YAU ALGEBRAS OF FEWNOMIAL SINGULARITIES

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Abstract. Following an approach of S.S-T.Yau, we investigate finite-dimensional solvable Lie algebras associated with isolated hypersurface singularities. More precisely, we deal with Yau algebras of isolated hypersurface singularities defined by polynomials with the number of monomials equal to the number of variables. For certain classes of such singularities, we show that the analytic isomorphism type of singularity is determined by its Yau algebra. Our results extend a similar statement for simple singularities recently established by A.Elashvili and the present author. We also touch upon a number of related results and open problems concerned with Yau algebras of fewnomials.

Key words: isolated hypersurface singularity, moduli algebra, Milnor number, derivation, vector field, Pursell-Shanks theorem, solvable Lie algebra, Yau algebra, fewnomial

Introduction

The aim of this paper is to present several results about the finite-dimensional Lie algebras associated with germs of *isolated hypersurface singularities* (IHS) defined by polynomials with the number of monomials equal to the number of variables. Recall that following S.S.-T.Yau [24], with any IHS germ $X = X(f) = \{f = 0\}$ one associates the Lie algebra of derivations $L(X) = \mathrm{Der}_{\mathbb{C}}(A(X), A(X))$ of the factor-algebra $A(X) = O_n/(f, df)$, where O_n is the algebra of convergent power series in n indeterminates, $f \in O_n$, and (f, df) is the ideal in O_n generated by f and all of its partial derivatives $\partial_i f = \frac{\partial f}{\partial x_i}, i = 1, \ldots, n$. According to S.S.-T.Yau, $L(X)$ is a finite-dimensional solvable Lie algebra which is often called the *Lie algebra of singularity* X [24]. Following [26] we call it the *Yau algebra* of X in order to distinguish from Lie algebras of other types appearing in singularity theory [4], [3], [16].

There exist a number of interesting general problems concerned with Yau algebras of IHS. The most natural one, called the *recognition problem* (cf. [24], [6]), is to find a characterization of the class of solvable Lie algebras arising in such way. This problem appeared quite difficult and no substantial progress has been reached up to now in the general setting. However, it seems worthy of mentioning that Yau algebras have been computed for several important classes of isolated singularities including simple singularities in the sense of Arnol'd and simple elliptic singularities [6], [22]. Eventually, this enabled A.Elashvili and the present author to reveal several specific structural properties of Yau algebras of simple singularities, which may hopefully serve as a pattern for solving the recognition problem for special classes of singularities [12], [13].

Another conceptual problem is to find out, to which extent does a Yau algebra determine the analytic or topological structure of singularity. In particular, it is natural to wonder which analytic and/or topological invariants of a given singularity can be restored from its Yau algebra. This circle of topics can be referred to as *restoration problem.* Since derivations of function algebras are analogs of vector fields on smooth manifolds, this problem is much in the spirit of the classical theorem of L.Pursell and M.Shanks stating that the Lie algebra of smooth vector fields on a smooth manifold determines the diffeomorphism type of manifold [21]. It is precisely the paradigm which we follow in the sequel and in order to refer to this classical result in a wide sense, it is convenient to speak of *Pursell-Shanks theorem* (PST) and *Pursell-Shanks paradigm* (PSP).

Accepting this terminology, the main topic of this paper can be described as investigating the PS-paradigm in the context of Yau algebras. The main results (Theorem 4.1, Theorem 5.1) yield analogs of PST for some classes of isolated singularities containing simple singularities in the sense of Arnold and other classes of singularities considered in [13]. Moreover, our results suggest a plausible conjecture about the range of validity of Pursell-Shanks type theorems in the context of Yau algebras .

In order to formulate this conjecture, it appears convenient to use a concept of fewnomial introduced by A.Khovanski [18] in the form adjusted to our situation. Namely, we say that a polynomial P in n variables is a *fewnomial* if the number of monomials appearing in P does not exceed n. It is easy to show that, except for certain trivial cases, a fewnomial in n variables can define an IHS only if it has exactly n monomials, in which case we speak of *fewnomial (isolated) singularity.* In other words, fewnomial singularities are those which can be defined by n-nomials in n indeterminates. Simple singularities are obviously fewnomial in this sense. Actually, they can be defined by binomials so it appears natural to consider also the class of binomial (isolated) singularities.

Our first observation is that using the results and methods of [13] it appears possible to prove an analog of PST for Yau algebras of binomial singularities (Theorem 4.1 below). At the same time, from the results of [24], [6] it follows that there exist isolated singularities defined by trinomials in two variables which are analytically non-isomorphic but have isomorphic Yau algebras (cf. [22]). Thus we conclude that there is no hope for analogs of PST outside the class of fewnomial singularities. Taking into account that, in the case of two variables, fewnomial singularities are the binomial ones, we arrive at conjecture that PST holds exactly in the class of fewnomial singularities. In case if this appears true, we achieve a reasonable description of PST paradigm in the context of Yau algebras. After presenting some general evidence in favour of this conjecture we establish it for a certain class of fewnomial singularities containing the class of binomial singularities (Theorem 5.1), which is the second main result of this paper.

It should be noted that systematic study of Lie algebras of isolated hypersurface singularities has been started by S.S.-T.Yau and his collaborators in eighties (see, e.g., [24], [25], [6], [22]). A detailed survey of results obtained in that period can be found in [6]. However neither in [6] nor in other related publications, Yau algebras have been considered in relation to the PS-paradigm.

The only papers in this spirit seem to be those by A.Elashvili and the present author [12], [13], where it was established that PST holds for simple singularities in the sense of Arnold and some other series of singularities. The present paper extends the approach and results of [12], [13] to wider classes of singularities introduced below.

The paper is organized as follows. After recalling a few basic concepts and auxiliary general results in section 1, in section 2 we briefly discuss isolated singularities defined by fewnomials. In particular, we introduce some classes of fewnomials and fewnomial singularities playing central role in the sequel, such as Pham singularities and binomial singularities. We also show that the isolated singularities defined by binomials appear in three series $P_{*,*}, D_{*,*}, T_{*,*}$ indexed by pairs of natural numbers. Specifically, the first series consists of *Pham singularities* $\{x_1^{k_1} + x_2^{k_2} = 0\}$, the second series is defined by polynomials $x_1^{k_1}x_2 + x_2^{k_2}$, and the third one by polynomials $x_1^{k_1}x_2 + x_1x_2^{k_2}$, where $2 \leq k_1, k_2 \in \mathbb{Z}$. Further classes of isolated singularities defined by fewnomials are obtained by taking direct sums of binomial singularities and Pham singularities.

In the third section we develop auxiliary computational tools and give a detailed description of Yau algebras for the three infinite series of isolated singularities indicated above. We also present here the results about Yau algebras of Pham singularities obtained in [13]. The explicit results collected in this section are crucial for our discussion.

In the fourth section we prove that, except for just one pair, binomial isolated singularities are analytically classified by their Yau algebras (Theorem 4.1). This follows from the computations performed in section 2 by a sort of structural analysis of arising Lie algebras.

In the fifth section we present a similar result for certain fewnomials in arbitrary number of variables (Theorem 5.1). In conclusion we briefly discuss several problems and conjectures suggested by our approach.

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1. Generalities on singularities and Lie algebras

We present here necessary definitions and auxiliary results concerning hypersurface germs with isolated singularities and derivations of Lie algebras. To give a consistent description of the background and setting we begin with recalling necessary concepts and constructions from singularity theory.

Let $\mathbb{C}_n$ be the algebra of complex polynomials in n variables. Denote by O_n the algebra of germs of holomorphic functions in n variables at the origin which is naturally identified with the algebra of convergent power series in n indeterminates with complex coefficients. For a polynomial $f \in \mathbb{C}_n$, denote by X the germ at the origin of $\mathbb{C}^n$ of hypersurface $X = \{f = 0\} \subset \mathbb{C}^n$.

We say that X is a germ of isolated hypersurface singularity if the origin is an isolated zero of the gradient of f. The local (function) algebra of X is defined as the (commutative associative) algebra $F(X) \cong O_n/(f)$, where (f) is the principal ideal generated by the germ of f at the origin. Further, denote by (f, df) the ideal in O_n generated by f and all of its partial derivatives. Recall that, for an isolated singularity $X = X(f) = \{f = 0\}$ as above, from the Hilbert's Nullstellensatz immediately follows that the factor-algebra $A(X) \cong O_n/(f, df)$ is finite dimensional. This factor-algebra is called the *moduli algebra* of X. An important result of J.Mather and S.S.-T.Yau states that the analytic isomorphism type of an isolated hypersurface singularity is determined by the isomorphism class of its moduli algebra [19].

Remark 1.1. As is well known, the moduli algebra $A(X)$ can serve as a base space of versal deformation of singularity X [4]. Its (complex) dimension $\tau(X)$ is often called the Tyurina number of X [4].

In many problems it is necessary to have an explicit basis of $A(X)$. It is well known and easy to prove that there always exist bases consisting of monomials. Such bases are called monomial bases and will be often used in the sequel. We are basically interested in the so-called simple singularities [4] which consist of two series $A_k : \{x^{k+1} = 0\} \subset \mathbb{C}$, $D_k : \{x^2y + y^{k-1} = 0\} \subset \mathbb{C}^2$ and three exceptional singularities E_6, E_7, E_8 defined in $\mathbb{C}^2$ by polynomials $x^3 + y^4, x^3 + xy^3, x^3 + y^5$, respectively. Monomial bases in moduli algebras of simple singularities are given in [4]. In order to be able to compare singularities defined by polynomials of different number of variables, several equivalence relations are used in singularity theory.

Two IHS are called (analytically) equivalent if they are isomorphic as germs of algebraic varieties [4]. It is often convenient to use another equivalence relation between IHS. If $f \in \mathbb{C}_n$ defines an IHS $X = X(f)$ then it is obvious that $g = f + x_{n+1}^2$ also defines an IHS in $\mathbb{C}^{n+1}$ which is called stabilization of X. Two singularities are called stably equivalent if they can be obtained as iterated stabilizations of the same IHS. It is easy to see that the moduli algebra is not changed under taking suspensions so stably equivalent singularities have isomorphic moduli algebras [4].

As was already mentioned, for our purposes it is sufficient to deal with homogeneous and quasihomogenous polynomials. Recall that a polynomial $f \in \mathbb{C}_n$ is called quasihomogeneous (qh) if there exist positive rational numbers $w_1, \dots, w_n$ (called weights of indeterminates x_j) and d such that, for each monomial $\prod x_j^{k_j}$ appearing in f with nonzero coefficient, one has $\sum w_j k_j = d$. The number d is called the quasihomogeneous degree (w-degree) of f with respect to weights w_j and denoted $w \deg f$. Obviously, without loss of generality one can assume that $w \deg f = 1$ and we will often do so in the sequel. The collection $(\mathbf{w}; d) = (w_1, \dots, w_n; d)$ is called the quasihomogeneity type (qh-type) of f. As is well known, for such an f, one has $df = \sum w_j x_j \partial_j f$ (Euler formula). Hence in this case $f \in (df)$. Moreover, the w-degree defines natural gradings on $F(X)$ and $A(X)$ called qh-gradings. Thus one can introduce the Poincaré polynomials with respect to these gradings and in many cases they can be explicitly computed in terms of qh-type (see, e.g., [15], [4]).

If singularity X is defined by quasihomogeneous polynomial f, then the Tyurina number $\tau(X)$ coincides with the *Milnor number* $\mu(X)$ which is defined as $\dim_{\mathbb{C}} M(X)$, where $M(X) = O_n/(df)$ is the so-called *Milnor algebra* of X [19]. The equality $\tau(X) = \mu(X)$ for quasihomogeneous polynomial f immediately follows from the aforementioned fact that f belongs to the ideal (df) generated by its derivatives [4]. Thus in the quasihomogeneous case $A(X) \cong M(X)$. As is well known, $\mu(X)$ a topological invariant of germ X which plays important role in many problems of singularity theory [4]. In quasihomogeneous case the Milnor number can be computed by a simple formula which will be repeatedly used in the sequel.

Proposition 1.1. *For an isolated hypersurface singularity X defined by a quasihomogeneous polynomial of $(\mathbf{w}; d)$ type, one has*

$$\tau(X) = \mu(X) = \prod_{i=1}^{n} \frac{d - w_i}{w_i}. \tag{1.1}$$

Remark 1.2. We wish to emphasize that throughout the whole paper we only deal with singularities defined by quasihomogeneous polynomials. Thus in our setting there is no difference between the moduli algebra and Milnor algebra, and the Milnor number $\mu(X)$ can be computed by the formula (1.1). Also, we often write $A(f)$ and $M(f)$ instead of $A(X)$ and $M(X)$ where this cannot cause misunderstanding.

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $\operatorname{Der} A$ with the bracket defined by the commutator of linear endomorphisms. In particular, for a singularity X as above, one can consider the Lie algebras $DF(X) = \operatorname{Der} F(X)$ and $DA(X) = \operatorname{Der} A(X)$. Since by the aforementioned result of L.Pursell and M.Shanks the algebra $C^{\infty}(M)$ of smooth functions on a smooth manifold M is completely determined by the Lie algebra of its derivations, one can wonder if the same holds for algebras $F(X)$ and $A(X)$. Notice that if this is the case, then by the mentioned result of J.Mather and S.S.-T.Yau the corresponding Lie algebra determines the analytic isomorphism type of the singularity considered. As was shown by H.Hauser and G.Müller, for an isolated hypersurface singularity X, the Lie algebra $DF(X)$ indeed determines the analytic type of X [16]. In other words, PST holds for Lie algebras of the form $DF(X)$.

Elegant as it is, this result is not quite effective because $DF(X)$ is an infinite-dimensional Lie algebra which is difficult to investigate and work with. At the same time $DA(X)$ is typically a finite-dimensional Lie algebra and its structural constants may be found in an algorithmic way. Moreover, S.S.-T.Yau showed that, for any isolated hypersurface singularity X, $DA(X)$ is a solvable Lie algebra [25]. Thus one may hope to identify such Lie algebras in concrete cases using a plethora of known results on classification of solvable and nilpotent Lie algebras. Moreover, some natural numerical invariants of such Lie algebras can be effectively computed and it is natural to try to relate them to the numerical invariants of the singularity considered.

These are the two main directions of research which we pursue in this paper. Notice at once that there are no a priori reasons why an analog of the result of H.Hauser and G.Müller may hold for $DA(X)$ because this is a much smaller algebra than $DF(X)$. Actually, as was shown in [12], for simple singularities A_6 and D_5 one has $DA(X) \cong DA(Y)$ but A_6 is not of course analytically isomorphic to D_5. So it came as a sort of surprise for us when it turned out that $DA(X)$ is a complete invariant for all simple singularities with Milnor number bigger than 6 [12]. Actually, this fact served as an impetus for further research in this direction. In the present paper we concentrate on investigation of Lie algebras of the form $DA(X)$ for isolated singularities defined by binomials. For clarity and convenience, it seems appropriate to explicitly present the main concept and related terminology in a separate definition.

Definition 1.1. Let $X = \{f = 0\}$ be a germ of isolated hypersurface singularity at the origin of $\mathbb{C}^n$ defined by complex polynomial $f \in \mathbb{C}_n$. The Lie algebra $\operatorname{Der} A(X)$ of derivations of the moduli algebra $A(X) = O_n/(f, df)$ is called the Yau algebra of X and denoted $L(X)$. Its dimension is called the Yau number of X and denoted $\lambda(X)$.

Two technical remarks are now in order. Firstly, elements of $L(X)$ can be represented as holomorphic vector fields $V = \sum h_i \partial_i, h_i \in O_n$ considered with the standard action on $O_n : Vg = \sum h_i \partial_i g$. Such a vector field V defines an element of $L(X)$ if and only if it leaves the ideal (f, df) invariant, i.e., for each $g \in (f, df)$, one has $Vg \in (f, df)$. It is obvious that in such case the standard action V can be pulled down to $A(X)$ and defines a derivation $\hat{V}$ of $A(X)$ since the Leibniz rule is trivially fulfilled for $\hat{V}$. We often omit the "hat" and denote the corresponding element of $L(X)$ simply by V.

Secondly, the coefficients h_i in vector field presentation of an element $V \in L(X)$ can be reduced modulo the ideal (f, df) which implies that one can always construct a basis $V_j = \sum_{i=1}^n h_j^i \partial_i, j = 1, \ldots, \lambda(X)$, in $L(X)$ such that all coefficients h_j^i are monomials. However an important caveat is appropriate here: there need not always exist a basis consisting of elements of the form "monomial $\times\, \partial_i$" so, in general, one needs to take linear combinations of elementary vector fields ∂_i with monomial coefficients.

In order to describe the structural properties of Lie algebras of hypersurface singularities in an appropriate way, we will make use of various notions and results from the theory of Lie algebras. All algebraic definitions and results used in the sequel can be found, e.g., in [8]. For convenience of the reader some of the most frequently used concepts and results are collected in the rest of this section.

Recall that a Cartan subalgebra C in Lie algebra L is defined as a maximal commutative subalgebra consisting of semi-simple elements. If L is the Lie algebra of an algebraic Lie group then all Cartan subalgebras are pairwise conjugated, hence of the same dimension r which is called the rank $\mathrm{rk}L$ of Lie algebra L [8]. The index $\mathrm{ind}L$ defined as the minimal codimension of orbits in its coadjoint representation $\mathrm{ad}^* : L \to \mathrm{End}(L^*)$, where L^* is the space dual to L [9]. It is well known that $\dim L + \mathrm{ind} L$ is always even and the maximal dimension of commutative subalgebras in a Lie algebra does not exceed $\frac{1}{2}(\dim L + \mathrm{ind} L)$.

We will basically deal with solvable and nilpotent Lie algebras so for completeness we recall the corresponding definitions. Given a Lie algebra L, introduce two series of ideals: $L_{(*)} = \{L_{(i)}\}, L^{(*)} = \{L^{(i)}\}, L_{(0)} = L^{(0)} = L, L_{(1)} = L^{(1)} = [L, L], L_{(i)} = [L, L_{(i-1)}], L^{(i)} = [L^{(i-1)}, L^{(i-1)}], i = 2, 3, \ldots$. Lie algebra is called nilpotent if the series $L_{(*)}$ (the lower central series of L) contains only a finite number of non-zero ideals. Lie algebra is called solvable if the series $L^{(*)}$ contains a finite number of non-zero ideals. According to Engel's theorem, Lie algebra is nilpotent if and only if all operators $\mathrm{ad}\, a : L \to L$ are nilpotent for $a \in L$ [8]. Another general result states that a solvable algebraic Lie algebra can be decomposed into semi-direct sum of a Cartan subalgebra and maximal nilpotent ideal $N(L)$ (the latter is called the nilpotent radical of L).

The following concepts and results enable one to compute the Yau algebras of many concrete singularities we are going to consider. Let A, B be associative algebras over a field F of characteristic zero which in the sequel will be either $\mathbb{R}$ or $\mathbb{C}$. Recall that the multiplication algebra $M(A)$ of A is defined as the subalgebra of endomorphisms of A generated by the identity element and left and right multiplications by elements of A. The centroid $C(A)$ is the set of endomorphisms of A which commute with all elements of $M(A)$. Clearly, $C(A)$ is a unital subalgebra of $\mathrm{End} A$. The following statement is a particular case of a general result from [7].

Proposition 1.2. *(cf. [7]) Let $S = A \otimes_F B$ be a tensor product of finite-dimensional associative algebras with units. Then*

$$DerS \cong (\mathrm{Der}\, A) \otimes C(B) + C(A) \otimes (\mathrm{Der}\, B). \tag{1.2}$$

We will only use this result for commutative associative algebras with unit, in which case the centroid coincides with the algebra itself. Thus for commutative associative algebras A, B one has:

$$\mathrm{Der}(A \otimes B) \cong (\mathrm{Der}\, A) \otimes B + A \otimes (\mathrm{Der}\, B). \tag{1.3}$$

The latter formula will be repeatedly used in the sequel.

Finally, we wish to notice that, for a quasihomogeneous IHS, one can obtain a natural grading on $L(X)$ by putting the weight of ∂_j equal to $-w_j$ [15]. Thus the weight of a vector field of the form $x_k^m \partial_j$ is equal to $mw_k - w_j$, which obviously defines a grading on $L(X)$ compatible with the standard one on $A(X)$ in the sense that the action of $L(X)$ becomes the action of graded Lie algebra.

2. Isolated singularities defined by fewnomials

Analyzing the results obtained in [6], [22], [13], the author came to a conjectural description of the class of IHS which gives a natural range of validity of Pursell-Shanks type theorems fro Yau algebras. In order to give a concise formulation of the corresponding results it is convenient to use the setting of the so-called *fewnomials* introduced in [18]. Let us first establish precise terminology which will be different from the setting of [18] where the term "fewnomial" has been introduced.

Let P be a polynomial in n variables. We'll say that P is a fewnomial if the number of monomials entering in P does not exceed n. Obviously, the number of monomials in P may depend on the system of coordinates. In order to obtain a rigorous concept we'll only admit linear changes of coordinates and say that P (or rather its germ at the origin) is a k-nomial if k is the smallest natural number such that P becomes a k-nomial after (possibly) a linear change of coordinates. For linguistic flexibility it is convenient to say in such case that the *nomiality* $\operatorname{nom} P$ of P is equal to k. Nomiality may be considered as a sort of elementary complexity measure of polynomials which appears relevant in some problems of enumerative algebraic geometry [18].

An isolated hypersurface singularity X is called k-nomial if there exists an IHS Y analytically isomorphic to X which can be defined by a k-nomial and k is the smallest such number. It turns out than, except for some noninteresting cases, a singularity defined by a fewnomial P can be isolated only if $\operatorname{nom} P = n$, i.e., if P is a n-nomial in n variables. We formulate this result separately for further reference.

Proposition 2.1. *A k-nomial P in n variables which does not contain monomials of order less than three, cannot have an isolated critical point at the origin if $k < n$.*

Proof. Fix a number $1 \leq\leq n$ and consider monomials of the form $x_i^k x_j, k \geq 2$, entering in P. If there are no such monomials at all, then the whole axis Ox_i consists of critical points of P. Thus, for each i, there exists a monomial of such type entering in P. For each i, fix a monomial of such form with the minimal $j = j(i)$. Since there are no monomials of degree two, two monomials of such type chosen for two different numbers $i \neq j$ cannot coincide. This obviously implies that the number of monomials in P cannot be less than the number of coordinates n, which gives the result.

Remark 2.1. Using terminology of [4], the requirement that there are no quadratic terms can be expressed by saying that P is of (maximal) corank n at the origin. The reason why we have to exclude quadratic terms, is that otherwise the formulation given above would not be correct. Indeed, a stabilization of A_1 singularity can be defined by a polynomial in $2k$ variables of the form $x_1x_2 + ... + x_{2k-1}x_{2k}$ which contains only k monomials. Notice also that Pham polynomials give evident examples of n-nomials with isolated singularity at the origin of $\mathbb{C}^n$.

Remark 2.2. Elementary as it appears, Proposition 2.1 up to our knowledge have never appeared in the literature. If fewnomial P is quasihomogeneous and defines an isolated singularity, then, for $n \leq 6$, one can obtain much more precise information about possible collections of monomials entering in P (see, e.g., [4], [1]). Those results imply, in particular, that all such collections contain not less than n monomials. However, not every fewnomial is quasihomogeneous so our Proposition 2.1 is not a formal consequence of the mentioned results from [4], [1].

The above observations suggest a simple description of the range of validity of Pursell-Shanks type theorems in the context of Yau algebras. To do this in a concise and convenient form we introduce some ad hoc terminology. We say that an IHS in $\mathbb{C}^n$ is *fewnomial* if it can be defined by an n-nomial in n variables and we say that it is a *qh-fewnomial* singularity if it can be defined by a quasihomogeneous fewnomial. Since in [6] one can find examples of trinomials in two variables which cannot be classified by their Yau algebras and similar examples can be constructed in all dimensions using stabilization, we conclude that there is no hope for a Pursell-Shanks type theorem outside the class of fewnomial singularities.

Notice that Pham singularities and D_{**} series are fewnomial (actually, even qh-fewnomial) and in [13] it was shown that an analog of PST holds for those singularities. Thus we are led to a conjecture that a Pursell-Shanks type theorem may hold for the class of fewnomial singularities or at least for the class of qh-fewnomial singularities.

In the sequel we'll prove that this conjecture is true for certain classes of qh-fewnomial singularities. Notice that a direct sum of isolated qh-fewnomial singularities is also a qh-fewnomial singularity. Moreover, according to Proposition 1.2 Yau algebras of direct sums can be easily computed. For this reason our strategy will be to establish PST for certain series of fewnomial singularities and then extend it to direct sums of singularities from those series.

Our nearest aim is to show that PST holds for isolated singularities of binomials in two variables so we now turn to considering binomial singularities of corank 2. It is well known that such singularities appear in three series $P_{*,*}, D_{*,*}, T_{*,*}$ described in Section 1 (cf. [4]).

Proposition 2.2. *Each binomial isolated singularity is analytically equivalent to one from the three series $P_{*,*}, D_{*,*}, T_{*,*}$.*

The proof is based on the description of normal forms of binomials in two variables presented in [4] (Section 13, p.179). Notice first that by our previous proposition such a singularity is necessarily defined by a binomial in two variables. According to loc. cit., up to changes of coordinates, binomials in two variables can be written in normal forms belonging to one of the three series: $x^a + y^b, x^a y + y^b, x^a y + xy^b, a, b \in \mathbb{Z}_+, a, b \geq 2$, which implies the desired result. It follows that all of them are quasihomogeneous.

Summing up, all binomial isolated singularities are defined by binomials in two variables and they are automatically quasihomogeneous. There are three series of such binomials indicated above. Simple singularities in the sense of Arnold obviously belong to the class of binomial singularities. Pham singularities are qh-fewnomial.

3. Yau algebras of binomial singularities

The results presented in this paper rely on a few basic computations which are described in this section. They will enable us, in particular, to compute Yau algebras for binomial isolated singularities . We begin with introducing a certain formalism suggested in [13] which facilitates further considerations. We describe this formalism in bigger generality than necessary for this paper having in mind that it may appear useful for further investigation of Yau algebras .

Let n be a natural number and $\kappa = (k_1+1, \dots, k_n+1)$ a non-negative multi-index with $k_i \geq 2, 1 \leq i \leq n$. Denote by $V = V_\kappa$ the vector space spanned by a basis indexed by collections $(a_1, \dots, a_n; i)$, where $1 \leq i \leq n$, $0 \leq a_j \leq k_j - 1$ for $j \neq i$, $1 \leq j \leq n$, and $1 \leq a_i \leq k_i - 1$. It is easy to see that the dimension of this space is equal to $n\sigma_n(k_1, \dots, k_n) - \sigma_{n-1}(k_1, \dots, k_n)$, where σ_j denotes the j-th symmetric function of n variables. Indeed, this immediately follows from the identity

$$\sum_{i=1}^{n}(x_i - 1)\prod_{j\neq i} x_j = n\sigma_n(x_1, \dots, x_n) - \sigma_{n-1}(x_1, \dots, x_n).$$

Identifying each basis vector with its index, introduce a bilinear operation on V by

$$[(a_1, \dots, a_n; i), (b_1, \dots, b_n; j)] =$$
$$-a_j(a_1 + b_1, \dots, a_{j-1} + b_{j-1}, a_j + b_j - 1, a_{j+1} + b_{j+1}, \dots, a_n + b_n; i)+$$
$$b_i(a_1 + b_1, \dots, a_{i-1} + b_{i-1}, a_i + b_i - 1, a_{i+1} + b_{i+1}, \dots, a_n + b_n; j). \quad (*)$$

We show now that this operation actually defines a Lie algebra structure on V. The skew-symmetry is obvious so only Jacobi identity should be verified. To this end we relate V with the Lie algebra of Pham singularity defined by polynomial $P_\kappa = \sum x_j^{k_j+1}$. Since this is a direct sum of simple singularities A_{k_j}, the moduli algebra $A(P_\kappa)$ is isomorphic to the tensor product of moduli algebras of A_{k_j} singularities which are well known [4].

Using Proposition 1.2, the tensor structure of the moduli algebra for P_κ, and the above description of $L(A_k)$ we can now identify the Yau algebra of P_κ with the vector space V introduced above. Using the vector field notation for elements of $L(P_\kappa)$ it is easy to check that an explicit isomorphism between $L(P_\kappa)$ and $V = V_\kappa$ is established by the correspondence:

$$\prod x_k^{a_k}\partial_j \mapsto (a_1, \dots, a_n; j).$$

Comparing the commutators in $L(P_\kappa)$ and in V we see that they coincide, which immediately implies that the bilinear operation introduced above satisfies Jacobi identity and defines thus a Lie algebra structure on V_κ. As a by-product we obtain a formula for the Yau number of Pham singularity.

Proposition 3.1.

$$\lambda(P_\kappa) = n\sigma_n(k_1, \dots, k_n) - \sigma_{n-1}(k_1, \dots, k_n). \quad (3.1)$$

Notice that we have also obtained an explicit basis of $L(P_\kappa)$ as a complex vector space. Such bases will be also indicated for the two other series of binomial singularities. They appear extremely useful for comparing Yau algebras.

Consider now the Yau algebra of D_{k_1,k_2}-singularity defined by the polynomial $f = x_1^{k_1}x_2 + x_2^{k_2}$. As is well known (see, e.g., [4]), its moduli algebra A is of dimension $k_2(k_1 - 1) + 1$ and has monomial basis of the form

$$\{X_1^{a_1}X_2^{k_2}, 0 \leq a_1 \leq k_1 - 2; 0 \leq a_2 \leq k_2 - 1; X_1^{k_1-1}\}. \quad (*)$$

Here and in the sequel the class of a function g in the moduli algebra $A(f)$ is denoted by the corresponding capital letter G.

Then it is easy to verify the following identities in the moduli algebra:

$$X_1^{k_1-1}X_2 = 0, \tag{3.2}$$

$$X_1^{k_1} + k_2X_2^{k_2-1} = 0. \tag{3.3}$$

From the formulae (3.2, 3.3) we get:

$$X_1^{k_1+i} = -k_2X_1^iX_2^{k_2-1},\ 0 \le i \le k_1-2, \tag{3.4}$$

$$X_1^m = 0,\ m \ge 2k_1-1, \tag{3.5}$$

$$X_2^m = 0,\ m \ge k_2. \tag{3.6}$$

As usual, in order to define a derivation d of A it suffices to indicate its values on the generators X_1, X_2 which can be written in the basis (*). Thus using the Einstein notation we can write

$$dX_j = d^j_{i_1,i_2}X_1^{i_1}X_2^{i_2} + d^j_{k_1-1,0}X_1^{k_1-1},\ j = 1, 2.$$

Using the relations (3.2 - 3.6) one now easily finds conditions defining a derivation of A.

Lemma 3.1. *In order that a linear transformation d defines a derivation of $A(f)$ it is necessary and sufficient that*

$$d^1_{0,0} = d^1_{0,1} = \ldots = d^1_{0,k_2-3} = 0;$$
$$d^2_{0,0} = d^2_{1,0} = \ldots = d^2_{k_1-2,0} = 0;$$

$$k_1d^1_{1,0} = (k_2-1)d^2_{0,1}, k_1d^1_{2,0} = (k_2-1)d^2_{1,1}, \ldots, k_1d^1_{k_1-1,0} = (k_2-1)d^2_{k_1-2,1},$$
$$(k_1-1)d^1_{0,k_2-2} = k_2d^2_{k_1-1,0}.$$

Using this lemma we easily obtain the following description of the Yau algebra in question.

Proposition 3.2. *The dimension of Yau algebra $L(D_{k_1,k_2})$ is equal to*

$$\lambda(D_{k_1,k_2}) = 2k_1k_2 - 2k_1 - 3k_2 + 5. \tag{3.7}$$

The derivations represented by the following vector fields form a basis in $L(D_{k_1,k_2})$:

$$(k_2-1)x_1\partial_1+k_1x_2\partial_2, (k_2-1)x_1^2\partial_1+k_1x_1x_2\partial_2, \ldots, (k_2-1)x_1^{k_1-1}\partial_1+k_1x_1^{k_1-2}x_2\partial_2,$$
$$k_2x_2^{k_2-2}\partial_1 + (k_1-1)x_1^{k_1-1}\partial_2;$$
$$x_1^{a_1}x_2^{a_2}\partial_1, x_2^{k_2-1}\partial_1, 1 \le a_1 \le k_1-2,\ 1 \le a_2 \le k_2-1, x_1^{b_1}x_2^{b_2}\partial_2,\ 0 \le b_1 \le k_1-2,$$
$$2 \le b_2 \le k_2-1.$$

Finally, we compute the Yau number and construct an explicit basis in Yau algebra of a T_{k_1,k_2}-singularity defined by the polynomial $x_1^{k_1}x_2 + x_1x_2^{k_2}$. The computations in this case are quite similar to ones for $D_{*,*}$ series but more lengthy and tedious, so we omit the details and present only the final result.

Proposition 3.3. *The dimension of Yau algebra $L(T_{k_1,k_2})$ is equal to*

$$\lambda(T_{k_1,k_2}) = 2k_1k_2 - 2(k_1+k_2) + 6. \tag{3.8}$$

The derivations represented by the following vector fields form a basis in $L(T_{k_1,k_2})$:

$$x_1^{i_1}x_2^{i_2}\partial_1, 2 \le i_1 \le k_1-1, 1 \le i_2 \le k_2-2; x_1^{i_1}x_2^{k_2-1}\partial_1, 1 \le i_1 \le k_1-1;$$

$$x_1^{j_1}x_2^{j_2}\partial_2, 1 \le j_1 \le k_1-2; 1 \le j_2 \le k_2-1; x_1^{k_1-1}x_2^{j_2}\partial_2, 1 \le j_2 \le k_2-1;$$

$$(k_2-1)x_1x_2^i\partial_1 + (k_1-1)x_2^{i+1}\partial_2, 0 \le i \le k_2-2;$$

$$(k_2-1)x_1^{j+1}\partial_1 + (k_1-1)x_1^jx_2\partial_2, 0 \le j \le k_1-2;$$

$$k_2x_1x_2^{k_2-2}\partial_1 + k_1x_1^{k_1-1}\partial_2,$$

$$k_2x_2^{k_2-2}\partial_1 + k_1x_1^{k_1-2}\partial_2;$$

$$k_2x_2^{k_2-1}\partial_1 + k_1x_1^{k_1-2}\partial_2.$$

The Cartan sublagebra in this case is generated by derivation $(k_2-1)x_1\partial_1 + (k_1-1)x_2\partial_2$ and the elements of the above basis are its eigenvectors. It is then straightforward to calculate the corresponding eigenvalues.

Having obtained these explicit results, we can already make a number of useful observations about Yau algebras of fewnomials. In particular, we become able to clarify some interesting issues concerned with Yau algebras of quasihomogeneous singularities. Proposition 1.1 shows that the dimension of moduli algebra of such a singularity X is determined by its quasihomogeneity type. It is thus natural to wonder if the same holds for the dimension of $L(X)$. It is now easy to give a simple example showing that this is not always true.

To this end consider the two singularities defined by polynomials $P_{a_1,b_1} = x_1^{a_1} + x_2^{b_1}$ and $D_{a_2,b_2} = x_1^{a_2}x_2 + x_2^{b_2}$. As indicated in [4], their quasihomogeneity types are $(\frac{1}{a_1}, \frac{1}{b_1})$ and $\frac{b_2-1}{a_2b_2}, \frac{1}{b_2}$, respectively. Taking any natural b, q and putting $b_2 = b_1 = b$, $a_1 = qb$, $a_2 = q(b-1)$ we obtain hat the two above polynomials have the same quasihomegeneity type $(\frac{1}{qb}, \frac{1}{b})$.

From the formulae for the dimension of $L(X)$ presented above we get that $\dim L(P_{a_1,b_1}) = 2qb^2 - 3b(q+1) + 4$ and $\dim L(D_{a_2,b_2}) = 2qb^2 - 2q(2b-1) - 3b + 5$. The difference between the two dimensions is equal to $q(b-2)-1$ so we see that they only coincide for $q = 1, b = 3$. Thus we see that the quasihomogeneity type does not determine the dimension of Yau algebra even for fewnomials.

This conclusion suggests a number of natural questions. First of all, one may wonder if there is a typical value of $\dim L(X)$ for fewnomial singularities with a fixed Newton diagram which means that all fewnomials which have a given Newton diagram and satisfy certain non-degeneracy condition with respect to the diagram should have the same value of $\lambda(X)$. If so, one could hope to express this typical value in terms of the prescribed Newton diagram. This seems especially plausible for qh-fewnomial singularities since their Newton diagrams are well studied.

It is also interesting to find the maximal and minimal values of $\dim L(X)$ for fewnomials with a fixed quasihomogeneity type and characterize singularities for which the extremal values are attained. For some qh-fewnomial types, we have verified that the Pham singularity has the maximal value of $\dim L(X)$ within its quasihomogeneity type. However this is definitely not always so. For example, the two singularities $X_1 = X(x_1^4 + x_2^2)$ (suspension of A_3 singularity) and $X_2 = X(x_2x_2 + x_2^4)$ (D_2 singularity) singularities are both quasihomogeneous of type $(1/4, 1/2; 1)$. In both cases $\lambda(X_i)$ can be computed by formulae (3.1), (3.7) and we get that $\dim L(A_3) = 2 < \dim L(D_2) = 3$. Thus $\lambda(A_3)$ is not maximal in the qh-type $(1/4, 1/2; 1)$. Actually in this case $\lambda(A_3)$ realizes the minimal value while $\lambda(D_2)$ gives the maximal value.

All these problems of course make sense for arbitrary quasihomogeneous singularities but seem quite difficult in general setting (cf. examples and comments given in [13]). At the same time they look much more accessible for the class of qh-fewnomial IHS so it seems natural to attack them first for this class of singularities.

4. Pursell-Shanks theorem for binomial singularities

We pass now to a precise formulation and outline of the proof of the first main result which yields an analog of PST for binomial singularities.

Theorem 4.1. *Binomial isolated singularities with the Milnor number bigger than* 6 *are classified by their Yau algebras.*

Before presenting an outline of the proof let us describe its scheme. In short, we show first that Yau algebras classify singularities within each of those series and then compare the series pair-wise, i.e., we have to prove that, except just one pair, no singularity from one series is analytically isomorphic to a singularity in another series. Correspondingly, proof naturally consists of six $(= 3 + 3)$ independent steps and its detailed exposition is sufficiently lengthy. For this reason, we'll only present in some detail comparison of $P_{*,*}$ and $D_{*,*}$ series. This pair is chosen because here arises the only exceptional case, namely: $L(A_6) \cong L(D_5)$ while A_6 and D_5 are of course nonisomorphic.

Thus we describe in some detail the following three steps. First, we show that the isomorphism type of $L(P_\kappa)$ determines values of parameters k_1, k_2 up to the order. Next, we verify the same for D_{**} series, which shows that singularities within each of these two series are classified by their Yau algebras. Finally, we show that, for $\mu > 6$, no Pham singularity can be isomorphic to a D_{k_1,k_2} singularity, which yields the result for the first two series. The three remaining steps are completely similar to the steps described below so we omit them.

Proof of theorem 4.1. (1) We begin with considering the Pham series. As was shown above, if $X = X(P_k)$ is defined by a polynomial in n variables then $\mathrm{rk} L(X) = n$. This implies that two such Lie algebras can be only isomorphic if they are defined by polynomials of the same number of variables. For our purposes it is sufficient to consider two Pham singularities of the type $(2; k_1, k_2\kappa)$

and without losing generality we may assume that $k_1 \geq k_2$. We will show that k_1 is an invariant of $L(X)$. Notice that k_1 has the biggest modulus (absolute value) among the eigenvalues of all basic operators $E_i = x_i\partial_i$. Notice that the spectra of all those operators are real and nonnegative and the smallest eigenvalue of all E_i is equal to one. These data can be used to obtain a numerical invariant of algebra $L(X)$ as follows.

Introduce a norm on $A(X)$. This induces the operator norm on $L(X)$ and we may consider the unit sphere $S \subset L(X)$. For each derivation $T \in L(X)$ denote by $\Lambda(T)$ (respectively, $\lambda(T)$) the maximal (respectively, minimal nonzero) modulus of eigenvalues of T. Consider now the maximum M of the ratio $r(T) = \Lambda(T)/\lambda(T)$ for all $T \in S$ (this maximum is attained since S is compact). Obviously, M being defined as a ratio does not depend on the choice of norm on $A(X)$. Thus it is an invariant of Lie algebra $L(P_k)$. Moreover, since all minimal moduli of eigenvalues of basic operators E_i are equal it follows that $\lambda(T)$ is constant on S. Thus the maximum of the ratio $r(T)$ is achieved on the operator which an eigenvalue with the maximal modulus over S. From the structure of spectra of the basic operators E_i described above and well known interlacing property of eigenvalues of linear combinations of operators [5] it is clear that this maximum is equal to $k_1 - 1$. Thus $k_1 - 1$ is an invariant of $L(X)$. Since in our case $\lambda(X) = k_1 + k_2 + 1$ it becomes clear that the pair (k_1, k_2) is determined up to the order by the isomorphism class of $L(P_{k_1,k_2})$. Thus the first step is completed.

(2) In order to deal with D_{**} series let us introduce a numerical invariant of Lie algebras of the form $L(D_{k_1,k_2})$. As was mentioned, on $L(D_{k_1,k_2})$ there exists a $\mathbb{Z}$-grading defined by the vector field $E = (k_2 - 1)x_1\partial_1 + k_1x_2\partial_2$ (Euler field). Instead of E we could take any other semi-simple element of $L(D_{k_1,k_2})$. Since the set of semi-simple elements in $L(D_{k_1,k_2})$ is one-dimensional, any two such gradings coincide up to a (complex) multiple. Notice that E is regular, in the sense that its orbit in the adjoint representation has the maximal dimension. The invariant we are after is now defined as follows.

Take a homogeneous element $a \in L(D_{k_1,k_2})$ of positive degree. Then the operator $\operatorname{ad} a$ is nilpotent. Let $n(a)$ denote its nilpotency index with respect to this grading. It is clear that $n(a)$ does not change if the grading is multiplied by a complex number. Define $n(L)$ as the maximal number among $n(a)$ over the set of all (non-zero) homogeneous nilpotent elements. Then it is obvious that $n(L)$ is an invariant of graded Lie algebras. We will compute this invariant for $L(D_{k_1,k_2})$ and show that it distinguishes such algebras.

For computing $n(L(D_{k_1,k_2}))$ it is useful to notice that there is a natural Lie-Rinehart structure on the pair $A(D_{k_1,k_2}), L(D_{k_1,k_2})$ and use some well-known properties of such structures [8]. Consider the elements $S = (k_2 - 1)x_1^2\partial_1 + k_1x_1x_2\partial_2$ and $T = x_2^2\partial_2$. By direct computation it is not difficult to show that $n(S) = k_2 - 2$ and $n(T) = 2k_1 - 5$. Moreover, from the description of the homogeneous components for Euler grading given in Section 2 it follows that no other element can have nilpotency index bigger than $N = \max(n(S), n(T))$. Thus $n(D_{k_1,k_2}) = k_2 - 2$ if $k_2 \geq 2k_1 - 5$ and $n(D_{k_1,k_2}) = 2k_1 - 5$ if $2k_1 - 5 > a_2 - 2$.

Suppose now that $L(D_{k_1,k_2}) \cong L(D_{m_1,m_2})$ for some pairs $(k_1, k_2), (m_1, m_2) \in \mathbb{Z}_+^2$. In order to show that $(k_1, k_2) = (m_1, m_2)$ we proceed as follows. First of all, in such a case we have $\dim L(D_{k_1,k_2}) = \dim L(D_{m_1,m_2})$ and using the explicit formula for the Yau number of D_{k_1,k_2} singularity we get equation

$$2k_1k_2 - 2k_1 - 3k_2 = 2m_1m_2 - 2m_1 - 3m_2. \tag{4.1}$$

Moreover, we have $n(D_{k_1,k_2}) = n(D_{m_1,m_2})$. Taking into account the above formulas for $n(D_{k_1,k_2})$ it is obvious that there are four logically possible relations between the parameters k_1, k_2, m_1, m_2: 1) $k_2 = m_2$; 2) $k_1 = m_1$; 3) $k_2 - 2 = 2m_1 - 5$; 4) $m_2 - 2 = 2k_1 - 5$. In the first two cases, substituting the relations in (4.1) we immediately get that the second parameters also coincide. The last two cases are symmetric so it is sufficient to prove the result when $2k_1 - 5 > k_2 - 2$ and $m_2 - 2 \geq 2m_1 - 5$. Thus in this case we have $m_2 - 2 = 2k_1 - 5$, hence $2k_1 = m_2 + 3$. Transforming the left hand side of (4.1) and substituting $m_2 + 3$ instead of $2k_1$ we get:

$$2k_1(k_2 - 1) - 3k_2 = (m_2 + 3)(k_2 - 1) - 3k_2 = m_2k_2 - m_2 - 3.$$

This obviously gives the equation

$$m_2k_2 - m_2 - 3 = 2m_1(m_2 - 1) - 3m_2.$$

Taking the number 3 to the right hand side and factoring the latter we get:

$$m_2(k_2 - 1) = (m_2 - 1)(2m_1 - 3).$$

Let us now rewrite the last relation in the form:

$$k_2 - 1 = \frac{m_2 - 1}{m_2}(2m_1 - 3).$$

Since in the left hand side we have an integer and $m_2 - 1$ is relatively prime with m_2, it follows that $2m_1 - 3$ is divisible by m_2. However by assumption $m_2 \geq 2m_1 - 3$ so we conclude that $2m_1 - 3 = m_2$, hence $k_2 = m_2$. The rest of the proof goes in a completely similar way. It follows that singularities of D_{**} series are classified by their Yau algebras, which completes the proofof the second step.

(3) Suppose now that $L(P_{k_1,k-2}) \cong L(D_{m_1,m_2}))$ for certain values of parameters k_i, m_j. Then their ranks should be equal. Remembering that for a Pham singularity of the type $n; \kappa$ the rank of Lie algebra equals n and for D_{**} series this rank always equals one, we get that n necessarily equals one. Thus we should only compare $D_{*,*}$ singularities with A_k singularities. Actually, the necessary comparison has already been done in [13] and we reproduce the argument here for completeness and convenience of the reader.

Since the Cartan subalgebras are one-dimensional in both cases, it is easy to arrive at the desired conclusion by comparing the spectra of semi-simple operators $\operatorname{ad} h$. Notice that the arithmetic structure of these spectra does not depend on the choice of such an element. Recall that for A_k series the spectrum consists of a single arithmetic progression. From the description of the homogeneous components of $L(D_{m_1,m_2})$ given in Section 2 it is clear that the spectrum of $\operatorname{ad} h$ reduces to a single arithmetic progression only if $m_1 = 2$. In other words, it is sufficient to compare simple singularities for A_k and D_m series. As was shown in [12], PST holds for simple singularities except just one pair, A_6 and D_5, which completes the proof of the third step. Thus theorem 4.1 is completely proven for the two series $P_{*,*}$ and $D*, *$. As was mentioned, the comparison of those two series with the third one is essentially analogous so we omit the rest of the proof.

This result gives the desired analog of PST for binomial singularities. Since there exist trinomials in two variables which cannot be classified by their Yau algebras [22] we conclude that, for polynomials in two variables, PST holds exactly in the class of binomial singularities. It is now natural and tempting to find out if n-nomials in n variables are classified by their Yau algebras and we make another step in this direction in the next section.

5. Pursell-Shanks theorem for fewnomial singularities

Using the foregoing results and considerations we can obtain now an analog of PST for certain isolated singularities defined by fewnomials in arbitrary number of variables. Recall that the direct sum of singularities $X_1 = X(f_1) \subset \mathbb{C}^m$ and $X_2 = X(f_2) \subset \mathbb{C}^n$ is defined as the singularity $X = X(f_1(x) + f_2(y)) \subset \mathbb{C}^{m+n}, x \in \mathbb{C}^m, y \in \mathbb{C}^n$. In particular, a suspension of an IHS is the same as its direct sum with A_1 singularity.

Obviously, the classes of fewnomial singularities and qh-fewnomial singularities are closed with respect to taking direct sums. Moreover, for direct sums it is easy to compute and compare the Yau algebras using Proposition 1.2. Let us say that an IHS is a *decomposable* fewnomial singularity if it is a direct sum of (an arbitrary amount of) binomial and Pham singularities. Since each Pham singularity is a direct sum of several binomial singularities and an A_k singularity, the same class arises if one considers direct sums of binomial and A_k singularities. Notice also that such singularities are necessarily quasi-homogeneous. It turns out that PST holds for Yau algebras of decomposable fewnomial singularities. The "raison d'être" of this result can be described as follows.

Recall that, as was proven in [13], PST holds for Yau algebras of Pham singularities. The proof presented in [13] uses only two properties of Pham singularities, namely, that: (1) they are directs sums of A_k singularities, and (2) PST holds for A_k singularities. The crucial fact on which relies the proof presented in [13], is that the Yau numbers and spectra of Cartan algebras of summand singularities can be restored from the Yau algebra of their direct sum. Combining this fact with PST for simple singularities we obtained in [13] an analog of PST for those "semi-simple" singularities. Now that we have proven PST for binomial singularities, the same argument becomes applicable to direct sums of binomial singularities and A_k singularities, i.e., to the class of decomposable fewnomial singularities.

The argument runs in two steps. Firstly, the information on spectra of Cartan subalgebras in Yau algebras of binomial singularities presented in Section 3 is used to prove that the spectra of summands $X(f_1)$ and $X(f_2)$ can be restored, up to the order, from the Yau algebra of direct sum $X(f_1) \oplus X(f_2)$. Secondly, using the proof of PST for binomial singularities presented above, we show that the isomorphism class of a direct sum under consideration determines the isomorphism classes of summands up to the order. Since we already have PST for summands, this implies that the isomorphism type of a direct sum of such type is determined by its Yau algebra.

Theorem 5.1. *Decomposable fewnomial singularities with the Milnor number bigger than* 6 *are classified by their Yau algebras.*

From the point of view of proving our "PST for fewnomials" conjecture on the range of validity of PST for Yau algebras , this result is not a big progress because the class of decomposable fewnomial singularities is quite special. We present it only to provide an application of our Theorem 4.1 and fix the state-of-the-art in this topic. Its proof outlined above is also quite specific so we omit the details. Actually, the same argument involving Cartan spectra enables one to prove this conjecture for a wider class of *generic* qh-fewnomial singularities for which Cartan spectra can be expressed in terms of quasihomogeneous weights and degree. This and other extensions of the results of the last two sections will be considered elsewhere.

6. Concluding remarks

In conclusion we briefly mention some open problems and perspectives connected with our results. As was already mentioned, there is a bunch of natural problems concerned with Yau algebras of qh-fewnomial singularities. As was shown in Section 2, the qh-type of singularity does not determine the dimension of $L(X)$ even for qh-fewnomial singularities. In fact, the infinite series of such examples presented in Section 2 shows that the variation of values of $\dim L(X)$ within a given qh-type can become arbitrarily big.

Thus it is interesting to estimate the modulus of variation of $\lambda(X)$ in terms of qh-weights. Clearly, this is closely related to the problem of finding the exact upper and lower bounds for the Yau number within a given qh-fewnomial type. A still more general problem is to describe the whole spectrum of possible values of $\lambda(X)$ within a given qh-fewnomial type. Some of the above problems are closely related to the results of [23], [2] and [15] but we were not able to derive their solutions from the existing results. For simple qh-fewnomial types, these problems can be successfully attacked using the results presented above.

Moreover, there exists a good evidence that some developments in the above topics are possible if one restricts attention to fewnomials with a fixed Newton diagram. Then one may hope to express various invariants of the Yau algebra of a typical function with a given Newton diagram P in terms of the geometry of P. Results of such type concerned with computing Milnor numbers are well known in singularity theory [4], [20]. Since the topological type of a generic (non-degenerate) singularity with a given Newton diagram is completely determined by the geometry of diagram (cf., e.g., [20]), it is highly plausible that the Yau number of such a singularity can be computed in terms of the given diagram. Specifically, this conjecture is supported by the expressions for the Yau numbers given in Propositions 3.1, 3.7, 3.8 since they look very much like the mixed areas of certain polygons associated with the Newton diagram. Thus an intriguing open problem is to express the Yau numbers of fewnomial singularities in terms of mixed volumes.

Summing up, the topics discussed in the present paper give rise to a variety of natural problems and the author intends to continue research along these lines.

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