BIFURCATION ANALYSIS OF PERIODIC ORBITS OF MAPS IN MATLAB

W. GOVAERTS*, R. KHOSHSIAR GHAZIANI†, YU. A. KUZNETSOV‡, AND H. G. E. MEIJER§

Abstract. We discuss new and improved algorithms for the bifurcation analysis of fixed points and periodic orbits (cycles) of maps and their implementation in MATCONT, a MATLAB toolbox for continuation and bifurcation analysis of dynamical systems.

This includes the numerical continuation of fixed points of iterates of the map with one control parameter, detecting and locating their bifurcation points (i.e., LP, PD and NS), and their continuation in two control parameters, as well as detection and location of all codimension 2 bifurcation points on the corresponding curves. For all bifurcations of codim 1 and 2, the critical normal form coefficients are computed, both numerically with finite directional differences and using symbolic derivatives of the original map. Using a parameter-dependent center manifold reduction, explicit asymptotics are derived for bifurcation curves of double and quadruple period cycles rooted at codim 2 points of cycles with arbitrary period. These asymptotics are implemented into the software and allow one to switch at codim 2 points to the continuation of the double and quadruple period bifurcations.

We provide several examples, in particular a juvenile/adult Leslie–Gower competition model from mathematical biology.

Key words. bifurcations of fixed points, cycles, normal forms, branch switching

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1. Introduction.

To fix the notation we consider a smooth map

$$x \mapsto f(x, \alpha), \tag{1.1}$$

where $x \in \mathbb{R}^n$ is a state variable vector and $\alpha \in \mathbb{R}^p$ is a parameter vector. Write the K-th iterate of (1.1) at some parameter value as

$$x \mapsto f^{(K)}(x, \alpha), \quad f^{(K)} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n, \tag{1.2}$$

where

$$f^{(K)}(x, \alpha) = \underbrace{f(f(f(\cdots f}_{K \text{ times}}(x, \alpha), \alpha), \alpha), \alpha).$$

The study of (1.1) usually starts with the analysis of fixed points. Numerically we continue fixed points of this map, i.e. solutions to the equation

$$F(x, \alpha) \equiv f(x, \alpha) - x = 0. \tag{1.3}$$

with one control parameter.

*Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281-S9, B-9000 Gent, Belgium, Willy.Govaerts@UGent.be.

†Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281-S9, B-9000 Gent, Belgium, and Department of Mathematics, Faculty of Science, Shahrekord University, PO Box 115, Shahrekord, Iran, Reza.KhoshsiarGhaziani@UGent.be.

‡Department of Mathematics, Utrecht University, Budapestlaan 6, 3584 CD Utrecht, The Netherlands, and Institute of Mathematical Problems of Biology, Russian Academy of Sciences, Pushchino, Moscow Region, 142290 Russia, kuznet@math.uu.nl.

§Department of Mathematics, Utrecht University, Budapestlaan 6, 3584 CD Utrecht, The Netherlands, hmeijer@math.uu.nl.

While varying one parameter, one may encounter codimension 1 bifurcations of fixed points, i.e., critical parameter values where the stability of the fixed point changes. The eigenvalues of the Jacobian matrix $A = f_x$ of f are called multipliers. The fixed point is asymptotically stable if $|\mu| < 1$ for every multiplier μ. If there exists a multiplier μ with $|\mu| > 1$, then the fixed point is unstable. While following a curve of fixed points, three codimension 1 singularities can generically occur, namely a *limit point* (*fold*, `LP`) with a multiplier $+1$, a *period-doubling* (*flip*, `PD`) point with a multiplier -1 and a *Neimark-Sacker* (`NS`) point with a conjugate pair of complex multipliers $e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$. Encountering such a bifurcation one may use the formulas for the normal form coefficients derived via the center manifold reduction, see e.g. [14], §5.4, to analyse the bifurcation. Generically, the curve of fixed points turns at an `LP`. In a `PD` point, generically, a cycle of period two bifurcates from the fixed point of f that changes stability. This bifurcation can be *supercritical* or *subcritical*, denoting the appearance of stable or unstable cycles for parameter values larger or smaller than the critical one, respectively. The continuation of this cycle can be reduced to the continuation of a fixed point of $f^{(2)}(x, \alpha) = f(f(x, \alpha), \alpha)$, the second iterate of the map. Typically, at an `NS` point a finitely-smooth closed invariant curve of (1.1) is born, while the primary fixed point changes stability. This bifurcation can also be thought of as an instrument to describe more complex (e.g., two-frequency oscillations) behavior in continuous-time nonlinear dynamical systems, when the resulting map comes from intersections between a periodic orbit and a certain plane, i.e. the Poincaré map.

A branch point (`BP`) is a point where the Jacobian matrix $[F_x(x, \alpha), F_\alpha(x, \alpha)]$ of (1.3) is rank deficient. This is a nongeneric situation in one-parameter problems where the implicit function theorem cannot be applied to ensure the existence of a unique smooth branch of solutions. However, it is encountered often in practical problems that exhibit some form of symmetry (equivariance).

When two control parameters are allowed to vary, eleven codimension 2 bifurcations can be met in generic families of maps (1.1), where curves of codimension 1 bifurcations intersect or meet tangentially. We proceed through listing smooth normal forms of the codim 2 bifurcation points and discussing their relationship with the original maps. The critical normal form coefficients for all generic codim 2 bifurcation points have been derived earlier in [17, 18] using a combined reduction-normalization technique. Note that the full verification of the genericity of a codim 2 bifurcation requires not only establishing its nondegeneracy at the critical parameter values but also a careful analysis of how the system depends on parameters. This "transversality" issue, which includes the expression of the canonical unfolding parameters of the normal form in terms of the original map parameters, received little attention in the up to date literature. When available, this information could be used to approximate codim 1 bifurcation curves that emanate from the codim 2 points.

There are several standard software packages supporting bifurcation analysis of iterated maps. Orbits of maps and one-dimensional invariant manifolds of saddle fixed points can be computed and visualized using DYNAMICS [22] and DsTool [2]. Location and continuation of fixed-point bifurcations is implemented in AUTO [8] and the LBFP-version of LOCBIF [20]. The latter program computes the critical normal form coefficient at `LP` points and locates some codim 2 bifurcations along branches of codim 1 fixed points and cycles. CONTENT [15] was the first software that computed the critical normal forms coefficients for all three codim 1 bifurcations of fixed points and cycles and allowed to continue these bifurcations in two parameters and to detect all eleven codim 2 singularities along them. Branch switching at PD and BP points is also implemented in AUTO, LOCBIF, and CONTENT. However, only trivial branch switching is possible at codim 2 points and only for two (cusp and 1:1 resonance) of eleven codim 2 bifurcations the critical normal form coefficients are computed by the current version of CONTENT. No software supports switching at codim 2 points to the continuation of the double- and quadruple-period bifurcation curves.

In the present paper we describe how CL_MATCONT, that previously could support only ODEs [6, 7], continues fixed points of an iterate of a map and handles the above mentioned bifurcations. The paper is organized as follows. In Section 2, we first give a list of all generic codim 1 and codim 2 bifurcation points of period-K orbits. Then we provide a (parameter-dependent) normal form for each case in the minimal possible phase dimension. In Section 3, we discuss the continuation of fixed points of the iterate map (1.2), as well as the computation of a new solution curve that emanates from a branch point. Continuation of fold, flip and Neimark-Sacker curves of a cycle is presented in Section 4. The algorithms described there are similar to those used in CONTENT [4]. In Section 5, we consider the branch switching at codim 2 points, where we use parameter-dependent center-manifold reduction. In Section 6 we present some algorithmic and numerical details, including the computation of partial derivatives of (1.2) up to and including order 5, both using symbolic partial derivatives of (1.1) and finite differences. In Section 7 we demonstrate the use of the algorithms and implementation. Finally, in Section 8 we draw some conclusions and mention a few problems for future work.

2. Codim 1 and 2 bifurcations of fixed points and cycles. Assume that for some $\alpha = \alpha_0$ and $K \geq 1$

$$g(x, \alpha) = f^{(K)}(x, \alpha) \tag{2.1}$$

has a fixed point $x = x_0$, that is not a fixed point of $f^{(J)}(\cdot, \alpha_0)$ for $1 \leq J < K$ when $K > 1$. In other words, x_0 is a fixed point or a cycle with minimal period K of $f(\cdot, \alpha_0)$. If the Jacobian matrix $A = g_x(x_0, \alpha_0)$ has no eigenvalue λ with $|\lambda| = 1$, i.e., for a hyperbolic fixed point, the dynamics near the origin is topologically equivalent to that of the linear map $x \mapsto Ax$ (Grobman-Hartman Theorem). If eigenvalues with $|\lambda| = 1$ are present, the Center Manifold Theorem guarantees the existence of local stable, unstable and center invariant manifolds near the fixed point for parameter values close to α_0. On the stable and unstable manifolds, the local dynamics is still determined by the linear part of the map. In contrast, the dynamics in the center manifolds depends on both linear and nonlinear terms. Not all nonlinear terms are equally important, since some of them can be eliminated by an appropriate smooth and smoothly depending on parameters coordinate transformation that puts the map restricted to the center manifold into a normal form, at least up to some order. Nonhyperbolic fixed points bifurcate, i.e., the dynamics near such points changes topologically under parameter variations. The birth of extra invariant objects, such as cycles or tori, is described by a parameter-dependent normal form of the restriction of g to a center manifold. Even though neither the center manifold nor the normal form on it are unique, the qualitative conclusions do not depend on the choices that are made.

Assuming sufficient smoothness of g, we write its Taylor expansion about (x_0, α_0)

Table 2.1
Smooth normal forms for generic codim 1 bifurcations of fixed points on center manifolds.

	Eigenvectors	Normal form	Critical coefficients
LP	$Aq = q$ $A^T p = p$	$w \mapsto \beta + w + aw^2$ $+O(w^3), \quad w \in \mathbb{R}$	$a = \frac{1}{2}\langle p, B(q,q)\rangle$
PD	$Aq = -q$ $A^T p = -p$	$w \mapsto -(1+\beta)w + bw^3$ $+O(w^4), \quad w \in \mathbb{R}$	$b = \frac{1}{6}\langle p, C(q,q,q) + 3B(q, h_{200})\rangle$ $h_{200} = (I_n - A)^{-1}B(q,q)$
NS	$Aq = \mathrm{e}^{i\theta_0} q$ $A^T p = \mathrm{e}^{-i\theta_0} p$ $\mathrm{e}^{i\nu\theta_0} \neq 1$ $\nu = 1, 2, 3, 4.$	$w \mapsto we^{i\theta}\left(1 + \beta + d\lvert w\rvert^2\right)$ $+ O(\lvert w\rvert^4), \quad w \in \mathbb{C}$	$d = \frac{1}{2} e^{-i\theta_0}\langle p, C(q,q,\bar{q})$ $+ 2B(q, h_{1100})$ $+ B(\bar{q}, h_{2000})\rangle$ $h_{1100} = (I_n - A)^{-1}B(q,\bar{q})$ $h_{2000} = (e^{2i\theta_0} I_n - A)^{-1}B(q,q)$

as
(2.2)

$$\begin{aligned} g(x_0 + x, \alpha_0 + \alpha) &= x_0 + Ax + \tfrac{1}{2}B(x,x) + \tfrac{1}{6}C(x,x,x) \\ &+ \tfrac{1}{24}D(x,x,x,x) + \tfrac{1}{120}E(x,x,x,x,x) \\ &+ J_1\alpha + \tfrac{1}{2}J_2(\alpha,\alpha) \\ &+ A_1(x,\alpha) + \tfrac{1}{2}B_1(x,x,\alpha) + \tfrac{1}{6}C_1(x,x,x,\alpha) + \tfrac{1}{24}D_1(x,x,x,x,\alpha) \\ &+ \tfrac{1}{2}A_2(x,\alpha,\alpha) + \tfrac{1}{4}B_2(x,x,\alpha,\alpha) + \tfrac{1}{12}C_2(x,x,x,\alpha,\alpha) \\ &+ \cdots, \end{aligned}$$

where all functions are multilinear forms of their arguments and the dots denote higher order terms in x and α. In particular, $A = g_x(x_0, \alpha)$ and the components of the multilinear functions B and C are given by

$$B_i(x,y) = \sum_{j,k=1}^{n} \frac{\partial^2 g_i(x_0,\alpha_0)}{\partial\xi_j \partial\xi_k} x_j y_k, \quad C_i(x,y,z) = \sum_{j,k,l=1}^{n} \frac{\partial^3 g_i(x_0,\alpha_0)}{\partial\xi_j \partial\xi_k \partial\xi_l} x_j y_k z_l, \tag{2.3}$$

for $i = 1, 2, \ldots, n$. From now on, I_n is the unit $n \times n$ matrix and $\|x\| = \sqrt{\langle x, x\rangle}$, where $\langle u, v\rangle = \bar{u}^T v$ is the standard scalar product in $\mathbb{C}^n$ (or $\mathbb{R}^n$).

Assume first that $\alpha \in \mathbb{R}$, i.e. the map under consideration depends on one control parameter. Well-known facts about three generic one-parameter (codim 1) bifurcations of fixed points of g and, thus, K-periodic orbits (cycles) of f, are summarized in Table 2.1. It is assumed that the critical eigenvalues are simple and no other eigenvalue of A with $|\lambda| = 1$ exists. In all cases, $\beta = \beta(\alpha)$ is a new real control parameter with critical value 0, and the eigenvectors are normalized such that $\langle p, q\rangle = 1$.

Generically, the following events happen in the state space near a bifurcation, see, for example [14]. When the control parameter crosses the critical value corresponding to a fold (LP) bifurcation, two fixed points of g collide and disappear, provided $a \neq 0$. This implies collision of two period-K cycles of the original map f. When the control parameter crosses the critical value corresponding to a flip (PD) bifurcation and $b \neq 0$, a cycle of period 2 for g bifurcates from the fixed point, that is a cycle of period $2K$ for map f. Finally, provided $c = \Re(d) \neq 0$, a unique closed invariant curve for g around the fixed point appears on the center manifold, when the control parameter crosses the critical value corresponding to the Neimark-Sacker (NS) bifurcation. For the original map, this means the appearance of K disjoint curves, cyclically shifted by f.

TABLE 2.2
Generic codim 2 bifurcations of cycles.

	Name	Bifurcation conditions
CP	cusp	$\lambda_1 = 1,\ a = 0$
DPD	degenerate flip	$\lambda_1 = -1,\ b = 0$
CH	Chenciner bifurcation	$\lambda_{1,2} = e^{\pm i\theta_0},\ c = 0$
R1	1:1 resonance	$\lambda_1 = \lambda_2 = 1$
R2	1:2 resonance	$\lambda_1 = \lambda_2 = -1$
R3	1:3 resonance	$\lambda_{1,2} = e^{\pm i\theta_0},\ \theta_0 = \frac{2\pi}{3}$
R4	1:4 resonance	$\lambda_{1,2} = e^{\pm i\theta_0},\ \theta_0 = \frac{\pi}{2}$
LPPD	fold-flip	$\lambda_1 = 1,\ \lambda_2 = -1$
LPNS	fold-NS	$\lambda_1 = 1,\ \lambda_{2,3} = e^{\pm i\theta_0}$
PDNS	flip-NS	$\lambda_1 = -1, \lambda_{2,3} = e^{\pm i\theta_0}$
NSNS	double NS	$\lambda_{1,2} = e^{\pm i\theta_0},\ \lambda_{3,4} = e^{\pm i\theta_1}$

2.1. Codimension-2 cases. Now assume that $\alpha \in \mathbb{R}^2$, i.e. $p = 2$. Eleven codim 2 bifurcations of cycles that can be met in generic two-parameter families of maps are listed in Table 2.2.

Below we give normal forms to which the restriction of a generic map $g(x, \alpha) = f^{(K)}(x, \alpha)$ to the parameter-dependent center manifold can be transformed near the corresponding bifurcation by smooth invertible coordinate and parameter transformations. We only incorporate the parameter-dependent part if branch switching to local bifurcations as in section 5 is involved. The $\mathcal{O}$-symbol denotes higher order terms in phase-variables, the coefficients of which may also depend on parameters. But the qualitative picture is determined by the lowest order terms listed below. We refer to [14], Ch. 9, and [17, 18] for details, including explicit expressions for all critical normal form coefficients which are not repeated in the present paper. If a complex critical eigenvalue λ is involved, it is always assumed that $\lambda^\nu \neq 1$ for $\nu = 1, 2, 3, 4$.

2.1.1. CP. The critical smooth normal form on the center manifold at a *cusp bifurcation* is

$$w \mapsto w + dw^3 + \mathcal{O}(|w|^4), \quad w \in \mathbb{R}, \tag{2.4}$$

where, generically, $d \neq 0$. Under this condition, a generic two-parameter unfolding of this singularity has two fold curves in the parameter plane which form a cuspidal wedge. For nearby parameter values, the map g has up to three fixed points that pairwise collide along the fold curves. In the direct product of the state and the parameter spaces, there is one smooth fold curve, so no branch switching is needed.

2.1.2. DPD. Near a *degenerate flip* bifurcation the restriction of the map g to the parameter-dependent center manifold is smoothly equivalent to the normal form

$$w \mapsto -(1 + \beta_1)w + \beta_2 w^3 + c_2 w^5 + \mathcal{O}(|w|^6), \quad w \in \mathbb{R}, \tag{2.5}$$

where, generically, the coefficient $c_2 \neq 0$, while (β_1, β_2) are smooth functions of α which can serve as new unfolding parameters. The fixed point $w = 0$ of the map (2.5) exhibits a flip bifurcation for $\beta_1 = 0$. It is well-known that from the point $\beta = 0$, corresponding to the degenerate flip bifurcation, a fold curve of double-period cycles emanates. The asymptotic expression for this curve in (2.5) is given by

$$(w, \beta_1, \beta_2) = (\varepsilon, -c_2\varepsilon^4 + \mathcal{O}(\varepsilon^5), -2c_2\varepsilon^2 + \mathcal{O}(\varepsilon^3)). \tag{2.6}$$

2.1.3. CH. If $e^{i\nu\theta_0} \neq 1$ for $\nu = 1, 2, \ldots, 6$, the critical smooth normal form on the center manifold at the *Chenciner bifurcation* can be written as

$$z \mapsto e^{i\theta_0} z + c_1 z|z|^2 + c_2 z|z|^4 + \mathcal{O}(|z|^6), \quad z \in \mathbb{C}, \tag{2.7}$$

where $\Re(e^{-i\theta_0}c_1) = 0$ but, generically, $\Re(e^{-i\theta_0}c_2) + \frac{1}{2}\Im(e^{-i\theta_0}c_1)^2 \neq 0$. A generic two-parameter unfolding of this singularity has a complicated bifurcation set due to the "collision" and destruction of two closed invariant curves of different stability born via the sub- and super-critical Neimark-Sacker bifurcations, respectively. There are no cycle bifurcation curves rooted at this bifurcation.

2.1.4. R1. The restriction of the map at a 1:1 *resonance* to the corresponding center manifold can be written in the form

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} w_1 + w_2 \\ w_2 + a_1 w_1^2 + b_1 w_1 w_2 \end{pmatrix} + \mathcal{O}(\|w\|^3), \quad w \in \mathbb{R}^2. \tag{2.8}$$

Generically, a Neimark-Sacker bifurcation curve of the fixed point meets tangentially the fold bifurcation curve. The local branch switching problem is trivial here, since both curves correspond to fixed points of g. The full bifurcation diagram near the codim 2 point is complicated and involves global bifurcations, e.g. tangencies of stable and unstable invariant manifolds of saddle fixed points of g and destruction of a closed invariant curve born via the Neimark-Sacker bifurcation.

2.1.5. R2. Near a 1:2 *resonance* the restriction of the map g to the parameter-dependent center manifold is smoothly equivalent to the normal form

$$\begin{aligned} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &\mapsto \begin{pmatrix} -w_1 + w_2 \\ \beta_1 w_1 + (-1+\beta_2) w_2 + C_1(\beta) w_1^3 + D_1(\beta) w_1^2 w_2 \end{pmatrix} \\ &+ \quad \mathcal{O}(\|w\|^4), \quad w \in \mathbb{R}^2. \end{aligned} \tag{2.9}$$

that depends on two control parameters (β_1, β_2). If $C_1 < 0$, then there is a Neimark-Sacker curve of fixed points of g with double period that emanates from the flip bifurcation curve $\beta_2 = 0$ of fixed points. It has the following asymptotic expression

$$(w_1^2, w_2, \beta_1, \beta_2) = \left(-\frac{1}{C_1}, 0, 1, \left(2 + \frac{D_1}{C_1}\right)\right)\varepsilon + \mathcal{O}(\varepsilon^2). \tag{2.10}$$

There are also global bifurcations associated with the destruction of closed invariant curves.

2.1.6. R3. At a 1:3 *resonance*, the restriction of the map g to the parameter-dependent center manifold is smoothly equivalent to the normal form

$$z \mapsto (e^{2i\pi/3} + \beta) z + B_1 \bar{z}^2 + C_1 z|z|^2 + \mathcal{O}(|z|^4), \quad z \in \mathbb{C}. \tag{2.11}$$

A generic unfolding of this singularity has a period-3 saddle cycle that does not bifurcate for nearby parameter values, although it merges with the primary fixed point as the parameters approach `R3`. Only global bifurcations related to the destruction of a closed invariant curve born via the primary Neimark-Sacker bifurcation occur in a neighborhood of this codim 2 point.

Note that the period-3 cycle becomes neutral near this bifurcation. Recall that a saddle cycle is called *neutral* if the corresponding fixed point has a pair of real eigenvalues with product 1. This singularity is important in analyzing global bifurcations of invariant manifolds of cycles. Moreover, the curve of neutral period-3 saddle cycles may turn into a true Neimark-Sacker bifurcation at `R1` or `R2`. Therefore, we give here an asymptotic of this curve.

First we need a vector field for which the time-1 flow approximates the third iterate of the map, i.e.

$$\widetilde{g}(\eta,\widetilde{\beta}) = \widetilde{\beta}\eta + \bar{\eta}^2 + C_0\eta^2\bar{\eta} + \mathcal{O}(|\eta|^4), \tag{2.12}$$

where

$$\widetilde{\beta} = 3\mathrm{e}^{-2i\pi/3}\beta,\ z = \frac{1}{|B_1(\beta)|}\mathrm{e}^{i\arg(B_1(\beta))/3}\eta,\ C_0 = \frac{1}{3}\left(\mathrm{e}^{-2i\pi/3}C_1/|B_1|^2 - 1\right).$$

We write $C_0 = a + ib$ and for $\eta = \rho\mathrm{e}^{i\phi}$ the neutral saddle curve has the following asymptotic expression

$$(\rho,\phi,\beta_1,\beta_2) = \left(\varepsilon, s(\pi/6 - a\varepsilon/3), -2a\varepsilon^2, s\varepsilon - b\varepsilon^2\right) + \mathcal{O}(\varepsilon^3), \tag{2.13}$$

where $s = \pm 1$.

2.1.7. R4. Near a 1:4 *resonance* the restriction of the map g to the parameter-dependent center manifold is smoothly equivalent to the normal form

$$z \mapsto (i+\beta)z + C_1(\beta)z^2\bar{z} + D_1(\beta)\bar{z}^3 + \mathcal{O}(|z|^4), \quad z \in \mathbb{C}. \tag{2.14}$$

For this bifurcation we do not only need this parameter-dependent normal form, but also an approximation of its 4th iterate by a unit-time shift along orbits of a vector field

$$\widetilde{g}(\eta,\widetilde{\beta}) = \widetilde{\beta}\eta + A_0(\beta)\eta^2\bar{\eta} + \bar{\eta}^3 + \mathcal{O}(|\eta|^4), \tag{2.15}$$

where $\eta \in \mathbb{C}$ and $\widetilde{\beta} = \tilde{\beta}_1 + i\tilde{\beta}_2, \tilde{\beta}_i \in \mathbb{R}$. Here the scaling

$$z = \frac{1}{\sqrt{|D_1(\beta)|}}\mathrm{e}^{i\arg(D_1(\beta))/4}\eta$$

is used and

$$A_0(\beta) = -i\frac{C_1(\beta)}{|D_1(\beta)|}.$$

Moreover, we have

$$\begin{pmatrix} \widetilde{\beta}_1 \\ \widetilde{\beta}_2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}. \tag{2.16}$$

There are three possible branch switches for this bifurcation. Let $a = \Re(A_0(0))$ and $b = \Im(A_0(0))$. If $\Delta \equiv a^2 + b^2 - 1 > 0$, then there are two half-lines $l_{1,2}$ of a limit-point curve of cycles with four times the original period. If

$$|b| > \frac{(1+a^2)}{\sqrt{1-a^2}},$$

then there is a curve n_1 along which a cycle of four times the primary period exhibits a Neimark-Sacker bifurcation. Using $\eta = re^{i\phi}$ we have the following approximations

(2.17)
$$\begin{aligned} l_{1,2} : (r^2, \phi, \widetilde{\beta}_1, \widetilde{\beta}_2) =& \left(\varepsilon, \frac{1}{4}\arctan\left(\frac{ab \pm \sqrt{\Delta}}{b^2-1}\right) + \mathcal{O}(\varepsilon), \right. \\ & \left. \frac{-a\Delta \mp b\sqrt{\Delta}}{a^2+b^2}\varepsilon, \frac{-b\Delta \pm a\sqrt{\Delta}}{a^2+b^2}\varepsilon \right) + \mathcal{O}(\varepsilon^2) \\ n_1 : (r^2, \phi, \widetilde{\beta}_1, \widetilde{\beta}_2) =& \ (\varepsilon + \mathcal{O}(\varepsilon^2), \operatorname{sign}(b)\arccos(a)/4 + \mathcal{O}(\varepsilon), \\ & -2a\varepsilon + \mathcal{O}(\varepsilon^2), -(b - \operatorname{sign}(b)\sqrt{1-a^2})\varepsilon + \mathcal{O}(\varepsilon^2)). \end{aligned}$$

Taking into account (2.16), we obtain expressions for β. If, in the formula for n_1, we replace $\operatorname{sign}(b)$ by $-\operatorname{sign}(b)$, then this gives the asymptotic for a neutral saddle singularity of the period-4 cycle.

Generically, there are also global bifurcations near R4.

2.1.8. LPPD. Near a *fold-flip* bifurcation, the restriction of the map g to the parameter-dependent center manifold is smoothly equivalent to the normal form

(2.18)
$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} \beta_1 + (1+\beta_2)w_1 + a(\beta)w_1^2 + b(\beta)w_2^2 + c_1(\beta)w_1^3 + c_2(\beta)w_1w_2^2 \\ -w_2 + e(\beta)w_1w_2 + c_3(\beta)w_1^2w_2 + c_4(\beta)w_2^3 \end{pmatrix}$$
$$+ \quad \mathcal{O}(\|w\|^4), \quad w \in \mathbb{R}^2.$$

A new branch predicted by (2.18) for a generic map g is a Neimark-Sacker of double period that exists if $be > 0$ and has the asymptotic expression

$$(x, y^2, \beta_1, \beta_2) = \left(-\frac{c_4}{e}, 1, -b, -\frac{2b + ec_2 - 2(a+e)c_4}{e}\right)\varepsilon + \mathcal{O}(\varepsilon^2). \tag{2.19}$$

As for the majority of the considered cases, there are also global bifurcations near this codim 2 point.

2.1.9. LPNS. For a *fold – Neimark-Sacker* bifurcation, the critical normal form on the center manifold is given by

$$\begin{pmatrix} w \\ z \end{pmatrix} \mapsto \begin{pmatrix} w + sz\bar{z} + w^2 + cx^3 \\ e^{i\theta_0}z + awz + bzw^2 \end{pmatrix} + \mathcal{O}(\|(w,z)\|^4), \quad (w,z) \in \mathbb{R}\times\mathbb{C}. \tag{2.20}$$

Depending on the coefficient values, several bifurcation scenarios are possible, which all involve only global phenomena.

2.1.10. PDNS. Near a *flip – Neimark-Sacker* bifurcation, the restriction of the map g to the parameter-dependent center manifold is smoothly equivalent to the parameter-dependent normal form

(2.21)
$$\begin{pmatrix} w \\ z \end{pmatrix} \mapsto \begin{pmatrix} -(1+\beta_1)w + c_1(\beta)w^3 + c_2(\beta)w|z|^2 \\ e^{i\theta(\beta)}(1+\beta_2)z + c_3(\beta)w^2z + c_4(\beta)z|z|^2 \end{pmatrix} + \mathcal{O}(\|(w,z)\|^4),$$
$$(w,z) \in \mathbb{R}\times\mathbb{C},$$

where $\theta(0) = \theta_0$. Besides global bifurcations, a Neimark-Sacker bifurcation curve of double period for g is rooted at $\beta = 0$; it is always present. The asymptotic expression

TABLE 3.1
Detection of codim 1 bifurcations of cycles.

Bifurcation	Test function(s)
LP	$t_{n+1} = 0, \det \begin{pmatrix} F_X \\ t^T \end{pmatrix} \neq 0$
PD	$\det(A + I_n) = 0$
NS	$\det(A \odot A - I_m) = 0$
BP	$\det \begin{pmatrix} F_X \\ t^T \end{pmatrix} = 0$

of this curve is given by

$$(w^2, z, \beta_1, \beta_2) = \big(1, 0, c_1, \mathrm{sign}(c_1)\Re(e^{-i\theta_0}c_3)\big)\,\varepsilon + \mathcal{O}(\varepsilon^2). \tag{2.22}$$

2.1.11. NSNS. For a *double Neimark-Sacker bifurcation*, provided $l\theta_0 \neq j\theta_1$ for integer l and j with $l + j \leq 4$, the critical normal form on the center manifold is

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} e^{i\theta_0} z_1 + c_1 z_1|z_1|^2 + c_2 z_1|z_2|^2 \\ e^{i\theta_1} z_2 + c_3 z_2|z_1|^2 + c_4 z_2|z_2|^2 \end{pmatrix} + \mathcal{O}(\|z\|^4), \quad z \in \mathbb{C}^2. \tag{2.23}$$

Depending on the coefficient values, several bifurcation scenarios are possible in parameter-dependent unfoldings, which all involve only global phenomena. For some of them, one has to take into account fourth- and fifth-order terms.

3. Continuation of cycles.

3.1. Defining system and singularities. The iteration of (1.1) gives rise to a sequence of points $\{x^1, x^2, x^3, \ldots, x^{K+1}\}$ in which $x^{J+1} = f(x^J, \alpha)$. Each point x of a cycle of period K then satisfies the fixed point equation for the K-th iterate

$$f^{(K)}(x, \alpha) - x = 0,$$

that we rewrite using (2.1) as

$$g(x, \alpha) - x = 0. \tag{3.1}$$

As in CONTENT, branches of period-K cycles are computed in CL_MATCONT by a variant of the Gauss-Newton continuation algorithm [1] applied to (3.1), see [6, 7].

To detect the bifurcations introduced in Section 2, as well as branch points of (3.1), we use in CL_MATCONT the standard test functions listed in Table 3.1, where t is the tangent vector to the curve (3.1) in the X-space for $X = (x, \alpha)^T$, $F(X) = g(x, \alpha) - x$, $A = g_x$ is the Jacobian matrix of $g = f^{(K)}$, and $\odot$ is the *bialternate matrix product*, see e.g. [12], §4.4.4. We notice that $\det(A \odot A - I_m)$, where $m = \frac{1}{2}n(n-1)$, also vanishes at neutral saddles. We distinguish true Neimark-Sacker bifurcations from neutral saddles when processing the NS-points.

3.2. Branch switching. In this section we consider the approximation of a new cycle curve that emanates from a branch point BP for (3.1). The same algorithm is used to switch at a PD-point for the period-K cycle to the period-$2K$ cycle, since it corresponds to a branch point for $f^{(2K)}(x, \alpha) - x = 0$. The method is similar to that for branch points of equilibria and is presented here only for completeness; it is also used in CONTENT.

A solution $X_0 = X(s_0)$ of

$$F(X) = g(x, \alpha) - x = 0 \tag{3.2}$$

is called a *simple singular point* if $F_X(X_0)$ has rank $n-1$. For system (3.2), we have $F_X^0 = [g_x(x_0, \alpha_0) - I_n, g_\alpha(x_0, \alpha_0)]$, and $X_0 = (x_0, \alpha_0)$ is a simple singular point if and only if, either

$$\dim N(g_x(x_0, \alpha_0) - I_n) = 1, \quad g_\alpha(x_0, \alpha_0) \in R(g_x(x_0, \alpha_0) - I_n)$$

or

$$\dim N(g_x(x_0, \alpha_0) - I_n) = 2, \quad g_\alpha(x_0, \alpha_0) \notin R(g_x(x_0, \alpha_0) - I_n).$$

The first case is a codimension 2 situation, the second case has codimension 4, so for practical purposes we consider only the first case.

Suppose we have a solution branch $X(s)$ and let $X_{s_0} = (x_0, \alpha_0)$ be a simple singular point. Then $N(F_X^0)$ is two-dimensional and can be written as span $\{\phi_1, \phi_2\}$ where $\phi_1, \phi_2 \in \mathbb{R}^n$ are linearly independent. Also, $N([F_X^0]^T)$ is one-dimensional and is spanned by a vector $\psi \in \mathbb{R}^{n+1}$. Let F_{YY}^0 be the bilinear form in the Taylor expansion of F about X_0. If $Y(s)$ is any solution branch of (3.2) with $Y(s_0) = X_0$, then $Y_s(s_0)$ can be written as $Y_s(s_0) = \alpha\phi_1 + \beta\phi_2$ for some $\alpha, \beta \in \mathbb{R}$. Differentiating the identity $F(Y(s)) = 0$ twice and computing the scalar product with ψ at s_0, we get

$$\langle \psi, F_{YY}^0(\alpha\phi_1 + \beta\phi_2)(\alpha\phi_1 + \beta\phi_2)\rangle = 0$$

or, equivalently,

$$c_{11}\alpha^2 + 2c_{12}\alpha\beta + c_{22}\beta^2 = 0, , \tag{3.3}$$

where $c_{jk} = \langle \psi, F_{YY}^0 \phi_j \phi_k\rangle$ for $j, k = 1, 2$.

Equation (3.3) is called the *algebraic bifurcation equation* (*ABE*). The case $c_{12}^2 - c_{11}c_{22} < 0$ is impossible, since at least one branch goes through X_0. Thus, generically, $c_{12}^2 - c_{11}c_{22} > 0$, and Equation (3.3) has two real nontrivial, independent solution pairs, (α_1, β_1) and (α_2, β_2), which are unique up to scaling. In this case we have a *simple branch point*, where two distinct branches pass through X_0.

The above procedure allows one to compute the normalized tangent vectors $Y_{1s}(s_0), Y_{2s}(s_0)$ of the two branches that pass through X_0. Now if

$$|\langle Y_{1s}(s_0), X_s(s_0)\rangle| < |\langle Y_{2s}(s_0), X_s(s_0)\rangle|$$

then we conclude that $Y_{1s}(s_0)$ is the tangent vector to the new branch; otherwise, $Y_{2s}(s_0)$ is the tangent vector.

4. Continuation of codimension one bifurcation curves.

In CL_MATCONT `LP`, `PD`, and `NS` curves for period-K cycles are computed by the mentioned Gauss-Newton continuation algorithm applied to *minimally extended defining systems*, cf. [12]. These systems were first implemented, together with the *standard extended defining systems*, in CONTENT [4]. We have adopted in CL_MATCONT the most robust and efficient methods tested there. Here follows a brief description of the defining systems, since we need to refer to them in Section 6.

The *limit point curve* and *period-doubling curve* are both defined by the following system

$$\left\{ \begin{array}{rcl} g(x,\alpha) - x & = & 0, \\ s(x,\alpha) & = & 0, \end{array} \right. \tag{4.1}$$

where $(x,\alpha) \in \mathbb{R}^{n+2}$, g is given by (2.1), while s is obtained by solving one of the algebraic systems

$$\begin{pmatrix} g_x(x,\alpha) \mp I_n & w_{bor} \\ v_{bor}^T & 0 \end{pmatrix} \begin{pmatrix} v \\ s \end{pmatrix} = \begin{pmatrix} 0_n \\ 1 \end{pmatrix}, \tag{4.2}$$

where $w_{bor}, v_{bor} \in \mathbb{R}^n$ are chosen such that the matrix in (4.2) is nonsingular. One should take the "$-$" sign in (4.2) for the `LP`-curve and the "$+$" sign for the `PD`-curve. The derivatives of s can be obtained easily from the derivatives of $g_x(x,\alpha)$:

$$s_z = -w^T (g_x)_z v, \tag{4.3}$$

where z is a state variable or an active parameter and w is obtained by solving

$$\begin{pmatrix} g_x^T(x,\alpha) \mp I_n & v_{bor} \\ w_{bor}^T & 0 \end{pmatrix} \begin{pmatrix} w \\ s \end{pmatrix} = \begin{pmatrix} 0_n \\ 1 \end{pmatrix}. \tag{4.4}$$

We note that the quantities called s in (4.2) and (4.4) are the same since they are both equal to the bottom right element of the inverse of the square matrix in (4.2).

The *Neimark-Sacker* and *neutral-saddle curves* are defined by the following system

$$\left\{ \begin{array}{rcl} g(x,\alpha) - x & = & 0 \\ s_{i_1 j_1}(x,\alpha,\kappa) & = & 0 \\ s_{i_2 j_2}(x,\alpha,\kappa) & = & 0, \end{array} \right. \tag{4.5}$$

i.e., by $n+2$ equations for the $(n+3)$ unknowns $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^2$, $\kappa \in \mathbb{R}$. Here $(i_1, j_1, i_2, j_2) \in \{1,2\}$ and $s_{i,j}$ are the components of S:

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

which is obtained by solving

$$\begin{pmatrix} (g_x)^2(x,\alpha) - 2\kappa g_x + I_n & W_{bor} \\ V_{bor}^T & O \end{pmatrix} \begin{pmatrix} V \\ S \end{pmatrix} = \begin{pmatrix} 0_{n,2} \\ I_2 \end{pmatrix}, \tag{4.6}$$

where $V_{bor}, W_{bor} \in \mathbb{R}^{n\times 2}$ are chosen (and can be adapted) so that the matrix in (4.6) is nonsingular. Along the Neimark-Sacker curve, κ is the real part of the critical multipliers $e^{\pm i\theta}$. The derivatives of s_{ij} can be obtained easily from the derivatives of $g_x(x,\alpha)$ as before.

Table 4.1 specifies test functions used in CL_MATCONT to detect and locate relevant codim 2 singularities along the codim 1 bifurcation curves. Here a and b are the critical normal form coefficients given in Table 2.1, where q and p should be replaced by the vectors v and w, respectively, computed in (4.2) and (4.4). The test function c is given by $c = \Re(d)$, where d is also specified in Table 2.1. Matrix A_1 is defined as $A_1 = A|_{N^\perp}$, where $A = g_x$ and $N^\perp$ is the orthogonal complement in $\mathbb{R}^n$ of the

Table 4.1
Detection of codim 2 bifurcations of cycles.

	LP	PD	NS
CP	$a = 0$		
DPD		$b = 0$	
CH			$c = 0$
R1	$\langle w, v\rangle = 0$		$\det(A - I_n) = \kappa - 1 = 0$
R2		$\langle w, v\rangle = 0$	$\det(A + I_n) = \kappa + 1 = 0$
R3			$\kappa + \frac{1}{2} = 0$
R4			$\kappa = 0$
LPPD	$\det(A + I_n) = 0$	$\det(A - I_n) = 0$	
LPNS	$\det(A \odot A - I_m) = 0$		$\det(A - I_n) = 0, \kappa - 1 \neq 0$
PDNS		$\det(A \odot A - I_m) = 0$	$\det(A + I_n) = 0, \kappa + 1 \neq 0$
NSNS			$\det(A_1 \odot A_1 - I_{m_1}) = 0$

two-dimensional eigenspace associated with the pair of multipliers with unit product of A^T; $m_1 = \frac{1}{2}(n-2)(n-3)$.

It is possible and sometimes necessary to adapt the defining system while continuing a bifurcation curve, i.e., to update the auxiliary variables used in the defining system of the computed branch. The bordering vectors v_{bor} and w_{bor} may require updating since they must ensure that the matrices in (4.2), (4.4) are nonsingular. Updating is done in CL_MATCONT by replacing v_{bor} and w_{bor} with the normalized vectors v, w computed in (4.2), (4.4), respectively. Updating of V and W in (4.6) is done similarly, while (i_1, j_1, i_2, j_2) are updated in such a way that the linearized system of (4.5) is as well-conditioned as possible.

5. Branch switching at codim 2 points. As it was indicated in Section 2.1, in codim 2 points branches of various codim 1 bifurcation curves are rooted. The problem of branch switching is thus to specify one starting point near the curve from which the continuation code converges to a point on the curve.

Here we address the problem of branch switching at codim 2 bifurcation points of maps, when the emanating curve corresponds to a local bifurcation. These cases involve degenerate flip, 1:2 resonance, 1:4 resonance, fold-flip and flip-Neimark-Sacker bifurcations only. The asymptotic expressions for the new curves for the parameter-dependent normal form are given in Section 2.1. Combining this information with a parameter-dependent center-manifold reduction, we will obtain appropriate initial continuation data for the original map.

In several cases there are also global bifurcations involved. We will not try to switch to those branches, since the continuation of these global bifurcations is not supported by the current version of CL_MATCONT.

Here we will focus only on the parameter-dependent computations and assume full knowledge of the critical center-manifold and the critical normal form coefficients, see [14],[17]. All cases may look alike, but all have their own subtle features. The solvability conditions imposed coincide with the transversality of the original family to the bifurcation manifold.

5.1. Parameter-Dependent Center-Manifold Reduction. In all our cases, the map $g(x, \alpha) : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^n$, where g is defined by (2.1), satisfies $g(x_0, \alpha_0) = x_0$, and its Jacobian matrix $A = g_x(x_0, \alpha_0)$ has at most 3 multipliers on the unit circle. Furthermore we know a parameter-dependent smooth normal form $G(w, \beta)$ for the corresponding bifurcation, see section 2.1. Then we assume a relation

$$\alpha - \alpha_0 = V(\beta) = v_{10}\beta_1 + v_{01}\beta_2 + \frac{1}{2}v_{20}\beta_1^2 + v_{11}\beta_1\beta_2 + \frac{1}{2}v_{02}\beta_2^2 + \mathcal{O}(\|\beta\|^3) \tag{5.1}$$

between the original and the unfolding parameters. Note that V incorporates linear scalings. The analysis will show that v_{20} and v_{11} need not be computed in the cases under consideration. Occasionally, we interpret $\beta_2 = \bar{\beta}_1$ as one complex parameter; in such cases: $v_{01} = \bar{v}_{10} \in \mathbb{C}^2$.

To find a parameter-dependent center-manifold as the graph of $x = x_0 + H(w, \beta)$ we make a Taylor expansion of the *homological equation*

$$g(x_0 + H(w,\beta), \alpha_0 + V(\beta)) = H(G(w,\beta), \beta) \tag{5.2}$$

in w and β at $(w, \beta) = (0, 0)$. All coefficients must vanish and this leads to a solution for H and V. A similar technique was introduced in [3], §11, to switch at codim 2 bifurcations of equilibria in ODEs.

We expand g as in (2.2) and write

$$H(w,\beta) = \sum_{|\mu|+|\nu|\geq 1} h_{\mu,\nu} w^\mu \beta^\nu, \tag{5.3}$$

where μ, ν are multi-indices.

It will be convenient to introduce some notation. Let p denote an eigenvector of A^T corresponding to the eigenvalue -1 of A. We will then write $\Gamma : \mathbb{R}^{n+2} \to \mathbb{R}^n$ for $\Gamma(q, v) = \langle p, A_1(q, v) + B(q, (I_n - A)^{-1}J_1 v)\rangle$ and $\gamma_i = \Gamma(q, e_i)$ for the evaluation of Γ on the standard basis vectors in $\mathbb{R}^2$. If $\gamma_i \neq 0$ for $i = 1, 2$ then $s_1 = \frac{1}{(\gamma_1^2+\gamma_2^2)}(\gamma_1, \gamma_2)^T$ and $s_2 = (-\gamma_2, \gamma_1)^T$ compose a new orthogonal basis in $\mathbb{R}^2$.

5.2. Degenerate Flip. We start with the linear part of $V(\beta)$. The homological equation (5.2) provides the following systems to be solved

$$(A - I_n)[h_{010},\ h_{001}] = -J_1[v_{10},\ v_{01}],$$

where $[a,\ b]$ is a $n \times 2$-matrix with columns $a, b \in \mathbb{R}^n$. This can be solved formally with $[h_{010},\ h_{001}] = (I_n - A)^{-1}J_1[v_{10},\ v_{01}]$ and next, we obtain

$$\begin{aligned}(A + I_n)[h_{110},\ h_{101}] &= -[q,\ 0] - A_1(q, [v_{10},\ v_{01}]) - B(q, [h_{010},\ h_{001}]) \\ &= -[q,\ 0] - A_1(q, [v_{10},\ v_{01}]) - B(q, (I_n - A)^{-1}J_1[v_{10},\ v_{01}]).\end{aligned}$$

First we remark that the systems are singular and the RHS must be orthogonal to the adjoint eigenvector p. Second, we see that the operator $\Gamma(q, v)$ appears naturally. The above systems can now be rewritten as

$$[\gamma_1,\ \gamma_2][v_{10},\ v_{01}] = [-1,\ 0].$$

The general solution is given by

$$v_{10} = -s_1 + \delta_1 s_2, \quad v_{01} = \delta_2 s_2, \quad \delta_1, \delta_2 \in \mathbb{R}.$$

The constants δ_1, δ_2 can only be fixed at a higher order, so we proceed with

$$
\begin{aligned}
(A-I_n)h_{210} = \; & +2h_{200} \\
& -[B_1(q,q,v_{10}) + B(h_{200},h_{010}) + A_1(h_{200},v_{10}) \\
& + \;2B(q,h_{110}) + C(q,q,h_{010})] \,, \\
(A-I_n)h_{201} = \; & -[B_1(q,q,v_{01}) + B(h_{200},h_{001}) + A_1(h_{200},v_{01}) \\
& + \;2B(q,h_{101}) + C(q,q,h_{001})] \,, \\
(A+I_n)h_{310} = \; & -3h_{300} - [D(q,q,q,h_{010}) + 3C(q,q,h_{110}) + C_1(q,q,q,v_{10}) \\
& +3B(h_{110},h_{200}) + B(h_{300},h_{010}) + 3B_1(h_{200},q,v_{10}) \\
& +A_1(h_{300},v_{10}) + 3C(h_{200},q,h_{010}) + 3B(h_{210},q)] \,, \\
(A+I_n)h_{301} = \; & 6q - [D(q,q,q,h_{001}) + 3C(q,q,h_{101}) + C_1(q,q,q,v_{01}) \\
& +3B(h_{101},h_{200}) + B(h_{300},h_{001}) + 3B_1(h_{200},q,v_{01}) \\
& +A_1(h_{300},v_{01}) + 3C(h_{200},q,h_{001}) + 3B(h_{201},q)] \,.
\end{aligned}
\tag{5.4}
$$

Since v_{10}, v_{01} appear linearly in these equations (via the multilinear functions), we have a linear system to be solved for δ_1, δ_2.

Now the linear part of $V(\beta)$ is obtained, but from the asymptotic expression (2.6) for the curve we notice that the approximation requires v_{02} as well. We have

$$
\begin{aligned}
(A-I_n)h_{002} &= -J_1 v_{02} - z_1, \\
(A+I_n)h_{102} &= -B(q,h_{002}) - A_1(q,v_{02}) - z_2,
\end{aligned}
$$

where

$$
\begin{aligned}
z_1 &= B(h_{001},h_{001}) + J_2(v_{01},v_{01}) + 2A_1(h_{001},v_{01}), \\
z_2 &= C(q,h_{001},h_{001}) + 2B(h_{101},h_{001}) + 2A_1(h_{101},v_{01}) \\
&\quad + 2B_1(q,h_{001},v_{01}) + A_2(q,v_{01},v_{01}).
\end{aligned}
$$

As before, the first equation can be solved formally and we substitute the result into the second. So formally we have $h_{002} = (I_n - A)^{-1}(z_1 + J_1 v_{02})$ and the solvability condition for the second equation yields

$$
(\gamma_1 \; \gamma_2) v_{02} = -\langle p, z_2 + B(q,(I_n - A)^{-1} z_1) \rangle.
$$

Thus we find $v_{02} = -\langle p, z_2 + B(q,(I_n - A)^{-1} z_1 \rangle s_1 + \delta_3 s_2$. The constant δ_3 can be found solving the systems at orders $w^{2,3}\beta_2^2$ of the homological equation, which are

$$
\begin{aligned}
(A-I_n)h_{202} \;=\; & -\{B_1(q,q,v_{02}) + A_1(h_{200},v_{02}) + 2B(h_{102},q) \\
& + C(q,q,h_{002}) + B(h_{200},h_{002})\} \\
& -[D(q,q,h_{001},h_{001}) + 4C(q,h_{101},h_{001}) + C(h_{001},h_{001},h_{200}) \\
& +2B(h_{201},h_{001}) + 2B(h_{101},h_{101}) + 2C_1(q,q,h_{001},v_{01}) \\
& +2B_1(h_{200},h_{001},v_{01}) \\
& +4B_1(q,h_{101},v_{01}) + 2A_1(h_{201},v_{01}) + B_2(q,q,v_{01},v_{01}) \\
& + A_2(h_{200},v_{01},v_{01})] \,,
\end{aligned}
\tag{5.5}
$$

and

(5.6)

$$
\begin{aligned}
(A+I_n)h_{302} \quad = \quad & -\{D(q,q,q,h_{002}) + 3C(q,q,h_{102}) + 3C(h_{200},h_{002},q) \\
& +3B(h_{202},q) + 3B(h_{200},h_{102}) + B(h_{300},h_{002}) \\
& C_1(q,q,q,v_{02}) + 3B_1(h_{200},q,v_{02}) + A_1(h_{300},v_{02})\} + 12h_{101} \\
& -[E(q,q,q,h_{001},h_{001}) + 6D(q,q,h_{101},h_{001}) \\
& +3D(h_{001},h_{001},h_{200},q) \\
& +C(h_{001},h_{001},h_{300}) + 6C(h_{101},h_{101},q) + 6C(h_{001},h_{101},h_{200}) \\
& +6C(h_{001},h_{201},q) + 6B(h_{201},h_{101}) + 2B(h_{301},h_{001}) \\
& +2D_1(q,q,q,h_{001},v_{01}) + 6C_1(q,q,h_{101},v_{01}) \\
& +6C_1(q,h_{001},h_{200},v_{01}) + 6B_1(h_{201},q,v_{01}) + 2B_1(h_{300},h_{001},v_{01}) \\
& +6B_1(h_{200},h_{101},v_{01}) + 2A_1(h_{301},v_{01}) + C_2(q,q,q,v_{01},v_{01}) \\
& +3B_2(h_{200},q,v_{01},v_{01}) + A_2(h_{300},v_{01},v_{01})] \,.
\end{aligned}
$$

Notice that the expressions between $\{\dots\}$ in equations (5.5) and (5.6) have the same structure as in (5.4). Thus we do not require any extra solvability conditions. The part between $[\dots]$ involves known quantities and is easily evaluated. To find δ_3 also the terms involving h_{002} and h_{102} between $\{\dots\}$ depending on z_1 and z_2 have to be computed.

5.3. 1:2 Resonance. As before, the first four linear systems are

$$
\begin{aligned}
&(A-I_n)[h_{0010},\ h_{0001}] = -J_1[v_{10},\ v_{01}], \\
&(A+I_n)[h_{1010},\ h_{1001}] = [q_1,\ 0] - A_1(q_0,[v_{10},\ v_{01}]) - B(q_0,[h_{0010},\ h_{0001}]).
\end{aligned}
$$

As for the degenerate flip, we use the formal solution

$$
[h_{0010},\ h_{0001}] = (I_n - A)^{-1} J_1 [v_{10},\ v_{01}].
$$

The solution for v_{10} and v_{01} is now

$$
v_{10} = s_1 + \delta_1 s_2, \quad v_{01} = \delta_2 s_2, \quad \delta_1, \delta_2 \in \mathbb{R}.
$$

The two remaining systems at linear order in phase variables are

$$
(A+I_n)[h_{0110},\ h_{0101}] = [h_{1010},\ q_1 + h_{1001}] - A_1(q_1,[v_{10},\ v_{01}]) - B(q_1,[h_{0010},\ h_{0001}]).
$$

Now we insert the s_i and write

$$
\begin{aligned}
Q_1 =& \langle p_1, A_1(q_0,s_1) + B(q_0,(A-I_n)^{-1}J_1 s_1)\rangle, Q_2 = \Gamma(q_1,s_1) \\
Q_3 =& \langle p_1, A_1(q_0,s_2) + B(q_0,(A-I_n)^{-1}J_1 s_2)\rangle, Q_4 = \Gamma(q_1,s_2).
\end{aligned}
$$

A little algebra shows that

$$
\delta_1 = -\left(\frac{Q_1+Q_2}{Q_3+Q_4}\right), \quad \delta_2 = \frac{1}{Q_3+Q_4}.
$$

One can check that the transversality of the original family of maps to the bifurcation manifold coincides with the condition $\gamma_1\gamma_2(Q_3+Q_4) \neq 0$.

5.4. 1:3 Resonance. We follow a slightly different procedure here. We want to find $V(\beta) = v\beta + \bar{v}\bar{\beta}$, where $\beta = \beta_1 + i\beta_2$. Then we treat β and $\bar{\beta}$ as independent variables which makes it slightly easier to find the solutions. As the final $V(\beta)$ should be real, it follows that $v = v_{10} = \bar{v}_{01}$.

Let $\lambda = e^{2i\pi/3}$ and introduce $Aq = \lambda q,\ A^T p = \bar{\lambda} p,\ \langle p, q\rangle = 1$. As before, the first linear systems resulting from (5.2) are given by

$$\begin{aligned}(A - I_n)[h_{0010},\ h_{0001}] &= -J_1[v_{10},\ v_{01}],\\ (A - \lambda I_n)[h_{1010},\ h_{1001}] &= [q,\ 0] - A_1(q, [v_{10},\ v_{01}]) - B(q, [h_{0010},\ h_{0001}]),\end{aligned}$$

and two complex conjugated systems for h_{0101} and h_{0110}. With the same approach we will now find complex γ_i and rewriting the system for $v = v_{10} = \bar{v}_{01}$ we have $(\gamma_1, \gamma_2)v = 1, (\gamma_1, \gamma_2)\bar{v} = 0$, with $v = (\bar{\gamma}_2, -\bar{\gamma}_1)/(\gamma_1\bar{\gamma}_2 - \gamma_2\bar{\gamma}_1)$ as solution. Finally $x = zq + \bar{z}\bar{q}$ relates the coordinates of the normal form and the original map.

5.5. 1:4 Resonance. Replacing $\lambda = i$ we can repeat the procedure for the case of 1:3 resonance.

5.6. Fold-Flip. Let $Aq_{1,2} = \pm q_{1,2},\ A^T p_{1,2} = \pm p_{1,2},\ \langle p_1, q_1\rangle = \langle p_2, q_2\rangle = 1$. The necessary systems to solve from the homological equation (5.2) are

$$(A - I_n)[h_{0010},\ h_{0001}] = [q_1,\ 0] - J_1[v_{10},\ v_{01}], \tag{5.7}$$

$$\begin{aligned}(A - I_n)[h_{1010},\ h_{1001}] = {}& [h_{2000},\ q_1] - A_1(q_1, [v_{10},\ v_{01}])\\ & - B(q_1, [h_{0010},\ h_{0001}]),\end{aligned} \tag{5.8}$$

$$\begin{aligned}(A + I_n)[h_{0110},\ h_{0101}] = {}& [h_{1100},\ 0] - A_1(q_2, [v_{10},\ v_{01}])\\ & - B(q_2, [h_{0010},\ h_{0001}]).\end{aligned} \tag{5.9}$$

First remark that all matrices in the left-hand sides are singular. If we take $(\gamma_1, \gamma_2) = p_1^T J_1$ and form the orthogonal vectors s_1 and s_2 as before then $v_{10} = s_1 + \delta_1 s_2$ and $v_{01} = \delta_2 s_2$ solve system (5.7). Bordering the singular matrix $(A - I_n)$ one can solve for h_{0010} and h_{0001}. Any multiple of q_1 can be added to h_{0010} and h_{0001}, so we use $h_{0010} = (A - I_n)^{INV}(q_1 - J_1 v_{10}) + \delta_3 q_1$ and $h_{0010} = -(A - I_n)^{INV}(J_1 v_{01}) + \delta_4 q_1$. We will use this freedom to solve equations (5.8) and (5.9) simultaneously for all δ's. Note that h_{2000} and h_{1100} are also found using bordered systems chosen, but such that $\langle p_1, h_{2000}\rangle = \langle p_2, h_{1100}\rangle = 0$.

Then we obtain the following 4−dimensional system

$$\begin{pmatrix} L & 0_{2\times 2}\\ 0_{2\times 2} & L\end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_3\\ \delta_2\\ \delta_4\end{pmatrix} = \begin{pmatrix} -\langle p_1, A_1(q_1, s_1) + B(q_1, (A - I_n)^{INV}(q_1 - J_1 s_1))\rangle\\ -\langle p_2, A_1(q_2, s_1) + B(q_2, (A - I_n)^{INV}(q_1 - J_1 s_1))\rangle\\ 1\\ 0\end{pmatrix}, \tag{5.10}$$

where L is defined by

$$L = \begin{pmatrix}\langle p_1, A_1(q_1, s_2) + B(q_1, (I_n - A)^{INV} J_1 s_2)\rangle & \langle p_1, B(q_1, q_1)\rangle\\ \langle p_2, A_1(q_2, s_2) + B(q_2, (I_n - A)^{INV} J_1 s_2)\rangle & \langle p_2, B(q_1, q_2)\rangle\end{pmatrix}. \tag{5.11}$$

Notice that $2a(0) = \langle p_1, B(q_1, q_1)\rangle$ and that q_1 can be chosen such that $e(0) = \langle p_2, B(q_1, q_2)\rangle = 1$. The condition $\gamma_1\gamma_2 \det(L) \neq 0$ is equivalent with the transversality to the bifurcation manifold of the family $g(x, \alpha)$.

5.7. Flip-Neimark-Sacker. Introduce $Aq_1 = q_1,\ A^T p_1 = p_1,\ \langle p_1, q_1\rangle = 1$, and $Aq_2 = e^{i\theta_0} q_2,\ A^T p_2 = e^{-i\theta_0} p_2,\ \langle p_2, q_2\rangle = 1$. The linear systems obtained from the homological equation (5.2) are

$$\begin{aligned}
(A - I_n)[h_{00010},\ h_{00001}] &= -J_1[v_{10},\ v_{01}],\\
(A + I_n)[h_{10010},\ h_{10001}] &= [-q_1,\ 0] - A_1(q_1, [v_{10},\ v_{01}]) - B(q_1, [h_{00010},\ h_{00001}]),\\
(A - e^{i\theta_0} I_n)[h_{01010},\ h_{01001}] &= [0,\ q_2 e^{i\theta_0}] - A_1(q_2, [v_{10},\ v_{01}]) - B(q_2, [h_{00010},\ h_{00001}]).
\end{aligned}$$

The same approach as for the degenerate flip and 1:2-resonance cases is to substitute the formal solution of the first equation into the second and we write

$$v_{10} = -s_1 + \delta_1 s_2, \quad v_{01} = \delta_2 s_2,$$

where the constants δ_i are to be found from the last equation. We compute

$$Q_i = \langle p_2, A_1(q_2, s_i) + B(q_2, (I_n - A)^{-1} J_1 s_i)\rangle$$

for $i = 1, 2$. To obtain the derivative of the modulus and not the argument of the complex multiplier, we proceed similar to [24], Appendix, but adapt to the case of maps. Then we find the following real solution for δ_i

$$\delta_1 = \frac{\Re(e^{-i\theta_0} Q_1)}{\Re(e^{-i\theta_0} Q_2)},\ \delta_2 = -\frac{1}{\Re(e^{-i\theta_0} Q_2)}. \tag{5.12}$$

6. Algorithmic and numerical details. In this section we consider the computation of the derivatives and tensor-vector products, which are not only necessary for the continuation, but also for the computation of the critical normal form coefficients at codim 1 and 2 bifurcation points.

6.1. Recursive formulas for derivatives of iterates of maps.

6.1.1. Derivatives with respect to phase variables. The iteration of (1.1) gives rise to a sequence of points

$$\{x^1, x^2, x^3, \dots, x^{K+1}\},$$

where $x^{J+1} = f^{(J)}(x^1, \alpha)$ for $J = 1, 2, \dots, K$. Suppose that symbolic derivatives of f up to order 5 can be computed at each point. We write

$$A(x^J)_{i,j} = \frac{\partial f_i}{\partial x_j}(x^J),\ B(x^J)_{i,j,k} = \frac{\partial^2 f_i}{\partial x_j \partial x_k}(x^J),\ C(x^J)_{i,j,k,l} = \frac{\partial^3 f_i}{\partial x_j \partial x_k \partial x_l}(x^J),$$

and similarly for $D(x^J)$ and $E(x^J)$.

We want to find recursive formulas for the derivatives of the composition (1.2), i.e. the coefficients of the multilinear functions in (2.2) that we now denote by $A^{(J)}, B^{(J)}$, and $C^{(J)}$ to indicate the iterate explicitly:

$$(A^{(J)})_{i,j} = \frac{\partial (f^{(J)}(x^1))_i}{\partial x_j},\ (B^{(J)})_{i,j,k} = \frac{\partial^2 (f^{(J)}(x^1))_i}{\partial x_j \partial x_k},\ (C^{(J)})_{i,j,k,l} = \frac{\partial^3 (f^{(J)}(x^1))_i}{\partial x_j \partial x_k \partial x_l},$$

and $D^{(J)}$ and $E^{(J)}$ are analogously defined. What follows is a straightforward application of the Chain Rule.

For $J = 1$ we have $A^{(1)} = A(x^1), B^{(1)} = B(x^1)$ and $C^{(1)} = C(x^1)$ and these are known. Now, as

$$A^{(J)}_{i,j} = \sum_k \frac{\partial f_i}{\partial x_k}(f^{(J-1)}(x^1))\frac{\partial (f^{(J-1)}(x^1))_k}{\partial x_j} = \sum_l A(x^J)_{i,k}A^{(J-1)}_{k,j}, \tag{6.1}$$

we see that

$$(F(x,\alpha))_x = A(x^K)A(x^{K-1})\cdots A(x^1) - I_n, \tag{6.2}$$

where $F(x,\alpha) = f^{(K)}(x,\alpha) - x$.

For the second order derivatives we first write $B^{(J)}$ once in coordinates

$$\begin{aligned} B^{(J)}_{i,j,k} &= \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} f_i(f^{(J-1)}(x)) \\ &= \sum_{l,m} \frac{\partial^2 f_i}{\partial x_l \partial x_m}(x^J)\frac{\partial (f^{(J-1)})_m}{\partial x_j}\frac{\partial (f^{(J-1)})_l}{\partial x_k} + \sum_l \frac{\partial f_i}{\partial x_l}(x^J)\frac{\partial^2 (f^{(J-1)})_l}{\partial x_j \partial x_k}. \end{aligned}$$

For any two vectors q_1 and q_2, we can multiply the previous expression by $(q_1)_j(q_2)_k$ and sum over (k,l) to obtain

$$B^{(J)}(q_1,q_2) = B(x^J)(A^{(J-1)}q_1, A^{(J-1)}q_2) + A(x^J)B^{(J-1)}(q_1,q_2). \tag{6.3}$$

As $A(x^J)$ and $B(x^J)$ are known, (6.3) allows to compute the multilinear form $B^{(K)}(q_1,q_2)$ recursively.

Let $q_i, i = 1,2,3,4,5$, be given vectors. Multilinear forms with higher order derivatives can be computed with

$$\begin{aligned} C^{(J)}(q_1,q_2,q_3) = \quad & C(x^J)(A^{(J-1)}q_1, A^{(J-1)}q_2, A^{(J-1)}q_3) + \\ & B(x^J)(B^{(J-1)}(q_1,q_2), A^{(J-1)}q_3)^* + \\ & A(x^J)(C^{(J-1)}(q_1,q_2,q_3)), \end{aligned} \tag{6.4}$$

where * means that all combinatorially different terms have to be included, i.e.,

$$\begin{aligned} B(x^J)(B^{(J-1)}(q_1,q_2), A^{(J-1)}q_3)^* = B(x^J)&(B^{(J-1)}(q_1,q_2), A^{(J-1)}q_3) + \\ B(x^J)&(B^{(J-1)}(q_1,q_3), A^{(J-1)}q_2) + \\ B(x^J)&(B^{(J-1)}(q_2,q_3), A^{(J-1)}q_1). \end{aligned}$$

For $D^{(J)}$ we get

$$\begin{aligned} D^{(J)}(q_1,q_2,q_3,q_4) = \quad & D(x^J)(A^{(J-1)}q_1, A^{(J-1)}q_2, A^{(J-1)}q_3, A^{(J-1)}q_4) + \\ & C(x^J)(B^{(J-1)}(q_1,q_2), A^{(J-1)}q_3, A^{(J-1)}q_4)^* + \\ & B(x^J)(B^{(J-1)}(q_1,q_2), B^{(J-1)}(q_3,q_4))^* + \\ & B(x^J)(C^{(J-1)}(q_1,q_2,q_3)), A^{(J-1)}q_4)^* + \\ & A(x^J)D^{(J-1)}(q_1,q_2,q_3,q_4). \end{aligned} \tag{6.5}$$

Finally, for $E^{(J)}$ holds

$$\begin{aligned} E^{(J)}(q_1,q_2,q_3,q_4,q_5) = \quad & E(x^J)(A^{(J-1)}q_1, A^{(J-1)}q_2, A^{(J-1)}q_3, A^{(J-1)}q_4, A^{(J-1)}q_5) + \\ & D(x^J)(B^{(J-1)}(q_1,q_2), A^{(J-1)}q_3, A^{(J-1)}q_4, A^{(J-1)}q_5)^* + \\ & C(x^J)(B^{(J-1)}(q_1,q_2), B^{(J-1)}(q_3,q_4), A^{(J-1)}q_5)^* + \\ & C(x^J)(C^{(J-1)}(q_1,q_2,q_3), A^{(J-1)}q_4, A^{(J-1)}q_5)^* + \\ & B(x^J)(C^{(J-1)}(q_1,q_2,q_3), B^{(J-1)}(q_4,q_5))^* + \\ & B(x^J)(D^{(J-1)}(q_1,q_2,q_3,q_4))(A^{(J-1)}q_5)^* + \\ & A(x^J)(E^{(J-1)}(q_1,q_2,q_3,q_4,q_5)). \end{aligned} \tag{6.6}$$

The multilinear forms $A^{(K)}(q_1)$, $B^{(K)}(q_1, q_2)$, $C^{(K)}(q_1, q_2, q_3)$, $D^{(K)}(q_1, q_2, q_3, q_4)$ and $E^{(K)}(q_1, q_2, q_3, q_4, q_5)$ are then used in the computations of the normal form coefficients for codim 1 and codim 2 bifurcations of period-K cycles and also in the branch switching.

6.1.2. Derivatives with respect to parameters. If enough symbolic derivatives of f are available, then MATCONT computes the expressions involving J_1 and A_1 in (2.2) symbolically. The idea is as follows. Taking the derivative of (2.1) with respect to α_k, gives

$$\frac{\partial(f^{(J)}(x^1,\alpha))}{\partial\alpha_k} = \frac{\partial f}{\partial\alpha_k}(x^J,\alpha) + \frac{\partial f}{\partial x}(x^J,\alpha)\frac{\partial(f^{(J-1)}(x^1,\alpha))}{\partial\alpha_k}, \tag{6.7}$$

which is recursively computable. Also mixed derivatives, which are necessary for continuation and branch switching, can be found recursively:

$$\frac{\partial^2(f^{(J)}(x^1,\alpha))}{\partial\alpha_k\partial x} = \frac{\partial^2 f}{\partial\alpha_k\partial x}(x^J,\alpha) + \frac{\partial^2 f}{\partial x^2}(x^J,\alpha)\frac{\partial(f^{(J-1)}(x^1,\alpha))}{\partial\alpha_k}. \tag{6.8}$$

In fact, the recursion is not applied to (6.8) itself, but to its product with a fixed vector.

This is sufficient for all continuations of fixed points and their codimension 1 bifurcations. It is also sufficient for all cases of branch switching from codimension 2 points, except for the case of degenerate flip. For this case, we fall back to a finite difference approximation. Since it is only used in the prediction step for which high accuracy is not needed, this seems acceptable.

6.2. Recursive formulas for derivatives of the defining systems for continuation. For the continuation of fixed points and cycles we need the derivatives of (2.1) which can be computed from (6.2) and (6.7).

Now, we consider the derivatives of s (as defined in (4.1)) with respect to z, a state variable or parameter. The flip and Neimark-Sacker cases can be handled in a similar way. Let M be the matrix in (4.2). By taking derivatives of (4.2) with respect to z we obtain

$$M\begin{bmatrix} v_z \\ s_z \end{bmatrix} + \begin{bmatrix} A_z^{(K)} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v \\ s \end{bmatrix} = 0. \tag{6.9}$$

Using (4.4) we obtain

$$s_z = -w^T(A^{(K)})_z v. \tag{6.10}$$

If z represents one of the state variables, then $s_{x_i} = -\langle w, B^{(K)}(e_i, v)\rangle$ as computed in section 6.1. When z is a parameter α_k we can write

$$s_{\alpha_k} = \sum_{J=1}^{K} C_J, \tag{6.11}$$

where

$$C_J = -w^T f_x(x^K)\ \cdots\ (f_x(x^J))_{\alpha_k} f_x(x^{J-1})\ \cdots\ f_x(x^1)v \tag{6.12}$$

where $J = 1, \ldots, K$. In this expression

$$(f_x(x^J))_{\alpha_k} = [f_x(f^{(J)}(x^1,\alpha))]_{\alpha_k} = f_{x\alpha}(x^J,\alpha) + B(x^J)T_J \tag{6.13}$$

where T_J is a vector, that can be recursively defined by

$$T_J = f_{\alpha_k}(x^{J-1}, \alpha) + A(x^{J-1})T_{J-1}, \quad T_1 = 0. \tag{6.14}$$

Summarizing, for the computation of s_α we need to compute $f_x, f_{\alpha_k}, f_{xx}, f_{x\alpha_k}$ in all iteration points $x^1, \ldots, x^K$, and given these compute T_J for $J = 1, \ldots K$. Then

$$C_J = -w^T A(x^K) \cdots (f_{x\alpha_k}(x^J) + B(x^J)T_J)A(x^{J-1}) \cdots A(x^1)v \tag{6.15}$$

and s_{α_k} is computed via (6.11).

6.3. Numerical computation of the directional derivatives. If symbolic derivatives of the original map are not available, then finite differences have to be used. However not the full tensors are needed, but the multilinear forms evaluated on vectors which can be computed with directional derivatives and central finite differences. This is an option of last resort and which is not reliable for very high iterates. For a general discussion of directional derivatives we refer to [14]. Here we only explain how we choose an increment h in the computation of the directional derivatives $Aq, B(q,q), C(q,q,q), D(q,q,q,q)$ and $E(q,q,q,q,q)$ for a given function $f : \mathbb{R}^n \to \mathbb{R}^n$. In fact, we want to choose h to minimize the combination of truncation and roundoff errors in the computation of the multilinear functions A, B, C, D and E. The analysis for A, B, and C is similar to that in [13], Appendix D.

We start with the Taylor expansion of $f(x \pm hq)$ w.r.t. h

$$f(x \pm hq) = f(x) \pm hf_x q + \frac{h^2}{2} f_{xx}qq \pm \frac{h^3}{6} f_{xxx}qqq + \mathcal{O}(h^4). \tag{6.16}$$

A little algebra yields

$$f_x q = \frac{f(x+hq) - f(x-hq)}{2h} - \frac{h^2}{6} f_{xxx}qqq + \mathcal{O}(h^4). \tag{6.17}$$

An unavoidable consequence of using numerical differentiation formulas like (6.17) is roundoff error. Taking into account this error and ignoring the $\mathcal{O}(h^4)$ term, the approximation formula (6.17) can be written as

$$f_x q = \left(\frac{f(x+hq) - f(x-hq)}{2h} \right)_{fl} + e_t(h) + e_r(h), \tag{6.18}$$

where $_{fl}$ is the floating point approximation, $e_r(h)$ the roundoff error and $e_t(h)$ is the truncation error. The total error $R(h)$ is bounded by

$$\|R(h)\| \leq \|e_t(h)\| + \|e_r(h)\|. \tag{6.19}$$

The roundoff error comes from the subtraction $f(x+hq) - f(x-hq)$. If C_0 is the norm of f and ϵ_m is machine precision, then $\|e_r(h)\| \leq C_1 C_0 \epsilon_m / 2h$, where C_1 is a modest constant. For the truncation error we assume that $\|f_{xxx}qqq\|$ is of the order of f, so $\|e_t(h)\| \leq h^2 C_0 C_2 / 6$. If also C_2 is a modest constant, then the choice $h_{min} \approx \epsilon_m^{1/3}$ minimizes $\|R(h)\|$. Using double-precision we have $\epsilon_m \approx 10^{-15}$ and thus $h \equiv h_1 = 10^{-5}$, which is the default value in CL_MATCONT.

Performing a similar procedure for derivatives of order k yields $h_k \approx (\epsilon_m)^{1/(k+2)} = (h_1)^{3/(k+2)}$ as an optimal stepsize. In CL_MATCONT, the *Increment* ($= h_1$) can be adjusted by the user. The increments of the higher-order derivatives are then adapted according to the above formulas.

7. Examples. Here we give two examples illustrating the developed techniques and their implementation in CL_MATCONT.

7.1. Generalized Hénon map. Consider the map

$$F : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ \alpha - \beta x_1 - x_2^2 + r x_1 x_2 \end{pmatrix}, \tag{7.1}$$

where r is in principle not too large. This map appears in numerous theoretical studies of degenerate homoclinic bifurcations. For bifurcation analysis of this map with $r = 0$ we refer to [9] and to [10] in general. It is known that this planar map exhibits the first four treated codim 2 bifurcations. Let us start with the 1:2 resonance and the fold-flip. For these cases we can apply the algorithms analytically, i.e., with r as a parameter. We note that $q = (1, -1)^T$ in both cases is an eigenvector of the Jacobian matrix corresponding to eigenvalue -1.

For the 1:2 resonance we have $(x_1, x_2) = (0, 0)$ and $(\alpha_0, \beta_0) = \left(\frac{4(3-r)}{(2-r)^2}, \frac{2+r}{2-r}\right)$. The critical center manifold reduction yields, see [10],

$$C_1 = -\frac{1+r}{2}, \quad D_1 = \frac{1}{4}(6 + 5r + r^2). \tag{7.2}$$

Applying the algorithm from Section 5.3, we find

$$v_{10} = \begin{pmatrix} \frac{-2}{2-r} \\ -1 \end{pmatrix}, \quad v_{01} = \begin{pmatrix} \frac{-4(3-r)}{(2-r)^2} \\ -\frac{-2}{2-r} \end{pmatrix}, \quad \tilde{p} = \begin{pmatrix} 1 \\ -\frac{2+r+r^2}{2(1+r)} \end{pmatrix} \varepsilon, \tag{7.3}$$

So that our prediction for the NS bifurcation curve of the period-2 cycle is

$$(x, y) = \left(\frac{2}{2-r}, \frac{2}{2-r}\right) + \sqrt{\frac{2\varepsilon}{1+r}}\, q,$$

$$(\alpha, \beta) = \left(\frac{4(3-r)}{(2-r)^2}, \frac{2+r}{2-r}\right) + \left(\frac{2(4+3r^2-r^3)}{(1+r)(2-r)^2}, \frac{2r^2}{(1+r)(2-r)}\right) \varepsilon.$$

The fold-flip bifurcation occurs for $(x_1, x_2) = (0, 0)$ at $(\alpha_0, \beta_0) = (0, -1)$. The critical center manifold reduction yields

$$a = \frac{1}{2}(1-r), \quad b = \frac{1}{2}(1+r), \quad c_2 = -\frac{1}{4}(1-r), \quad c_4 = \frac{1}{4}(1+r)^2. \tag{7.4}$$

Then applying the algorithm from Section 5.6, we find

$$v_{10} = \begin{pmatrix} -2 \\ \frac{r^2}{2-r} \end{pmatrix}, \quad v_{01} = \begin{pmatrix} 0 \\ \frac{-2}{2-r} \end{pmatrix}, \quad \tilde{p} = \begin{pmatrix} -1 \\ \frac{(1-r)r^2}{2(1+r)} \end{pmatrix} \varepsilon. \tag{7.5}$$

Therefor our prediction for the NS bifurcation curve of the period-2 cycle emanating here is

$$(x, y) = (0, 0) + \sqrt{\frac{2\varepsilon}{1+r}}\, q, \quad (\alpha, \beta) = (0, -1) + \left(2, \frac{-2r^2}{(1+r)(2-r)}\right) \varepsilon.$$

Let us compare the predictions with the exact expressions for these curves. Consider the following set

$$\alpha = (1+\beta)(\beta - 1 - r + r^2)/r^2, \tag{7.6}$$

$$x_1 = \frac{r(\beta+1) + \sqrt{(r-2)(\beta+1)(2+r-\beta(2-r))}}{2r}, \tag{7.7}$$

$$x_2 = \frac{r(\beta+1) - \sqrt{(r-2)(\beta+1)(2+r-\beta(2-r))}}{2r}, \tag{7.8}$$

when

$$\frac{1}{4}(3-r)(\beta+1)^2 \leq \alpha \leq \frac{5+4r+r^2+2\beta(3+r)+\beta^2(5+2r-r^2)}{2(2+2r+r^2)}.$$

It consists of two different pieces, where a Neimark-Sacker bifurcation of a cycle of period 2 occurs. If we take $0 \leq \varepsilon \ll 1$ and consider the linear approximations of (7.6) near $\beta = -1 - \frac{r^2}{2-r}\tilde{\varepsilon}$ and $\beta = \left(\frac{2+r}{2-r}\right) + \frac{r^2}{2-r}\tilde{\varepsilon}$ we find for the 1:2 resonance

$$\begin{aligned}
(\alpha, \beta) &= \left(\frac{4(3-r)}{(2-r)^2}, \frac{2+r}{2-r}\right) + \left(\frac{4+3r^2-r^3}{(2-r)^2}, \frac{r^2}{2-r}\right)\tilde{\varepsilon} + \mathcal{O}(\tilde{\varepsilon}^2).\\
(x, y) &= \left(\frac{2}{2-r}, \frac{2}{2-r}\right) + \sqrt{\tilde{\varepsilon}}q + \mathcal{O}(\tilde{\varepsilon})
\end{aligned}$$

and for the fold-flip bifurcation

$$\begin{aligned}
(\alpha, \beta) &= (0, -1) + \left(1+r, \frac{-r^2}{2-r}\right)\tilde{\varepsilon} + \mathcal{O}(\tilde{\varepsilon}^2).\\
(x, y) &= (0, 0) - \sqrt{\tilde{\varepsilon}}q + \mathcal{O}(\tilde{\varepsilon})
\end{aligned}$$

So up to positive factor our results coincide up to first order in ε.

For the other two cases a numerical approach is more illuminating. There is a period-doubling curve of period 4 cycles along which there are two degenerate flips near $(\alpha, \beta) = (3, -1)$. For $r = 0$ we found a fold curve and a period-doubling curve of cycles of period 4. Then we produced the approximations to the fold curves of the 8-th iterates in the degenerate flip points and from these easily continued the fold curves of cycles of period 8. In Figure 7.1(a) we show the continuation results and also the approximation curves. For the 1:4 resonance we used $r = -0.1$. For this specific value of r, the $1:4$-resonance involves the mentioned local branches. In Figure 7.1(b) we show the continuation results and also the approximate curves.

7.2. Leslie–Gower competition model. The origin of this example is in [16, 19, 11]. It has been found in biological experiments that two species of flour beetles can coexist under strong competition for the same food. This was rather unexpected at the time and several models were built to explain this phenomenon. One of the ideas proposed in [11] and [23] was to use an age-structured population model. For general background we refer to [5]; the model that we actually use is a four-dimensional map

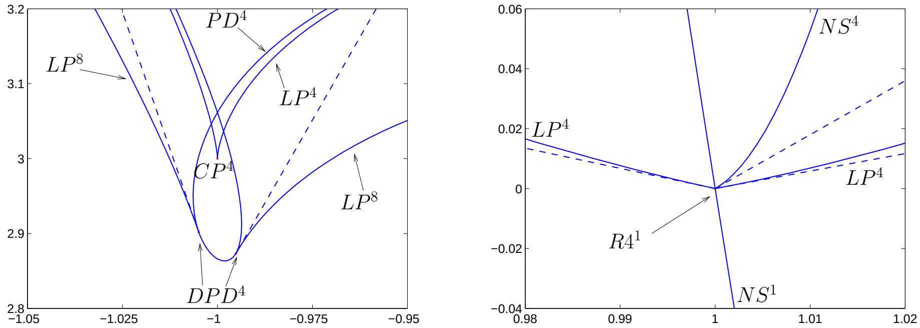

FIG. 7.1. *The straight lines are computed with* CL_MATCONT*, the dashed lines are the predictions from the switching algorithms.*

M_{LG} (7.9) with 14 parameters described in [23]:

$$
M_{LG}: \begin{pmatrix} j \\ a \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} j' \\ a' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \dfrac{b_1 a}{(1 + c_{jj}j + c_{ja}a + c_{jy}y + c_{jz}z)} \\ (1-\mu_j)j + (1-\mu_a)a \\ \dfrac{b_2 z}{(1 + c_{yj}j + c_{ya}a + c_{yy}y + c_{yz}z)} \\ (1-\mu_y)y + (1-\mu_z)z \end{pmatrix} \tag{7.9}
$$

This is the Leslie–Gower competition model for the interaction between the juveniles (j) and adults (a) of one species of the flour beetle *Tribolium* and the juveniles (y) and adults (z) of another species for the same food. Each species has its own juvenile recruitment rate $b_1 > 0$, $b_2 > 0$, juvenile death rate μ_j and μ_y, and adult death rate μ_a and μ_z. For biological reasons we have

$$
0 < \mu_j, \mu_a, \mu_y, \mu_z < 1. \tag{7.10}
$$

The other coefficients $c_{jj}, c_{ja}, c_{jy}, c_{jz}$ and $c_{yj}, c_{ya}, c_{yy}, c_{yz}$ describe the competition. They are all strictly positive. By assumption, the competition does not affect the adults of either species. As in [23], we will study the influence of the coefficients c_{yj} and c_{jy} on the behavior of M_{LG} in a case where all other parameters are fixed. In other words, we study the role of the competition between juveniles alone.

The origin $(0,0,0,0)$ is a fixed point of (7.9) but is of little interest. The model also has 'horizontal' fixed points, i.e., fixed points of the form $(j^*, a^*, 0, 0)$ given by

$$
j^* = \frac{b_1(1-\mu_j) - \mu_a}{\mu_a c_{jj} + c_{ja}(1-\mu_j)}, \quad a^* = \frac{1-\mu_j}{\mu_a} j^* = \frac{b_1(1-\mu_j)^2 - \mu_a(1-\mu_j)}{\mu_a(\mu_a c_{jj} + c_{ja}(1-\mu_j))}, \tag{7.11}
$$

where $j^*, a^* > 0$ (i.e., biologically meaningful) when

$$
b_1(1-\mu_j) > \mu_a.
$$

Similarly, there are unique 'vertical' fixed points of the form $(0,0,y^*,z^*)$ given by

$$
y^* = \frac{b_2(1-\mu_y) - \mu_z}{\mu_z c_{yy} + c_{yz}(1-\mu_y)}, \quad z^* = \frac{1-\mu_y}{\mu_z} y^* = \frac{b_2(1-\mu_y)^2 - \mu_z(1-\mu_y)}{\mu_z(\mu_z c_{yy} + c_{yz}(1-\mu_y))} \tag{7.12}
$$

TABLE 7.1
Parameter values for the Leslie–Gower model.

$b_1 = 20$	$c_{jj} = 0.36$	$b_2 = 18$	$c_{ja} = 0.55$	$c_{jz} = 0.23$	$\mu_j = 0.23$
$\mu_a = 0.72$	$c_{ya} = 0.08$	$c_{yy} = 0.18$	$c_{yz} = 0.26$	$\mu_y = 0.29$	$\mu_z = 0.98$

These are biologically meaningful if $y^*, z^* > 0$, i.e., when

$$b_2(1-\mu_y) > \mu_z.$$

Furthermore there exists a unique coexistence fixed point (j^*, a^*, y^*, z^*) with

$$\begin{aligned} j^* &= \frac{\gamma(b_2\beta - 1) - (b_1\alpha - 1)\eta}{\delta\gamma - \epsilon\eta}, \\ a^* &= \alpha\left(\frac{\gamma(b_2\beta - 1) - (b_1\alpha - 1)\eta}{\delta\gamma - \epsilon\eta}\right), \\ y^* &= \frac{-\epsilon(b_2\beta - 1) + (b_1\alpha - 1)\delta}{\delta\gamma - \epsilon\eta}, \\ z^* &= \beta\left(\frac{-\epsilon(b_2\beta - 1) + (b_1\alpha - 1)\delta}{\delta\gamma - \epsilon\eta}\right), \end{aligned}$$

provided

$$H \equiv \delta\gamma - \epsilon\eta \neq 0, \tag{7.13}$$

where

$$\alpha = \frac{1-\mu_j}{\mu_a}, \quad \beta = \frac{1-\mu_y}{\mu_z}, \quad \epsilon = c_{jj} + c_{ja}\alpha$$

and

$$\gamma = c_{jy} + c_{jz}\beta, \quad \delta = c_{yj} + c_{ya}\alpha, \quad \eta = c_{yy} + c_{yz}\beta.$$

The equation $H = 0$ defines a hyperbola in (c_{yj}, c_{jy}) space.

We will study the overall dynamics of the model for the parameter values specified in Table 7.1. The parameters c_{jy} and c_{yj} will vary.

First we consider the horizontal and the vertical fixed points and their stability. For all values of c_{jy} and c_{yj}, the fixed point obtained from (7.11)

$$F_H = (21.50285631, 22.99611022, 0, 0)$$

remains unchanged since c_{jy} and c_{yj} do not appear in (7.11). For the given model parameters, the horizontal fixed point is stable if $c_{yj} > c_{yj0}$, where $c_{yj0} = 0.474477674$. It is biologically plausible that the horizontal fixed point is stable only if the juveniles of the first species suppress the juveniles of the second species to a sufficient degree. The vertical fixed point (7.12) is stable if $c_{jy} > c_{jy0}$, where $c_{jy0} = 0.4571312026$.

Now we consider the coexistence fixed point (j^*, a^*, y^*, z^*), starting the continuation from

$$F_C = (16.42912, 17.570032, 28.871217, 20.916902),$$

where $c_{yj} = c_{jy} = 0$. This fixed point bifurcates into vertical and horizontal fixed points respectively, when one of c_{jy} and c_{yj} is varied and the other variable is fixed at 0.

In the model this means that one species drives another to extinction. Continuation of the coexistence fixed point, where c_{jy} is the free parameter leads to `BP` and `PD` bifurcations at $c_{jy} = c_{yj0}$ and $c_{jy} = 0.170849$, respectively. The coexistence fixed points bifurcate into vertical fixed points at the `BP`. The coexistence fixed point is stable before the `BP` and unstable afterwards, this reconfirms the above analytical results. If we continue the coexistence fixed points with the free parameter c_{yj}, it bifurcates into the horizontal fixed point at another `BP`. The coexistence fixed point is stable before this `BP` and unstable afterwards.

The solutions to the equation $H = 0$, where H is given by (7.13), are the parameter values for which the existence and uniqueness of the coexistence fixed point are not guaranteed. In the present context, where only c_{yj} and c_{jy} vary, this leads to the condition

$$c_{yj}c_{jy} + ac_{yj} + bc_{jy} - c = 0, \tag{7.14}$$

where $a = 0.1666326531$, $b = 0.0855555552$, and $c = 0.3350275226$, which indeed defines a hyperbola in (c_{yj}, c_{jy}) space. It is not hard to prove that the point (c_{yj0}, c_{jy0}) lies on the hyperbola.

Now we return to the stability of the coexistence fixed points. The coexistence fixed point is unstable outside the rectangle

$$S_1 = \{(c_{yj}, c_{jy}) : 0 \leq c_{yj} \leq c_{yj0}, \quad 0 \leq c_{jy} \leq c_{jy0}\}.$$

Figure 7.2 shows the hyperbola $H = 0$ and the rectangle S_1.

Inside S_1 the stability properties of the coexistence fixed point are more complicated. By numerical continuation we find that there is an interior region in which the coexistence fixed points are unstable. This region is bounded by the `PD` curve, where the stability changes. The projection of the `PD` curve on the (c_{yj}, c_{jy})-plane goes twice through the point (c_{yj0}, c_{jy0}). Indeed, the `PD` curve has two fold-flip points, where $(c_{yj}, c_{jy}) = (c_{yj0}, c_{jy0})$. Moreover, there are two degenerate period-doubling points `DPD` on the `PD` curve: $(c_{yj}, c_{jy}) = (0.210138, 0.383143)$ and $(c_{yj}, c_{jy}) = (454279, 0.297779)$. The branches of fold curves of the second iterate can be computed by switching at the `DPD` points. These curves emanate tangentially to the `PD` curve.

The region where there are stable fixed points of the second iterate is bounded by the two fold curves of the second iterate and the lower left part of the `PD` curve. From the applications point of view, this is the most interesting region because it shows that indeed the two species can coexist even when the competition is strong. We note that if both c_{yj} and c_{jy} are larger than 0.5 then the horizontal fixed points, the vertical fixed points and the fixed points of the second iterate are all stable. The `PD` curve and the fold curves of the second iterate are given in Figure 7.2.

It can be shown analytically that there is a straight line of coexistence fixed points for the fixed parameter values $(c_{yj}, c_{jy}) = (c_{yj0}, c_{jy0})$ which bifurcates to the horizontal and vertical fixed points where $c_{jy} = c_{jy0}$ and $c_{yj} = c_{yj0}$ respectively. This straight line can be found numerically by switching to the fold curve in the fold-flip (`LPPD`) points of the flip curve since technically it is a curve of (degenerate) fold points of the original Leslie–Gower map. It contains a horizontal fixed point, a vertical fixed point and two coexistence fixed points of flip type.

8. Conclusions. We have described the implementation of continuation and normal form analysis of fixed points and cycles in MATCONT. We use minimally extended systems, which proved to be one of the best methods.

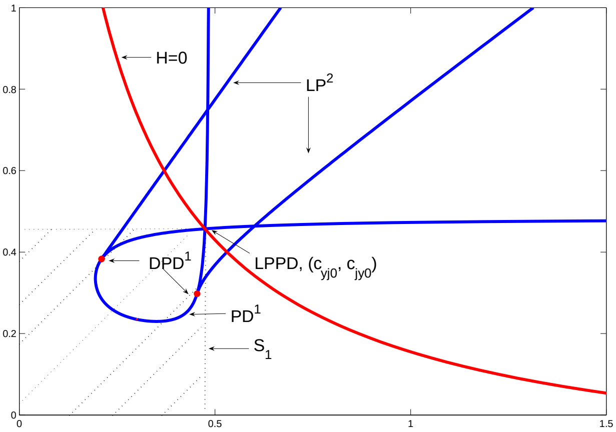

FIG. 7.2. *The flip curve PD^1, the fold curve of the second iterate LP^2, hyperbola $H = 0$ and the rectangle S_1 in (c_{yj}, c_{jy})-plane.*

Codim 1 bifurcation analysis is standard by now, while the analysis of codim 2 bifurcations, both quantitative and qualitative, on which this paper focuses, is new. Our implementation uses all aspects of the center-manifold reduction. First, the critical normal form coefficients are calculated automatically, which determines the type of the unfolding. Second, when nondegeneracy and transversality are checked, predictions about new codim 1 branches are made.

We mention a few problems for future work. We use the iterated maps in case of cycles. When we are dealing with stiff systems and small basins of attraction, a BVP-approach using extended systems for the fixed point, i.e., $f(x^1) - x^2 = 0, \ldots, f(x^n) - x^1 = 0$, might be more efficient. Another idea is to use automatic differentiation as an alternative to symbolic derivatives. One reason is that symbolic derivatives may not always be available. The second is that preliminary evidence suggests that the computation of normal form coefficients for high iterates is faster when automatic differentiation is used. Then we did not discuss global bifurcations. As a first step, it is well worth the effort to incorporate in our package some of the recent algorithms, [25] to compute (un)stable manifolds. If a transverse intersection appears to be present, then using the defining systems as in [26], approximations of homoclinic orbits may be computed.

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