More on the order of prolongations and restrictions

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Abstract

For the analysis of multi-grid methods applied to finite difference discretizations, two definitions of orders of intergrid transfer operators are being used. The order is either defined in terms of the symbol of the transfer operator, i.e., its Fourier transform, or as the order of polynomials being preserved. In [*J. Comput. Appl. Math.*, 32(3) (1990), pp. 423–429], Hemker showed that the second definition is stronger than the first one. In this note, we show that both definitions are even equivalent.

Key words: Multi-grid methods

1 Prolongations and restrictions

For given invertible $\boldsymbol{A} \in \mathbb{Z}^{d\times d}$ and $\boldsymbol{b} \in \mathbb{R}^d$, we consider fine and coarse grids $\mathbb{Z}^d$ and $\boldsymbol{A}\mathbb{Z}^d + \boldsymbol{b}$. Examples are $\boldsymbol{A} = 2Id$ ("standard coarsening"), $\boldsymbol{A} = \operatorname{diag}(q_1, \ldots, q_d)$ with one of more $q_m = 1$ ("semi-coarsening"), $\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ ("red-black coarsening"), and $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$ that shows up in $\sqrt{3}$ subdivision schemes ([1]). A vector $\boldsymbol{b} \neq 0$ allows for the use of "staggered coarse grids".

For rapidly decreasing $(p_{\boldsymbol{\ell}})_{\boldsymbol{\ell}\in\mathbb{Z}^d}$, $(r_{\boldsymbol{\ell}})_{\boldsymbol{\ell}\in\mathbb{Z}^d}$, i.e., $\sup_{\boldsymbol{\ell}\in\mathbb{Z}^d}(1 + |\boldsymbol{\ell}|)^k|p_{\boldsymbol{\ell}}| < \infty$ for any $k \in \mathbb{N} = \{1, 2 \ldots\}$, and analogously for $(r_{\boldsymbol{\ell}})$, we define the *prolongation*

[⋆] Contribution in honour of the multi-grid pioneer Piet Hemker at the occasion of his retirement

$p : \ell_2(\boldsymbol{A}\mathbb{Z}^d + \boldsymbol{b}) \to \ell_2(\mathbb{Z}^d)$ and *restriction* $r : \ell_2(\mathbb{Z}^d) \to \ell_2(\boldsymbol{A}\mathbb{Z}^d + \boldsymbol{b})$ by, for $\boldsymbol{j} \in \mathbb{Z}^d$,

$$\begin{aligned}(pu)(\boldsymbol{j}) &= \sum_{\boldsymbol{\ell} \in \boldsymbol{A}\mathbb{Z}^d - \boldsymbol{j}} p_{\boldsymbol{\ell}} u(\boldsymbol{j} + \boldsymbol{\ell} + \boldsymbol{b}), \\ (ru)(\boldsymbol{A}\boldsymbol{j} + \boldsymbol{b}) &= \sum_{\boldsymbol{\ell} \in \mathbb{Z}^d} r_{\boldsymbol{\ell}} u(\boldsymbol{A}\boldsymbol{j} + \boldsymbol{\ell}),\end{aligned}$$

respectively. One easily verifies that the adjoint of a prolongation is a restriction and vice versa, and that

$$p = r^* \text{ if and only if } p_{\boldsymbol{\ell}} = \overline{r_{-\boldsymbol{\ell}}}.$$

Definition 1.1 *The* order *of p or r, notated by $m(p)$ or $m(r)$, respectively, is defined as the largest integer k such that $(pu|_{\boldsymbol{A}\mathbb{Z}^d+\boldsymbol{b}})(\boldsymbol{j}) = u(\boldsymbol{j})$ or $(ru|_{\mathbb{Z}^d})(\boldsymbol{A}\boldsymbol{j}+\boldsymbol{b}) = |\det(\boldsymbol{A})| \times u(\boldsymbol{A}\boldsymbol{j}+\boldsymbol{b})$ for all polynomials $u \in P_{k-1}$ and $\boldsymbol{j} \in \mathbb{Z}^d$, where $P_{-1} := \emptyset$.*

Above definition of $m(p)$ has been used in [2] for the analysis of multi-grid methods. It extends to irregular and bounded grids. When multi-grid is applied to consistent finite difference discretizations of elliptic problems of order $2m$, grid independent convergence rates were shown for prolongations p and restrictions r that satisfy $m(p) + m(r^*) > 2m$.

2 Transformation to Fourier space

As is well-known, the Fourier transform

$$(\mathcal{F}u)(\boldsymbol{\xi}) := (2\pi)^{-d/2} \sum_{\boldsymbol{j} \in \mathbb{Z}^d} u(\boldsymbol{j}) e^{-i\boldsymbol{j}\cdot\boldsymbol{\xi}}$$

defines isomorphism between $\ell_2(\mathbb{Z}^d)$ and $L_2(\mathbb{T}^d)$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Its adjoint $\mathcal{F}^* = \mathcal{F}^{-1}$ is given by

$$(\mathcal{F}^*v)(\boldsymbol{j}) = (2\pi)^{-d/2} \int_{(-\pi,\pi)^d} v(\boldsymbol{\xi}) e^{i\boldsymbol{j}\cdot\boldsymbol{\xi}} d\boldsymbol{\xi}.$$

We will need the following lemma:

Lemma 2.1

$$\sum_{\beta \in \mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d} e^{i\boldsymbol{\beta}\cdot\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha}} = \begin{cases} |\det(\boldsymbol{A})| & \textit{when } \boldsymbol{\alpha} \bmod \boldsymbol{A}^T\mathbb{Z}^d = 0, \\ 0 & \boldsymbol{\alpha} \in \mathbb{Z}^d \textit{ otherwise.} \end{cases}$$

Before we give a proof, note that for $\boldsymbol{A} = \operatorname{diag}(q_1, \ldots, q_d)$, $\mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d = \prod_{m=1}^d \mathbb{Z}/q_m\mathbb{Z}$, and the result follows directly from the formula for geometrical sums. For general invertible $\boldsymbol{A}$, representatives of $\mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d$ are given by $\boldsymbol{A}([0,1)^d) \cap \mathbb{Z}^d$, see Figure 1.

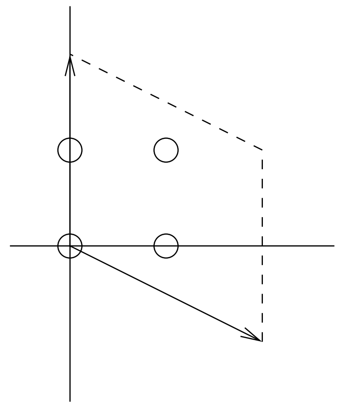


Fig. 1. Representatives of $\mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d$ for $\boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$

Proof For $\boldsymbol{\alpha} \operatorname{mod} \boldsymbol{A}^T\mathbb{Z}^d = 0$ the sum is equal to $\#\mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d$, which is equal to $\lim_{M\to\infty} \frac{\#\boldsymbol{A}([0,M)^d)\cap\mathbb{Z}^d}{\#\boldsymbol{A}([0,M)^d\cap\mathbb{Z}^d)} = |\det(\boldsymbol{A})|$.

Now let $\boldsymbol{\alpha} \operatorname{mod} \boldsymbol{A}^T\mathbb{Z}^d \neq 0$. We have

$$\sum_{\boldsymbol{\beta}\in\mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d} e^{i\boldsymbol{\beta}\cdot\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha}} = \lim_{M\to\infty} \frac{\sum_{\boldsymbol{\beta}\in[0,M)^d\cap\mathbb{Z}^d} e^{i\boldsymbol{\beta}\cdot\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha}}}{\#[0,M)^d\cap\mathbb{Z}^d \,/\, \#\mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d}. \tag{1}$$

There exists an $m \in \{1, \ldots, d\}$ with $(\boldsymbol{A}^{-T}\boldsymbol{\alpha})_m \notin \mathbb{Z}$. On the other hand, there exists a $0 \neq q \in \mathbb{N}$ with $q(\boldsymbol{A}^{-T}\boldsymbol{\alpha})_m \in \mathbb{Z}$. We conclude that for any $M \in q\mathbb{N}$, the numerator at the right hand side of (1) is zero, and thus that $\sum_{\boldsymbol{\beta}\in\mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d} e^{i\boldsymbol{\beta}\cdot\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha}} = 0$. $\square$

As the Fourier transform is a bijection between the rapidly decreasing functions and $C^\infty(\mathbb{T})$, the *symbols*

$$\hat{p}(\boldsymbol{\xi}) := \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} p_{\boldsymbol{\ell}} e^{i\boldsymbol{\ell}\cdot\boldsymbol{\xi}}, \quad \hat{r}(\boldsymbol{\xi}) := \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} r_{\boldsymbol{\ell}} e^{i\boldsymbol{\ell}\cdot\boldsymbol{\xi}},$$

are in $C^\infty(\mathbb{T})$. Defining the isomorphism $T : \ell_2(\boldsymbol{A}\mathbb{Z}^d + \boldsymbol{b}) \to \ell_2(\mathbb{Z}^d)$ by

$$(Tu)(\boldsymbol{j}) = u(\boldsymbol{A}\boldsymbol{j} + \boldsymbol{b}),$$

and thus $(T^*w)(\boldsymbol{A}\boldsymbol{j}+\boldsymbol{b}) = w(\boldsymbol{j})$, one may verify that for $v \in L_2(\mathbb{T}^d)$

$$
\begin{aligned}
(\mathcal{F}pT^*\mathcal{F}^*v)(\boldsymbol{\xi}) &= \hat{p}(\boldsymbol{\xi})v(\boldsymbol{A}^T\boldsymbol{\xi}),\\
(\mathcal{F}Tr\mathcal{F}^*v)(\boldsymbol{A}^T\boldsymbol{\xi}) &= |\det \boldsymbol{A}|^{-1} \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d/\boldsymbol{A}^T\mathbb{Z}^d} \hat{r}(\xi + \boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha})v(\xi+\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha}).
\end{aligned}
$$

Definition 2.2 *The* low frequency *(LF) and* high frequency *(LF)* orders $m_{\mathrm{LF}}(p)$ *and* $m_{\mathrm{HF}}(p)$ *of the prolongation* p *are defined as the largest integers for which*

$$
\begin{aligned}
&e^{i\boldsymbol{b}\cdot\boldsymbol{\xi}}\hat{p}(\boldsymbol{\xi}) - |\det(\boldsymbol{A})| = \mathcal{O}(|\boldsymbol{\xi}|^{m_{\mathrm{LF}}(p)}) \quad (\boldsymbol{\xi}\to 0),\\
&e^{i\boldsymbol{b}\cdot\boldsymbol{\xi}}\hat{p}(\boldsymbol{\xi}+\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha}) = \mathcal{O}(|\boldsymbol{\xi}|^{m_{\mathrm{HF}}(p)}) \quad (\boldsymbol{\xi}\to 0),\ \textit{for all}\ 0\neq\boldsymbol{\alpha}\in\mathbb{Z}^d/\boldsymbol{A}^T\mathbb{Z}^d.
\end{aligned}
$$

The low and high frequency orders of r *are defined as that of the prolongation* r^**.*

Obviously, in the definition of the high frequency order the factor $e^{i\boldsymbol{b}\cdot\boldsymbol{\xi}}$ can be deleted or replaced by $e^{i\boldsymbol{b}\cdot(\boldsymbol{\xi}+\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha})}$. It seems however not natural to redefine the symbol of p as $\xi \to e^{i\boldsymbol{b}\cdot\boldsymbol{\xi}}\hat{p}(\boldsymbol{\xi})$. Indeed, this function cannot be viewed as the Fourier transform of some operator, and in particular it is not necessarily $2\pi\boldsymbol{k}$ periodic for some $\boldsymbol{k}\in\mathbb{Z}^d$.

Above definitions of high and low frequency orders are used for the analysis of two-grid methods by means of Fourier transforms. Let $L_{\mathrm{f}} : \ell_2(\mathbb{Z}^d) \to \ell_2(\mathbb{Z}^d)$ and $L_{\mathrm{c}} : \ell_2(\boldsymbol{A}\mathbb{Z}^d+\boldsymbol{b}) \to \ell_2(\boldsymbol{A}\mathbb{Z}^d+\boldsymbol{b})$ be consistent discretizations of an elliptic differential operator of order $2m$ having constant coefficients. Writing

$$
\begin{aligned}
&\mathcal{F}(Id - pL_{\mathrm{c}}^{-1}rL_{\mathrm{f}})\mathcal{F}^* = \\
&\qquad Id - [\mathcal{F}pT^*\mathcal{F}^*e^{i\boldsymbol{b}\cdot\boldsymbol{\xi}}][e^{-i\boldsymbol{b}\cdot\boldsymbol{\xi}}\mathcal{F}TL_{\mathrm{c}}T^*\mathcal{F}^*e^{i\boldsymbol{b}\cdot\boldsymbol{\xi}}]^{-1}[e^{-i\boldsymbol{b}\cdot\boldsymbol{\xi}}\mathcal{F}Tr\mathcal{F}^*][\mathcal{F}L_{\mathrm{f}}\mathcal{F}^*],
\end{aligned}
$$

one can infer that two-grid convergence requires $m_{\mathrm{LF}}(p), m_{\mathrm{LF}}(r) > 0$ and $m_{\mathrm{HF}}(p)+m_{\mathrm{HF}}(r) \geq 2m$ (see [3]), where convergence with any smoother can be shown when in particular $m_{\mathrm{HF}}(p)+m_{\mathrm{HF}}(r) > 2m$. Compared to the condition $m(p) + m(r^*) > 2m$ mentioned earlier, the forthcoming Theorem 3.1 shows that here the conditions on the low frequency orders are milder (see however Remark 4.5).

Above analysis of a two-grid method via Fourier transforms only applies to differential operators with constant coefficients on $\mathbb{R}^d$ or $\mathbb{T}^d$ discretized with respect to regular grids. Yet, in [4,5], it was demonstrated that, under certain conditions, for problems on general domains with differential operators having smoothly variable coefficients, a worst case local analysis by Fourier transforms ("local mode analysis") provides an asymptotic upper bound for the two-grid convergence rate, provided the method is extended with local smoothing steps along the boundary, that require an asymptotically neglectable amount of additional work. Moreover, local mode analysis turns out to be an effective tool for the design or selection of efficient components of a multi-grid method ([6–8]).

3 A relation between the definitions of orders of intergrid transfer operators

The following theorem extends upon [3], in which, for standard coarsening and $\boldsymbol{b}=0$, it was shown that $\min(m_{\mathrm{LF}}(p), m_{\mathrm{HF}}(p)) \leq m(p)$ and $m_{\mathrm{LF}}(r) \leq m(r)$.

Theorem 3.1 $\min(m_{\mathrm{LF}}(p), m_{\mathrm{HF}}(p)) = m(p)$ *and* $m_{\mathrm{LF}}(r) = m(r)$.

Proof For $k \in \mathbb{N}$, $u \in P_{k-1}$, $\boldsymbol{\beta} \in \mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d$ and $\boldsymbol{j} \in \boldsymbol{A}\mathbb{Z}^d + \boldsymbol{\beta}$, we have

$$\begin{aligned}(pu|_{\mathbb{Z}^d})(\boldsymbol{j}) = \sum_{\boldsymbol{\ell}\in\boldsymbol{A}\mathbb{Z}^d-\boldsymbol{j}} p_{\boldsymbol{\ell}} u(\boldsymbol{j}+\boldsymbol{\ell}+\boldsymbol{b}) &= \sum_{\boldsymbol{\ell}\in\boldsymbol{A}\mathbb{Z}^d-\boldsymbol{\beta}} p_{\boldsymbol{\ell}} \sum_{|\boldsymbol{\nu}|\leq k-1} \tfrac{(\boldsymbol{\ell}+\boldsymbol{b})^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!}\partial^{\boldsymbol{\nu}} u(\boldsymbol{j}) \\ &= \sum_{|\boldsymbol{\nu}|\leq k-1} \tfrac{\partial^{\boldsymbol{\nu}} u(\boldsymbol{j})}{\boldsymbol{\nu}!} c_{\boldsymbol{\nu}\boldsymbol{\beta}},\end{aligned}$$

where

$$c_{\boldsymbol{\nu}\boldsymbol{\beta}} := \sum_{\boldsymbol{\ell}\in\boldsymbol{A}\mathbb{Z}^d-\boldsymbol{\beta}} p_{\boldsymbol{\ell}}(\boldsymbol{\ell}+\boldsymbol{b})^{\boldsymbol{\nu}}.$$

We conclude that $m(p) \geq k$ if and only if for all $|\boldsymbol{\nu}| \leq k-1$ and $\boldsymbol{\beta} \in \mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d$,

$$c_{\boldsymbol{\nu}\boldsymbol{\beta}} = \begin{cases} 1 & \text{when } \boldsymbol{\nu}=0, \\ 0 & \text{otherwise.}\end{cases} \tag{2}$$

On the other hand, for $\boldsymbol{\alpha} \in \mathbb{Z}^d/\boldsymbol{A}^T\mathbb{Z}^d$,

$$\begin{aligned}&e^{i\boldsymbol{b}\cdot\boldsymbol{\xi}}\hat{p}(\boldsymbol{\xi}+\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha}) = e^{i\boldsymbol{b}\cdot\boldsymbol{\xi}} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} p_{\boldsymbol{\ell}} e^{i\boldsymbol{\ell}\cdot(\boldsymbol{\xi}+\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha})} \\ &= \sum_{\boldsymbol{\beta}\in\mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d} \sum_{\boldsymbol{\ell}\in\boldsymbol{A}\mathbb{Z}^d-\boldsymbol{\beta}} p_{\boldsymbol{\ell}} e^{-i\boldsymbol{\beta}\cdot\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha}} e^{i(\boldsymbol{\ell}+\boldsymbol{b})\cdot\boldsymbol{\xi}} \\ &= \sum_{\boldsymbol{\beta}\in\mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d} \sum_{\boldsymbol{\ell}\in\boldsymbol{A}\mathbb{Z}^d-\boldsymbol{\beta}} p_{\boldsymbol{\ell}} e^{-i\boldsymbol{\beta}\cdot\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha}} \sum_{|\boldsymbol{\nu}|\leq k-1} \tfrac{\boldsymbol{\xi}^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!}(i(\boldsymbol{\ell}+\boldsymbol{b}))^{\boldsymbol{\nu}} + \mathcal{O}(|\boldsymbol{\xi}|^k) \quad (\boldsymbol{\xi}\to 0) \\ &= \sum_{|\boldsymbol{\nu}|\leq k-1} \tfrac{(i\boldsymbol{\xi})^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \sum_{\boldsymbol{\beta}\in\mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d} e^{-i\boldsymbol{\beta}\cdot\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha}} c_{\boldsymbol{\nu}\boldsymbol{\beta}} + \mathcal{O}(|\boldsymbol{\xi}|^k) \quad (\boldsymbol{\xi}\to 0),\end{aligned}$$

where for the third line we used that $\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}(1+|\boldsymbol{\ell}|)^k|p_{\boldsymbol{\ell}}| < \infty$. We conclude that $\min(m_{\mathrm{LF}}(p), m_{\mathrm{HF}}(p)) \geq k$ if and only if for all $|\boldsymbol{\nu}| \leq k-1$ and $\boldsymbol{\alpha} \in \mathbb{Z}^d/\boldsymbol{A}^T\mathbb{Z}^d$,

$$\sum_{\boldsymbol{\beta}\in\mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d} e^{-i\boldsymbol{\beta}\cdot\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha}} c_{\boldsymbol{\nu}\boldsymbol{\beta}} = \begin{cases} |\det(\boldsymbol{A})| & \text{when } \boldsymbol{\nu}=\boldsymbol{\alpha}=0, \\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

We will show that for any $\boldsymbol{\nu} \in \mathbb{N}^d$, the systems (2) and (3) are equivalent, which completes the proof of the first statement. For the matrix $\boldsymbol{B} = [e^{-i\boldsymbol{\beta}\cdot\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha}}]_{\alpha\in\mathbb{Z}^d/\boldsymbol{A}^T\mathbb{Z}^d,\beta\in\mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d}$, Lemma 2.1 shows that $\boldsymbol{B}\overline{\boldsymbol{B}}^T = |\det(\boldsymbol{A})|Id$ and, with $\boldsymbol{A}$ replaced by $\boldsymbol{A}^T$, that $\overline{\boldsymbol{B}}^T\boldsymbol{B} = |\det(\boldsymbol{A}^T)|Id$. We conclude that $\boldsymbol{B}$ is invertible, which proves the equivalence for $\boldsymbol{\nu} \neq 0$. Again Lemma 2.1 shows that $\sum_{\boldsymbol{\beta}\in\mathbb{Z}^d/\boldsymbol{A}\mathbb{Z}^d} e^{-i\boldsymbol{\beta}\cdot\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha}} = \begin{cases} |\det(\boldsymbol{A})| & \text{when } \boldsymbol{\alpha} = 0, \\ 0 & \text{otherwise,} \end{cases}$ which is the equivalence for $\boldsymbol{\nu} = 0$.

For $k \in \mathbb{N}$, $u \in P_{k-1}$, we have

$$\begin{aligned}(ru)(\boldsymbol{Aj}+\boldsymbol{b}) &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} r_{\boldsymbol{\ell}} u(\boldsymbol{Aj}+\boldsymbol{\ell}) = \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} r_{\boldsymbol{\ell}} \sum_{|\boldsymbol{\nu}|\le k-1} \tfrac{(\boldsymbol{\ell}-\boldsymbol{b})^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \partial^{\boldsymbol{\nu}} u(\boldsymbol{Aj}+\boldsymbol{b}) \\ &= \sum_{|\boldsymbol{\nu}|\le k-1} \tfrac{\partial^{\boldsymbol{\nu}} u(\boldsymbol{Aj}+\boldsymbol{b})}{\boldsymbol{\nu}!} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} r_{\boldsymbol{\ell}} (\boldsymbol{\ell}-\boldsymbol{b})^{\boldsymbol{\nu}},\end{aligned}$$

whereas, with $p = r^*$,

$$\begin{aligned}\overline{e^{i\boldsymbol{b}\cdot\boldsymbol{\xi}}\hat{p}(\boldsymbol{\xi})} = \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} r_{\boldsymbol{\ell}} e^{i(\boldsymbol{\ell}-\boldsymbol{b})\cdot\boldsymbol{\xi}} &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} r_{\boldsymbol{\ell}} \sum_{|\boldsymbol{\nu}|\le k-1} \tfrac{\boldsymbol{\xi}^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} (i(\boldsymbol{\ell}-\boldsymbol{b}))^{\boldsymbol{\nu}} + \mathcal{O}(|\boldsymbol{\xi}|^k) \quad (\boldsymbol{\xi} \to 0) \\ &= \sum_{|\boldsymbol{\nu}|\le k-1} \tfrac{(i\boldsymbol{\xi})^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} r_{\boldsymbol{\ell}} (\boldsymbol{\ell}-\boldsymbol{b})^{\boldsymbol{\nu}} + \mathcal{O}(|\boldsymbol{\xi}|^k) \quad (\boldsymbol{\xi} \to 0).\end{aligned}$$

So both $m(r) \geq k$ and $m_{\mathrm{LF}}(r) \geq k$ are equivalent to

$$\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} r_{\boldsymbol{\ell}} (\boldsymbol{\ell}-\boldsymbol{b})^{\boldsymbol{\nu}} = \begin{cases} |\det(\boldsymbol{A})| & \text{when } \boldsymbol{\nu} = 0, \\ 0 & \text{when } 0 < |\boldsymbol{\nu}| \le k-1, \end{cases}$$

which shows the second statement. □

Theorem 3.1 is a generalization of [4, Theorem 6.8] for standard coarsening and $\boldsymbol{b} = 0$. In the context of refinable functions, similar like results can be deduced by combining theorems from [9–11], with less elementary proofs though.

Note that Theorem 3.1 implies that $m(r) \geq m(r^*)$.

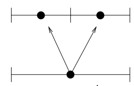
4 Examples

We give some simple examples of prolongations, and compute their low and high frequency orders.

Example 4.1 *(from [3] or [12])* $d = 1$, $\boldsymbol{A} = 2$, $\boldsymbol{b} = 0$. *Representatives of* $\mathbb{Z}/\boldsymbol{A}\mathbb{Z}$ *are* $\{0, 1\}$.

- *(linear interpolation)* $p_0 = 1$, $p_{\pm 1} = \frac{1}{2}$, *i.e, in the commonly used stencil notation,* $p = [\,\frac{1}{2}\ 1\ \frac{1}{2}\,]$. $\hat{p}(\xi) = 1+\cos(\xi) = 2+\mathcal{O}(\xi^2)$, $\hat{p}(\xi+\pi) = 1-\cos(\xi) = \mathcal{O}(\xi^2)$ *so* $m_{\mathrm{LF}}(p) = m_{\mathrm{HF}}(p) = 2$ *(and thus* $m(p) = 2$*).*
- *(cubic interpolation)* $p = [\,-\frac{1}{16}\ 0\ \frac{9}{16}\ 1\ \frac{9}{16}\ 0\ -\frac{1}{16}\,]$. $m_{\mathrm{LF}}(p) = m_{\mathrm{HF}}(p) = 3$.
- $p = [\,-\frac{1}{8}\ \frac{1}{2}\ \frac{5}{4}\ \frac{1}{2}\ -\frac{1}{8}\,]$. $m_{\mathrm{LF}}(p) = 4$, $m_{\mathrm{HF}}(p) = 2$.
- $p = [\,\frac{1}{8}\ \frac{1}{2}\ \frac{3}{4}\ \frac{1}{2}\ \frac{1}{8}\,]$. $m_{\mathrm{LF}}(p) = 2$, $m_{\mathrm{HF}}(p) = 4$.

Example 4.2 *(from [12])* $d = 1$, $\boldsymbol{A} = 2$, $\boldsymbol{b} = \frac{1}{2}$.

- *(constant interpolation)* $p_0 = p_{-1} = 1$ 
  $e^{i\xi/2}\hat{p}(\xi) = e^{i\xi/2}(1 + e^{-i\xi}) = 2\cos(\xi/2)$, $e^{i\xi/2}\hat{p}(\xi + \pi) = 2\sin(\xi/2)$, *so* $m_{\mathrm{LF}}(p) = 2$, $m_{\mathrm{HF}}(p) = 1$.
- *(linear interpolation)* $p_0 = p_{-1} = \frac{3}{4}$, $p_1 = p_{-2} = \frac{1}{4}$, $m_{\mathrm{LF}}(p) = 2$, $m_{\mathrm{HF}}(p) = 3$.

Example 4.3 $d = 2$, $\boldsymbol{A} = 2Id$, $\boldsymbol{b} = 0$. *Representatives of* $\mathbb{Z}^2/\boldsymbol{A}\mathbb{Z}^2$ *are* $\{(0,0),(1,0),(1,1)\}$. *Let* $p = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$ *(bilinear interpolation), then* $\hat{p}(\boldsymbol{\xi}) = (1 + \cos(\boldsymbol{\xi}_1))(1 + \cos(\boldsymbol{\xi}_2))$, *and so* $m_{\mathrm{LF}}(p) = m_{\mathrm{HF}}(p) = 2$.

Example 4.4 $d = 2$, $\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ *(red-black coarsening),* $\boldsymbol{b} = 0$. *Representatives of* $\mathbb{Z}^2/\boldsymbol{A}\mathbb{Z}^2$ *are* $\{(0,0),(0,1)\}$. *Let* $p_{(0,0)} = 1$, $p_{(\pm 1,0)} = p_{(0,\pm 1)} = \frac{1}{4}$ *(bilinear interpolation), then* $\hat{p}(\boldsymbol{\xi}) = 1+\frac{1}{2}(\cos(\boldsymbol{\xi}_1)+\cos(\boldsymbol{\xi}_2))$, $\hat{p}(\boldsymbol{\xi}+\boldsymbol{A}^{-T}2\pi[\,0\ 1\,]^T) = 1 - \frac{1}{2}(\cos(\boldsymbol{\xi}_1) + \cos(\boldsymbol{\xi}_2))$, *and so* $m_{\mathrm{LF}}(p) = m_{\mathrm{HF}}(p) = 2$.

Remark 4.5 *If* $p_0 = 1$ *and* $p_{\boldsymbol{\ell}} = 0$ *for* $0 \neq \boldsymbol{\ell} \in \boldsymbol{A}\mathbb{Z}^d$, *as is the case when* $\boldsymbol{b} = 0$ *and* p *is an interpolator, then, using Lemma 2.1 (with* $\boldsymbol{A}$ *reading as* $\boldsymbol{A}^T$*), we have*

$$\begin{aligned}\sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d/\boldsymbol{A}^T\mathbb{Z}^d} \hat{p}(\boldsymbol{\xi} + \boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha}) &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} p_{\boldsymbol{\ell}} e^{i\boldsymbol{\ell}\cdot\boldsymbol{\xi}} \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d/\boldsymbol{A}^T\mathbb{Z}^d} e^{i\boldsymbol{\ell}\cdot\boldsymbol{A}^{-T}2\pi\boldsymbol{\alpha}} \\ &= \sum_{\boldsymbol{\ell}\in\boldsymbol{A}\mathbb{Z}^d} p_{\boldsymbol{\ell}} e^{i\boldsymbol{\ell}\cdot\boldsymbol{\xi}} |\det(\boldsymbol{A})| = |\det(\boldsymbol{A})|,\end{aligned}$$

so that $m_{\mathrm{LF}}(p) \geq m_{\mathrm{HF}}(p)$ *(and even* $m_{\mathrm{LF}}(p) = m_{\mathrm{HF}}(p)$ *when* $\#\mathbb{Z}^d/\boldsymbol{A}^T\mathbb{Z}^d = 2$*). We conclude that in this case, the alternative conditions for multi-grid convergence, viz.* $m(p)+m(r^*) > 2m$ *on the one hand, and* $m_{\mathrm{LF}}(p), m_{\mathrm{LF}}(r) > 0$ *and* $m_{\mathrm{HF}}(p) + m_{\mathrm{HF}}(r) > 2m$ *on the other, are equivalent.*

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