Adaptive Frame Methods for Elliptic Operator Equations: The Steepest Descent Approach*

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Abstract

This paper is concerned with the development of adaptive numerical methods for elliptic operator equations. We are especially interested in discretization schemes based on wavelet frames. We show that by using three basic subroutines an implementable, convergent scheme can be derived, which, moreover, has optimal computational complexity. The scheme is based on adaptive steepest descent iterations. We illustrate our findings by numerical results for the computation of solutions of the Poisson equation with limited Sobolev smoothness on intervals in 1D and on L-shaped domains in 2D.

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1 Introduction

In recent years, wavelets have been very successfully applied to several tasks. In signal/image analysis/compression, wavelet schemes are by now already well–accepted and clearly compete with other methods. Moreover, wavelets have also been used in numerical analysis, especially for the treatment of elliptic operator equations. Current interest in particular centers around the development of *adaptive* discretization schemes. Based on the equivalence of Sobolev norms and weighted sequence norms of wavelet expansion coefficients, convergent adaptive wavelet schemes were designed for symmetric elliptic problems [6, 12, 23] as well as for nonsymmetric and stationary nonlinear problems [7, 8, 22].

Although quite convincing from the theoretical point of view, so far the potential of adaptive wavelet schemes has not been fully exploited in practice for the following reason. Usually, the operator under consideration is defined on a bounded domain or on a closed manifold, so that a construction of a suitable wavelet basis on this domain is needed. There

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exist by now several constructions such as, e.g., [3, 9, 17, 18, 29, 34]. None of them, however, seems to be fully satisfactory in the sense that, besides the relevant positive virtues, these bases do not exhibit reasonable quantitative stability properties. Moreover, the constructions in aforementioned references are all based on *non-overlapping* domain decomposition techniques, most of them requiring certain matching conditions on the parametric mappings, which in practical situations can be difficult to satisfy.

One possible way to circumvent this bottleneck is to use a slightly weaker concept, i.e., to work with (wavelet) frames. In general, a sequence $\mathcal{F} = \{f_n\}_{n\in\mathcal{N}}$ in a Hilbert space $\mathcal{H}$ is a *frame* for the Hilbert space $\mathcal{H}$ if

$$A_{\mathcal{F}}\|f\|_{\mathcal{H}}^2 \leq \sum_{n\in\mathcal{N}} \left|\langle f, f_n\rangle_{\mathcal{H}}\right|^2 \leq B_{\mathcal{F}}\|f\|_{\mathcal{H}}^2, \quad \text{for all } f \in \mathcal{H},$$

for suitable constants $0 < A_{\mathcal{F}} \leq B_{\mathcal{F}} < \infty$, see [5, 19] for further details. Every element of $\mathcal{H}$ has an expansion with respect to the frame elements, but this expansion is not necessarily unique. On the one hand, this redundancy may cause problems in numerical applications since it gives rise to a singular stiffness matrix. On the other hand, it has turned out that the construction of suitable frames on domains and manifolds is a much simpler task compared to that of stable multiscale bases, see [14, 33]. The idea is to write the domain or manifold as an *overlapping* union of subdomains, each of them being the smooth parametric image of a reference domain. By lifting a wavelet basis on the reference domain to the subdomains, and taking the union of these lifted bases, a frame is obtained. Due to their nature, we refer to such frames as *aggregated wavelet frames*. In recent studies, it has been shown that, despite of the singular stiffness matrix, a damped Richardson iteration can be generalized to the frame case in a very natural way [14, 33]. Then, by using the basic building blocks of the adaptive wavelet algorithms in [7], an implementable and asymptotically optimal convergent version of this scheme can be constructed.

This paper follows similar lines and can be interpreted as the continuation of the studies [14, 33]. Instead of using the classical Richardson iteration, here we are especially interested in the steepest descent method. As we will show, with this method again an asymptotically optimal convergent scheme can be derived. Its main advantage is that it releases the user of providing a damping parameter as with Richardson's method, which preferably is close to the optimal one. This, however, requires an accurate estimate of the largest and smallest non-zero eigenvalues of the stiffness matrix, where in the frame case in particular the smallest non-zero eigenvalue is hard to access. Although the steepest descent method requires more computational effort per iteration, in our numerical experiments it is as efficient as the Richardson iteration. Moreover, in case the Richardson damping parameter is not optimally chosen, then the steepest descent method can even outperform the Richardson iteration, see Section 4.

The steepest descent method for the adaptive solution of infinite–dimensional systems has also been studied in [4], however, there the results are restricted to the basis case, whereas we are concerned with frames.

This paper is organized as follows. In Section 2, we discuss the scope of problems we shall be concerned with and summarize the basic concepts of frame discretizations. Then, in Section 3, we introduce the adaptive steepest descent method and establish its convergence and optimality. Finally, in Section 4, we present numerical experiments for the special case of the Poisson equation on an interval in 1D and on an L–shaped domain in 2D. The results fully confirm the expected convergence and optimality for both the Richardson and the steepest descent iterations. A comparison between the two schemes is also discussed.

2 Preliminaries

In this section, we briefly describe the scope of problems we shall be concerned with. Moreover, we recall the basic concepts of frame discretization schemes for operator equations.

We consider linear operator equations

$$\mathcal{L}u = f, \tag{2.1}$$

where we will assume $\mathcal{L}$ to be a boundedly invertible operator from some Hilbert space H into its normed dual H', i.e.,

$$\|\mathcal{L}u\|_{H'} \eqsim \|u\|_H, \quad u \in H. \tag{2.2}$$

Here '$a \eqsim b$' means that both quantities can be uniformly bounded by some constant multiple of each other. Likewise, '$\lesssim$' indicates inequalities up to constant factors. We write out such constants explicitly only when their value matters. Since $\mathcal{L}$ is assumed to be boundedly invertible, (2.1) has a unique solution u for any $f \in H'$. In the sequel, we shall focus on the important special case where

$$a(v, w) := \langle \mathcal{L}v, w \rangle \tag{2.3}$$

defines a *symmetric* bilinear form on H, $\langle \cdot, \cdot \rangle$ corresponding to the dual pairing of H' and H. We will always assume that $a(\cdot, \cdot)$ is *elliptic* in the sense that

$$a(v, v) \eqsim \|v\|_H^2, \tag{2.4}$$

which is easily seen to imply (2.2).

Typical examples are variational formulations of second order elliptic boundary value problems on a domain $\Omega \subset \mathbb{R}^d$ such as the Poisson equation

$$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.5}$$

In this case, $H = H_0^1(\Omega)$, $H' = H^{-1}(\Omega)$, and the corresponding bilinear form is given by

$$a(v, w) = \int_\Omega \nabla v \cdot \nabla w dx. \tag{2.6}$$

Thus typically H is a Sobolev space. Therefore, from now on, we will always assume that H and H', together with $L_2(\Omega)$, form a *Gelfand triple*, i.e.,

$$H \subset L_2(\Omega) \subset H' \tag{2.7}$$

with continuous and dense embeddings.

The design of adaptive wavelet or frame schemes in the aforementioned setting starts with a reformulation of (2.1) as an equivalent discrete problem on some sequence space $\ell_2(\mathcal{N})$. However, to perform this transformation, it will not be sufficient to work with a simple frame in L_2, since the operator $\mathcal{L}$ acts between Sobolev spaces. Similar to the classical wavelet case, we need specific norm equivalences of Sobolev norms and weighted sequence norms of frame coefficients. These can be realized by so–called Gelfand frames as introduced in [14]. Given a frame $\mathcal{F}$ in $\mathcal{H}$, one usually defines the corresponding operators of analysis and synthesis to be

$$F : \mathcal{H} \to \ell_2(\mathcal{N}), \quad f \mapsto \big(\langle f, f_n\rangle_{\mathcal{H}}\big)_{n\in\mathcal{N}}, \tag{2.8}$$

$$F^* : \ell_2(\mathcal{N}) \to \mathcal{H}, \quad \mathbf{c} \mapsto \sum_{n\in\mathcal{N}} c_n f_n. \tag{2.9}$$

The composition $S := F^*F$ is a boundedly invertible (positive and self–adjoint) operator, called the *frame operator*, and $\tilde{\mathcal{F}} := S^{-1}\mathcal{F}$ is again a frame for $\mathcal{H}$, the *canonical dual frame*. Then, a frame $\mathcal{F}$ for $\mathcal{H}$ is called a *Gelfand frame* for the Gelfand triple $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$, if $\mathcal{F} \subset \mathcal{B}$, $\tilde{\mathcal{F}} \subset \mathcal{B}'$ and there exists a Gelfand triple $\big(\mathcal{B}_d, \ell_2(\mathcal{N}), \mathcal{B}_d'\big)$ of sequence spaces such that

$$F^* : \mathcal{B}_d \to \mathcal{B},\ F^*\mathbf{c} = \sum_{n\in\mathcal{N}} c_n f_n \quad \text{and} \quad \tilde{F} : \mathcal{B} \to \mathcal{B}_d,\ \tilde{F}f = \big(\langle f, \tilde{f}_n\rangle_{\mathcal{B}\times\mathcal{B}'}\big)_{n\in\mathcal{N}} \tag{2.10}$$

are bounded operators.

Remark 2.1. i) For the applications we have in mind, clearly the case $(\mathcal{B}, \mathcal{H}, \mathcal{B}') = (H, L_2(\Omega), H')$, where H denotes some Sobolev space, is the most important one. Then, similar to the classical wavelet basis case, the spaces $\mathcal{B}_d$ and $\mathcal{B}_d'$ are weighted $\ell_2(\mathcal{N})$–spaces, see [14] for details.

ii) It can be shown that Gelfand frames are also Banach frames for the spaces $\mathcal{B}$ and $\mathcal{B}'$ in the sense of [28], see again [14] for details.

iii) A natural way to construct wavelet Gelfand frames on domains and manifolds is that by means of overlapping partitions of parametric images of unit cubes, see Section 4 and [14, 33] for details. We call such frames aggregated wavelet frames.

For the transformation of (2.1) into a discrete problem on $\ell_2(\mathcal{N})$, we have to assume that there exists an isomorphism $D_{\mathcal{B}} : \mathcal{B}_d \to \ell_2(\mathcal{N})$, so that its $\ell_2(\mathcal{N})$–adjoint $D_{\mathcal{B}}^* : \ell_2(\mathcal{N}) \to \mathcal{B}_d'$ is also an isomorphism. Then, the following lemma holds [14, 33].

Lemma 2.2. *Under the aforementioned assumptions on the frame, as well as* (2.3), (2.4) *on $\mathcal{L}$, the operator*

$$\mathbf{G} := (D_{\mathcal{B}}^*)^{-1} F\mathcal{L}F^* D_{\mathcal{B}}^{-1} \tag{2.11}$$

is a bounded operator from $\ell_2(\mathcal{N})$ to $\ell_2(\mathcal{N})$. Moreover $\mathbf{G} = \mathbf{G}^*$, and it is boundedly invertible on its range $\mathrm{ran}(\mathbf{G}) = \mathrm{ran}((D_{\mathcal{B}}^*)^{-1}F)$.

With

$$\mathbf{f} := (D_{\mathcal{B}}^*)^{-1} Ff, \tag{2.12}$$

we are therefore left with the problem to solve

$$\mathbf{G}\mathbf{u} = \mathbf{f}. \tag{2.13}$$

One natural way to solve (2.13) would be to use a damped Richardson iteration. Indeed, the following theorem can be shown [14, 33].

Theorem 2.3. *Let $\mathcal{L}$ satisfy* (2.3) *and* (2.4)*. Then with* $\mathbf{G}$ *and* $\mathbf{f}$ *as in* (2.11) *and* (2.12)*, respectively, the solution u of* (2.1) *can be computed as*

$$u = F^* D_{\mathcal{B}}^{-1}\mathbf{u} \tag{2.14}$$

with $\mathbf{u}$ *given by*

$$\mathbf{u} = \left(\alpha \sum_{n=0}^{\infty} (\mathbf{I} - \alpha\mathbf{G})^{\mathbf{n}}\right)\mathbf{f}, \tag{2.15}$$

with $0 < \alpha < 2/\|\mathbf{G}\|_{\ell_2(\mathcal{N})\to\ell_2(\mathcal{N})}$.

Observe that the computation of (2.15) is indeed nothing but an infinite damped Richardson iteration

$$\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + \alpha(\mathbf{f} - \mathbf{G}\mathbf{u}^{(i)}), \tag{2.16}$$

starting with $\mathbf{u}^{(0)} = \mathbf{0}$. This scheme has been analyzed in [14, 33]. In this paper, we use a different approach and work with a version of the *steepest descent scheme*, see the following section.

3 The Steepest Descent Scheme

In this section, we introduce and analyze a steepest descent scheme for the solution of (2.13). In Subsection 3.1, we explain the basic setting, and we prove a perturbation theorem for this scheme. Then, in Subsection 3.2, we derive an implementable version and show its asymptotically optimal convergence.

3.1 Basic Setting

The first step is to introduce a natural energy (semi)–norm on $\ell_2(\mathcal{N})$. In the following, we write $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ for $\|\cdot\|_{\ell_2(\mathcal{N})}$, or $\|\cdot\|_{\ell_2(\mathcal{N})\to\ell_2(\mathcal{N})}$, and $\langle\cdot,\cdot\rangle_{\ell_2(\mathcal{N})}$, respectively. We set $\langle\!\langle\cdot,\cdot\rangle\!\rangle := \langle\mathbf{G}\cdot,\cdot\rangle$ and the semi–norm $|\!|\!|\cdot|\!|\!| := \langle\!\langle\cdot,\cdot\rangle\!\rangle^{\frac{1}{2}}$. With $\mathbf{G}^\dagger$ being the Moore-Penrose pseudo inverse of $\mathbf{G}$, and

$$\mathbf{Q} : \ell_2(\mathcal{N}) \to \operatorname{ran}\mathbf{G}$$

being the orthogonal projector onto $\operatorname{ran}\mathbf{G}$, for any $\mathbf{v} \in \ell_2(\mathcal{N})$ we have

$$\|\mathbf{G}^\dagger\|^{-\frac{1}{2}}\|\mathbf{Q}\mathbf{v}\| \le |\!|\!|\mathbf{v}|\!|\!| \le \|\mathbf{G}\|^{\frac{1}{2}}\|\mathbf{Q}\mathbf{v}\|, \quad \|\mathbf{G}^\dagger\|^{-\frac{1}{2}}|\!|\!|\mathbf{v}|\!|\!| \le \|\mathbf{G}\mathbf{v}\| \le \|\mathbf{G}\|^{\frac{1}{2}}|\!|\!|\mathbf{v}|\!|\!|. \tag{3.1}$$

Then, the steepest descent scheme reads as follows:

Proposition 3.1. *Let* $\mathbf{w}$ *be an approximation for* $\mathbf{u}$ *with* $\mathbf{r} := \mathbf{f} - \mathbf{G}\mathbf{w} \neq 0$*. Then, with* $\kappa(\mathbf{G}) := \|\mathbf{G}\|\|\mathbf{G}^\dagger\|$*, for*

$$\tilde{\mathbf{w}} := \mathbf{w} + \frac{\langle \mathbf{r}, \mathbf{r} \rangle}{\langle \mathbf{G}\mathbf{r}, \mathbf{r} \rangle}\mathbf{r} \tag{3.2}$$

we have

$$\|\mathbf{u} - \tilde{\mathbf{w}}\| \leq \frac{\kappa(\mathbf{G}) - 1}{\kappa(\mathbf{G}) + 1} \|\mathbf{u} - \mathbf{w}\|.$$

The proof is a standard argument on the convergence of iterative descent methods. In the following, we will often use $\mathbf{r}$ as a shorthand notation for the residual $\mathbf{f} - \mathbf{G}\mathbf{w}$.

It is clear that (3.2) cannot be implemented directly since infinite sequences and biinfinite matrices are involved. Therefore the challenging task is to transform (3.2) into an implementable version. This will be done in the next section. One has to replace the infinite sequences by finite ones without destroying the overall convergence of the scheme. The basic tool for this is the following perturbation result.

Proposition 3.2. *For any* $\lambda \in (\frac{\kappa(\mathbf{G})-1}{\kappa(\mathbf{G})+1}, 1)$*, there exists a* $\delta = \delta(\lambda) > 0$ *small enough, such that if* $\|\tilde{\mathbf{r}} - \mathbf{r}\| \leq \delta\|\tilde{\mathbf{r}}\|$ *and* $\|\mathbf{z} - \mathbf{G}\tilde{\mathbf{r}}\| \leq \delta\|\tilde{\mathbf{r}}\|$*, then with*

$$\tilde{\mathbf{w}} := \mathbf{w} + \frac{\langle \tilde{\mathbf{r}}, \tilde{\mathbf{r}} \rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle}\tilde{\mathbf{r}},$$

we have

$$\|\mathbf{u} - \tilde{\mathbf{w}}\| \leq \lambda\|\mathbf{u} - \mathbf{w}\|,$$

and $|\frac{\langle \tilde{\mathbf{r}}, \tilde{\mathbf{r}} \rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle}| \lesssim 1$*. If, for some* $\eta > 0$*, in addition* $\|\tilde{\mathbf{r}} - \mathbf{r}\| \leq \eta$*, then* $\|(\mathbf{I} - \mathbf{Q})(\tilde{\mathbf{w}} - \mathbf{w})\| \leq C_3\eta$*, with some absolute constant* $C_3 > 0$*.*

Proof. Eq. (3.1) implies that $\langle \mathbf{G}\mathbf{r}, \mathbf{r} \rangle \eqsim \|\mathbf{r}\|^2$. The first step is to show that for a sufficiently small $\bar{\delta}$, and any $0 < \delta \leq \bar{\delta}$

$$\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle \eqsim \|\mathbf{r}\|^2 \qquad \text{and} \qquad \|\tilde{\mathbf{r}}\| \eqsim \|\mathbf{r}\| \tag{3.3}$$

holds. We have

$$\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle = \langle \mathbf{z} - \mathbf{G}\tilde{\mathbf{r}} + \mathbf{G}\tilde{\mathbf{r}}, \tilde{\mathbf{r}} \rangle \leq \|\mathbf{z} - \mathbf{G}\tilde{\mathbf{r}}\|\|\tilde{\mathbf{r}}\| + \|\mathbf{G}\tilde{\mathbf{r}}\|\|\tilde{\mathbf{r}}\| \leq (\delta + \|\mathbf{G}\|)\|\tilde{\mathbf{r}}\|^2$$

and

$$\|\tilde{\mathbf{r}}\|^2 \eqsim \langle \mathbf{G}\tilde{\mathbf{r}}, \tilde{\mathbf{r}} \rangle = \langle \mathbf{G}\tilde{\mathbf{r}} - \mathbf{z} + \mathbf{z}, \tilde{\mathbf{r}} \rangle \leq \|\mathbf{G}\tilde{\mathbf{r}} - \mathbf{z}\|\|\tilde{\mathbf{r}}\| + \langle \mathbf{z}, \tilde{\mathbf{r}} \rangle \leq \delta\|\tilde{\mathbf{r}}\|^2 + \langle \mathbf{z}, \tilde{\mathbf{r}} \rangle,$$

which implies the first equivalence in (3.3). The second one can be proved in a similar fashion. From (3.3), we infer that

$$\left| \frac{\langle \tilde{\mathbf{r}}, \tilde{\mathbf{r}} \rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle} - \frac{\langle \mathbf{r}, \mathbf{r} \rangle}{\langle \mathbf{G}\mathbf{r}, \mathbf{r} \rangle} \right| \lesssim \delta,$$

since

$$\begin{aligned}
\frac{\langle \tilde{\mathbf{r}}, \tilde{\mathbf{r}} \rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle} - \frac{\langle \mathbf{r}, \mathbf{r} \rangle}{\langle \mathbf{G}\mathbf{r}, \mathbf{r} \rangle} &= \frac{\langle \tilde{\mathbf{r}}, \tilde{\mathbf{r}} \rangle - \langle \mathbf{r}, \mathbf{r} \rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle} + \langle \mathbf{r}, \mathbf{r} \rangle \Big[\frac{1}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle} - \frac{1}{\langle \mathbf{G}\mathbf{r}, \mathbf{r} \rangle} \Big] \\
&= \frac{\langle \tilde{\mathbf{r}}, \tilde{\mathbf{r}} \rangle - \langle \mathbf{r}, \mathbf{r} \rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle} + \frac{\langle \mathbf{r}, \mathbf{r} \rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle \langle \mathbf{G}\mathbf{r}, \mathbf{r} \rangle} [\langle \mathbf{G}\mathbf{r}, \mathbf{r} \rangle - \langle \mathbf{z}, \tilde{\mathbf{r}} \rangle] \\
&= \frac{\langle 2(\tilde{\mathbf{r}} - \mathbf{r}), \mathbf{r} \rangle + \|\tilde{\mathbf{r}} - \mathbf{r}\|^2}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle} + \frac{\langle \mathbf{r}, \mathbf{r} \rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle \langle \mathbf{G}\mathbf{r}, \mathbf{r} \rangle} [\langle \mathbf{G}\mathbf{r}, \mathbf{r} - \tilde{\mathbf{r}} \rangle + \langle \mathbf{G}\tilde{\mathbf{r}} - \mathbf{z}, \tilde{\mathbf{r}} \rangle + \langle \mathbf{G}(\mathbf{r} - \tilde{\mathbf{r}}), \tilde{\mathbf{r}} \rangle].
\end{aligned}$$

Writing

$$\frac{\langle \tilde{\mathbf{r}}, \tilde{\mathbf{r}} \rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle}\tilde{\mathbf{r}} - \frac{\langle \mathbf{r}, \mathbf{r} \rangle}{\langle \mathbf{G}\mathbf{r}, \mathbf{r} \rangle}\mathbf{r} = \Big[\frac{\langle \tilde{\mathbf{r}}, \tilde{\mathbf{r}} \rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle} - \frac{\langle \mathbf{r}, \mathbf{r} \rangle}{\langle \mathbf{G}\mathbf{r}, \mathbf{r} \rangle}\Big]\mathbf{r} + \frac{\langle \tilde{\mathbf{r}}, \tilde{\mathbf{r}} \rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle}[\tilde{\mathbf{r}} - \mathbf{r}], \tag{3.4}$$

we find that $\|\frac{\langle \tilde{\mathbf{r}}, \tilde{\mathbf{r}} \rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle}\tilde{\mathbf{r}} - \frac{\langle \mathbf{r}, \mathbf{r} \rangle}{\langle \mathbf{G}\mathbf{r}, \mathbf{r} \rangle}\mathbf{r}\| \lesssim \delta\|\mathbf{r}\| \lesssim \delta|\!|\!|\mathbf{u} - \mathbf{w}|\!|\!|$, which, together with Proposition 3.1, completes the proof of the first statement.

From (3.4) and $(\mathbf{I} - \mathbf{Q})\mathbf{r} = 0$, we have $(\mathbf{I} - \mathbf{Q})(\tilde{\mathbf{w}} - \mathbf{w}) = \frac{\langle \tilde{\mathbf{r}}, \tilde{\mathbf{r}} \rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle}(\mathbf{I} - \mathbf{Q})(\tilde{\mathbf{r}} - \mathbf{r})$, which by $|\frac{\langle \tilde{\mathbf{r}}, \tilde{\mathbf{r}} \rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}} \rangle}| \lesssim 1$ and $\|\mathbf{I} - \mathbf{Q}\| \leq 1$ completes the proof of the second statement. $\square$

3.2 Numerical Realization

Obviously, the steepest descent scheme in Proposition 3.1 cannot be implemented since neither infinite sequences nor biinfinite matrices can be handled. Therefore the task is to replace the scheme (3.2) by an implementable one. The guideline given by Proposition 3.2 is to approximate the infinite expressions by finite ones within some sufficiently small relative tolerance.

In the sequel, we shall make the following basic assumptions. Let Σ_N denote the (non-linear) subspace of $\ell_2(\mathcal{N})$ consisting of all vectors with at most N nonzero coordinates. Given $\mathbf{v} \in \ell_2(\mathcal{N})$, we introduce the error of approximation

$$\sigma_N(\mathbf{v}) := \inf_{\mathbf{w} \in \Sigma_N} \|\mathbf{v} - \mathbf{w}\|. \tag{3.5}$$

Clearly this infimum is attained for $\mathbf{w}$ being a best N-term approximation for $\mathbf{v}$, i.e., a vector from Σ_N that agrees with $\mathbf{v}$ in those coordinates on which $\mathbf{v}$ takes its N largest values in modulus. Such a best N-term approximation for $\mathbf{v}$ will be denoted as $\mathbf{v}_N$. Note that it is not necessarily unique.

For some $s > 0$, we assume that

$$\sup_{N \in \mathbb{N}} N^s \sigma_N(\mathbf{u}) < \infty. \tag{3.6}$$

Eq. (3.6) describes how well the solution $\mathbf{u}$ to (2.13) can be approximated by the elements of Σ_N. Essentially, (3.6) is a regularity assumption on the exact solution u to (2.1). Indeed, in the wavelet basis case, it is well–known that the convergence order of best N–term approximation is determined by the maximum of the polynomial order and a specific Besov regularity of the object that we want to approximate [20]. For aggregated wavelet frames the same holds true, see [33]. Specifically, when H is a Sobolev space of order t over an n-dimensional domain, and the aggregated wavelet frame has order d, then $s = \frac{d-t}{n}$ if not limited by the Besov regularity. Fortunately, recent studies indicate that for the solution of elliptic operator equations this Besov regularity is quite large, see, e.g., [10, 11, 13], and, moreover, that in many cases it is much larger than the Sobolev regularity that governs the convergence rate of non-adaptive schemes.

The concept of best N–term approximation is closely related to the *weak ℓ_τ–spaces* $\ell_\tau^w(\mathcal{N})$. Given some $0 < \tau < 2$, $\ell_\tau^w(\mathcal{N})$ is defined as

$$\ell_\tau^w(\mathcal{N}) := \{\mathbf{c} \in \ell_2(\mathcal{N}) : \ |\mathbf{c}|_{\ell_\tau^w} := \sup_{n \in \mathbb{N}} n^{1/\tau}|\gamma_n(\mathbf{c})| < \infty\}, \tag{3.7}$$

where $\gamma_n(\mathbf{c})$ is the nth largest coefficient in modulus of $\mathbf{c}$. Then, for each $s > 0$,

$$\sup_N N^s \sigma_N(\mathbf{v}) \eqsim |\mathbf{v}|_{\ell^w_\tau}, \tag{3.8}$$

where here, and for the remainder of this paper, s and τ are always related according to

$$\tau = (\frac{1}{2} + s)^{-1}.$$

The expression $|\mathbf{v}|_{\ell^w_\tau}$ defines only a quasi-norm since it does not necessarily satisfy the triangle inequality. Yet, for each $0 < \tau < 2$, there exists a $C_1(\tau) > 0$ with

$$|\mathbf{v} + \mathbf{w}|_{\ell^w_\tau} \leq C_1(\tau) \left(|\mathbf{v}|_{\ell^w_\tau} + |\mathbf{w}|_{\ell^w_\tau}\right) \qquad (\mathbf{v}, \mathbf{w} \in \ell^w_\tau(\mathcal{N})). \tag{3.9}$$

We refer to [6, 20] for further details on the quasi–Banach spaces $\ell^w_\tau(\mathcal{N})$.

For some s^* larger than any s for which (3.6) can be expected (i.e., $s > \frac{d-t}{n}$), we assume the existence of the following three subroutines:

- **APPLY**$[\mathbf{w}, \varepsilon] \to \mathbf{z}_\varepsilon$. Determines for $\varepsilon > 0$ and a finitely supported $\mathbf{w}$, a finitely supported $\mathbf{z}_\varepsilon$ with
$$\|\mathbf{G}\mathbf{w} - \mathbf{z}_\varepsilon\| \leq \varepsilon. \tag{3.10}$$
Moreover, for any $s < s^*$, $\#\mathrm{supp}\,\mathbf{z}_\varepsilon \lesssim \varepsilon^{-1/s}|\mathbf{w}|^{1/s}_{\ell^w_\tau}$, where the number of arithmetic operations and storage locations used by this call is bounded by some absolute multiple of $\varepsilon^{-1/s}|\mathbf{w}|^{1/s}_{\ell^w_\tau} + \#\mathrm{supp}\,\mathbf{w} + 1$.

- **RHS**$[\varepsilon] \to \mathbf{f}_\varepsilon$. Determines for $\varepsilon > 0$, a finitely supported $\mathbf{f}_\varepsilon$ with $\|\mathbf{f} - \mathbf{f}_\varepsilon\| \leq \varepsilon$. Moreover, for any $s < s^*$, if $\mathbf{u} \in \ell^w_\tau(\mathcal{N})$, then $\#\mathrm{supp}\,\mathbf{f}_\varepsilon \lesssim \varepsilon^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau}$, where the number of arithmetic operations and storage locations used by the call is bounded by some absolute multiple of $\varepsilon^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau} + 1$.

- **COARSE**$[\mathbf{w}, \varepsilon] \to \mathbf{w}_\varepsilon$. Determines for a finitely supported $\mathbf{w}$, a finitely supported $\mathbf{w}_\varepsilon$, such that
$$\|\mathbf{w} - \mathbf{w}_\varepsilon\| \leq \varepsilon. \tag{3.11}$$
Moreover, $\#\mathrm{supp}\,\mathbf{w}_\varepsilon \lesssim \inf\{N : \sigma_N(\mathbf{w}) \leq \varepsilon\}$, and **COARSE** can be arranged to take a number of arithmetic operations and storage locations that is bounded by an absolute multiple of $\#\mathrm{supp}\,\mathbf{w} + \max\{\log(\varepsilon^{-1}\|\mathbf{w}\|), 1\}$.

Using that $\mathbf{G} : \ell_2(\mathcal{N}) \to \ell_2(\mathcal{N})$ is bounded, the properties of **APPLY** and **RHS** imply the following:

Proposition 3.3. *For any $s \in (0, s^*)$, $\mathbf{G} : \ell^w_\tau(\mathcal{N}) \to \ell^w_\tau(\mathcal{N})$ is bounded. For $\mathbf{z}_\varepsilon :=$ **APPLY**$[\mathbf{w}, \varepsilon]$ and $\mathbf{f}_\varepsilon :=$ **RHS**$[\varepsilon]$, we have $|\mathbf{z}_\varepsilon|_{\ell^w_\tau} \lesssim |\mathbf{w}|_{\ell^w_\tau}$ and $|\mathbf{f}_\varepsilon|_{\ell^w_\tau} \lesssim |\mathbf{u}|_{\ell^w_\tau}$, uniformly over $\varepsilon > 0$ and all finitely supported $\mathbf{w}$.*

Proof. Since the proof in [33] is incomplete, we include a proof here. We first show that for $s \in (0, s^*)$, $\mathbf{G} : \ell^w_\tau(\mathcal{N}) \to \ell^w_\tau(\mathcal{N})$ is bounded. Let $C > 0$ be a constant such that for

$\mathbf{z}_\varepsilon := \mathbf{APPLY}[\mathbf{w}, \varepsilon]$, $\operatorname{supp} \mathbf{z}_\varepsilon \leq C\varepsilon^{-1/s}|\mathbf{w}|_{\ell^w_\tau}^{1/s}$. Let $\mathbf{v} \in \ell^w_\tau(\mathcal{N})$ and $N \in \mathbb{N}$ be given. For $\bar{\varepsilon} := C^s|\mathbf{v}_N|_{\ell^w_\tau} N^{-s}$, let $\mathbf{z}_{\bar{\varepsilon}} := \mathbf{APPLY}[\mathbf{v}_N, \bar{\varepsilon}]$. Then, by (3.8),

$$\begin{aligned}\|\mathbf{G}\mathbf{v} - \mathbf{z}_{\bar{\varepsilon}}\| &\leq \|\mathbf{G}\mathbf{v}_N - \mathbf{z}_{\bar{\varepsilon}}\| + \|\mathbf{G}\|\|\mathbf{v} - \mathbf{v}_N\| \\ &\lesssim C^s|\mathbf{v}_N|_{\ell^w_\tau} N^{-s} + \|\mathbf{G}\| N^{-s}|\mathbf{v}|_{\ell^w_\tau} \lesssim N^{-s}|\mathbf{v}|_{\ell^w_\tau}.\end{aligned}$$

Since $\#\operatorname{supp} \mathbf{z}_\varepsilon \leq N$, from again (3.8) we infer that $|\mathbf{G}\mathbf{v}|_{\ell^w_\tau} \lesssim |\mathbf{v}|_{\ell^w_\tau}$.

By using that for any $\mathbf{v} \in \ell^w_\tau(\mathcal{N})$, and finitely supported $\mathbf{z}$, we have

$$|\mathbf{z}|_{\ell^w_\tau} \lesssim |\mathbf{v}|_{\ell^w_\tau} + (\#\operatorname{supp} \mathbf{z})^s\|\mathbf{v} - \mathbf{z}\| \tag{3.12}$$

[6, Lemma 4.11], for finitely supported $\mathbf{w}$, $\varepsilon > 0$, and with $\mathbf{z}_\varepsilon := \mathbf{APPLY}[\mathbf{w}, \varepsilon]$, we have $|\mathbf{z}_\varepsilon|_{\ell^w_\tau} \lesssim |\mathbf{G}\mathbf{w}|_{\ell^w_\tau} + (\#\operatorname{supp} \mathbf{z}_\varepsilon)^s\varepsilon \leq |\mathbf{G}\mathbf{w}|_{\ell^w_\tau} + C^s|\mathbf{w}|_{\ell^w_\tau} \lesssim |\mathbf{w}|_{\ell^w_\tau}$. Similarly, for $\mathbf{f}_\varepsilon := \mathbf{RHS}[\varepsilon]$, we have $|\mathbf{f}_\varepsilon|_{\ell^w_\tau} \lesssim |\mathbf{G}\mathbf{u}|_{\ell^w_\tau} + (\#\operatorname{supp} \mathbf{f}_\varepsilon)^s\varepsilon \leq |\mathbf{u}|_{\ell^w_\tau}$. □

Thanks to the properties of **COARSE** we have the following:

Proposition 3.4. *Let $\mu > 1$ and $s > 0$. Then for any $\varepsilon > 0$, $\mathbf{v} \in \ell^w_\tau(\mathcal{N})$, and finitely supported $\mathbf{w}$ with*

$$\|\mathbf{v} - \mathbf{w}\| \leq \varepsilon,$$

for $\overline{\mathbf{w}} := \mathbf{COARSE}[\mu\varepsilon, \mathbf{w}]$ it holds that

$$\#\operatorname{supp} \overline{\mathbf{w}} \lesssim \varepsilon^{-1/s}|\mathbf{v}|_{\ell^w_\tau}^{1/s},$$

obviously $\|\mathbf{v} - \overline{\mathbf{w}}\| \leq (1 + \mu)\varepsilon$, and

$$|\overline{\mathbf{w}}|_{\ell^w_\tau} \lesssim |\mathbf{v}|_{\ell^w_\tau}.$$

Proof. Let N be the smallest integer such that $\|\mathbf{v}_N - \mathbf{v}\| \leq (\mu - 1)\varepsilon$ for a best N–term approximation $\mathbf{v}_N$ of v. Then $\#\operatorname{supp} \mathbf{v}_N \lesssim \varepsilon^{-1/s}|\mathbf{v}|_{\ell^w_\tau}^{1/s}$. Furthermore $\|\mathbf{v}_N - \mathbf{w}\| \leq \|\mathbf{v}_N - \mathbf{v}\| + \|\mathbf{v} - \mathbf{w}\| \leq (\mu - 1 + 1)\varepsilon = \mu\varepsilon$, and so $\#\operatorname{supp} \overline{\mathbf{w}} \lesssim \#\operatorname{supp} \mathbf{v}_N$. The last statement follows from an application of (3.12). □

Let us briefly discuss the assumptions we made on **APPLY**, **RHS** and **COARSE**. The approximate matrix-vector product **APPLY** can be implemented in the way as introduced in [6, §6.4]. Then the question whether **APPLY** has the assumed properties reduces to the question how well $\mathbf{G}$ can be approximated by sparse matrices constructed by dropping small entries. This can be quantified by the concept of s^*-compressibility, meaning that if $\mathbf{G}$ is s^*-compressible, then **APPLY** has the assumed properties with that value of s^*. For the basis case, for both differential operators and singular integral operators, and for sufficiently smooth wavelets with sufficiently many vanishing moments in relation to their approximation order, in [35] it was shown that $\mathbf{G}$ is s^*-compressible with s^* larger than any $s = 1/\tau - 1/2$ for which $\mathbf{u} \in \ell^w_\tau(\mathcal{N})$ can be expected. This result extends to aggregated wavelet frames, see [33] for details.

Above, we silently assumed that the remaining entries from the sparse approximations for $\mathbf{G}$ are exactly available. Generally, however, these entries have to be approximated by numerical quadrature. For the basis case, in [24, 25] it was verified that these remaining entries can be approximated within a sufficiently small tolerance by quadrature rules that, on average over each row and column, take only $\mathcal{O}(1)$ operations per entry, showing that the "fully discrete" **APPLY** has the required properties. The development of suitable numerical quadrature is more complicated in the aggregated wavelet frame case, since in overlap regions, pairs of frame elements can be piecewise smooth with respect to uncorrelated partitions. Despite of this, in a forthcoming paper we will show that relatively easy implementable quadrature exists that realizes the above $\mathcal{O}(1)$ condition also in the general aggregated wavelet frame case. In the nice setting of the numerical examples in this paper, all entries of $\mathbf{G}$ are exactly available at unit cost, so that the question of numerical quadrature does not play a role.

Concerning **RHS**, for some $s < s^*$, let $\mathbf{u} \in \ell^w_\tau(\mathcal{N})$. Then Proposition 3.3 shows that $\mathbf{f} = \mathbf{G}\mathbf{u} \in \ell^w_\tau(\mathcal{N})$ with $|\mathbf{f}|_{\ell^w_\tau} \lesssim |\mathbf{u}|_{\ell^w_\tau}$. So (3.8) shows that indeed for any $\varepsilon > 0$, there exists an $\mathbf{f}_\varepsilon$ with $\|\mathbf{f} - \mathbf{f}_\varepsilon\| \leq \varepsilon$ and $\#\operatorname{supp} \mathbf{f}_\varepsilon \lesssim \varepsilon^{-1/s} |\mathbf{u}|^{1/s}_{\ell^w_\tau}$. The question how to construct such an $\mathbf{f}_\varepsilon$ in $\mathcal{O}(\varepsilon^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau} + 1)$ operations cannot be answered in general, and therefore depends on the right-hand side at hand.

Finally, a routine **COARSE** with the aforementioned properties can be based on binary binning, see [1, 33] for details.

We are going to solve $\mathbf{G}\mathbf{u} = \mathbf{f}$ with an approximate steepest descent method. Unless $\mathcal{F}$ is a basis, $\mathbf{G}$ has a non-trivial kernel, meaning that, as with any iterative method, a component of the error in a current approximation $\mathbf{w}$ that is in $\ker(\mathbf{G})$ will never be reduced in subsequent iterations. Although such components do not influence the resulting approximation $w := F^* D_{\mathcal{B}}^{-1}\mathbf{w}$ because $\ker(F^* D_{\mathcal{B}}^{-1}) = \ker(\mathbf{G})$, in principal they may cause an unbounded increase of $|\mathbf{w}|_{\ell^w_\tau}$ as the iteration proceeds, making the cost of calls of **APPLY** possibly uncontrollable. Under the assumption given below, we will nevertheless be able to control these cost, which allow us to show optimality of the method.

Assumption 3.5. For any $s \in (0, s^*)$, $\mathbf{Q}$ is bounded on $\ell^w_\tau(\mathcal{N})$.

On the one hand, it is a very difficult theoretical problem to prove that $\mathbf{Q}$ is bounded on $\ell^w_\tau(\mathcal{N})$ for all $s \in (0, s^*)$ for aggregated wavelet frames. On the other hand, this condition can be indirectly verified numerically as we will show in Section 4, by observing the optimal convergence of **SOLVE**. According to [33, Remark 3.13], the boundedness of $\mathbf{Q}$ on $\ell^w_\tau(\mathcal{N})$ for all $s \in (0, s^*)$ is (almost) a necessary requirement for the scheme to behave optimally. Moreover, not restricting our analysis to wavelet frames and to differential equations, there exist frames, for example time–frequency localized Gabor frames (and more generally all intrinsically polynomially localized frames [14, 21]), for which the boundedness of the corresponding $\mathbf{Q}$ has been proven rigorously, see [14, Theorem 7.1 in Section 7]. Therefore, for specific operator equations, optimality of **SOLVE** based on, e.g., Gabor frame discretizations is justified theoretically.

Remark 3.6. For cases in which Assumption 3.5 might not be valid, one can apply a modified algorithm that contains a recurrent inexact application of a projector to reduce components in $\ker(\mathbf{G})$, similar to the algorithm **modSOLVE** in [33] based on Richardson iteration. Although also for this algorithm optimal computational complexity can be shown, even with a simpler proof, we focus on the algorithm without this projector, since we expect it to have better quantitative properties.

Now we are in the position to formulate our inexact steepest descent scheme. The first step is to establish a routine that computes an approximate residual of the current approximation $\mathbf{w}$ for $\mathbf{u}$ within a sufficiently small tolerance ζ such that either, in view of Proposition 3.2, the relative error in this approximate residual is below some prescribed tolerance δ, or the residual itself, being a measure of the error in $\mathbf{w}$, is below some other prescribed tolerance ε. In view of controlling the components of the approximations in $\ker(\mathbf{G})$, the tolerance ζ should be in any way below some third input parameter ξ.

$\mathbf{RES}[\mathbf{w}, \xi, \delta, \varepsilon] \to [\tilde{\mathbf{r}}, \nu]$:

$\zeta := 2\xi$
`do` $\zeta := \zeta/2$
  $\tilde{\mathbf{r}} := \mathbf{RHS}[\zeta/2] - \mathbf{APPLY}[\mathbf{w}, \zeta/2]$
`until` $\nu := \|\tilde{\mathbf{r}}\| + \zeta \leq \varepsilon$ `or` $\zeta \leq \delta\|\tilde{\mathbf{r}}\|$

Theorem 3.7. *The routine* **RES** *has the following properties.*

i) $[\tilde{\mathbf{r}}, \nu] = \mathbf{RES}[\mathbf{w}, \xi, \delta, \varepsilon]$ *terminates with* $\nu \geq \|\mathbf{r}\|$, $\nu \gtrsim \min\{\xi, \varepsilon\}$ *and* $\|\mathbf{r} - \tilde{\mathbf{r}}\| \leq \xi$.

ii) If, for $s \leq \breve{s} < s^*$, *with, as always,* $\tau = (\frac{1}{2} + s)^{-1}$ *and* $\breve{\tau} = (\frac{1}{2} + \breve{s})^{-1}$, $\mathbf{u} \in \ell^w_\tau(\mathcal{N})$, *then*

$$\#\mathrm{supp}\,\tilde{\mathbf{r}} \lesssim \min\{\xi, \nu\}^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau} + \min\{\xi, \nu\}^{-1/\breve{s}}|\mathbf{w}|^{1/\breve{s}}_{\ell^w_{\breve{\tau}}}, \tag{3.13}$$

$$\min\{\xi, \nu\}^{(\breve{s}/s)-1}|\tilde{\mathbf{r}}|_{\ell^w_{\breve{\tau}}} \lesssim |\mathbf{u}|^{\breve{s}/s}_{\ell^w_\tau} + \min\{\xi, \nu\}^{(\breve{s}/s)-1}|\mathbf{w}|_{\ell^w_{\breve{\tau}}}, \tag{3.14}$$

and the number of arithmetic operations and storage locations required by the call is bounded by some absolute multiple of

$$\min\{\xi, \nu\}^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau} + \min\{\xi, \nu\}^{-1/\breve{s}}[|\mathbf{w}|^{1/\breve{s}}_{\ell^w_{\breve{\tau}}} + \xi^{1/\breve{s}}(\#\mathrm{supp}\,\mathbf{w} + 1)].$$

iii) In addition, if **RES** *terminates with* $\nu > \varepsilon$, *then* $\|\mathbf{r} - \tilde{\mathbf{r}}\| \leq \delta\|\tilde{\mathbf{r}}\|$, $\nu \leq (1 + \delta)\|\tilde{\mathbf{r}}\|$, *and* $\nu \leq \frac{1+\delta}{1-\delta}\|\mathbf{r}\|$.

Proof. Let us start by proving i). If at evaluation of the until-case, $\zeta > \delta\|\tilde{\mathbf{r}}\|$, then $\|\tilde{\mathbf{r}}\| + \zeta < (\delta^{-1} + 1)\zeta$. Since ζ is halved in each iteration, we infer that, if not by $\zeta \leq \delta\|\tilde{\mathbf{r}}\|$, **RES** will terminate by $\|\tilde{\mathbf{r}}\| + \zeta \leq \varepsilon$.

Since after any evaluation of $\tilde{\mathbf{r}}$ inside the algorithm, $\|\tilde{\mathbf{r}} - \mathbf{r}\| \leq \zeta$, any value of ν determined inside the algorithm is an upper bound on $\|\mathbf{r}\|$.

If the do–loop terminates in the first iteration, then $\nu \geq \xi$. In the other case, let $\tilde{\mathbf{r}}^{\mathrm{old}} := \mathbf{RHS}[\zeta] - \mathbf{APPLY}[\mathbf{w}, \zeta]$. We have $\|\tilde{\mathbf{r}}^{\mathrm{old}}\| + 2\zeta > \varepsilon$ and $2\zeta > \delta\|\tilde{\mathbf{r}}^{\mathrm{old}}\|$, so that

$$\nu \geq \zeta > (2\delta^{-1} + 2)^{-1}(\|\tilde{\mathbf{r}}^{\mathrm{old}}\| + 2\zeta) > \frac{\delta\varepsilon}{2 + 2\delta},$$

and i) is shown.

The next step is to establish part ii). For any finitely supported $\mathbf{v}$, we have

$$|\mathbf{v}|_{\ell^w_{\breve{\tau}}} \leq (\#\mathrm{supp}\,\mathbf{v})^{\breve{s}-s}|\mathbf{v}|_{\ell^w_\tau}. \tag{3.15}$$

So for $\mathbf{g} := \mathbf{RHS}[\zeta]$, from $\#\mathrm{supp}\,\mathbf{g} \lesssim \zeta^{-1/s}|\mathbf{u}|_{\ell_\tau^w}^{1/s}$ and $|\mathbf{g}|_{\ell_\tau^w} \lesssim |\mathbf{u}|_{\ell_\tau^w}$, we have $\zeta^{(\breve{s}/s)-1}|\mathbf{g}|_{\ell_{\breve{\tau}}^w} \lesssim |\mathbf{u}|_{\ell_\tau^w}^{\breve{s}/s}$. With ζ, $\tilde{\mathbf{r}}$, and ν having their values at termination, the properties of **APPLY**, cf. Proposition 3.3, now show that

$$\#\mathrm{supp}\,\tilde{\mathbf{r}} \lesssim \zeta^{-1/s}|\mathbf{u}|_{\ell_\tau^w}^{1/s} + \zeta^{-1/\breve{s}}|\mathbf{w}|_{\ell_{\breve{\tau}}^w}^{1/\breve{s}},$$

and

$$\zeta^{(\breve{s}/s)-1}|\tilde{\mathbf{r}}|_{\ell_{\breve{\tau}}^w} \lesssim |\mathbf{u}|_{\ell_\tau^w}^{\breve{s}/s} + \zeta^{(\breve{s}/s)-1}|\mathbf{w}|_{\ell_{\breve{\tau}}^w}.$$

Therefore, (3.13) and (3.14) follow from these expressions once we have shown that $\zeta \gtrsim \min\{\xi, \nu\}$. When the do–loop terminates in the first iteration, we have $\zeta \gtrsim \xi$. In the other case, with $\tilde{\mathbf{r}}^{\mathrm{old}}$ as above, we have $\delta\|\tilde{\mathbf{r}}^{\mathrm{old}}\| < 2\zeta$, and so from $\|\tilde{\mathbf{r}} - \tilde{\mathbf{r}}^{\mathrm{old}}\| \leq \zeta + 2\zeta$, we infer $\|\tilde{\mathbf{r}}\| \leq \|\tilde{\mathbf{r}}^{\mathrm{old}}\| + 3\zeta < (2\delta^{-1} + 3)\zeta$, so that $\nu < (2\delta^{-1} + 4)\zeta$.

To complete the proof of ii), it remains to estimate the number or arithmetic operations. Again the properties of **APPLY** and that of **RHS** together with the geometric decrease of ζ inside the algorithm, imply that the total cost can be bounded by some multiple of $\zeta^{-1/s}|\mathbf{u}|_{\ell_\tau^w}^{1/s} + \zeta^{-1/\breve{s}}|\mathbf{w}|_{\ell_{\breve{\tau}}^w}^{1/\breve{s}} + K(\#\mathrm{supp}\,\mathbf{w} + 1)$, with K being the number of calls of **APPLY** that were made. Taking into account its initial value, and the geometric decrease of ζ inside the algorithm, we have $K(\#\mathrm{supp}\,\mathbf{w}+1) = K\xi^{-1/\breve{s}}\xi^{1/\breve{s}}(\#\mathrm{supp}\,\mathbf{w}+1) \lesssim \zeta^{-1/\breve{s}}\xi^{1/\breve{s}}(\#\mathrm{supp}\,\mathbf{w}+1)$. Since we have already shown that $\zeta \gtrsim \min\{\xi, \nu\}$, this finishes the proof of ii).

Finally, let us check iii). Suppose that **RES** terminates with $\nu > \varepsilon$, and thus with $\zeta \leq \delta\|\tilde{\mathbf{r}}\|$. Then obviously $\|\mathbf{r} - \tilde{\mathbf{r}}\| \leq \delta\|\tilde{\mathbf{r}}\|$.

From $\|\tilde{\mathbf{r}}\| \leq \|\mathbf{r} - \tilde{\mathbf{r}}\| + \|\mathbf{r}\| \leq \delta\|\tilde{\mathbf{r}}\| + \|\mathbf{r}\|$, we have $\|\tilde{\mathbf{r}}\| \leq \frac{\|\mathbf{r}\|}{1-\delta}$, and so we arrive at $\nu = \|\tilde{\mathbf{r}}\| + \zeta \leq (1+\delta)\|\tilde{\mathbf{r}}\| \leq \frac{1+\delta}{1-\delta}\|\mathbf{r}\|$. □

The routine **RES** is the basic building block for our fundamental algorithm which reads as follows.

Algorithm 1. SOLVE$[\omega, \varepsilon] \to \mathbf{w}$*:*
% Input should satisfy $\omega \geq \|\mathbf{Q}\mathbf{u}\|$.
% Let λ *and* $\delta = \delta(\lambda)$ *be constants as in Proposition 3.2.*
% Fix some constants $\mu > 1$, $\beta \in (0,1)$.
% Let K, M *be the smallest integers with* $\beta^K\omega \leq \varepsilon$, $\lambda^M \leq \frac{1-\delta}{1+\delta}\frac{\beta}{(1+3\mu)\kappa(\mathbf{G})}$, *respectively.*

$\mathbf{w}_0 := 0$; $\omega_0 := \omega$
`for` $i := 1$ `to` K `do`
  $\bar{\mathbf{w}}_i := \mathbf{w}_{i-1}$; $\omega_i := \beta\omega_{i-1}$; $\xi_i := \frac{\omega_i}{(1+3\mu)C_3 M}$ *%* C_3 *from Proposition 3.2*
  `while` *with* $[\tilde{\mathbf{r}}_i, \nu_i] := \mathbf{RES}[\bar{\mathbf{w}}_i, \xi_i, \delta, \frac{\omega_i}{(1+3\mu)\|\mathbf{G}^\dagger\|}]$, $\nu_i > \frac{\omega_i}{(1+3\mu)\|\mathbf{G}^\dagger\|}$ `do`
    $\mathbf{z}_i := \mathbf{APPLY}[\tilde{\mathbf{r}}_i, \delta\|\tilde{\mathbf{r}}_i\|]$
    $\bar{\mathbf{w}}_i := \bar{\mathbf{w}}_i + \frac{\langle\tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_i\rangle}{\langle\mathbf{z}_i, \tilde{\mathbf{r}}_i\rangle}\tilde{\mathbf{r}}_i$
  `enddo`
  $\mathbf{w}_i := \mathbf{COARSE}[\bar{\mathbf{w}}_i, \frac{3\mu\omega_i}{1+3\mu}]$
`endfor`

It turns out that Algorithm 1 indeed converges with the optimal order. This is confirmed by the following theorem which is the main result of this paper.

Theorem 3.8. *i) If $\omega \geq \|\mathbf{Q}\mathbf{u}\|$, then $\mathbf{w} := \mathbf{SOLVE}[\omega, \varepsilon]$ terminates with $\|\mathbf{Q}(\mathbf{u} - \mathbf{w})\| \leq \varepsilon$.*

ii) For any $\eta \in (0, s^*)$, let $\breve{s} = s^* - \frac{\eta}{2}$, $\breve{\tau} = (\frac{1}{2} + \breve{s})^{-1}$, and let the constant β inside **SOLVE** *satisfy*

$$\beta < \min\{1, [C_1(\breve{\tau})C_2(\breve{\tau})|\mathbf{I} - \mathbf{Q}|_{\ell^w_{\breve{\tau}} \to \ell^w_{\breve{\tau}}}]^{2(s^*-\eta)/\eta}\}.$$

Then if for some $s \in (0, s^* - \eta]$, $\mathbf{u} \in \ell^w_\tau(\mathcal{N})$, then $\#\mathrm{supp}\,\mathbf{w} \lesssim \varepsilon^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau}$ and, when $\varepsilon \lesssim \omega \lesssim \|\mathbf{u}\|$, the number of arithmetic operations and storage locations required by the call is bounded by some absolute multiple of the same expression.

Proof. The first step is to prove i). Let us consider the ith iteration of the for-loop. Assume that

$$\|\mathbf{Q}(\mathbf{u} - \mathbf{w}_{i-1})\| \leq \omega_{i-1}, \tag{3.16}$$

which holds by assumption for $i = 1$. The inner loop terminates after not more than $M + 1$ calls of **RES**. Indeed, suppose that this is not the case, then the first $M + 1$ calls of **RES** do not terminate because the first condition in the until-clause is satisfied, and so Theorem 3.7 iii), Proposition 3.2, (3.1) and assumption (3.16) show that the $(M + 1)$th call outputs a ν_i with

$$\begin{aligned}
\nu_i &\leq \frac{1+\delta}{1-\delta}\|\mathbf{f} - \mathbf{G}\bar{\mathbf{w}}_i\| = \frac{1+\delta}{1-\delta}\|\mathbf{G}(\mathbf{u} - \bar{\mathbf{w}}_i)\| \leq \frac{1+\delta}{1-\delta}\|\mathbf{G}\|^{\frac{1}{2}}|\!|\!|\mathbf{u} - \bar{\mathbf{w}}_i|\!|\!| \\
&\leq \frac{1+\delta}{1-\delta}\|\mathbf{G}\|^{\frac{1}{2}}\lambda^M|\!|\!|\mathbf{u} - \mathbf{w}_{i-1}|\!|\!| \leq \frac{1+\delta}{1-\delta}\|\mathbf{G}\|^{\frac{1}{2}}\lambda^M\|\mathbf{G}\|^{\frac{1}{2}}\|\mathbf{Q}(\mathbf{u} - \mathbf{w}_{i-1})\| \\
&\leq \frac{\omega_i}{(1+3\mu)\|\mathbf{G}^\dagger\|}
\end{aligned}$$

by definition of M, which gives a contradiction.

With $\hat{\mathbf{w}}_i$ denoting $\bar{\mathbf{w}}_i$ at termination of the inner loop, we have by (3.1) and the properties of **RES**

$$\|\mathbf{Q}(\mathbf{u} - \hat{\mathbf{w}}_i)\| \leq \|\mathbf{G}^\dagger\|^{\frac{1}{2}}|\!|\!|\mathbf{u} - \hat{\mathbf{w}}_i|\!|\!| \leq \|\mathbf{G}^\dagger\|\|\mathbf{G}(\mathbf{u} - \hat{\mathbf{w}}_i)\| \leq \|\mathbf{G}^\dagger\|\nu_i \leq \frac{\omega_i}{1+3\mu}, \tag{3.17}$$

so that, by the properties of **COARSE**,

$$\|\mathbf{Q}(\mathbf{u} - \mathbf{w}_i)\| \leq \tfrac{\omega_i}{1+3\mu} + \tfrac{3\mu\omega_i}{1+3\mu} = \omega_i,$$

showing convergence, and by definition of K completes the proof of the first statement.

The proof of ii) follows the lines of the proof of [33, Theorem 3.12]. In our case where $\mathbf{G}$ has possibly a non-trivial kernel, generally, due to the errors in $\mathrm{ran}(\mathbf{I} - \mathbf{Q})$, we have no convergence of $\hat{\mathbf{w}}_i$ to $\mathbf{u}$ for $i \to \infty$, and as a consequence, we are not able to bound $|\mathbf{w}_i|_{\ell^w_\tau}$ by some absolute multiple of $|\mathbf{u}|_{\ell^w_\tau}$. Instead we prove a weaker result (3.21), that, however, suffices to conclude optimal computational complexity. By part i) of Theorem 3.7, Proposition 3.2 and the definition of the ξ_i,

$$\|(\mathbf{I} - \mathbf{Q})(\hat{\mathbf{w}}_i - \mathbf{w}_{i-1})\| \leq C_3 M\xi_i = \tfrac{\omega_i}{1+3\mu}. \tag{3.18}$$

Since $\mathbf{Q}$ is bounded on ℓ_2, and by Assumption 3.5, it is bounded on $\ell^w_{\breve{\tau}}$, an interpolation argument, cf. [20, (4.24)], shows that it is bounded on ℓ^w_τ, uniformly in $\tau \in [\breve{\tau}, 2]$. Let N_i be the smallest integer such that

$$\|\mathbf{Qu} - (\mathbf{Qu})_{N_i}\| \leq \tfrac{\omega_i}{1+3\mu}, \tag{3.19}$$

where $(\mathbf{Qu})_N$ denotes the best N-term approximation for $\mathbf{Qu}$. Then, using the assumption $\mathbf{u} \in \ell^w_\tau(\mathcal{N})$, (3.8) shows that

$$N_i \lesssim \omega_i^{-1/s} |\mathbf{Qu}|^{1/s}_{\ell^w_\tau} \lesssim \omega_i^{-1/s} |\mathbf{u}|^{1/s}_{\ell^w_\tau},$$

and so, using (3.15),

$$\omega_i^{(\breve{s}/s)-1} |(\mathbf{Qu})_{N_i}|_{\ell^w_{\breve{\tau}}} \lesssim |\mathbf{u}|^{(\breve{s}/s)-1}_{\ell^w_\tau} |(\mathbf{Qu})_{N_i}|_{\ell^w_\tau} \lesssim |\mathbf{u}|^{(\breve{s}/s)-1}_{\ell^w_\tau} |\mathbf{Qu}|_{\ell^w_\tau} \lesssim |\mathbf{u}|^{\breve{s}/s}_{\ell^w_\tau}. \tag{3.20}$$

From (3.17), (3.18) and (3.19), we get

$$\|(\mathbf{Qu})_{N_i} + (\mathbf{I} - \mathbf{Q})\mathbf{w}_{i-1} - \hat{\mathbf{w}}_i\| \leq \tfrac{3\omega_i}{1+3\mu}.$$

From Proposition 3.4, with $\mathbf{v}$ reading as $(\mathbf{Qu})_{N_i} + (\mathbf{I} - \mathbf{Q})\mathbf{w}_{i-1}$ and by using that $\mu > 1$, it follows that $\mathbf{w}_i := \mathbf{COARSE}[\hat{\mathbf{w}}_i, \frac{3\mu\omega_i}{1+3\mu}]$ satisfies

$$\begin{aligned}
|\mathbf{w}_i|_{\ell^w_{\breve{\tau}}} &\leq C_2(\breve{\tau}) |(\mathbf{Qu})_{N_i} + (\mathbf{I} - \mathbf{Q})\mathbf{w}_{i-1}|_{\ell^w_{\breve{\tau}}} \\
&\leq C_1(\breve{\tau}) C_2(\breve{\tau}) |(\mathbf{Qu})_{N_i}|_{\ell^w_{\breve{\tau}}} + C_1(\breve{\tau}) C_2(\breve{\tau}) |(\mathbf{I} - \mathbf{Q})|_{\ell^w_{\breve{\tau}} \leftarrow \ell^w_{\breve{\tau}}} |\mathbf{w}_{i-1}|_{\ell^w_{\breve{\tau}}}
\end{aligned}$$

by (3.9), and so by (3.20),

$$\omega_i^{(\breve{s}/s)-1} |\mathbf{w}_i|_{\ell^w_{\breve{\tau}}} \leq C |\mathbf{u}|^{\breve{s}/s}_{\ell^w_\tau} + C_1(\breve{\tau}) C_2(\breve{\tau}) |(\mathbf{I} - \mathbf{Q})|_{\ell^w_{\breve{\tau}} \leftarrow \ell^w_{\breve{\tau}}} \beta^{(\breve{s}/s)-1} \omega_{i-1}^{(\breve{s}/s)-1} |\mathbf{w}_{i-1}|_{\ell^w_{\breve{\tau}}}$$

for some absolute constant $C > 0$. The assumption on β made in the theorem shows that

$$C_1(\breve{\tau}) C_2(\breve{\tau}) |(\mathbf{I} - \mathbf{Q})|_{\ell^w_{\breve{\tau}} \leftarrow \ell^w_{\breve{\tau}}} \beta^{(\breve{s}/s)-1} < 1,$$

from which we conclude by a geometric series argument that

$$\omega_i^{(\breve{s}/s)-1} |\mathbf{w}_i|_{\ell^w_{\breve{\tau}}} \lesssim |\mathbf{u}|^{\breve{s}/s}_{\ell^w_\tau}, \tag{3.21}$$

which, as we emphasize here, holds uniformly in i. Moreover, knowing this, Proposition 3.4 and (3.20) show that

$$\begin{aligned}
\#\operatorname{supp} \mathbf{w}_i &\lesssim \omega_i^{-1/\breve{s}} |(\mathbf{Qu})_{N_i} + (\mathbf{I} - \mathbf{Q})\mathbf{w}_{i-1}|^{1/\breve{s}}_{\ell^w_{\breve{\tau}}} \\
&\lesssim \omega_i^{-1/s} \big(\omega_i^{(\breve{s}/s)-1} \big[|(\mathbf{Qu})_{N_i}|_{\ell^w_{\breve{\tau}}} + |\mathbf{I} - \mathbf{Q}|_{\ell^w_{\breve{\tau}} \to \ell^w_{\breve{\tau}}} |\mathbf{w}_{i-1}|_{\ell^w_{\breve{\tau}}}\big]\big)^{1/\breve{s}} \\
&\lesssim \omega_i^{-1/s} |\mathbf{u}|^{1/s}_{\ell^w_\tau},
\end{aligned} \tag{3.22}$$

again uniformly in i.

For any computed ν_i in the inner loop, Theorem 3.7 i) shows that $\frac{\omega_i}{(1+3\mu)\|\mathbf{G}^\dagger\|} \lesssim \nu_i$. At termination of the inner loop we have $\nu_i \lesssim \omega_i$, whereas for any evaluation of **RES** that does not lead to termination, Theorem 3.7 iii) and Proposition 3.2 show that

$$\nu_i \leq \frac{1+\delta}{1-\delta}\|\mathbf{f} - \mathbf{G}\bar{\mathbf{w}}_i\| \lesssim |\!|\!|\mathbf{u} - \bar{\mathbf{w}}_i|\!|\!| \leq |\!|\!|\mathbf{u} - \mathbf{w}_{i-1}|\!|\!| \lesssim \omega_{i-1}.$$

We conclude that

$$\nu_i \eqsim \omega_i,$$

uniformly in i and over all computations of ν_i in the inner loop.

Inside the body of the inner loop, we have that the tolerance for the call of **APPLY** satisfies $\delta\|\tilde{\mathbf{r}}_i\| \geq \frac{\delta\nu_i}{1+\delta}$ by Theorem 3.7 iii) and, by Proposition 3.2, that $|\frac{\langle \tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_i\rangle}{\langle \mathbf{z}_i, \tilde{\mathbf{r}}_i\rangle}| \lesssim 1$. By (3.21) and the fact that the number of iterations of the inner loop is uniformly bounded, Theorem 3.7 ii) shows that

$$\omega_i^{(\breve{s}/s)-1}|\tilde{\mathbf{r}}_i|_{\ell^w_{\breve{\tau}}} \lesssim |\mathbf{u}|_{\ell^w_\tau}^{\breve{s}/s}, \quad \omega_i^{(\breve{s}/s)-1}|\bar{\mathbf{w}}_i|_{\ell^w_{\breve{\tau}}} \lesssim |\mathbf{u}|_{\ell^w_\tau}^{\breve{s}/s}.$$

With this result and (3.22), Theorem 3.7 ii) and the properties of **APPLY** (with s reading as $\breve{s}$) show that

$$\#\mathrm{supp}\, \tilde{\mathbf{r}}_i \lesssim \omega_i^{-1/s}|\mathbf{u}|_{\ell^w_\tau}^{1/s}, \quad \#\mathrm{supp}\, \mathbf{z}_i \lesssim \omega_i^{-1/s}|\mathbf{u}|_{\ell^w_\tau}^{1/s}, \quad \#\mathrm{supp}\, \bar{\mathbf{w}}_i \lesssim \omega_i^{-1/s}|\mathbf{u}|_{\ell^w_\tau}^{1/s}.$$

By using these results concerning the lengths of the supports and the $\ell^w_{\breve{\tau}}$-norms, again Theorem 3.7 ii) and the properties of **APPLY** and **COARSE** show that the number of arithmetic operations and storage locations required for the computation of $\mathbf{w}_i$ starting from $\mathbf{w}_{i-1}$ is bounded by an absolute multiple of $\omega_i^{-1/s}|\mathbf{u}|_{\ell^w_\tau}^{1/s} + \max\{\log(\omega_i^{-1}\|\hat{\mathbf{w}}_i\|), 1\}$. From $\log(\omega_i^{-1}\|\hat{\mathbf{w}}_i\|) \lesssim \omega_i^{-1/\breve{s}}\|\hat{\mathbf{w}}_i\|^{1/\breve{s}} \lesssim \omega_i^{-1/\breve{s}}|\hat{\mathbf{w}}_i|_{\ell^w_{\breve{\tau}}}^{1/\breve{s}} \lesssim \omega_i^{-1/s}|\mathbf{u}|_{\ell^w_\tau}^{1/s}$, as well as $1 \lesssim \omega^{-1/s}\|\mathbf{u}\|^{1/s} \lesssim \omega_i^{-1/s}|\mathbf{u}|_{\ell^w_\tau}^{1/s}$ by assumption, the geometric decrease of the ω_i, and $\omega_K \gtrsim \varepsilon$, which, in case $K = 0$, is an assumption, the proof is completed. $\square$

4 Numerical Experiments

After the construction of a convergent and asymptotically optimal steepest descent algorithm, we now investigate the practical applicability of the scheme. Moreover we want to compare it with the adaptive scheme based on the damped Richardson iteration. The version of the latter scheme appearing in [14, 33] has been proven to converge and, under Assumption 3.5, to be also asymptotically optimal. Unfortunately its concrete implementation has shown to be rather inefficient and therefore not well suited for comparison. For this reason, we will compare the results of our adaptive frame algorithm **SOLVE** with those obtained with the Richardson iteration based method from [7]. This scheme can be shown to converge also in the case of a wavelet frame discretization, but a proof of its optimality has not been achieved yet. Nevertheless we will see that it is in fact optimal in practice. Again the routines **RHS**, **APPLY**, and **COARSE** are the basic building blocks for its implementation which reads as follows.

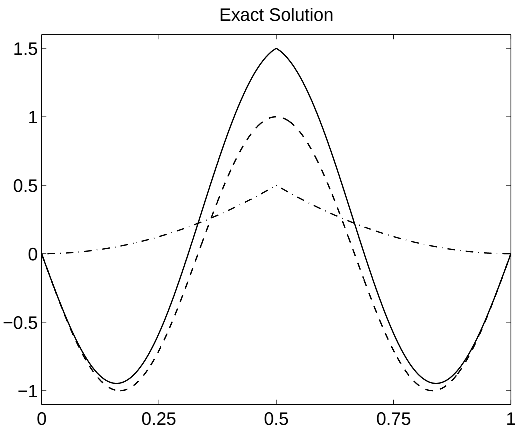


Figure 1: Exact solution (solid line) for the one–dimensional example being the sum of the dashed and dash–dotted functions.

Algorithm 2. $\mathbf{CDD2SOLVE}[\eta, \varepsilon] \to \mathbf{w}$*:*
% Input should satisfy $\eta \geq \|\mathbf{Qu}\|$*.*
% Define the parameters $\alpha_{opt} := \frac{2}{\|\mathbf{G}\|+\|\mathbf{G}^\dagger\|^{-1}}$ *and* $\rho := \frac{\kappa(\mathbf{G})-1}{\kappa(\mathbf{G})+1}$*.*
% Let θ *and* K *be constants with* $2\rho^K < \theta < 1/2$*.*

$\mathbf{w} := 0$;
`while` $\eta > \varepsilon$ `do`
  `for` $j := 1$ `to` K `do`
    $\mathbf{w} := \mathbf{w} + \alpha_{opt}\left(\mathbf{RHS}[\frac{\rho^j\eta}{2\alpha K}] - \mathbf{APPLY}[\mathbf{w}, \frac{\rho^j\eta}{2\alpha K}]\right)$;
  `endfor`
  $\eta := 2\rho^K\eta/\theta$;
  $\mathbf{w} := \mathbf{COARSE}[\mathbf{w}, (1-\theta)\eta]$;
`enddo`

For the discretization we use aggregated wavelet frames on suitable overlapping domain decompositions, as the union of local wavelet bases lifted to the subdomains. As such local bases we use piecewise linear wavelets with complementary boundary conditions from [16], with order of polynomial exactness $d = 2$ and with $\tilde{d} = 2$ vanishing moments. In particular, we impose here homogenous boundary conditions on the primal wavelets and free boundary conditions on the duals. We will test the algorithms both on 1D and 2D Poisson problems.

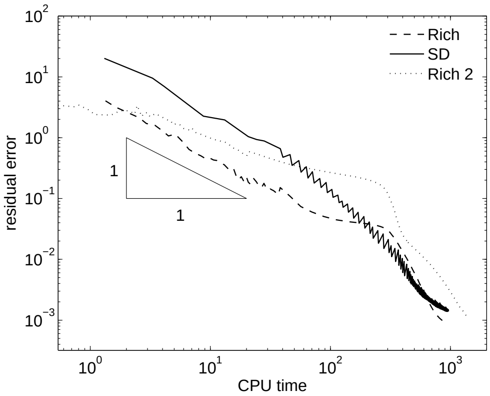
4.1 Poisson Equation on the Interval

We consider the variational formulation of the following problem of order $2t = 2$ on the interval $\Omega = (0, 1)$, i.e., $n = 1$, with homogenous boundary conditions

$$-u'' = f \quad \text{on } \Omega, \quad u(0) = u(1) = 0. \tag{4.1}$$

The right-hand side f is given as the functional defined by $f(v) := 4v(\frac{1}{2}) + \int_0^1 g(x)v(x)dx$, where

$$g(x) = -9\pi^2 \sin(3\pi x) - 4.$$



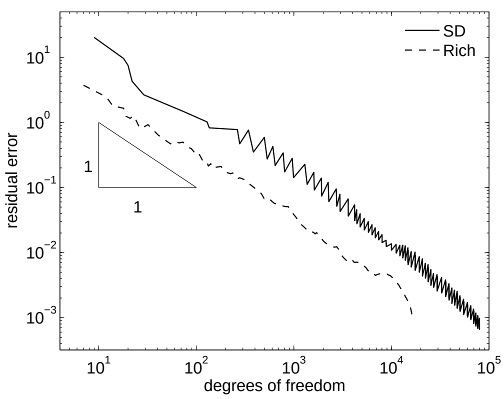


Figure 2: *Left:* Convergence histories of **SOLVE** and **CDD2SOLVE** with respect to CPU time. Two tests for **CDD2SOLVE** with different fixed damping parameters are shown. *Right:* Convergence histories with respect to the support size of the iterands.

The solution is consequently given by

$$u(x) = -\sin(3\pi x) + \begin{cases} 2x^2 & , \ x \in [0, \frac{1}{2}) \\ 2(1-x)^2, & x \in [\frac{1}{2}, 1] \end{cases} ,$$

see Figure 1. As an overlapping domain decomposition we choose $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = (0, 0.7)$ and $\Omega_2 = (0.3, 1)$. Associated to this decomposition we construct our aggregrated wavelet frames just as the union of the local bases. It is shown in [14, 33] that such a system is a (Gelfand) frame for $H_0^t(\Omega)$ and that it can provide a suitable characterization of Besov spaces in terms of wavelet coefficients. On the one hand, the solution u is contained in $H_0^{s+1}(\Omega)$ only for $s < \frac{1}{2}$. This means that linear methods can only converge with limited order. On the other hand, it can be shown that $u \in B_\tau^s(L_\tau(\Omega))$ for any $s > 0$, $1/\tau = s - 1/2$, so that the wavelet frame coefficients $\mathbf{u}$ associated with u define a sequence in ℓ_τ^w for any $s < \frac{d-t}{n}$, see [20, 33]. This ensures that the choice of wavelets with suitable order d can allow for any order of convergence in adaptive schemes like that presented in this paper, in the sense that the error is $O(N^{-s})$ where N is the number of unknowns. Due to our choice of piecewise linear wavelets with order $d = 2$, the optimal rate of convergence is bound to be $s = \frac{d-t}{n} = 1$. We will show that the numerical experiments confirm this expected rate.

We have tested the adaptive wavelet algorithms **CDD2SOLVE** with parameters $\alpha_{opt} \approx 0.52$, $\theta = 2/7$, $K = 83$, and with initial $\eta = 64.8861$, and **SOLVE** with parameters $\delta = 1$, $\mu = 1.0001$, $\beta = 0.9$, $M = C_3 = 1$, $K = 134$, $\omega_0 = 64.8861$. The parameters M, C_3 have been chosen in such a way to produce an optimal response of the numerical results. The numerical results in Figure 2 illustrate the optimal computational complexity of both schemes. In particular, we show that for a suboptimal choice of the damping parameter ($\alpha^* = 0.2 \leq \alpha_{opt} \approx 0.52$ in this specific test) **SOLVE** outperforms **CDD2SOLVE**. In practice, the wrong guess of the damping parameter can even spoil convergence and/or optimality.

In order to speed up computational time, we have implemented a caching strategy for the entries of the stiffness matrix involved. Due to limited memory resources one is then

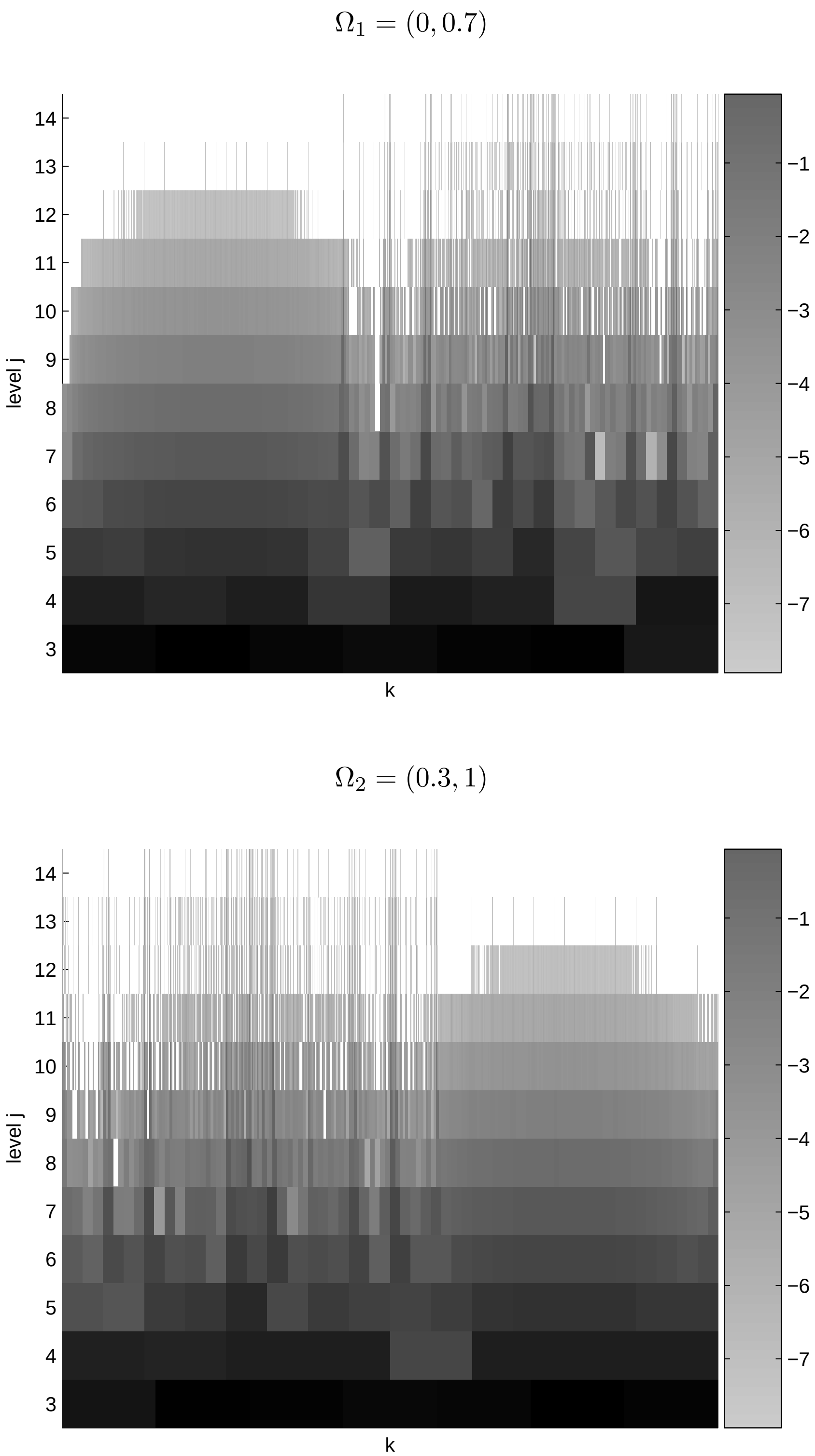


Figure 3: Distribution of active wavelet frame elements in Ω_1 and Ω_2.

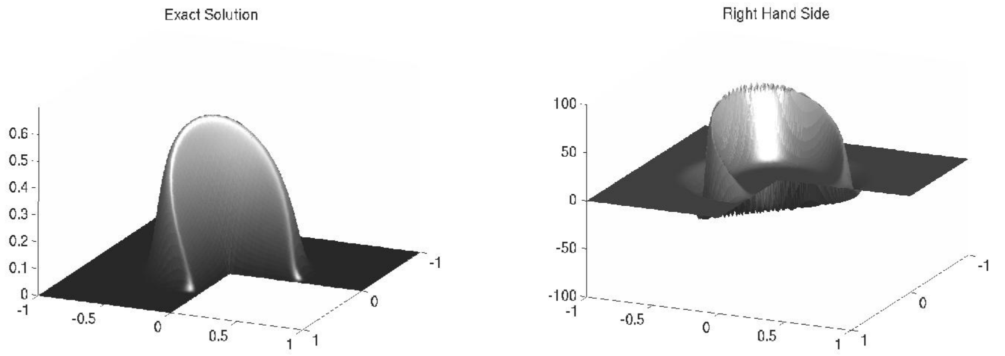


Figure 4: Exact solution (left) and right–hand side for the two–dimensional Poisson equation in an L–shaped domain.

forced to fix in advance a certain maximal number of frame elements that can be taken into account during the iteration process, which means that we are solving a truncated problem. Thus, for the computation of the residuals, only wavelets up to a fixed scale are used. For small accuracies, the actually computed residuals may then deviate from the true ones. This effect shows up in the CPU time histories displayed in Figure 2 for the area of small error tolerances. However, we observe that the influence of the truncation is almost negligible in the 1D–case, since here the finest refinement level can be chosen very high.

Finally, Figure 3 illustrates the distribution of the active wavelet frame elements used by the steepest descent scheme, each of them corresponding to a colored rectangle. The two overlapping subintervals are treated separately. For both patches one observes that the adaptive scheme detects the singularity of the solution. The chosen frame elements arrange in a tree–like structure with large coefficients around the singularity, while on the smooth part the coefficients are equally distributed, and along a fixed level they are here of similar size.

4.2 Poisson Equation on the L-shaped Domain

We consider the model problem of the variational formulation of Poisson's equation in two spatial dimensions:

$$-\Delta u = f \text{ in } \Omega, \quad u_{|\Omega} = 0. \tag{4.2}$$

The problem will be chosen in such a way that the application of *adaptive* algorithms pays off most, as it is the case for domains with reentrant corners. Here, the reentrant corners themselves lead to singular parts in the solutions, forcing them to have a limited Sobolev regularity, even for smooth right–hand sides. For instance, considering the L-shaped domain $\Omega = (-1,1)^2 \backslash [0,1) \times [0,1)$, and $f \in L_2(\Omega)$, the solution u is known to be of the form

$$u = \kappa \mathcal{S} + \bar{u},$$

where $\bar{u} \in H^2(\Omega) \cap H_0^1(\Omega)$, κ is a generally non-zero constant, and, with respect to polar coordinates (r,θ) related to the reentrant corner,

$$\mathcal{S}(r,\theta) := \zeta(r) r^{2/3} \sin(\frac{2}{3}\theta),$$

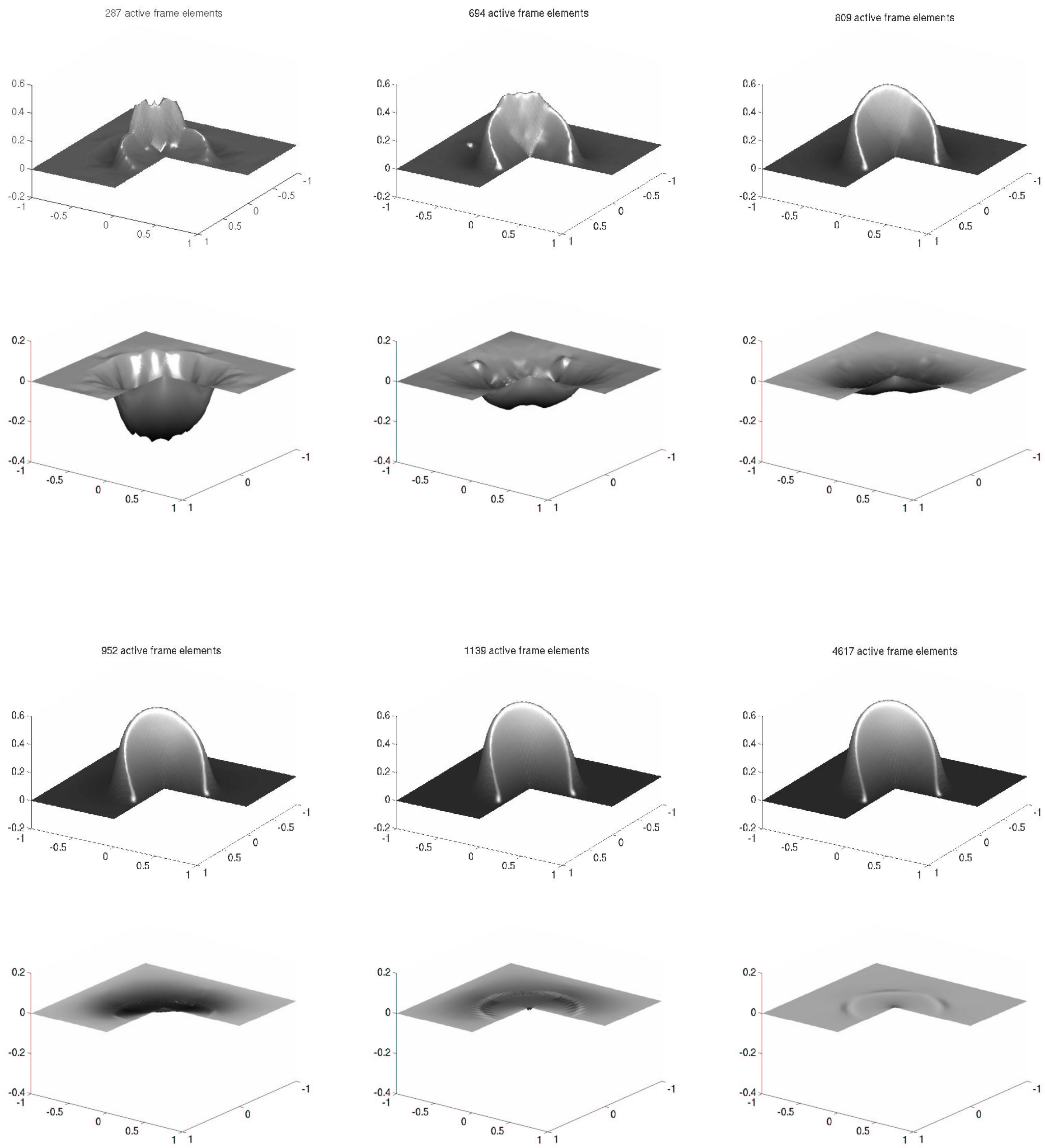


Figure 5: Approximations and corresponding pointwise errors produced by the adaptive steepest descent algorithm, using piecewise linear frame elements. *Upper part:* Approximations with 287, 694, and 809 frame elements. *Lower part:* Approximations with 952, 1139, and 4617 frame elements.

where $\zeta \in C^\infty(\Omega)$ is a truncation function. We use $\mathcal{S}$ as exact solution, which is shown together with the corresponding right–hand side in Figure 4. It is well-known that $\mathcal{S} \in H^s(\Omega)$ for $s < 5/3$ only, but it is contained in every Besov space $B^s_\tau(L_\tau(\Omega))$, where $s > 0$, $1/\tau = (s-1)/2 + 1/2$, see [10]. As previously recalled, the convergence rate of a uniform refinement strategy is determined by the Sobolev regularity of the solution, while in the context of adaptive schemes it depends on the Besov regularity [11]. In particular, considering piecewise linear approximation, the best possible convergence rate in the $H^1(\Omega)$-norm for uniform refinement strategies is $\mathcal{O}(N^{-(\frac{5}{3}-1)/2})$, with N being the number of unknowns, whereas our adaptive frame scheme gives the optimal rate $\mathcal{O}(N^{-1/2})$. The latter can be shown, in view of our assumptions on the **APPLY** routine, provided that the wavelets used in the construction of the aggregated frame are smooth enough and have sufficiently many vanishing moments, see [33] for a detailed discussion of this relation.

More generally, assuming f is sufficiently smooth, with piecewise polynomial approximation of order d, a further expansion of u into more singularity functions associated to the corners of the domain shows that with the adaptive scheme the optimal rate $\mathcal{O}(N^{-(d-1)/2})$ is reached, whereas with uniform refinement strategies the rate is always restricted to $\mathcal{O}(N^{-(\frac{5}{3}-1)/2})$.

For our numerical experiments, we will use an aggregated wavelet frame. With $\Omega_1 = (-1,0)\times(-1,1)$, $\Omega_2 = (-1,1)\times(-1,0)$, and $\square = (0,1)^2$, let κ_i be affine bijections between $\square$ and Ω_i $(i = 1,2)$. For $\Psi^\square$ being a piecewise linear wavelet basis as mentioned above for $H^1_0(\square)$, we set $\mathcal{F} = \cup_{i=1}^2 \kappa_i(\Psi^\square)$. Although this construction is in the spirit of that from [14, 33], we cannot conclude from the theory developed there that $\mathcal{F}$ is actually a frame. The difficulty is that there does not exist a partition of unity with respect to the open cover $\Omega_1 \cup \Omega_2$ of Ω. The non-overlapping parts of Ω_1 and Ω_2 are infinitely close at the reentrant corner. We give a direct proof that nevertheless $\mathcal{F}$ is a frame.

Using that $\kappa_i(\Psi^\square)$ are frames (even bases) for $H^1_0(\Omega_i)$, it is sufficient to show that

$$\|u\|^2_{H^1(\Omega)} \eqsim \inf_{u_1\in H^1_0(\Omega_1),\, u_2\in H^1_0(\Omega_2), u=u_1+u_2} \|u_1\|^2_{H^1(\Omega_1)} + \|u_2\|^2_{H^1(\Omega_2)},$$

uniformly in $u \in H^1_0(\Omega)$. Let $\phi : [0, \frac{3\pi}{2}] \to \mathbb{R}_{\geq 0}$ be a smooth function with $\phi(\theta) = 1$ for $\theta \leq \frac{\pi}{2}$ and $\phi(\theta) = 0$ for $\theta \geq \pi$. Writing $u = u_1 + u_2$ where $u_2(x,y) = u(x,y)\phi(\theta(x,y))$ with $(r(x,y), \theta(x,y))$ being the polar coordinates of (x,y) with respect to the reentrant corner, we have that $u_i \in H^1_0(\Omega_i)$ $(i = 1,2)$. Since Ω is a Lipschitz domain, with $\delta(x,y)$ denoting the distance of $(x,y) \in \Omega$ to the boundary, we know that for $u \in H^1_0(\Omega)$, $\delta^{-1}u \in L^2(\Omega)$ with $\|\delta^{-1}u\|_{L_2(\Omega)} \lesssim \|u\|_{H^1(\Omega)}$, uniformly in u, see [27, Theorem 1.4.4.4]. Since furthermore

$$\nabla(u\phi) = \phi\nabla u + \frac{u}{r}\left(-\sin(\theta)\frac{\partial\phi}{\partial\theta}, \cos(\theta)\frac{\partial\phi}{\partial\theta}\right)^T,$$

and $r(x,y) \geq \delta(x,y)$, we conclude that $\|u_i\|_{H^1(\Omega)} \lesssim \|u\|_{H^1(\Omega)}$ uniformly in u, which completes the proof of $\mathcal{F}$ being a frame for $H^1_0(\Omega)$.

We have tested the adaptive wavelet algorithms **CDD2SOLVE** with parameters $\alpha_{opt} \approx 0.238$, $\theta = 2/7$, $K = 488$, and with initial $\eta = 158.8$, and **SOLVE** with parameters $\delta = 1$, $\mu = 1.0001$, $\beta = 0.9$, $M = C_3 = 1$, $K = 92$, $\omega_0 = 158.8$. In Figure 5 we show some of the approximations and the corresponding pointwise differences to the exact solution produced

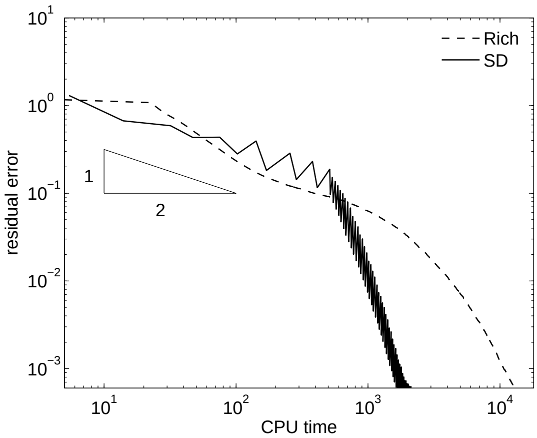


Figure 6: Convergence histories of **SOLVE** and **CDD2SOLVE** with respect to CPU time.

by our steepest descent scheme. The numerical results in Figure 6 illustrate the optimal convergence of both schemes for moderate accuracies. In order to get good approximations of the true residuals, here we need a significantly higher number of frame elements than in the one–dimensional example discussed in Section 4.1. Therefore, due to the restriction on the number of frame elements, we observe a deviation of the computed residuals from the true ones sooner in the iteration process.

Remark 4.1. For the damped Richardson and the steepest descent scheme, optimality has been theoretically proven only under Assumption 3.5. As previously mentioned, according to [33, Remark 3.13], the boundedness of $\mathbf{Q}$ on $\ell_\tau^w(\mathcal{N})$ for all $s \in (0, s^*)$ is (almost) a necessary requirement for the scheme to behave optimally. So our numerical results can also be seen as a possible indirect confirmation of such boundedness.

Remark 4.2. We have restricted our numerical tests to piecewise linear wavelets with complementary boundary conditions as defined in [16]. Of course, one may consider higher order wavelets, but one would encounter the following drawback. For the wavelet bases constructed in [15, 16], as soon as the order increases, their condition number increases also rather significantly, spoiling the benefits due to adaptivity. Nevertheless, very recently new bases have been constructed [2, 26, 32] that exhibit significantly better condition numbers and they will probably allow for very efficient implementations with high order bases.

5 Conclusion

In this paper we have presented a new optimally convergent adaptive scheme for the numerical solution of elliptic operator equations, based on redundant frame discretizations. The scheme is based on approximated iterations of steepest descent type. We have shown that the search of the damping parameter can be executed adaptively at each iteration, allowing for better practical usability compared to the damped Richarson iteration. There, the optimal damping parameter can often only be guessed, since the estimation of the lowest non–zero eigenvalue of the stiffness matrix is difficult in the case of frame discretizations. The use of frames instead of Riesz bases does not spoil the optimal convergence of the scheme that can be theoretically proved and numerically verified. Moreover, the construction of wavelet systems on domains with complicated geometry is extremely simplified by considering frames instead of Riesz bases. The numerical implementation is also significantly simplified.

The results included in [30, 31] illustrate that frames can be naturally used for domain decomposition methods, where the overlapping patches induce a Schwarz alternating iteration. Together with adaptive schemes and the implementation of well-conditioned high order bases [2, 26, 32], we expect that this line of research will produce a significant breakthrough for numerical schemes based on frame decompositions.

References

[1] A. Barinka, *Fast computation tools for adaptive wavelet schemes*, Ph.D. thesis, RWTH Aachen, 2005.

[2] K. Bittner, *Biorthogonal spline wavelets on the interval*, preprint, 2006.

[3] C. Canuto, A. Tabacco, and K. Urban, *The wavelet element method part I: Construction and analysis*, Appl. Comput. Harmon. Anal. **6** (1999), 1–52.

[4] C. Canuto and K. Urban, *Adaptive optimization in convex Banach spaces*, SIAM J. Numer. Anal. **42** (2005), no. 5, 2043–2075.

[5] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, 2003.

[6] A. Cohen, W. Dahmen, and R. DeVore, *Adaptive wavelet methods for elliptic operator equations — Convergence rates*, Math. Comp. **70** (2001), 27–75.

[7] ———, *Adaptive wavelet methods II: Beyond the elliptic case*, Found. Comput. Math. **2** (2002), no. 3, 203–245.

[8] ———, *Adaptive wavelet schemes for nonlinear variational problems*, SIAM J. Numer. Anal. **41** (2003), no. 5, 1785–1823.

[9] A. Cohen and R. Masson, *Wavelet adaptive method for second order elliptic problems: Boundary conditions and domain decomposition*, Numer. Math. **86** (2000), no. 2, 193–238.

[10] S. Dahlke, *Besov regularity for elliptic boundary value problems on polygonal domains*, Applied Mathematics Letters **12** (1999), 31–38.

[11] S. Dahlke, W. Dahmen, and R. DeVore, *Nonlinear approximation and adaptive techniques for solving elliptic operator equations*, Multiscale Wavelet Methods for Partial Differential Equations (W. Dahmen, A. Kurdila, and P. Oswald, eds.), Academic Press, San Diego, 1997, pp. 237–283.

[12] S. Dahlke, W. Dahmen, R. Hochmuth, and R. Schneider, *Stable multiscale bases and local error estimation for elliptic problems*, Appl. Numer. Math. **23** (1997), 21–48.

[13] S. Dahlke and R. DeVore, *Besov regularity for elliptic boundary value problems*, Commun. Partial Differ. Equations **22** (1997), no. 1&2, 1–16.

[14] S. Dahlke, M. Fornasier, and T. Raasch, *Adaptive frame methods for elliptic operator equations*, Bericht 2004-3, FB 12 Mathematik und Informatik, Philipps-Universität Marburg, 2004, To appear in Adv. Comput. Math.

[15] W. Dahmen, A. Kunoth, and K. Urban, *Biorthogonal spline–wavelets on the interval — Stability and moment conditions*, Appl. Comput. Harmon. Anal. **6** (1999), 132–196.

[16] W. Dahmen and R. Schneider, *Wavelets with complementary boundary conditions — Function spaces on the cube*, Result. Math. **34** (1998), no. 3–4, 255–293.

[17] ______, *Composite wavelet bases for operator equations*, Math. Comp. **68** (1999), 1533–1567.

[18] ______, *Wavelets on manifolds I. Construction and domain decomposition*, SIAM J. Math. Anal. **31** (1999), 184–230.

[19] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, 1992.

[20] R. DeVore, *Nonlinear approximation*, Acta Numerica **7** (1998), 51–150.

[21] M. Fornasier and K. Gröchenig, *Intrinsic localization of frames*, Constructive Approximation (2005), to appear.

[22] T. Gantumur, *An optimal adaptive wavelet method for nonsymmetric and indefinite elliptic problems*, Tech. Report 1343, Utrecht University, January 2006, Submitted.

[23] T. Gantumur, H. Harbrecht, and R.P. Stevenson, *An optimal adaptive wavelet method without coarsening of the iterands*, Tech. Report 1325, Utrecht University, March 2005, To appear in *Math. Comp.*

[24] T. Gantumur and R.P. Stevenson, *Computation of differential operators in wavelet coordinates*, Tech. Report 1306, Utrecht University, August 2004, To appear in *Math. Comp.*

[25] ______, *Computation of singular integral operators in wavelet coordinates*, Computing **76** (2006), 77–107.

[26] L. Gori, L. Pezza, and F. Pitolli, *Recent results on wavelet bases on the interval generated by GP refinable functions.*, Appl. Numer. Math. **51** (2004), no. 4, 549–563.

[27] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Monographs and Studies in Mathematics, vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[28] K. Gröchenig, *Describing functions: atomic decompositions versus frames*, Monatsh. Math. **112** (1991), no. 1, 1–42.

[29] H. Harbrecht and R.P. Stevenson, *Wavelets with patchwise cancellation properties*, Tech. Report 1311, Utrecht University, October 2004, To appear in *Math. Comp.*

[30] P. Oswald, *Frames and space splittings in Hilbert spaces*, Survey lectures on multilevel schemes for elliptic problems in Sobolev spaces. http://www.faculty.iu-bremen.de/poswald/bonn1.pdf, 1997.

[31] ______, *Multilevel frames and Riesz bases in Sobolev spaces*, Survey lectures on multilevel schemes for elliptic problems in Sobolev spaces http://www.faculty.iu-bremen.de/poswald/bonn2.pdf, 1997.

[32] M. Primbs, *Stabile biorthogonale Spline-Waveletbasen auf dem Intervall*, Ph.D. thesis, Universität Duisburg-Essen, 2006.

[33] R.P. Stevenson, *Adaptive solution of operator equations using wavelet frames*, SIAM J. Numer. Anal. **41** (2003), no. 3, 1074–1100.

[34] ______, *Composite wavelet bases with extended stability and cancellation properties*, Tech. Report 1304, Utrecht University, July 2004, Submitted.

[35] ______, *On the compressibility of operators in wavelet coordinates*, SIAM J. Math. Anal. **35** (2004), no. 5, 1110–1132.

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