.21}$$

Then the call of this routine gives

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_{\mathcal{T}_{i+1}^0}\|_{\mathbf{X}} &\leq \Lambda_\Psi^{1/2}\|\mathbf{u} - \mathbf{u}_{\mathcal{T}_{i+1}^0}\|_\Psi \\ &\leq \kappa_\Psi^{1/2}(\eta^{-1} + 1)\delta\varepsilon_i(1 + \rho) = \kappa_\Psi^{1/2}(\eta^{-1} + 1)\varepsilon_{i+1} \end{aligned} \tag{7.22}$$

By (7.16),(7.22) and the definition of $N$, the proof of the first part of the theorem is completed.

Let us analyse the computational complexity of the algorithm. Since we start with the triangulation $\mathcal{T}_0^0$, where $\sharp\mathcal{T}_0^0 \lesssim 1$ and by assumption $\varepsilon_0 \lesssim \|p\|_Q \leq \|p\|_{\mathcal{A}^s(Q)}$ we have

$$\sharp\mathcal{T}_0^0 \lesssim \varepsilon_0^{-1/s}(\|p\|_{\mathcal{A}^s(Q)}^{1/s} + \|\mathbf{u}\|_{\mathcal{A}^s(\mathbf{X})}^{1/s}). \tag{7.23}$$

For $i > 0$, before the call of **DEREFINE-Q** it holds $\|p - p_{\mathcal{T}_{i-1}^M}\|_Q \leq \delta\varepsilon_{i-1}$. After the call of this routine, its properties guarantee that for any triangulation $\check{\mathcal{T}}$ such that

$$\inf_{p_{\check{\mathcal{T}}} \in Q_{\check{\mathcal{T}}}} \|p - p_{\check{\mathcal{T}}}\|_Q \leq (\sqrt{t_1}\rho - 1)\delta\varepsilon_{i-1} \tag{7.24}$$

we have

$$\sharp\mathcal{T}^p \lesssim T_2 \sharp\check{\mathcal{T}}. \tag{7.25}$$

Since $p \in \mathcal{A}^s(Q)$, we conclude that

$$\sharp\mathcal{T}^p \leq T_2(\sharp\mathcal{T}_0 + ((\sqrt{t_1}\rho - 1)\delta\varepsilon_{i-1})^{-1/s}\|p\|_{\mathcal{A}^s(Q)}^{1/s} \lesssim \varepsilon_i^{-1/s}\|p\|_{\mathcal{A}^s(Q)}^{1/s} \tag{7.26}$$

Similar, since $\mathbf{u} \in \mathcal{A}^s(\mathbf{X})$, after the call of **DEREFINE-X** we have

$$\sharp\mathcal{T}^{\mathbf{u}} \lesssim \varepsilon_i^{-1/s}\|\mathbf{u}\|_{\mathcal{A}^s(\mathbf{X})}^{1/s} \tag{7.27}$$

Apparently, the smallest common refinement $\mathcal{T}_i^0$ constructed by the routine **SCR** satisfies

$$\sharp\mathcal{T}_i^0 \lesssim \varepsilon_i^{-1/s}(\|p\|_{\mathcal{A}^s(Q)}^{1/s} + \|\mathbf{u}\|_{\mathcal{A}^s(\mathbf{X})}^{1/s}). \tag{7.28}$$

In the internal for-loop, before the call of **INNERELLIPTIC**$[\mathcal{T}_i^{j-1}, \mathbf{f}, p_{\mathcal{T}_i^{j-1}}, \mathbf{u}_{\mathcal{T}_i^{j-1}}, \bar{\varepsilon}_0^{j-1}, \gamma^j\varepsilon_i]$, using (7.18) and (7.19) or, when $j = 1$, (7.22), and that $1 \leq j \leq M \lesssim 1$, we obtain

$$\|\mathbf{u}^{p_{\mathcal{T}_i^{j-1}}} - \mathbf{u}_{\mathcal{T}_i^{j-1}}\|_{\mathbf{X}} \leq \|\mathbf{u}^{p_{\mathcal{T}_i^{j-1}}} - \mathbf{u}\|_{\mathbf{X}} + \|\mathbf{u} - \mathbf{u}_{\mathcal{T}_i^{j-1}}\|_{\mathbf{X}} \leq \bar{\varepsilon}_0^{j-1} \lesssim \gamma^j\varepsilon_i. \tag{7.29}$$

In view of the properties of **INNERELLIPTIC**, after the call of this routine it holds

$$\sharp\mathcal{T}_i^j \lesssim (\eta^j\varepsilon_i)^{-1/s}c_f^{1/s} + \sharp\mathcal{T}_i^{j-1} \tag{7.30}$$

Since $\sum_{j=1}^M(\eta^j\varepsilon_i)^{-1/s} \lesssim (\eta^M\varepsilon_i)^{-1/s}$ we conclude

$$\begin{aligned}\sharp\mathcal{T}_i^M &\lesssim (\eta^M\varepsilon_i)^{-1/s}c_f^{1/s} + \sharp\mathcal{T}_i^0\\ &\lesssim \varepsilon_i^{-1/s}(\|\mathbf{u}\|_{\mathcal{A}^s(\mathbf{X})}^{1/s} + \|p\|_{\mathcal{A}^s(Q)}^{1/s} + c_f^{1/s})\end{aligned} \tag{7.31}$$

where in the last inequality we have used (7.28) or, when $i = 0$, (7.23).

For $i > 0$ the call of **DEREFINE-Q**$[\mathcal{T}_{i-1}^M, p_{\mathcal{T}_{i-1}^M}, \rho\delta\varepsilon_{i-1}]$ requires

$$\text{number of flops} \lesssim \sharp\mathcal{T}_i^M + \max\{0, \log((\rho\delta\varepsilon_i)^{-1}\|p_{\mathcal{T}_i^M}\|_Q\} \tag{7.32}$$

Due to $\log((\rho\delta\varepsilon_i)^{-1}\|p_{\mathcal{T}_i^M}\|_Q) \leq (\rho\delta\varepsilon_i)^{-1/s}\|p_{\mathcal{T}_i^M}\|_Q^{1/s} \lesssim \varepsilon_{i+1}^{-1/s}\|p\|_{\mathcal{A}^s(Q)}^{1/s}$, and the fact that similar results are valid for the call of **DEREFINE-X**, we find that the calls of these routines including **SCR** require

$$\text{a number of flops} \lesssim \varepsilon_i^{-1/s}(\|\mathbf{u}\|_{\mathcal{A}^s(\mathbf{X})}^{1/s} + \|p\|_{\mathcal{A}^s(Q)}^{1/s} + c_f^{1/s}) \tag{7.33}$$

Since the $j$th call of **INNERELLIPTIC** requires a number of flops $\lesssim (\eta^j\varepsilon_i)^{-1/s}c_f^{1/s} + \sharp\mathcal{T}_i^j$, we find that the whole internal for-loop needs

$$\text{a number of flops } \lesssim \varepsilon_i^{-1/s}(\|\mathbf{u}\|_{\mathcal{A}^s(\mathbf{X})}^{1/s} + \|p\|_{\mathcal{A}^s(Q)}^{1/s} + c_f^{1/s}). \tag{7.34}$$

Finally, recalling that

$$N = \min\{j \in \mathbb{N} : (1 + \kappa_\Psi^{1/2}(\eta^{-1} + 1))\varepsilon_{j+1} \leq \varepsilon\} \tag{7.35}$$

we have $\varepsilon_i \gtrsim \varepsilon$, and for the whole algoritm $[\mathcal{T}, \mathbf{u}_\mathcal{T}, p_\mathcal{T}]$:=**AFEMSTOKESSOLVER**$[\mathbf{f}, \varepsilon]$ it holds that both

$$\text{the number of flops and} \sharp\mathcal{T} \lesssim \varepsilon^{-1/s}(\|\mathbf{u}\|_{\mathcal{A}^s(\mathbf{X})}^{1/s} + \|p\|_{\mathcal{A}^s(Q)}^{1/s} + c_f^{1/s}) \tag{7.36}$$

completing the proof. $\diamondsuit$

## 8. Convergent Adaptive Finite Element Algorithm for Inner Elliptic Problem

In this section we develop the **INNERELLIPTIC** routine that approximates the inverse of $\mathscr{A}$ in (7.9), or equivalently, that constructs the approximation to the solution of (8.1)

$$\begin{cases} \text{Given a triangulation } \bar{\mathcal{T}},\ p_{\bar{\mathcal{T}}} \in Q_{\bar{\mathcal{T}}},\ \mathbf{f} \in \mathbf{X}' \\ \text{find } \mathbf{u} \in \mathbf{X} \text{ such that} \\ a(\mathbf{u}^{p_{\bar{\mathcal{T}}}}, \mathbf{v}) = \mathbf{f}(\mathbf{v}) - b(\mathbf{v}, p_{\bar{\mathcal{T}}}) \text{ for all } \mathbf{v} \in \mathbf{X}. \end{cases}$$

With $\mathcal{T} \supset \bar{\mathcal{T}}$, defining $\mathscr{A}_\mathcal{T} : \mathbf{X}_\mathcal{T} \to (\mathbf{X}_\mathcal{T})' \supset \mathbf{X}'$ by $(\mathscr{A}_\mathcal{T}\mathbf{u}_\mathcal{T})(\mathbf{v}_\mathcal{T}) = a(\mathbf{u}_\mathcal{T}, \mathbf{v}_\mathcal{T})$ for $\mathbf{u}_\mathcal{T}, \mathbf{v}_\mathcal{T} \in \mathbf{X}_\mathcal{T}$, $\mathscr{A}_\mathcal{T}^{-1}(\mathbf{f} - \mathscr{B}^* p_{\bar{\mathcal{T}}})$ is the *Galerkin approximation* of $\mathbf{u}^{p_{\bar{\mathcal{T}}}}$. One easily verifies that for $\mathbf{f} \in \mathbf{X}'$, $\|\mathscr{A}_\mathcal{T}^{-1}\mathbf{f}\|_\mathbf{X} \leq \|\mathbf{f}\|_{\mathbf{X}'}$.

In [Ste05a], an adaptive algorithm of optimal complexity has been developed for a scalar elliptic problem. Here we generalize this algorithm for the system of elliptic equations (8.1), also adapting it to the special term $\mathscr{B}^* p_{\bar{\mathcal{T}}} \in \mathbf{X}'$ in the right-hand side. Another important issue we have to pay attention to here is the fact that our inner solver will be called every Uzawa iteration with a different initial triangulation $\bar{\mathcal{T}}$ and a different right-hand side.

We start the design of an adaptive algorithm for the above described problem with the development of an a posteriori error estimator. Let $\mathcal{T} \supset \bar{\mathcal{T}}$ and $\mathbf{w}_\mathcal{T} \in \mathbf{X}_\mathcal{T}$. In the following it is only needed that $p_{\bar{\mathcal{T}}} \in Q_\mathcal{T}$, for that reason we will write $p_\mathcal{T}$, $\mathbf{u}^{p_\mathcal{T}}$ instead of $p_{\bar{\mathcal{T}}}$, $\mathbf{u}^{p_{\bar{\mathcal{T}}}}$, respectively. For $\mathbf{f} \in [L_2(\Omega)]^2$, using integration by parts, we find

$$
\begin{aligned}
&a(\mathbf{u}^{p_\mathcal{T}} - \mathbf{w}_\mathcal{T}, \mathbf{v}) = \mathbf{f}(\mathbf{v}) - b(\mathbf{v}, p_\mathcal{T}) - a(\mathbf{w}_\mathcal{T}, \mathbf{v}) \\
&= \sum_{K \in \mathcal{T}} \int_K \{\mathbf{f} + \Delta \mathbf{w}_\mathcal{T}\}\mathbf{v} - \int_K \nabla p_\mathcal{T} \cdot \mathbf{v} + \int_{\partial K} p_\mathcal{T} \mathbf{v} \cdot \nu - \sum_{\ell \in \partial K} \int_\ell \sum_{i=1}^2 \frac{\partial(\mathbf{w}_\mathcal{T})_i}{\partial \nu_\ell} \mathbf{v}_i \qquad (8.1)\\
&= \sum_{K \in \mathcal{T}} \int_K \mathbf{R}_K \cdot \mathbf{v} - \sum_{\ell \in \mathcal{E}_\mathcal{T}} \int_\ell \mathbf{R}_\ell \cdot \mathbf{v}
\end{aligned}
$$

where $\mathbf{R}_K$ is an element residual

$$\mathbf{R}_K := (\mathbf{f} + \Delta \mathbf{w}_\mathcal{T} - \nabla p_\mathcal{T})_{|K} \tag{8.2}$$

and $\mathbf{R}_\ell$ denotes an edge residual

$$\mathbf{R}_\ell := [\begin{pmatrix} \frac{\partial(\mathbf{w}_\mathcal{T})_1}{\partial \nu_\ell} \\ \frac{\partial(\mathbf{w}_\mathcal{T})_2}{\partial \nu_\ell} \end{pmatrix} - p_\mathcal{T} \nu_\ell], \tag{8.3}$$

with $\nu_\ell$ being an unit vector orthogonal to $\ell$. We set up the a posteriori error estimator

$$\mathfrak{E}(\mathcal{T}, \mathbf{f}, \mathbf{w}_\mathcal{T}, p_\mathcal{T}) := (\sum_{K \in \mathcal{T}} \operatorname{diam}(K)^2 \|\mathbf{R}_K\|^2_{[L_2(K)]^2} + \sum_{\ell \in \mathcal{E}_\mathcal{T}} \operatorname{diam}(\ell) \|\mathbf{R}_\ell\|^2_{[L_2(\ell)]^2})^{1/2}. \tag{8.4}$$

Before stating the theorem about our error estimator, we need to recall one more result, that will be invoked in the proof. For $K \in \mathcal{T}$ and $\ell \in \mathcal{E}_\mathcal{T}$ we introduce the notations

$$\Omega_K := \{K' \in \mathcal{T}_j, K \cap K' \neq \emptyset\}, \quad \Omega_\ell := \{K' \in \mathcal{T}_j, \ell \subset K'\} \tag{8.5}$$

In [Ste05a] a piecewise linear (quasi)-interpolant was constructed on admissible triangulations for $H^1_0(\Omega)$ functions. We state the properties of its obvious generalization to $\mathbf{X}$ vector-fields in the following lemma.

**Lemma 8.1.** *Let $\mathcal{T}$ be an admissible triangulation of $\Omega$. Then there exists a linear mapping $I_\mathcal{T} : \mathbf{X} \to \mathbf{X}_\mathcal{T}$ such that*

$$\|\mathbf{v} - I_\mathcal{T} \mathbf{v}\|_{[H^s(K)]^2} \lesssim \operatorname{diam}(K)^{1-s} \|\mathbf{v}\|_{[H^1(\Omega_K)]^2} \quad (\mathbf{v} \in \mathbf{X},\ s = 0, 1,\ K \in \mathcal{T}) \tag{8.6}$$

$$\|\mathbf{v} - I_\mathcal{T} \mathbf{v}\|_{[L_2(\ell)]^2} \lesssim \operatorname{diam}(\ell)^{1/2} \|\mathbf{v}\|_{[H^1(\Omega_\ell)]^2} \quad (\mathbf{v} \in \mathbf{X},\ \ell \in \mathcal{E}_\mathcal{T}). \tag{8.7}$$

**Theorem 8.2.** *Let $\mathcal{T}$ be an admissible triangulation and $\tilde{\mathcal{T}}$ be a refinement of $\mathcal{T}$. Assume that $\mathbf{f} \in \mathbf{Y}_\mathcal{T}$, and let $\mathbf{u}^{p_\mathcal{T}} = \mathscr{A}^{-1}(\mathbf{f} - \mathscr{B}^* p_\mathcal{T})$ and let $\mathbf{u}^{p_\mathcal{T}}_\mathcal{T} = \mathscr{A}^{-1}_\mathcal{T}(\mathbf{f} - \mathscr{B}^* p_\mathcal{T})$, $\mathbf{u}^{p_\mathcal{T}}_{\tilde{\mathcal{T}}} = \mathscr{A}^{-1}_{\tilde{\mathcal{T}}}(\mathbf{f} - \mathscr{B}^* p_\mathcal{T})$ be its Galerkin approximations on the triangulations $\mathcal{T}$ and $\tilde{\mathcal{T}}$ respectively.*

*i) If $\mathcal{V}_{\tilde{\mathcal{T}}}$ contains a point inside $K \in \mathcal{T}$, then*

$$\|\mathbf{u}^{p_\mathcal{T}}_{\tilde{\mathcal{T}}} - \mathbf{u}^{p_\mathcal{T}}_\mathcal{T}\|^2_{[H^1(K)]^2} \gtrsim \operatorname{diam}(K)^2 \|\mathbf{R}_K\|^2_{[L_2(K)]^2} \tag{8.8}$$

*ii) With $K_1, K_2 \in \mathcal{T}$, such that $\ell := K_1 \cap K_2 \in \mathcal{E}_{\mathcal{T}}$, if $\mathcal{V}_{\tilde{\mathcal{T}}}$ contains a point in the interior of both $K_1$ and $K_2$, then*

$$\|\mathbf{u}^{p_{\mathcal{T}}}_{\tilde{\mathcal{T}}} - \mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}\|^2_{[H^1(K_1\cup K_2)]^2} \gtrsim \operatorname{diam}(\ell)\|\mathbf{R}_\ell\|^2_{[L_2(\ell)]^2} \tag{8.9}$$

*iii) Let for some $F \subset \mathcal{T}$ and $G \subset \mathcal{E}_{\mathcal{T}}$, the refinement $\tilde{\mathcal{T}}$ satisfies the condition i) or ii) for all $K \in F$ and $\ell \in G$, respectively, then*

$$\|\mathbf{u}^{p_{\mathcal{T}}}_{\tilde{\mathcal{T}}} - \mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}\|^2_{\mathbf{X}} \geq C_L^2\{\sum_{K\in F}\operatorname{diam}(K)^2\|\mathbf{R}_K\|^2_{[L_2(K)]^2} + \sum_{\ell\in G}\operatorname{diam}(K)^2\|\mathbf{R}_\ell\|^2_{[L_2(\ell)]^2}\} \tag{8.10}$$

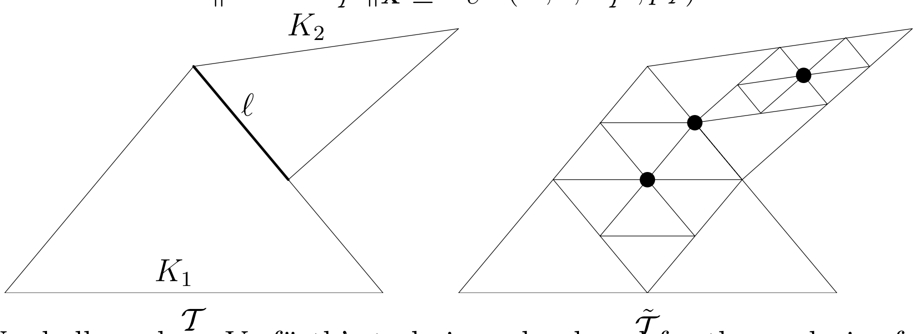
*for some absolute constant $C_L > 0$.*

*iv)*

$$C_L\mathfrak{E}(\mathcal{T}, \mathbf{f}, \mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}, p_{\mathcal{T}}) \leq \|\mathbf{u}^{p_{\mathcal{T}}} - \mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}\|_{\mathbf{X}} \tag{8.11}$$

*v) there exists an absolute constant $C_U$ such that even for any $\mathbf{f} \in [L_2(\Omega)]^2$,*

$$\|\mathbf{u}^{p_{\mathcal{T}}} - \mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}\|_{\mathbf{X}} \leq C_U\mathfrak{E}(\mathcal{T}, \mathbf{f}, \mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}, p_{\mathcal{T}}). \tag{8.12}$$




*Proof.* We shall use here Verfürth's technique developed for the analysis of a posteriori error estimators. Actually, one who is familiar with this technique will recognize that it has been used in [BMN02] to prove similar results for conforming triangulations.

(i) To prove the local lower bounds i)-ii), we notice that, due to the conditions on $\tilde{\mathcal{T}}$, for all $K \in \mathcal{T}$, $\ell \in \mathcal{E}_{\mathcal{T}}$ there exist functions $\psi_K$, $\psi_\ell$, which are continuous piecewise linear with respect to $\tilde{\mathcal{T}}$, with the properties

$$\operatorname{supp}(\psi_K) \subset K,\ \operatorname{supp}(\psi_\ell) \subset \Omega_\ell,\ \int_K \psi_K \eqsim \operatorname{meas}(K),\ \int_\ell \psi_\ell \eqsim \operatorname{meas}(\ell), \tag{8.13}$$

$$0 \leq \psi_K \leq 1,\ x \in K,\ 0 \leq \psi_\ell \leq 1,\ x \in \ell, \tag{8.14}$$

$$\begin{aligned} &\|\psi_K\|_{L_2(K)} \lesssim \operatorname{meas}(K)^{1/2}, \\ &\|\psi_\ell\|_{L_2(\Omega_\ell)} \lesssim \operatorname{diam}(\ell). \end{aligned} \tag{8.15}$$

We have

$$\begin{aligned} a(\mathbf{u}^{p_{\mathcal{T}}}_{\tilde{\mathcal{T}}} - \mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}, \mathbf{v}) &= \mathbf{f}(\mathbf{v}) + b(p_{\mathcal{T}}, \mathbf{v}) - a(\mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}, \mathbf{v}) \\ &= \sum_{K\in\mathcal{T}}\int_K \mathbf{R}_K\cdot\mathbf{v} - \sum_{\ell\in\mathcal{E}_{\mathcal{T}}}\int_\ell \mathbf{R}_\ell\cdot\mathbf{v} \quad (\mathbf{v}\in\mathbf{X}_{\tilde{\mathcal{T}}}). \end{aligned} \tag{8.16}$$

Now by, for $K$ as in (i), choosing $\mathbf{v} := \mathbf{R}_K\psi_K$, that, thanks to $\mathbf{R}_K$ being a constant vector-field and the properties (8.13)-(8.15) of $\psi_K$ is in $\mathbf{X}_{\tilde{\mathcal{T}}}$, and using Bernstein inequality (5.4), we have

$$\begin{aligned}\|\mathbf{R}_K\|^2_{L_2(K)^2} \eqsim \int_K \mathbf{R}_K \cdot \mathbf{v} &= a(\mathbf{u}^{p_\mathcal{T}}_{\tilde{\mathcal{T}}} - \mathbf{u}^{p_\mathcal{T}}_{\mathcal{T}}, \mathbf{v}) \\ &\lesssim \|\mathbf{u}^{p_\mathcal{T}}_{\tilde{\mathcal{T}}} - \mathbf{u}^{p_\mathcal{T}}_{\mathcal{T}}\|_{H^1(K)^2} \mathrm{diam}(K)^{-1}\|\mathbf{v}\|_{L_2(K)^2} \\ &\lesssim \mathrm{diam}(K)^{-1}\|\mathbf{R}_K\|_{L_2(K)^2}\|\mathbf{u}^{p_\mathcal{T}}_{\tilde{\mathcal{T}}} - \mathbf{u}^{p_\mathcal{T}}_{\mathcal{T}}\|_{H^1(K)^2}.\end{aligned} \tag{8.17}$$

(ii) In a similar way, let for $\ell$ as in (ii), $\mathbf{v} := \mathbf{R}_\ell\psi_\ell$, then after some manipulations using (i), we find

$$\begin{aligned}\int_\ell \mathbf{R}_\ell \cdot \mathbf{v} &= a(\mathbf{u}^{p_\mathcal{T}}_{\tilde{\mathcal{T}}} - \mathbf{u}^{p_\mathcal{T}}_{\mathcal{T}}, \mathbf{v}) + \sum_{K\in\Omega_\ell}\int_K \mathbf{R}_K\cdot\mathbf{v} \\ &\lesssim \sum_{K\in\Omega_\ell}\{\|\mathbf{u}^{p_\mathcal{T}}_{\tilde{\mathcal{T}}} - \mathbf{u}^{p_\mathcal{T}}_{\mathcal{T}}\|_{H^1(K)^2}\mathrm{diam}(K)^{-1}\|\mathbf{v}\|_{L_2(K)^2} + \|\mathbf{R}_K\|_{L_2(K)^2}\|\mathbf{v}\|_{L_2(K)^2} \\ &\lesssim \sum_{K\in\Omega_\ell}\{\|\mathbf{u}^{p_\mathcal{T}}_{\tilde{\mathcal{T}}} - \mathbf{u}^{p_\mathcal{T}}_{\mathcal{T}}\|_{H^1(K)^2}\mathrm{diam}(K)^{-1}\|\mathbf{v}\|_{L_2(K)^2} \\ &\lesssim \sum_{K\in\Omega_\ell}\|\mathbf{u}^{p_\mathcal{T}}_{\tilde{\mathcal{T}}} - \mathbf{u}^{p_\mathcal{T}}_{\mathcal{T}}\|_{H^1(K)^2}\mathrm{diam}(\ell)^{-1/2}\|\mathbf{R}_\ell\|_{L_2(\ell)^2}\end{aligned} \tag{8.18}$$

Noting that $\int_\ell \mathbf{R}_\ell\cdot\mathbf{v} \eqsim \|\mathbf{R}_\ell\|_{L_2(\ell)^2}$ the statement follows.

iii) Follows from i) and ii).

iv) Let $\tilde{\mathcal{T}}$ be a refinement of $\mathcal{T}$ such that it satisfies conditions i),ii) for all $K \in \mathcal{T}$ and for all $\ell \in \mathcal{E}_\mathcal{T}$, respectively. The statement follows from iii) and Pythagoras

$$\|\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}^{p_\mathcal{T}}_{\mathcal{T}}\|^2_\mathbf{X} = \|\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}^{p_\mathcal{T}}_{\tilde{\mathcal{T}}}\|^2_\mathbf{X} + \|\mathbf{u}^{p_\mathcal{T}}_{\tilde{\mathcal{T}}} - \mathbf{u}^{p_\mathcal{T}}_{\mathcal{T}}\|^2_\mathbf{X} \tag{8.19}$$

v) Using the Galerkin orthogonality

$$a(\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}^{p_\mathcal{T}}_{\mathcal{T}}, \mathbf{v}_\mathcal{T}) = 0, \text{ for all } \mathbf{v}_\mathcal{T} \in \mathbf{X}_\mathcal{T}, \tag{8.20}$$

equation (8.1), Cauchy-Schwarz inequality and Lemma 8.1 we find

$$\begin{aligned}&a(\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}^{p_\mathcal{T}}_{\mathcal{T}}, \mathbf{v}) = a(\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}^{p_\mathcal{T}}_{\mathcal{T}}, \mathbf{v} - I_\mathcal{T}\mathbf{v}) = \\ &= \sum_{K\in\mathcal{T}}\int_K \mathbf{R}_K\cdot(\mathbf{v} - I_\mathcal{T}\mathbf{v}) - \sum_{\ell\in\mathcal{E}_\mathcal{T}}\int_\ell \mathbf{R}_\ell\cdot(\mathbf{v} - I_\mathcal{T}\mathbf{v}) \\ &\le \sum_{K\in\mathcal{T}}\|\mathbf{R}_K\|_{[L_2(K)]^2}\|\mathbf{v} - I_\mathcal{T}\mathbf{v}\|_{[L_2(K)]^2} + \sum_{\ell\in\mathcal{E}_\mathcal{T}}\|\mathbf{R}_\ell\|_{[L_2(\ell)]^2}\|\mathbf{v} - I_\mathcal{T}\mathbf{v}\|_{[L_2(\ell)]^2} \\ &\lesssim \sum_{K\in\mathcal{T}}\|\mathbf{R}_K\|_{[L_2(K)]^2}\mathrm{diam}(K)\|\mathbf{v}\|_{[H^1(\Omega_K)]^2} + \sum_{\ell\in\mathcal{E}_\mathcal{T}}\|\mathbf{R}_\ell\|_{[L_2(\ell)]^2}\mathrm{diam}^{1/2}(\ell)\|\mathbf{v}\|_{[H^1(\Omega_\ell)]^2} \\ &\le \{\sum_{K\in\mathcal{T}}\mathrm{diam}(K)^2\|\mathbf{R}_K\|^2_{[L_2(K)]^2} + \sum_{\ell\in\mathcal{E}_\mathcal{T}}\mathrm{diam}(\ell)\|\mathbf{R}_\ell\|^2_{[L_2(\ell)]^2}\}^{1/2}\|\mathbf{v}\|_\mathbf{X}\end{aligned} \tag{8.21}$$

Finally, invoking

$$\|\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}_\mathcal{T}^{p_\mathcal{T}}\|_\mathbf{X} = \sup_{\mathbf{v}\in\mathbf{X}} \frac{a(\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}_\mathcal{T}^{p_\mathcal{T}}, \mathbf{v})}{\|\mathbf{v}\|_\mathbf{X}}, \tag{8.22}$$

the upper bound follows. $\diamondsuit$

Based on our observations from the previous theorem, we define a routine that constructs a local refinement of a given triangulation, such that, as we will see later, an error reduction in our discrete approximation is ensured. In order to be able to apply Theorem 8.2, for the moment we will assume that $\mathbf{f} \in \mathbf{Y}_\mathcal{T}$ for any triangulation we encounter, i.e., that $\mathbf{f} \in \mathbf{Y}_{\mathcal{T}_0}$. Later, given $\mathbf{f} \in \mathbf{X}'$, on any triangulation $\mathcal{T}$ we will replace $\mathbf{f}$ by an approximation $\mathbf{f}_{\tilde{\mathcal{T}}} \in \mathbf{Y}_{\tilde{\mathcal{T}}}$ produced by the **RHS** routine. Here either $\tilde{\mathcal{T}} = \mathcal{T}$ or $\tilde{\mathcal{T}}$ is a proper refinement of $\mathcal{T}$ needed to get $\|\mathbf{f} - \mathbf{f}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}'}$ sufficiently small.

**Algorithm 8.3.** **REFINE**$[\mathcal{T},\mathbf{f}_\mathcal{T},p_\mathcal{T},\mathbf{w}_\mathcal{T},\theta]\to \tilde{\mathcal{T}}$

/* Input of the routine:

- admissible triangulation $\mathcal{T}$
- $\mathbf{f}_\mathcal{T} \in \mathbf{Y}_\mathcal{T}$, $p_\mathcal{T} \in Q_\mathcal{T}$
- $\mathbf{w}_\mathcal{T} \in \mathbf{X}_\mathcal{T}$
- $\theta \in (0,1)$

Select in $\mathcal{O}(\mathcal{T})$ operations $F \subset \mathcal{T}$ and $G \subset \mathcal{E}_\mathcal{T}$, such that

$$\{\sum_{K\in F} \operatorname{diam}(K)^2 \|\mathbf{R}_K\|^2_{[L_2(K)]^2} + \sum_{\ell\in G} \operatorname{diam}(K)^2 \|\mathbf{R}_\ell\|^2_{[L_2(\ell)]^2} \geq \theta^2 \mathfrak{E}(\mathcal{T}, \mathbf{f}_\mathcal{T}, \mathbf{w}_\mathcal{T}, p_\mathcal{T})^2 \tag{8.23}$$

Construct a refinement $\tilde{\mathcal{T}}$ of $\mathcal{T}$, such that for all $K \in F$ and $\ell \in G$ the conditions i)-ii) from Theorem 8.2 are satisfied. */

**Theorem 8.4.** *(Basic principle of adaptive error reduction) Let $\mathcal{T}$ be an admissible triangulation, assume that $\mathbf{f} \in \mathbf{Y}_\mathcal{T}$, and let $p_\mathcal{T} \in Q_\mathcal{T}$, $\mathbf{u}_\mathcal{T}^{p_\mathcal{T}} := \mathscr{A}_\mathcal{T}^{-1}(\mathbf{f} - \mathscr{B}^* p_\mathcal{T})$. Taking $\tilde{\mathcal{T}} =$**REFINE**$[\mathcal{T}, \mathbf{f}, p_\mathcal{T}, \mathbf{u}_\mathcal{T}^{p_\mathcal{T}}, \theta]$ then for $\mathbf{u}_{\tilde{\mathcal{T}}}^{p_\mathcal{T}} := \mathscr{A}_{\tilde{\mathcal{T}}}^{-1}(\mathbf{f} - \mathscr{B}^* p_\mathcal{T})$ the following error reduction holds*

$$\|\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}_{\tilde{\mathcal{T}}}^{p_\mathcal{T}}\|_\mathbf{X} \leq (1 - (\frac{C_L\theta}{C_U})^2)^{1/2} \|\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}_\mathcal{T}^{p_\mathcal{T}}\|_\mathbf{X} \tag{8.24}$$

*Proof.* Using Galerkin orthogonality, properties of **REFINE** and that of the a posteriori error estimator, we easily find

$$\begin{aligned}\|\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}_{\tilde{\mathcal{T}}}^{p_\mathcal{T}}\|^2_\mathbf{X} &= \|\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}_\mathcal{T}^{p_\mathcal{T}}\|^2_\mathbf{X} - \|\mathbf{u}_{\tilde{\mathcal{T}}}^{p_\mathcal{T}} - \mathbf{u}_\mathcal{T}^{p_\mathcal{T}}\|^2_\mathbf{X} \\ &\leq \|\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}_\mathcal{T}^{p_\mathcal{T}}\|^2_\mathbf{X} - C_L^2\theta^2\mathfrak{E}^2(\mathcal{T}, \mathbf{f}_\mathcal{T}, \mathbf{u}_\mathcal{T}^{p_\mathcal{T}}, p_\mathcal{T}) \\ &\leq (1 - (\frac{C_L\theta}{C_U})^2)^{1/2} \|\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}_\mathcal{T}^{p_\mathcal{T}}\|^2_\mathbf{X}\end{aligned} \tag{8.25}$$

$\diamondsuit$

Aiming at optimal computational complexity, we will solve the Galerkin problems approximately using the following routine

**Algorithm 8.5. GALSOLVE**$[\mathcal{T},\mathbf{f}_\mathcal{T},p_\mathcal{T},\mathbf{u}_\mathcal{T}^{(0)},\varepsilon]\to \mathbf{u}_\mathcal{T}^\varepsilon$

/* Input of the routine:

- an admissible triangulation $\mathcal{T}$
- $\mathbf{f}_\mathcal{T}\in \mathbf{X'}_\mathcal{T}(\supset \mathbf{X}')$, $p_\mathcal{T}\in Q_\mathcal{T}$
- $\mathbf{u}_\mathcal{T}^{(0)}\in \mathbf{X}_\mathcal{T}$
- $\varepsilon>0$

output: $\mathbf{u}_\mathcal{T}^\varepsilon\in \mathbf{X}_\mathcal{T}$ such that

$$\|\mathbf{u}_\mathcal{T}^{p_\mathcal{T}}-\mathbf{u}_\mathcal{T}^\varepsilon\|_\mathbf{X}\le\varepsilon \tag{8.26}$$

where $\mathbf{u}_\mathcal{T}^{p_\mathcal{T}}=\mathscr{A}_\mathcal{T}^{-1}(\mathbf{f}_\mathcal{T}-\mathscr{B}^*p_\mathcal{T})$.

The call of the algorithm takes $\lesssim \max\{1,\log(\varepsilon^{-1}\|\mathbf{u}_\mathcal{T}^{p_\mathcal{T}}-\mathbf{u}_\mathcal{T}^{(0)}\|_\mathbf{X})\}\sharp\mathcal{T}$ flops. */

An implementation of **GALSOLVE** that realizes the requirements is, for example, given by the application of Conjugate Gradients to the matrix-vector representation of $\mathscr{A}_\mathcal{T}\mathbf{u}_\mathcal{T}^{p_\mathcal{T}}=\mathbf{f}_\mathcal{T}-\mathscr{B}^*p_\mathcal{T}$ with respect to the wavelet basis $\Psi_\mathcal{T}$, that is well-conditioned uniformly in $\sharp\mathcal{T}$.

In the previous Theorem 8.4, the error reduction was based on the availability of the exact Galerkin solution $\mathbf{u}_\mathcal{T}^{p_\mathcal{T}}:=\mathscr{A}_\mathcal{T}^{-1}(\mathbf{f}-\mathscr{B}^*p_\mathcal{T})$ and the assumption that $\mathbf{f}\in\mathbf{Y}_\mathcal{T}$. Of course, in practice we will approximate $\mathbf{u}_\mathcal{T}^{p_\mathcal{T}}$ by some $\mathbf{u}_\mathcal{T}\in\mathbf{X}_\mathcal{T}$ using **GALSOLVE**, and the evaluation of the a posteriori error estimator and so the mesh refinement will be performed using this inexact Galerkin solution $\mathbf{u}_\mathcal{T}$. Furthermore, instead of making the unrealistic assumption that our right-hand side $\mathbf{f}\in\mathbf{Y}_\mathcal{T}$, we will replace $\mathbf{f}$ by an $\mathbf{f}_{\tilde{\mathcal{T}}}\in\mathbf{Y}_{\tilde{\mathcal{T}}}$ with $\tilde{\mathcal{T}}=\mathcal{T}$ or possibly $\tilde{\mathcal{T}}$ a refinement of $\mathcal{T}$. In the following, we will study how this influences the convergence of the adaptive approximations. We will start with investigating the stability of the error estimator.

**Lemma 8.6.** *For any admissible triangulation $\mathcal{T}$, $p_\mathcal{T}\in Q_\mathcal{T}$, $\mathbf{f}\in[L_2(\Omega)]^2$, $\mathbf{u}_\mathcal{T},\tilde{\mathbf{u}}_\mathcal{T}\in\mathbf{X}_\mathcal{T}$, it holds*

$$|\mathfrak{E}(\mathcal{T},\mathbf{f},\mathbf{u}_\mathcal{T},p_\mathcal{T})-\mathfrak{E}(\mathcal{T},\mathbf{f},\tilde{\mathbf{u}}_\mathcal{T},p_\mathcal{T})|\le C_S\|\mathbf{u}_\mathcal{T}-\tilde{\mathbf{u}}_\mathcal{T}\|_\mathbf{X}, \tag{8.27}$$

*with an absolute constant $C_S>0$.*

*Proof.* Using the triangle inequality twice, first for vectors and then for functions, we find

$$\begin{aligned}
&|\mathfrak{E}(\mathcal{T},\mathbf{f},\mathbf{u}_\mathcal{T},p_\mathcal{T})-\mathfrak{E}(\mathcal{T},\mathbf{f},\tilde{\mathbf{u}}_\mathcal{T},p_\mathcal{T})|\\
&=|(\sum_{K\in\mathcal{T}}\mathrm{diam}(K)^2\|\mathbf{R}_K(\mathbf{f},p_\mathcal{T})\|^2_{[L_2(K)]^2}+\sum_{\ell\in\mathcal{E}_\mathcal{T}}\mathrm{diam}(\ell)\|\mathbf{R}_\ell(\mathbf{u}_\mathcal{T},p_\mathcal{T})\|^2_{[L_2(\ell)]^2})^{1/2}\\
&-(\sum_{K\in\mathcal{T}}\mathrm{diam}(K)^2\|\mathbf{R}_K(\mathbf{f},p_\mathcal{T})\|^2_{[L_2(K)]^2}+\sum_{\ell\in\mathcal{E}_\mathcal{T}}\mathrm{diam}(\ell)\|\mathbf{R}_\ell(\tilde{\mathbf{u}}_\mathcal{T},p_\mathcal{T})\|^2_{[L_2(\ell)]^2})^{1/2}|\\
&\le(\sum_{\ell\in\mathcal{E}_\mathcal{T}}\mathrm{diam}(\ell)(\|\mathbf{R}_\ell(\mathbf{u}_\mathcal{T},p_\mathcal{T})\|_{[L_2(\ell)]^2}-\|\mathbf{R}_\ell(\tilde{\mathbf{u}}_\mathcal{T},p_\mathcal{T})\|_{[L_2(\ell)]^2})^2)^{1/2}\\
&\le(\sum_{\ell\in\mathcal{E}_\mathcal{T}}\mathrm{diam}(\ell)\|\mathbf{R}_\ell(\mathbf{u}_\mathcal{T}-\tilde{\mathbf{u}}_\mathcal{T},0)\|^2_{[L_2(\ell)]^2})^{1/2}\le C_S\|\mathbf{u}_\mathcal{T}-\tilde{\mathbf{u}}_\mathcal{T}\|_\mathbf{X},
\end{aligned} \tag{8.28}$$

where in the last line we have used that, for any edge $\ell$ of a triangle $K \in \mathcal{T}$, and any $\mathbf{w}_{\mathcal{T}} \in [P(K)]^2$ it holds

$$\|\mathbf{R}_\ell(\mathbf{w}_{\mathcal{T}}, 0)\|_{[L_2(\ell)]^2} \lesssim \operatorname{diam}(\ell)^{-1/2}\|\mathbf{w}_{\mathcal{T}}\|_{[H^1(K)]^2}. \tag{8.29}$$

$\diamondsuit$

**Theorem 8.7.** *Let $\mathcal{T}$ be an admissible triangulation, $\mathbf{f} \in \mathbf{X}'$, $p_{\mathcal{T}} \in Q_{\mathcal{T}}$, $\mathbf{u}^{p_{\mathcal{T}}} = \mathscr{A}^{-1}(\mathbf{f} - \mathscr{B}^* p_{\mathcal{T}})$, $\mathbf{f}_{\mathcal{T}} \in \mathbf{Y}_{\mathcal{T}}$, $\mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}} = \mathscr{A}^{-1}_{\mathcal{T}}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}})$, $\mathbf{u}_{\mathcal{T}} \in \mathbf{X}_{\mathcal{T}}$, and let $\tilde{\mathcal{T}} = \mathbf{REFINE}[\mathcal{T}, \mathbf{f}_{\mathcal{T}}, p_{\mathcal{T}}, \mathbf{u}_{\mathcal{T}}, \theta]$, $\mathbf{f}_{\tilde{\mathcal{T}}} \in \mathbf{X}'$, $\mathbf{u}^{p_{\mathcal{T}}}_{\tilde{\mathcal{T}}} = \mathscr{A}^{-1}_{\tilde{\mathcal{T}}}(\mathbf{f}_{\tilde{\mathcal{T}}} - \mathscr{B}^* p_{\mathcal{T}})$. Then it holds*

$$\|\mathbf{u}^{p_{\mathcal{T}}} - \mathbf{u}^{p_{\mathcal{T}}}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}} \le (1 - \frac{1}{2}(\frac{C_L\theta}{C_U})^2)^{1/2}\|\mathbf{u}^{p_{\mathcal{T}}} - \mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}\|_{\mathbf{X}} + 2C_S C_L\|\mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}} - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} + 3\|\mathbf{f} - \mathbf{f}_{\mathcal{T}}\|_{\mathbf{X}'} + \|\mathbf{f} - \mathbf{f}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}'} \tag{8.30}$$

Before we come to the proof, note that on $\mathcal{T}, \tilde{\mathcal{T}}$ we use the approximate right-hand sides $\mathbf{f}_{\mathcal{T}}, \mathbf{f}_{\tilde{\mathcal{T}}}$ respectively, and in the refinement routine **REFINE** we use an approximation $\mathbf{u}_{\mathcal{T}}$ instead of $\mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}} = \mathscr{A}^{-1}_{\mathcal{T}}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}})$, where we think of an approximation we got by the application of **GALSOLVE**.

*Proof.*

$$\begin{aligned}
&\|\mathscr{A}^{-1}(\mathbf{f} - \mathscr{B}^* p_{\mathcal{T}}) - \mathscr{A}^{-1}_{\tilde{\mathcal{T}}}(\mathbf{f}_{\tilde{\mathcal{T}}} - \mathscr{B}^* p_{\mathcal{T}})\|_{\mathbf{X}}\\
\le&\|\mathscr{A}^{-1}(\mathbf{f} - \mathscr{B}^* p_{\mathcal{T}}) - \mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}})\|_{\mathbf{X}} + \|\mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathscr{A}^{-1}_{\tilde{\mathcal{T}}}(\mathbf{f}_{\tilde{\mathcal{T}}} - \mathscr{B}^* p_{\mathcal{T}})\|_{\mathbf{X}}\\
\le&\|\mathbf{f} - \mathbf{f}_{\mathcal{T}}\|_{\mathbf{X}'} + \|\mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathscr{A}^{-1}_{\tilde{\mathcal{T}}}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}})\|_{\mathbf{X}} +\\
&\|\mathscr{A}^{-1}_{\tilde{\mathcal{T}}}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathscr{A}^{-1}_{\tilde{\mathcal{T}}}(\mathbf{f}_{\tilde{\mathcal{T}}} - \mathscr{B}^* p_{\mathcal{T}})\|_{\mathbf{X}}\\
\le&2\|\mathbf{f} - \mathbf{f}_{\mathcal{T}}\|_{\mathbf{X}'} + \|\mathbf{f} - \mathbf{f}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}'} + \|\mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathscr{A}^{-1}_{\tilde{\mathcal{T}}}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}})\|_{\mathbf{X}}
\end{aligned} \tag{8.31}$$

Now, to get an estimate for $\|\mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathscr{A}^{-1}_{\tilde{\mathcal{T}}}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}})\|_{\mathbf{X}}$, we will apply a similar analysis as in the proof of the Theorem 8.4. Using the properties of **REFINE**, Lemma 8.6 and Theorem 8.2 (iii, v) we find

$$\begin{aligned}
&\|\mathscr{A}^{-1}_{\tilde{\mathcal{T}}}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}\|_{\mathbf{X}}\\
\ge&C_L(\sum_{K\in\mathcal{T}} \operatorname{diam}(K)^2\|\mathbf{R}_K(\mathbf{f}_{\mathcal{T}}, p_{\mathcal{T}})\|^2_{[L_2(K)]^2} + \sum_{\ell\in\mathcal{E}_{\mathcal{T}}} \operatorname{diam}(\ell)^2\|\mathbf{R}_\ell(\mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}, p_{\mathcal{T}})\|^2_{[L_2(\ell)]^2})^{1/2}\\
\ge&C_L(\sum_{K\in\mathcal{T}} \operatorname{diam}(K)^2\|\mathbf{R}_K(\mathbf{f}_{\mathcal{T}}, p_{\mathcal{T}})\|^2_{[L_2(K)]^2} + \sum_{\ell\in\mathcal{E}_{\mathcal{T}}} \operatorname{diam}(\ell)^2\|\mathbf{R}_\ell(\mathbf{u}_{\mathcal{T}}, p_{\mathcal{T}})\|^2_{[L_2(\ell)]^2})^{1/2}\\
&-C_S\|\mathbf{u}_{\mathcal{T}} - \mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}\|_{\mathbf{X}})\\
\ge&C_L(\theta\mathfrak{E}(\mathcal{T}, \mathbf{f}_{\mathcal{T}}, \mathbf{u}_{\mathcal{T}}, p_{\mathcal{T}}) - C_S\|\mathbf{u}_{\mathcal{T}} - \mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}\|_{\mathbf{X}})\\
\ge&C_L(\theta\mathfrak{E}(\mathcal{T}, \mathbf{f}_{\mathcal{T}}, \mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}, p_{\mathcal{T}}) - 2C_S\|\mathbf{u}_{\mathcal{T}} - \mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}\|_{\mathbf{X}})\\
\ge&C_L(\frac{\theta}{C_U}\|\mathscr{A}^{-1}_{\mathcal{T}}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}\|_{\mathbf{X}} - 2C_S\|\mathbf{u}_{\mathcal{T}} - \mathbf{u}^{p_{\mathcal{T}}}_{\mathcal{T}}\|_{\mathbf{X}})
\end{aligned} \tag{8.32}$$

Thanks to the Galerkin orthogonality and the previous estimate we have

$$
\begin{aligned}
&\|\mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathscr{A}_{\tilde{\mathcal{T}}}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}})\|_{\mathbf{X}}^2 \\
=&\|\mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}}\|_{\mathbf{X}}^2 - \|\mathscr{A}_{\tilde{\mathcal{T}}}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}}\|_{\mathbf{X}}^2 \\
\leq&\|\mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}}\|_{\mathbf{X}}^2 \\
-&C_L^2(\frac{\theta}{C_U}\|\mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}}\|_{\mathbf{X}} - 2C_S\|\mathbf{u}_{\mathcal{T}} - \mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}}\|_{\mathbf{X}})^2 \\
\leq&\|\mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}}\|_{\mathbf{X}}^2 \\
-&C_L^2(\frac{1}{2}(\frac{\theta}{C_U})^2\|\mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}}\|_{\mathbf{X}}^2 - 4C_S^2\|\mathbf{u}_{\mathcal{T}} - \mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}}\|_{\mathbf{X}}^2) \qquad (8.33)\\
=&(1-\frac{1}{2}(\frac{\theta C_L}{C_U})^2)\|\mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}}\|_{\mathbf{X}}^2 + 4C_S^2C_L^2\|\mathbf{u}_{\mathcal{T}} - \mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}}\|_{\mathbf{X}}^2 \\
\leq&((1-\frac{1}{2}(\frac{\theta C_L}{C_U})^2)^{1/2}\|\mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}}) - \mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}}\|_{\mathbf{X}} + 2C_SC_L\|\mathbf{u}_{\mathcal{T}} - \mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}}\|_{\mathbf{X}})^2 \\
\leq&((1-\frac{1}{2}(\frac{\theta C_L}{C_U})^2)^{1/2}(\|\mathbf{u} - \mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}}\|_{\mathbf{X}} + \|\mathbf{f} - \mathbf{f}_{\mathcal{T}}\|_{\mathbf{X}'}) \\
+&2C_SC_L\|\mathbf{u}_{\mathcal{T}} - \mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}}\|_{\mathbf{X}})^2,
\end{aligned}
$$

where in (8.33) we have used that for any scalars $a, b$, $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$. Combination of last result with the first bound obtained in this proof completes the task. $\diamondsuit$

In Theorem 8.7 we estimated $\|\mathbf{u}^{p_{\mathcal{T}}} - \mathbf{u}_{\tilde{\mathcal{T}}}^{p_{\mathcal{T}}}\|_{\mathbf{X}}$, where $\mathbf{u}_{\tilde{\mathcal{T}}}^{p_{\mathcal{T}}} = \mathscr{A}_{\tilde{\mathcal{T}}}^{-1}(\mathbf{f}_{\tilde{\mathcal{T}}} - \mathscr{B}^* p_{\mathcal{T}})$, i.e., the exact Galerkin approximation. In the following Corollary we bound $\|\mathbf{u}^{p_{\mathcal{T}}} - \mathbf{u}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}}$, where we think of $\mathbf{u}_{\tilde{\mathcal{T}}} \in \mathbf{X}_{\tilde{\mathcal{T}}}$ being a sufficiently close approximation for $\mathbf{u}_{\tilde{\mathcal{T}}}^{p_{\mathcal{T}}}$, obtained by the **GALSOLVE** routine. As we will see, we get error reduction with a factor $\mu < 1$ in case we control the errors in the approximate Galerkin solutions and the approximate right-hand sides in a proper way.

**Corollary 8.8.** *(General adaptive error reduction estimate) For any* $\mu \in ((1-\frac{1}{2}(\frac{C_L\theta}{C_U})^2)^{1/2}, 1)$ *there exists a sufficiently small constant* $\delta > 0$ *such that if for* $\mathbf{f} \in \mathbf{X}'$*, an admissible triangulation* $\mathcal{T}$*,* $p_{\mathcal{T}} \in Q_{\mathcal{T}}$*,* $\mathbf{f}_{\mathcal{T}} \in \mathbf{Y}_{\mathcal{T}}$*,* $\mathbf{u}_{\mathcal{T}} \in \mathbf{X}_{\mathcal{T}}$*,* $\tilde{\mathcal{T}} = \mathbf{REFINE}[\mathcal{T}, \mathbf{f}_{\mathcal{T}}, \mathbf{u}_{\mathcal{T}}, \theta]$*,* $\mathbf{f}_{\tilde{\mathcal{T}}} \in \mathbf{X}'$*,* $\mathbf{u}_{\tilde{\mathcal{T}}} \in \mathbf{X}_{\tilde{\mathcal{T}}}$ *and* $\varepsilon > 0$*, with* $\mathbf{u}^{p_{\mathcal{T}}} = \mathscr{A}^{-1}(\mathbf{f} - \mathscr{B}^* p_{\mathcal{T}})$*,* $\mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}} = \mathscr{A}_{\mathcal{T}}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\mathcal{T}})$*,* $\mathbf{u}_{\tilde{\mathcal{T}}}^{p_{\mathcal{T}}} = \mathscr{A}_{\tilde{\mathcal{T}}}^{-1}(\mathbf{f}_{\tilde{\mathcal{T}}} - \mathscr{B}^* p_{\mathcal{T}})$*, we have that* $\|\mathbf{u}^{p_{\mathcal{T}}} - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \leq \varepsilon$ *and*

$$
\|\mathbf{u}_{\mathcal{T}}^{p_{\mathcal{T}}} - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} + \|\mathbf{f} - \mathbf{f}_{\mathcal{T}}\|_{\mathbf{X}'} + \|\mathbf{u}_{\tilde{\mathcal{T}}}^{p_{\mathcal{T}}} - \mathbf{u}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}} + \|\mathbf{f} - \mathbf{f}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}'} \leq 2(1+\mu)\delta\varepsilon, \tag{8.34}
$$

*then*

$$
\|\mathbf{u}^{p_{\mathcal{T}}} - \mathbf{u}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}} \leq \mu\varepsilon \tag{8.35}
$$

*Proof.* Using the triangle inequality, the conditions of the corollary and Theorem 8.7, we easily find

$$
\begin{aligned}
&\|\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}_{\tilde{\mathcal{T}}}\|_\mathbf{X} \leq \|\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}^{p_\mathcal{T}}_{\tilde{\mathcal{T}}}\|_\mathbf{X} + \|\mathbf{u}^{p_\mathcal{T}}_{\tilde{\mathcal{T}}} - \mathbf{u}_{\tilde{\mathcal{T}}}\|_\mathbf{X} \\
\leq&(1-\frac{1}{2}(\frac{C_L\theta}{C_U})^2)^{1/2}\|\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}^{p_\mathcal{T}}_{\mathcal{T}}\|_\mathbf{X} + 2C_SC_L\|\mathbf{u}^{p_\mathcal{T}}_{\mathcal{T}} - \mathbf{u}_\mathcal{T}\|_\mathbf{X} \\
+&3\|\mathbf{f}-\mathbf{f}_\mathcal{T}\|_{\mathbf{X}'} + \|\mathbf{f}-\mathbf{f}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}'} + \|\mathbf{u}^{p_\mathcal{T}}_{\tilde{\mathcal{T}}} - \mathbf{u}_{\tilde{\mathcal{T}}}\|_\mathbf{X} \\
\leq&\mu\|\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}_\mathcal{T}\|_\mathbf{X} + ((1-\frac{1}{2}(\frac{C_L\theta}{C_U})^2)^{1/2}-\mu)\|\mathbf{u}^{p_\mathcal{T}} - \mathbf{u}_\mathcal{T}\|_\mathbf{X} \\
+&\max\{2C_SC_L,3\}(\|\mathbf{u}^{p_\mathcal{T}}_{\mathcal{T}} - \mathbf{u}_\mathcal{T}\|_\mathbf{X} + \|\mathbf{f}-\mathbf{f}_\mathcal{T}\|_{\mathbf{X}'} + \|\mathbf{f}-\mathbf{f}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}'} + \|\mathbf{u}^{p_\mathcal{T}}_{\tilde{\mathcal{T}}} - \mathbf{u}_{\tilde{\mathcal{T}}}\|_\mathbf{X}) \\
\leq&\mu\varepsilon,
\end{aligned}
\tag{8.36}
$$

where we have chosen $\delta$ to satisfy

$$
\delta \leq \frac{\mu - (1-\frac{1}{2}(\frac{C_L\theta}{C_U})^2)^{1/2}}{2(1+\mu)\max\{2C_SC_L,3\}}. \ \diamondsuit
\tag{8.37}
$$

We are now ready to present the adaptive solver of the inner elliptic problem.

**Algorithm 8.9.** Adaptive Inner Solver
**INNERELLIPTIC**$[\bar{\mathcal{T}}, \mathbf{f}, p_{\bar{\mathcal{T}}}, \mathbf{u}_{\bar{\mathcal{T}}}, \varepsilon_0, \varepsilon] \to [\mathcal{T}, \mathbf{u}_\mathcal{T}]$
/*
Input parameters of the algorithm are: $\mathbf{f} \in \mathbf{X}'$, an admissible triangulation $\bar{\mathcal{T}}$, $p_{\bar{\mathcal{T}}} \in Q_{\bar{\mathcal{T}}}$, $\mathbf{u}_{\bar{\mathcal{T}}} \in \mathbf{X}_{\bar{\mathcal{T}}}$, and $\varepsilon_0 \geq \|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathbf{u}_\mathcal{T}\|_\mathbf{X}$. The parameter $\delta < 1/3$ is chosen such that it corresponds to a $\mu < 1$ as in Corollary 8.8.
*/

$\mathcal{T} := \bar{\mathcal{T}}$, $\mathbf{u}_\mathcal{T} := \mathbf{u}_{\bar{\mathcal{T}}}$
$\varepsilon_1 := \frac{\varepsilon_0}{1-3\delta}$
$[\mathcal{T}, \mathbf{f}_\mathcal{T}]$:=**RHS**$[\mathcal{T}, \mathbf{f}, \delta\varepsilon_1]$
$\mathbf{u}_\mathcal{T}$:=**GALSOLVE**$[\mathcal{T}, \mathbf{f}_\mathcal{T}, p_{\bar{\mathcal{T}}}, \mathbf{u}_\mathcal{T}, \delta\varepsilon_1]$
$\bar{N} := \min\{j \in \mathbb{N} : \mu^j\varepsilon_1 \leq \varepsilon\}$
`for` $k = 1, \ldots, \bar{N}$ `do`
  $\mathcal{T}$:=**REFINE**$[\mathcal{T}, \mathbf{f}_\mathcal{T}, p_{\bar{\mathcal{T}}}, \mathbf{u}_\mathcal{T}, \theta]$
  $[\mathcal{T}, \mathbf{f}_\mathcal{T}]$:=**RHS**$[\mathcal{T}, \mathbf{f}, \delta\mu^k\varepsilon_1]$
  $\mathbf{u}_\mathcal{T}$:=**GALSOLVE**$[\mathcal{T}, \mathbf{f}_\mathcal{T}, p_{\bar{\mathcal{T}}}, \mathbf{u}_\mathcal{T}, \delta\mu^k\varepsilon_1]$
`endfor`

**Theorem 8.10.** *(i) Algorithm* **INNERELLIPTIC**$[\bar{\mathcal{T}}, \mathbf{f}, p_{\bar{\mathcal{T}}}, \mathbf{u}_{\bar{\mathcal{T}}}, \varepsilon_0, \varepsilon] \to [\mathcal{T}, \mathbf{u}_\mathcal{T}]$ *terminates with*

$$
\|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathbf{u}_\mathcal{T}\|_\mathbf{X} \leq \varepsilon
\tag{8.38}
$$

*(ii) If for some $s > 0$ the pair* $(\mathbf{f}, \mathbf{RHS})$ *is $s$-optimal, and the algorithm is called with $\varepsilon_0 \lesssim \varepsilon$ then both the cardinality of the output triangulation and the number of flops required by the*

*algorithm satisfy*

$$\sharp\mathcal{T},\ \textit{number of flops} \lesssim \sharp\bar{\mathcal{T}} + \varepsilon^{-1/s} c_f^{1/s} \tag{8.39}$$

*Proof.* i) We are going to show that just before the $k$th call of **REFINE**

$$\|\mathscr{A}_{\mathcal{T}}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}}) - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \le \varepsilon_1 \mu^{k-1} \tag{8.40}$$

meaning that after the $k$th inner loop $\|\mathscr{A}_{\mathcal{T}}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}}) - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \le \varepsilon_1 \mu^{k}$, which, by definition of $\bar{N}$, proves the first part of the theorem.

For $k = 1$, after the first call $[\mathcal{T}, \mathbf{f}_{\mathcal{T}}]$ :=**RHS**$[\mathcal{T}, \mathbf{f}, \delta\varepsilon_1]$ it holds

$$\|\mathbf{f} - \mathbf{f}_{\mathcal{T}}\|_{\mathbf{X}'} \le \delta\varepsilon_1. \tag{8.41}$$

For the input $\mathbf{u}_{\mathcal{T}}$ we have $\|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \le (1-3\delta)\varepsilon_1$. Using the triangle inequality and the fact that $\mathscr{A}_{\mathcal{T}}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}})$ is the best approximation of $\mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}})$ from $\mathbf{X}_{\mathcal{T}}$ with respect to $\|\cdot\|_{\mathbf{X}}$, we find

$$\begin{aligned}
&\|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathscr{A}_{\mathcal{T}}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}})\|_{\mathbf{X}} \le \|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}})\|_{\mathbf{X}} \\
&+\|\mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}}) - \mathscr{A}_{\mathcal{T}}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}})\|_{\mathbf{X}} \\
&\le\|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}})\|_{\mathbf{X}} + \|\mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}}) - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \\
&\le 2\|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}})\|_{\mathbf{X}} + \|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \\
&\le 2\|\mathbf{f} - \mathbf{f}_{\mathcal{T}}\|_{\mathbf{X}} + \|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \le 2\delta\varepsilon_1 + (1-3\delta)\varepsilon_1 \le (1-\delta)\varepsilon_1
\end{aligned} \tag{8.42}$$

After the call $\mathbf{u}_{\mathcal{T}}$ :=**GALSOLVE**$[\mathcal{T}, \mathbf{f}_{\mathcal{T}}, p_{\bar{\mathcal{T}}}, \mathbf{u}_{\mathcal{T}}, \delta\varepsilon_1]$ we have

$$\|\mathscr{A}_{\mathcal{T}}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}}) - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \le \delta\varepsilon_1. \tag{8.43}$$

Together with the previous estimate it gives

$$\|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \le \|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathscr{A}_{\mathcal{T}}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}})\|_{\mathbf{X}} + \|\mathscr{A}_{\mathcal{T}}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}}) - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \le \varepsilon_1, \tag{8.44}$$

i.e., (8.40) is valid for $k = 1$.

Let us now assume that (8.40) is valid for some $k$. By the last calls of **RHS** and **GALSOLVE** for the current $\mathcal{T}$, $\mathbf{u}_{\mathcal{T}}$ and $\mathbf{f}_{\mathcal{T}}$ we have

$$\|\mathbf{f} - \mathbf{f}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}'} \le \delta\varepsilon_1\mu^{k-1}, \quad \|\mathscr{A}_{\tilde{\mathcal{T}}}^{-1}(\mathbf{f}_{\tilde{\mathcal{T}}} - \mathscr{B}^* p_{\bar{\mathcal{T}}}) - \mathbf{u}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}} \le \delta\varepsilon_1\mu^{k-1}. \tag{8.45}$$

The subsequent calls $\tilde{\mathcal{T}}$ :=**REFINE**$[\mathcal{T}, \mathbf{f}_{\mathcal{T}}, p_{\bar{\mathcal{T}}}, \mathbf{u}_{\mathcal{T}}, \theta]$, $[\tilde{\mathcal{T}}, \mathbf{f}_{\tilde{\mathcal{T}}}]$ :=**RHS**$[\tilde{\mathcal{T}}, \mathbf{f}, \delta\mu^k\varepsilon_1]$, and $\mathbf{u}_{\tilde{\mathcal{T}}}$ :=**GALSOLVE**$[\tilde{\mathcal{T}}, \mathbf{f}_{\tilde{\mathcal{T}}}, p_{\bar{\mathcal{T}}}, \mathbf{u}_{\mathcal{T}}, \delta\mu^k\varepsilon_1]$ result in

$$\|\mathbf{f} - \mathbf{f}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}'} \le \delta\varepsilon_1\mu^{k}, \quad \|\mathscr{A}_{\tilde{\mathcal{T}}}^{-1}(\mathbf{f}_{\tilde{\mathcal{T}}} - \mathscr{B}^* p_{\bar{\mathcal{T}}}) - \mathbf{u}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}} \le \delta\varepsilon_1\mu^{k}. \tag{8.46}$$

Therefore we obtain

$$\|\mathbf{f}-\mathbf{f}_{\mathcal{T}}\|_{\mathbf{X}'}+\|\mathscr{A}_{\mathcal{T}}^{-1}(\mathbf{f}_{\mathcal{T}}-\mathscr{B}^* p_{\bar{\mathcal{T}}})-\mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}}+\|\mathbf{f}-\mathbf{f}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}'}+\|\mathscr{A}_{\tilde{\mathcal{T}}}^{-1}(\mathbf{f}_{\tilde{\mathcal{T}}}-\mathscr{B}^* p_{\bar{\mathcal{T}}})-\mathbf{u}_{\tilde{\mathcal{T}}}\|_{\mathbf{X}} \le 2(1+\mu)\delta\varepsilon_1\mu^k. \tag{8.47}$$

Using the induction hypothesis, Corollary 8.8 shows that

$$\|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathbf{u}_{\tilde{\mathcal{T}}}\| \le \varepsilon_1\mu^k, \tag{8.48}$$

with which (8.40) is proven.

ii) Let us now analyse the computational complexity of the algorithm. For $k = 1, \dots, \bar{N}$, just before the call $\mathbf{GALSOLVE}[\mathcal{T}, \mathbf{f}_{\mathcal{T}}, \mathbf{u}_{\mathcal{T}}, \delta\mu^k\varepsilon_1]$ it holds that $\|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \le \mu^{k-1}\varepsilon_1$ and $\|\mathbf{f} - \mathbf{f}_{\mathcal{T}}\|_{\mathbf{X}'} \le \delta\mu^k\varepsilon_1$. Since, thanks to (8.42),

(8.49)
$$\|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathscr{A}_{\mathcal{T}}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}})\|_{\mathbf{X}} \le 2\|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathscr{A}^{-1}(\mathbf{f}_{\mathcal{T}} - \mathscr{B}^* p_{\bar{\mathcal{T}}})\|_{\mathbf{X}} + \|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \le (2\delta\mu^k + \mu^{k-1})\varepsilon_1,$$

we have that $\|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \le 2(\delta\mu^k + \mu^{k-1})\varepsilon_1$. Observing that $\frac{2(\delta\mu^k + \mu^{k-1})\varepsilon_1}{\delta\mu^k\varepsilon_1}$ is a constant, the cost of this call is $\lesssim \sharp\mathcal{T}$. Before the first call $\mathbf{GALSOLVE}[\mathcal{T}, \mathbf{f}_{\mathcal{T}}, \mathbf{u}_{\mathcal{T}}, \delta\varepsilon_1]$ we have $\|\mathbf{u}^{p_{\bar{\mathcal{T}}}} - \mathbf{u}_{\mathcal{T}}\|_{\mathbf{X}} \le \varepsilon_0 = (1 - 3\delta)\varepsilon_1$ and $\|\mathbf{f} - \mathbf{f}_{\mathcal{T}}\|_{\mathbf{X}'} \le \delta\varepsilon_1$. The same arguments show that also the cost of this call is $\lesssim \sharp\mathcal{T}$.

Recalling the properties of the routine **REFINE**, **RHS**, we can list the costs of each call.

$$\tilde{\mathcal{T}} := \mathbf{REFINE}[\mathcal{T}, \mathbf{f}_{\mathcal{T}}, \mathbf{u}_{\mathcal{T}}], \quad \sharp\tilde{\mathcal{T}}, \text{ flops} \lesssim \sharp\mathcal{T} \tag{8.50}$$

$$[\tilde{\mathcal{T}}, \mathbf{f}_{\mathcal{T}}] := \mathbf{RHS}[\mathcal{T}, \mathbf{f}, \delta\mu^k\varepsilon_1], \quad \sharp\tilde{\mathcal{T}}, \text{ flops} \lesssim \sharp\mathcal{T} + (\delta\mu^k\varepsilon_1)^{-1/s} c_f^{1/s} \tag{8.51}$$

Finally, since the for -loop runs for $\bar{N}$ iterations, being an absolute constant only dependent on $\varepsilon_0/\varepsilon$, we conclude that after the call $\mathbf{INNERELLIPTIC}[\bar{\mathcal{T}}, \mathbf{f}, p_{\bar{\mathcal{T}}}, \mathbf{u}_{\bar{\mathcal{T}}}, \varepsilon_0, \varepsilon] \to [\mathcal{T}, \mathbf{u}_{\mathcal{T}}]$, the cardinality of the output triangulation and the number of flops required by the algorithm satisfy

$$\sharp\mathcal{T}, \text{number of flops} \lesssim \sharp\bar{\mathcal{T}} + \varepsilon^{-1/s} c_f^{1/s} \quad \diamondsuit \tag{8.52}$$

**Conclusions.** In this paper we have designed an adaptive FEM algorithm for solving the Stokes problem. We were able to prove that this algorithm produces a sequence of adaptive approximations that converges with the optimal rate using the fact that the algorithm contains mesh derefinement routine.

Recently, in the work of Stevenson ([Ste05b]), an adaptive FEM method was constructed for solving elliptic problems that has optimal computational complexity not relying on a recurrent derefinement of the triangulations. Currently, it is open question whether this is possible for mixed problems.

In any case, in our view, availability of an efficient mesh optimization/derefinement routine is very useful for the adaptive solutions of nonstationary problems that often exhibit moving shock waves.

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