RANDOM β-EXPANSIONS WITH DELETED DIGITS

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Abstract. In this paper we define random β-expansions with digits taken from a given set of real numbers $A = \{a_1, \ldots, a_m\}$. We study a generalization of the greedy and lazy expansion and define a function K, that generates essentially all β-expansions with digits belonging to the set A. We show that K admits an invariant measure ν under which K is isomorphic to the uniform Bernoulli shift on A.

1. Introduction

Let $\beta > 1$ be a real number, and $A = \{a_1, \ldots, a_m\}$ a given set of real numbers. We assume that $a_1 < a_2 < \ldots < a_m$, and that $m \geq 2$. We are interested in algorithms that generate β-expansions of the form

$$x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i}, \tag{1}$$

with $b_i \in A$. Clearly, such an expansion is possible for points in the interval $[\frac{a_1}{\beta-1}, \frac{a_m}{\beta-1}]$, but not necesarily for all points in this interval; see [KSS]. There are two cases that have been extensively studied. The first is when $\beta = r$ is an integer, and $A = \{0, 1, \ldots, r-1\}$, leading to the well-known r-adic expansion of points in $[0, 1]$. Each point has a unique expansion except for points of the form k/r^n, $0 < k \leq r^n - 1$ which have exactly two expansions. The second is when $\beta > 1$ is a non-integer, and $A = \{0, 1, \ldots, \lfloor\beta\rfloor\}$, is *a complete digit set*. In this case, almost every $x \in [0, \lfloor\beta\rfloor/(\beta-1)]$ has a continuum number of expansions of the form

$$x = \sum_{k=1}^{\infty} \frac{a_k}{\beta^k}, \quad a_k \in \{0, 1, \ldots, \lfloor\beta\rfloor\}, k \geq 1,$$

see [EJK], [Si], [DV1]. There are two well-known algorithms producing β-expansions with a complete digit set, the *greedy* and the *lazy* algorithms. The greedy algorithm chooses at each step the largest possible digit, while the lazy chooses the smallest possible digit. Dynamically, the greedy algorithm is generated by iterating the map T_β defined on $[0, \lfloor\beta\rfloor/(\beta-1)]$ by

$$T_\beta(x) = \begin{cases} \beta x \pmod 1, & 0 \leq x < 1, \\ \beta x - \lfloor\beta\rfloor, & 1 \leq x \leq \lfloor\beta\rfloor/(\beta-1). \end{cases}$$

Similarly, the lazy algorithm is obtained by iterating the map L_β defined on the interval $[0, \lfloor\beta\rfloor/(\beta-1)]$ by

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$$L_\beta(x) = \beta x - d \quad \text{for } x \in \Delta(d),$$

where

$$\Delta(0) = \left[0, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}\right],$$

and

$$\begin{aligned}\Delta(d) &= \left(\frac{\lfloor\beta\rfloor}{\beta-1} - \frac{\lfloor\beta\rfloor - d + 1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1} - \frac{\lfloor\beta\rfloor - d}{\beta}\right] \\ &= \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{d-1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{d}{\beta}\right], \quad d \in \{1, 2, \dots, \lfloor\beta\rfloor\}.\end{aligned}$$

In order to capture, in a dynamical way, all possible expansions with a complete digit sets, the authors in [DK2] and [DV] considered a map K_β, defined on $\{0,1\}^{\mathbb{N}} \times [0, \frac{\lfloor\beta\rfloor}{\beta-1}]$, which gives rise to random β-expansions. The map K_β is defined as follows. We first partition the interval $[0, \frac{\lfloor\beta\rfloor}{\beta-1}]$ into $\lfloor\beta\rfloor$ switch regions, $S_1, \dots, S_{\lfloor\beta\rfloor}$, and $\lfloor\beta\rfloor + 1$ equality regions, $E_0, \dots, E_{\lfloor\beta\rfloor}$, where

$$\begin{aligned}E_0 &= \left[0, \frac{1}{\beta}\right), \quad E_{\lfloor\beta\rfloor} = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{\lfloor\beta\rfloor - 1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}\right], \\ E_k &= \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{k-1}{\beta}, \frac{k+1}{\beta}\right), \quad k = 1, \dots, \lfloor\beta\rfloor - 1, \\ S_k &= \left[\frac{k}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{k-1}{\beta}\right], \quad k = 1, \dots, \lfloor\beta\rfloor.\end{aligned}$$

On S_k, the greedy map assigns the digit k, while the lazy map assigns the digit $k-1$. On E_k both maps assign the same digit k. The elements $\omega \in \{0,1\}^{\mathbb{N}}$ determine which digit is chosen each time we can make a decision. The transformation $K_\beta : \{0,1\}^{\mathbb{N}} \times [0, \frac{\lfloor\beta\rfloor}{\beta-1}] \to \{0,1\}^{\mathbb{N}} \times [0, \frac{\lfloor\beta\rfloor}{\beta-1}]$ is then given by

$$K_\beta(\omega, x) = \begin{cases} (\omega, \beta x - k), & \text{if } x \in E_k,\ k = 0, \dots, \lfloor\beta\rfloor, \\ (\sigma(\omega), \beta x - k), & \text{if } x \in S_k \text{ and } \omega_1 = 1,\ k = 1, \dots, \lfloor\beta\rfloor, \\ (\sigma(\omega), \beta x - k + 1), & \text{if } x \in S_k \text{ and } \omega_1 = 0,\ k = 1, \dots \lfloor\beta\rfloor, \end{cases}$$

where $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is the left shift. The digits are defined by

$$d_1 = d_1(\omega, x) = \begin{cases} k, & \text{if } x \in E_k,\ k = 0, \dots, \lfloor\beta\rfloor, \\ & \text{or } (\omega, x) \in \{\omega_1 = 1\} \times S_k,\ k = 1, \dots, \lfloor\beta\rfloor, \\ k-1, & \text{if } (\omega, x) \in \{\omega_1 = 0\} \times S_k,\ k = 1, \dots, \lfloor\beta\rfloor, \end{cases}$$

and for $n \geq 1$, $d_n = d_n(\omega, x) = d_1(K_\beta^{n-1}(\omega, x))$. An element $x \in [0, \frac{\lfloor\beta\rfloor}{\beta-1}]$ has a unique β-expansion if and only if the orbit of x under the map T_β visits only the equality regions. Furthermore, in [DV] it is shown that there exists a unique K_β-invariant measure of maximal entropy ν_β such that the system $(\{0,1\}^{\mathbb{N}} \times [0, \frac{\lfloor\beta\rfloor}{\beta-1}], \mathcal{F} \times \mathcal{B}, \nu_\beta, K_\beta)$ is isomorphic to the uniform Bernoulli shift on $\lfloor\beta\rfloor + 1$ symbols, where $\mathcal{F}$ is the product σ-algebra on $\{0,1\}^{\mathbb{N}}$ and $\mathcal{B}$ is the Borel σ-algebra on $[0, \frac{\lfloor\beta\rfloor}{\beta-1}]$.

In [P], M. Pedicini defined an algorithm that generates expansions of the form $x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i}$, where the digits b_i belong to an arbitrary set of real numbers $A = \{a_1, a_2, \ldots, a_m\}$. His algorithm, which we call *greedy with deleted digits* is similar to the "classical greedy expansion" i.e. with a complete digit set, and is defined recursively as follows.
Let $x \in \left[\frac{a_1}{\beta-1}, \frac{a_m}{\beta-1}\right]$ and suppose the digits $b_1 = b_1(x), \ldots, b_{n-1} = b_{n-1}(x)$ are already defined, then $b_n = b_n(x)$ is the largest element of $\{a_1, \ldots, a_m\}$, such that

$$\frac{b_1}{\beta} + \ldots + \frac{b_n}{\beta^n} + \sum_{i=n+1}^{\infty} \frac{a_1}{\beta^i} \leq x. \tag{2}$$

Pedicini showed that if

$$\max_{1 \leq j \leq m-1} (a_{j+1} - a_j) \leq \frac{a_m - a_1}{\beta - 1}, \tag{3}$$

then every point x in $\left[\frac{a_1}{\beta-1}, \frac{a_m}{\beta-1}\right]$ has a greedy expansion of the form $x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i}$ with the digits b_i in A and satisfying (3). In Section 2, we give a dynamical way of generating this greedy expansion. This allows us to give simple dynamical proofs of some of the results proved in [P]. We also study a generalization of the lazy expansion and its relationship with the greedy expansion. In the third section, we define random β-expansions with deleted digits and show that the transformation K generating these expansions captures all possible β-expansions with a given digit set A. We also find a K-invariant measure such that (K, ν) is isomorphic to the uniform Bernoulli shift with digit set A.

2. The Greedy and Lazy Transformations with Deleted Digits

Let $\beta > 1$ be a real number. We call a set of numbers $A = \{a_1, \ldots, a_m\}$ with $a_1 < \ldots < a_m \in \mathbb{R}$ an *allowable digit set* for β, if it satisfies (3). Throughout the rest of the paper, we denote the interval $\left[\frac{a_1}{\beta-1}, \frac{a_m}{\beta-1}\right]$ by J_{a_1,a_m}, and we assume that A is an allowable digit set. We are seeking a transformation T on J_{a_1,a_m} such that if (1) is the greedy expansion of x as given by (2), then $T^n x = \sum_{i=1}^{\infty} \frac{b_{n+i}}{\beta^i}$. Now, if we rewrite condition (2), we see that for $k = 1, \dots, m-1$,

$$\begin{aligned} b_n = a_k \quad &\Leftrightarrow \quad \sum_{i=1}^{n-1} \frac{b_i}{\beta^i} + \frac{a_k}{\beta^n} + \sum_{i=n+1}^{\infty} \frac{a_1}{\beta^i} \le x < \sum_{i=1}^{n-1} \frac{b_i}{\beta^i} + \frac{a_{k+1}}{\beta^n} + \sum_{i=n+1}^{\infty} \frac{a_1}{\beta^i} \\ &\Leftrightarrow \quad \frac{a_1}{\beta^n(\beta-1)} + \frac{a_k}{\beta^n} \le \frac{1}{\beta^{n-1}} \sum_{i=1}^{\infty} \frac{b_{n-1+i}}{\beta^i} < \frac{a_1}{\beta^n(\beta-1)} + \frac{a_{k+1}}{\beta^n} \\ &\Leftrightarrow \quad \frac{a_1}{\beta-1} + \frac{a_k - a_1}{\beta} \le \sum_{i=1}^{\infty} \frac{b_{n-1+i}}{\beta^i} < \frac{a_1}{\beta-1} + \frac{a_{k+1}-a_1}{\beta}, \end{aligned}$$

and $b_n = a_m$ if and only if

$$\frac{a_1}{\beta-1} + \frac{a_m - a_1}{\beta} \le \sum_{i=1}^{\infty} \frac{b_{n-1+i}}{\beta^i} \le \frac{a_m}{\beta-1}.$$

In view of this we define the *greedy transformation* $T = T_{\beta,A}$ with allowable digit set A by

$$Tx = \begin{cases} \beta x - a_j, & \text{if } x \in \left[\frac{a_1}{\beta-1} + \frac{a_j - a_1}{\beta}, \frac{a_1}{\beta-1} + \frac{a_{j+1}-a_1}{\beta}\right), \\ & \text{for } j = 1, \dots, m-1, \\ \beta x - a_m, & \text{if } x \in \left[\frac{a_1}{\beta-1} + \frac{a_m - a_1}{\beta}, \frac{a_m}{\beta-1}\right]. \end{cases}$$

Notice that $T([\frac{a_1}{\beta-1} + \frac{a_m-a_1}{\beta}, \frac{a_m}{\beta-1}]) = [\frac{a_1}{\beta-1}, \frac{a_m}{\beta-1}] = J_{a_1,a_m}$. Furthermore, the assumption that A is an allowable digit set implies that for $j = 1, 2, \dots, m-1$, $T([\frac{a_1}{\beta-1} + \frac{a_j-a_1}{\beta}, \frac{a_1}{\beta-1} + \frac{a_{j+1}-a_1}{\beta})) = [\frac{a_1}{\beta-1}, \frac{a_1}{\beta-1} + a_{j+1} - a_j) \subseteq J_{a_1,a_m}$.

This shows that T maps the interval J_{a_1,a_m} onto itself.

Let

$$b_1 = b_1(x) = \begin{cases} a_j, & \text{if } x \in \left[\frac{a_1}{\beta-1} + \frac{a_j - a_1}{\beta}, \frac{a_1}{\beta-1} + \frac{a_{j+1}-a_1}{\beta}\right), \\ & \text{for } j = 1, \dots, m-1, \\ a_m, & \text{if } x \in \left[\frac{a_1}{\beta-1} + \frac{a_m - a_1}{\beta}, \frac{a_m}{\beta-1}\right]. \end{cases}$$

and set $b_n = b_n(x) = b_1(T^{n-1}x)$. Then, $Tx = \beta x - b_1$, and for any $n \ge 1$,

$$x = \sum_{i=1}^{n} \frac{b_i}{\beta^i} + \frac{T^n x}{\beta^n}.$$

Letting $n \to \infty$, it is easily seen that $x = \sum_{n=1}^{\infty} \frac{b_n}{\beta^n}$, with b_n satisfying (2). From the definition of the greedy map T, it is easy to see that the point $\frac{a_m}{\beta-1}$ is the only point whose greedy expansion eventually ends in the sequence $a_m, a_m, \dots$.

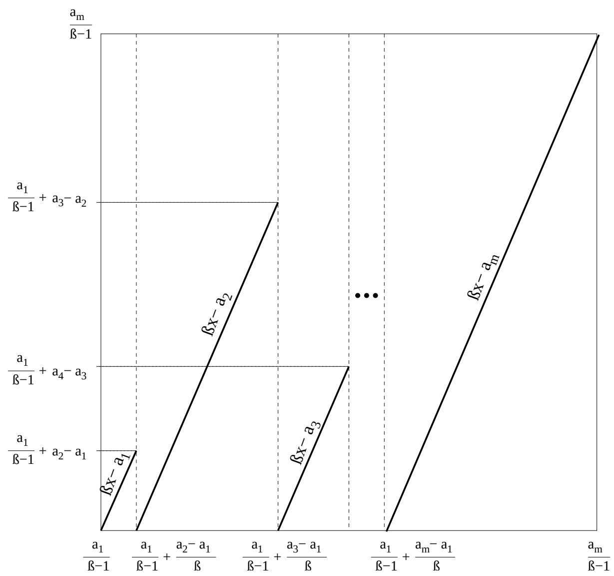

FIGURE 1. The greedy transformation with deleted digits.

In the next proposition we show that the usual order on $\mathbb{R}$ respects the lexicographical ordering $<_{lex}$ on the set of sequences.

Proposition 2.1. *Let A be an allowable digit set and suppose $x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i}$ and $y = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i}$ are the greedy expansions of x and y in base β and digits in A. Then*

$$x < y \Leftrightarrow (b_1, b_2, \dots) <_{lex} (d_1, d_2, \dots).$$

Proof. Suppose $x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i} < \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} = y$. Then $(b_1, b_2, \dots) \neq (d_1, d_2, \dots)$. Let k be the smallest integer, such that $b_k \neq d_k$. If $b_k > d_k$, then

$$y < \sum_{i=1}^{k-1} \frac{b_i}{\beta^i} + \frac{d_k + 1}{\beta^k} + \sum_{i=k+1}^{\infty} \frac{a_1}{\beta^i} \leq \sum_{i=1}^{k-1} \frac{b_i}{\beta^i} + \frac{b_k}{\beta^k} + \sum_{i=k+1}^{\infty} \frac{a_1}{\beta^i} \leq x,$$

contradicting the assumption that $x < y$. So,

$$(b_1, b_2, \dots) <_{lex} (d_1, d_2, \dots).$$

Conversely, if $(b_1, b_2, \dots) <_{lex} (d_1, d_2, \dots)$ and k is the first index such that $b_k < d_k$, then $T^{k-1}x < T^{k-1}y$, which implies that

$$x = \frac{b_1}{\beta} + \dots + \frac{b_{k-2}}{\beta^{k-2}} + \frac{1}{\beta^{k-1}} T^{k-1}x < \frac{b_1}{\beta} + \dots + \frac{b_{k-2}}{\beta^{k-2}} + \frac{1}{\beta^{k-1}} T^{k-1}y = y.$$

□

The next lemma states that condition (3) puts a restriction on the number of elements of the allowable set A.

Lemma 2.1. *Let $\beta > 1$ be a real number, and $A = \{a_1, \dots, a_m\}$ an allowable digit set. Then, $m \geq \lceil \beta \rceil$, where $\lceil \beta \rceil$ is the smallest integer greater or equal to β.*

Proof. By assumption, $a_{j+1} - a_j \leq \dfrac{a_m - a_1}{\beta - 1}$ for all $1 \leq j \leq m-1$. Summing over j gives

$$a_m - a_1 = \sum_{j=1}^{m-1} (a_{j+1} - a_j) \leq (m-1)\frac{a_m - a_1}{\beta - 1},$$

so $\beta \leq m$. Since m is an integer, one has that $m \geq \lceil \beta \rceil$. □

Remark 2.1. (i) In [P], Pedicini proved among other things that if $x = \sum_{n=1}^{\infty} \frac{b_n}{\beta^n}$ is the greedy expansion of x, and if $b_n = a_k \neq a_m$, then

$$b_{n+1} b_{n+2} \dots <_{lex} c_1 c_2 \dots, \tag{4}$$

where $\dfrac{a_1}{\beta - 1} + a_{k+1} - a_k = \sum_{n=1}^{\infty} \frac{c_n}{\beta^n}$ is the greedy expansion of $\dfrac{a_1}{\beta - 1} + a_{k+1} - a_k$. Using our approach we can give a simple proof of this result as follows. Since T is piecewise increasing, one easily sees that if $T^{n-1}x$ has first greedy digit $a_k \neq a_m$, then

$$T^n x < \frac{a_1}{\beta - 1} + a_{k+1} - a_k. \tag{5}$$

Since $T^n x = \sum_{i=1}^{\infty} \frac{b_{n+i}}{\beta^i}$ is the greedy expansion of $T^n x$, then from Proposition 2.1 we see that (4) is equivalent to (5).

(ii) In [KSS], Keane, Smorodinsky and Solomyak studied the size of the set

$$C_\beta = \{\sum_{n=1}^{\infty} \frac{b_n}{\beta^n} : b_n \in \{0, 1, 3\}\}.$$

They showed that if $\beta \leq 5/2$, then $C_\beta = [0, \dfrac{3}{\beta - 1}]$. This is exactly the case that the digit set $A = \{0, 1, 3\}$ is allowable for β. They also showed that C_β has Lebesgue measure zero if $\beta \geq 3$, and they exhibited a countable set of $\beta's$ in the interval $(5/2, 3)$ for which C_β has Lebesgue measure zero. In [PS], the Hausdorff dimension of C_β was obtained. See [PSS] for further generalizations.

As in the (classical) complete digit case (see [DK1], [EJ], [EJK], [KL]), we can define, recursively as well as dynamically, another algorithm generating β-expansions, called the *lazy algorithm*. Given a real number $\beta > 1$ and an allowable digit set $A = \{a_1, \dots, a_m\}$, we first define the lazy expansion recursively as follows. Let $x \in \left[\dfrac{a_1}{\beta - 1}, \dfrac{a_m}{\beta - 1}\right]$ and suppose the digits $c_1 = c_1(x), \dots, c_{n-1} = c_{n-1}(x)$ are already defined, then $c_n = c_n(x)$ is the smallest element of A such that

$$x \leq \sum_{i=1}^{n-1} \frac{c_i}{\beta^i} + \frac{c_n}{\beta^n} + \sum_{i=n+1}^{\infty} \frac{a_m}{\beta^i}. \tag{6}$$

It is not yet obvious that this leads to an expansion, but we will use this notation formally in order to derive a dynamical definition, where it will be clear that such an algorithm leads to an expansion. Rewriting condition (6), one sees that for $k = 2, \ldots, m$,

$$c_n = a_k \quad \Leftrightarrow \quad \sum_{i=1}^{n-1} \frac{c_i}{\beta^i} + \frac{a_{k-1}}{\beta^n} + \sum_{i=n+1}^{\infty} \frac{a_m}{\beta^i} < x \leq \sum_{i=1}^{n-1} \frac{c_i}{\beta^i} + \frac{a_k}{\beta^n} + \sum_{i=n+1}^{\infty} \frac{a_m}{\beta^i}$$

$$\Leftrightarrow \quad \frac{a_m}{\beta - 1} - \frac{a_m - a_{k-1}}{\beta} < \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{\beta^i} \leq \frac{a_m}{\beta - 1} - \frac{a_m - a_k}{\beta},$$

and $c_n = a_1$ if and only if

$$\frac{a_1}{\beta - 1} \leq \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{\beta^i} \leq \frac{a_m}{\beta - 1} - \frac{a_m - a_1}{\beta}$$

As in the greedy case, we want to define a transformation L on J_{a_1,a_m} such that if $x = \sum_{i=1}^{\infty} \frac{c_i}{\beta^i}$ is the lazy expansion of x, then $L^{n-1}x = \sum_{i=1}^{\infty} \frac{c_{n-1+i}}{\beta^i}$. In view of the above, we define the *lazy transformation* $L = L_{\beta,A}$ *with allowable digit set* A by

$$Lx = \begin{cases} \beta x - a_1, & \text{if } x \in \left[\frac{a_1}{\beta - 1}, \frac{a_m}{\beta - 1} - \frac{a_m - a_1}{\beta}\right], \\ \beta x - a_j, & \text{if } x \in \left(\frac{a_m}{\beta - 1} - \frac{a_m - a_{j-1}}{\beta}, \frac{a_m}{\beta - 1} - \frac{a_m - a_j}{\beta}\right], \\ & \text{for } j = 2, \ldots, m. \end{cases}$$

Notice that $L([\frac{a_1}{\beta - 1}, \frac{a_m}{\beta - 1} - \frac{a_m - a_1}{\beta}]) = [\frac{a_1}{\beta - 1}, \frac{a_m}{\beta - 1}] = J_{a_1,a_m}$. Furthermore, since A is allowable, then for $j = 2, \ldots, m$ one has

$$L\left(\left(\frac{a_m}{\beta - 1} - \frac{a_m - a_{j-1}}{\beta}, \frac{a_m}{\beta - 1} - \frac{a_m - a_j}{\beta}\right]\right) = \left(\frac{a_m}{\beta - 1} - (a_{j+1} - a_j), \frac{a_m}{\beta - 1}\right] \subseteq J_{a_1,a_m},$$

This shows that L maps the interval J_{a_1,a_m} onto itself. Let

$$c_1 = c_1(x) = \begin{cases} a_1 & \text{if } x \in \left[\frac{a_1}{\beta - 1}, \frac{a_m}{\beta - 1} - \frac{a_m - a_1}{\beta}\right], \\ a_j, & \text{if } x \in \left(\frac{a_m}{\beta - 1} - \frac{a_m - a_{j-1}}{\beta}, \frac{a_m}{\beta - 1} - \frac{a_m - a_j}{\beta}\right], \\ & \text{for } j = 2, \ldots, m, \end{cases}$$

and set $c_n = c_n(x) = c_1(L^{n-1}x)$. Then, $Lx = \beta x - c_1$, and for any $n \geq 1$,

$$x = \sum_{i=1}^{n} \frac{c_i}{\beta^i} + \frac{L^n x}{\beta^n}.$$

Letting $n \to \infty$, it is easily seen that $x = \sum_{n=1}^{\infty} \frac{c_n}{\beta^n}$, with c_n satisfying (6). ¿From the definition of the map L it is easily seen that the point $\frac{a_1}{\beta - 1}$ is the only point in the interval J_{a_1,a_m} whose lazy expansion eventually ends in the sequence $a_1, a_1, \ldots$.

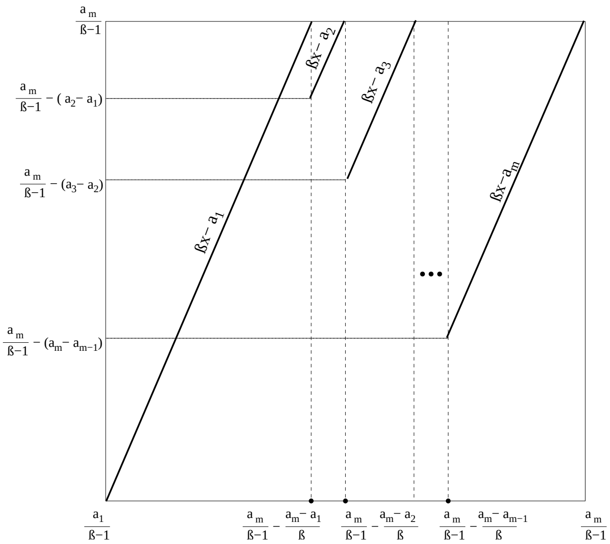

FIGURE 2. The lazy transformation with deleted digits.

Let $A = \{a_1, \ldots, a_m\}$ be a digit set. Consider the digit set $\bar{A} = \{\bar{a}_m, \ldots, \bar{a}_1\}$, where $\bar{a}_i = a_1 + a_m - a_i$. Notice that $\bar{a}_m = a_1$, $\bar{a}_1 = a_m$, and $\bar{a}_i - \bar{a}_{i+1} = a_{i+1} - a_i$, $i = 1, \ldots, m-1$. Thus, A is allowable if and only if $\bar{A}$ is allowable. We have the following proposition.

Proposition 2.2. *Let $A = \{a_1, \ldots, a_m\}$ be an allowable digit set, and let $T = T_{\beta,A}$ and $L = L_{\beta,\bar{A}}$ be the greedy and lazy transformations with digit set A and $\bar{A}$ respectively. Define $f : J_{a_1,a_m} \to J_{a_1,a_m}$ by*

$$f(x) = \frac{a_1 + a_m}{\beta - 1} - x.$$

Then, f is a continuous bijection satisfying $L \circ f = f \circ T$.

Proof. It is clear that f is a continuous bijection. It remains to show that $L \circ f = f \circ T$. To this end, let $x \in \left[\frac{a_1}{\beta - 1} + \frac{a_i - a_1}{\beta}, \frac{a_1}{\beta - 1} + \frac{a_{i+1} - a_1}{\beta}\right)$, then $Tx = \beta x - a_i$ and

$$f(Tx) = \frac{a_1 + a_m}{\beta - 1} - \beta x + a_i.$$

On the other hand,

$$f(x) \in \left(\frac{\bar{a}_m}{\beta - 1} - \frac{\bar{a}_m - \bar{a}_{i+1}}{\beta}, \frac{\bar{a}_m}{\beta - 1} - \frac{\bar{a}_m - \bar{a}_i}{\beta}\right].$$

Thus,

$$L(f(x)) = \beta f(x) - \bar{a}_i = \frac{a_1 + a_m}{\beta - 1} - \beta x + a_i = f(Tx).$$

A similar proof works for the case $x \in \left[\frac{a_1}{\beta - 1} + \frac{a_m - a_1}{\beta}, \frac{a_m}{\beta - 1}\right]$. $\square$

Remark 2.2. (i) From Proposition 2.2, we see that if $x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i}$ is the greedy expansion of x with digits in A, then $f(x) = \sum_{i=1}^{\infty} \frac{\bar{b}_i}{\beta^i}$ is the lazy expansion of $f(x)$ with digits in $\bar{A}$.

(ii) In a similar way as in the proof of Proposition 2.2, one can show that the function $g : \left[0, \frac{a_m - a_1}{\beta - 1}\right] \to \left[\frac{a_1}{\beta - 1}, \frac{a_m}{\beta - 1}\right]$, given by

$$g(x) = x + \frac{a_1}{\beta - 1}$$

is a continuous bijection satisfying $T \circ f = f \circ T'$, where T' is the greedy transformation for the same β, but with digit set $A' = \{0, a_2 - a_1, \ldots, a_m - a_1\}$.

3. Random β-expansions with Deleted Digits

Let $\beta > 1$, and A be a given allowable digit set for β. In this section, we will define a transformation whose iterates generate all possible β-expansions with digit set A. In order to do so, we will construct a random procedure similar to that studied in [DK2] and [DV] for the case when A is a complete digit set. We first superimpose the greedy transformation T and the lazy transformation L both with digit set A. In Figure 3, the greedy and lazy transformations for $\beta = 2.5$ and $A = \{3, 4.25, 6, 11, 14.5, 15\}$ are given. We see that there are regions in which the greedy and lazy transformation overlap, but there are also regions in which we can choose between a number of different digits. As in the classical case we will call the regions in which the two transformations overlap *equality regions*, since the digits assigned there are completely determined. The regions in which we can make a choice are called *switch regions*. In Figure 3 there are three equality regions, and seven switch regions. On four switch regions we have a choice of two possible digits, and on three switch regions we have a choice of three possible digits. The boundaries of these regions are given by the points specified by the transformations as given in the previous section (see Example 3.1).

To describe the general situation, we first define for $1 \leq j \leq m-1$ the *greedy partition points* by

$$g_j = \frac{a_1}{\beta - 1} + \frac{a_{j+1} - a_1}{\beta},$$

and the *lazy partition points* by

$$l_j = \frac{a_m}{\beta - 1} - \frac{a_m - a_j}{\beta}.$$

We set $l_0 = g_0 = \frac{a_1}{\beta - 1}$, and $l_m = g_m = \frac{a_m}{\beta - 1}$. Notice that $g_j \leq l_j$ for all $0 \leq j \leq m$. Furthermore, on $[g_{j-1}, g_j)$, the greedy transformation T is given by $Tx = \beta x - a_j$, and on $(l_{j-1}, l_j]$, the lazy transformation L is given by $Lx = \beta x - a_j$. If $l_j = g_{j+k}$ for some k, j with $j > 0$, $j + k < m$, then any expansion ending in the digits $a_j, a_m, a_m, \ldots$ has a corresponding representation ending in the digits $a_{j+k+1}, a_1, a_1, \ldots$.

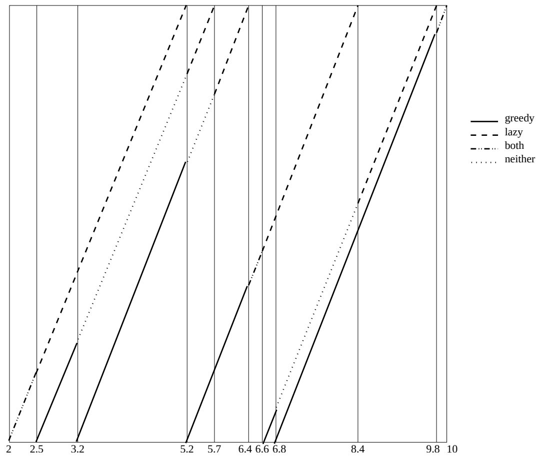

FIGURE 3. The greedy and lazy expansion for $\beta = 2.5$ and allowable digit set $A = \{3, 4.25, 6, 11, 14.5, 15\}$.

Using the points l_j, g_j, $j = 0, 1, \dots, m$, we can make a new partition of the interval $[\frac{a_1}{\beta-1}, \frac{a_m}{\beta-1}]$ in the following way. Let $\{p_n : 0 \le n \le 2m-1\}$ be the ordered sequence obtained when the greedy and the lazy partition points are written in increasing order. Notice that

$$p_0 = l_0 = g_0 = \frac{a_1}{\beta - 1} \quad \text{and} \quad p_{2m-1} = l_m = g_m = \frac{a_m}{\beta - 1}.$$

If for some j, k, we have $l_j = g_{j+k}$, then we write it once, and we consider it as a greedy partition point. Now consider the partition $\mathcal{P}$ consisting of intervals with endpoints two consecutive elements of this sequence, in such a way that the left endpoint of such an interval is included if it is given by a greedy partition point and excluded if it is given by a lazy partition point. Similarly, the right endpoint is excluded if it is given by a greedy partition point and included if it is given by a lazy partition point.

Example 3.1. Let $\beta = 2.5$ and consider the digit set $A = \{3, 4.25, 6, 11, 14.5, 15\}$. The greedy and lazy partition points are given by

$$g_1 = \tfrac{3}{1.5} + \tfrac{1.25}{2.5} = 2.5, \quad l_1 = \tfrac{15}{1.5} - \tfrac{12}{2.5} = 5.2,$$

$$g_2 = \tfrac{3}{1.5} + \tfrac{3}{2.5} = 3.2, \quad l_2 = \tfrac{15}{1.5} - \tfrac{10.75}{2.5} = 5.7,$$

$$g_3 = \tfrac{3}{1.5} + \tfrac{8}{2.5} = 5.2, \quad l_3 = \tfrac{15}{1.5} - \tfrac{9}{2.5} = 6.4,$$

$$g_4 = \tfrac{3}{1.5} + \tfrac{11.5}{2.5} = 6.6, \quad l_4 = \tfrac{15}{1.5} - \tfrac{4}{2.5} = 8.4,$$

$$g_5 = \tfrac{3}{1.5} + \tfrac{12}{2.5} = 6.8, \quad l_5 = \tfrac{15}{1.5} - \tfrac{0.5}{2.5} = 9.8,$$

and the partition $\mathcal{P}$ is then

$$\begin{aligned} \mathcal{P} \;=\; & \{[2; 2.5), [2.5; 3.2), [3.2; 5.2), [5.2; 5.7], (5.7; 6.4], (6.4; 6.6), \\ & [6.6; 6.8), [6.8; 8.4], (8.4; 9.8], (9.8, 10]\}. \end{aligned}$$

We now identify explicitly the intervals that are the equality regions and those that are the switch regions. If there is a $2 \le j \le m-1$ such that $l_{j-1} < g_j$, then since $g_i \le l_i$ for each i, we know that the interval (l_{j-1}, g_j) must belong to $\mathcal{P}$. On such an interval the greedy and lazy transformation overlap. This interval is therefore an equality region and we will call it E_{a_j}. Furthermore, we will write $[p_0, p_1) = [\frac{a_1}{\beta-1}, g_1) = E_{a_1}$ and $(p_{2m-2}, p_{2m-1}] = (l_{m-1}, \frac{a_m}{\beta-1}] = E_{a_m}$, since the two transformations always overlap in the first and last interval. In all the other cases the interval is a switch region. An interval with endpoints p_n and p_{n+1} is called $S_{a_j,\dots,a_{j+k}}$ if $j+k-1 = \max\{i : g_i \le p_n\}$ and $j-1 = \max\{i : l_i \le p_n\}$. So, the endpoints of $S_{a_j,\dots,a_{j+k}}$ are given by $\max\{g_{j+k-1}, l_{j-1}\}$ and $\min\{g_{j+k}, l_j\}$. In this way every element of the partition $\mathcal{P}$ is either an equality region or a switch region.

Example 3.2. If we take β and A as in the previous example, we have the following equality and switch regions:

$$\begin{array}{ll} P_1 = [2; 2.5) = E_3, & P_2 = [2.5; 3.2) = S_{3,4.25}, \\ P_3 = [3.2; 5.2) = S_{3,4.25,6}, & P_4 = [5.2; 5.7] = S_{4.25,6,11}, \\ P_5 = (5.7; 6.4] = S_{6,11}, & P_6 = (6.4; 6.6) = E_{11}, \\ P_7 = [6.6; 6.8) = S_{11,14.5}, & P_8 = [6.8; 8.4] = S_{11,14.5,15}, \\ P_9 = (8.4; 9.8] = S_{14.5,15}, & P_{10} = (9.8, 10] = E_{15}. \end{array}$$

In two special cases we can explicitly give the locations of the equality regions and switch regions.

Proposition 3.1. *Let $\beta > 1$ and $A = \{a_1, \dots, a_m\}$ an allowable digit set. Then,*

(i) *if $m = \lfloor \beta \rfloor + 1$, then $g_j \le l_j \le g_{j+1}$ for each $j \in \{1, \dots, m-2\}$, i.e. the greedy and lazy partition points alternate;*

(ii) *if $\lfloor \beta \rfloor = 1$ and $m > 2$, then $g_{m-1} < l_1$, i.e. the last greedy partition point is strictly smaller than the first lazy partition point.*

Proof. For (i), since $g_j \le l_j$ for all j, we only have to check that $l_j \le g_{j+1}$ for all $j \in \{1, \dots, m-2\}$. First observe that for $j \in \{1, \dots, m-2\}$ we have

$$a_{j+2} - a_j = (a_m - a_1) - \sum_{k=1}^{j-1} (a_{k+1} - a_k) - \sum_{k=j+2}^{m-1} (a_{k+1} - a_k).$$

By condition (3) we get

$$\begin{aligned} a_{j+2} - a_j &\ge (a_m - a_1) - (j-1)\frac{a_m - a_1}{\beta - 1} - (m-j-2)\frac{a_m - a_1}{\beta - 1} \\ &= [(\beta - 1) - (m-3)]\frac{a_m - a_1}{\beta - 1} \\ &\ge \frac{a_m - a_1}{\beta - 1}, \end{aligned}$$

since $m = \lfloor \beta \rfloor + 1$. So

$$\frac{a_{j+2} - a_j}{\beta} \ge \frac{a_m - a_1}{\beta - 1} - \frac{a_m - a_1}{\beta}$$

and therefore also

$$l_j = \frac{a_m}{\beta - 1} - \frac{a_m - a_j}{\beta} \le \frac{a_1}{\beta - 1} + \frac{a_{j+2} - a_1}{\beta} = g_{j+1}.$$

For (ii), just notice that $\beta - 1 < 1$, so that $a_m - a_1 < \dfrac{a_m - a_1}{\beta - 1}$. This means that

$$g_{m-1} = \frac{a_1}{\beta - 1} + \frac{a_m - a_1}{\beta} < \frac{a_m}{\beta - 1} - \frac{a_m - a_1}{\beta} = l_1. \qquad \square$$

Remark 3.1. Notice that, if the greedy partition points and lazy partition points are all different, the first part of the proposition above implies that the equality regions and switch regions alternate in the case $m = \lfloor \beta \rfloor + 1$. The second part of the proposition states that, if $\lfloor \beta \rfloor = 1$ and $m > 2$, then we have only two equality regions, namely E_{a_1} and E_{a_m} and all the other elements of the partition $\mathcal{P}$ are switch regions (see Figure 4).

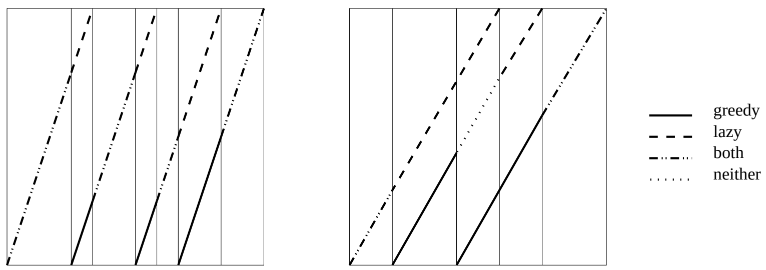

FIGURE 4. On the left we have $m = \lfloor \beta \rfloor + 1$ and on the right we have $\lfloor \beta \rfloor = 1$ and $m > 2$.

We have the following lemma.

Lemma 3.1. *Suppose $x \in J_{a_1,a_m}$ can be written in the form $x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i}$, where $b_i \in A$ for each $i \geq 1$. One has*

(a) *If $x \in E_{a_j}$, then $b_1 = a_j$.*

(b) *If $x \in S_{a_j,\dots,a_{j+k}}$ and $x \notin \{l_j : 1 \leq j \leq m-1\} \cap \{g_j : 1 \leq j \leq m-1\}$, then $b_1 \in \{a_j, \dots, a_{j+k}\}$.*

(c) *If $x = l_j = g_{j+k}$, then $b_1 \in \{a_j, \dots, a_{j+k+1}\}$.*

Proof. Since the proofs of the three statements are very similar, we will only prove the first one. Suppose $x \in E_{a_j}$ and that $b_1 = a_\ell < a_j$. Then $j \neq 1$, so the left endpoint of E_{a_j} is given by l_{j-1} and this point itself is not included in the interval. Furthermore,

$$x \leq \frac{a_\ell}{\beta} + \sum_{i=2}^{\infty} \frac{a_m}{\beta^i} = \frac{a_\ell}{\beta} + \frac{a_m}{\beta(\beta-1)} = \frac{a_m}{\beta - 1} - \frac{a_m - a_\ell}{\beta} = l_\ell \leq l_{j-1},$$

contradicting the fact that $x > l_{j-1}$. If on the other hand $b_1 = a_\ell > a_j$, then $j \neq m$. The right endpoint of E_{a_j} is then given by g_j, which itself is not included in the interval. We have

$$x \geq \frac{a_\ell}{\beta} + \sum_{i=2}^{\infty} \frac{a_1}{\beta^i} = \frac{a_\ell}{\beta} + \frac{a_1}{\beta(\beta-1)} = \frac{a_1}{\beta - 1} + \frac{a_\ell - a_1}{\beta} = g_{\ell-1} \geq g_j,$$

which also yields a contradiction. Thus, $b_1 = a_j$. $\square$

Remark 3.2. Notice that in the above lemma, if $x = l_j = g_{j+k}$, and $b_1 = a_j$, then $b_n = a_m$ for all $n \geq 2$. Hence the given expansion is the lazy expansion. Similarly, if $b_1 = a_{j+k+1}$, then $b_n = a_1$ for all $n \geq 2$. In defining the partition $\mathcal{P}$, we treat these points as greedy partition points. As a consequence expansions ending in $a_j, a_m, a_m, a_m, \ldots$ cannot be generated dynamically in the system defined below, but we can keep in mind that, if for some j, k we have $g_{j+k} = l_j$, then whenever we see an expansion of the form

$$x = \sum_{i=1}^{k} \frac{b_i}{\beta^i} + \frac{a_{j+k+1}}{\beta^{k+1}} + \sum_{i=k+2}^{\infty} \frac{a_1}{\beta^i},$$

where $b_i \in A$ for each $1 \leq i \leq k$, then the same element x, can also be written as

$$x = \sum_{i=1}^{k} \frac{b_i}{\beta^i} + \frac{a_j}{\beta^{k+1}} + \sum_{i=k+2}^{\infty} \frac{a_m}{\beta^i}.$$

This will ease the exposition.

We will now define a transformation generating the random β-expansions in the following way. If an element x lies in an equality region, just assign the digit given by the greedy transformation. To each of the switch regions, we assign a die with an appropriate number of sides. For example, to the switch region $S_{a_j,\ldots,a_{j+k}}$ we assign a $(k+1)$-sided die with the numbers $j, \ldots, j+k$ on it. If x lies in a switch region, we throw the corresponding die and let the outcome determine the digit we choose.

Suppose $\mathcal{P}$ contains Q switch regions. For each $1 \leq q \leq Q$, if the q-th switch region S_q is given by $S_{a_j,\ldots,a_{j+k}}$, let $A_q = \{a_j, \ldots, a_{j+k}\}$ and define the set $\Omega^{(q)}$ by $\Omega^{(q)} = \{j, \ldots, j+k\}^{\mathbb{N}}$. Let each of the sets $\Omega^{(q)}$ be equipped with the product σ-algebra and let $\sigma^{(q)}$ be the left shift on $\Omega^{(q)}$. Elements of $\Omega^{(q)}$ indicate a series of outcomes of the die associated with the region S_q and in this manner specify which digit we choose, each time an element hits S_q. Let $\Omega = \prod_{q=1}^{Q} \Omega^{(q)}$ and define the left shift on the q-th sequence by

$$\sigma_q : \Omega \to \Omega : (\omega^{(1)}, \ldots, \omega^{(Q)}) \mapsto (\omega^{(1)}, \ldots, \omega^{(q-1)}, \sigma^{(q)}(\omega^{(q)}), \omega^{(q+1)}, \ldots, \omega^{(Q)}).$$

On the set $\Omega \times J_{a_1,a_m}$ we define the function $K = K_{\beta,A}$ as follows:

$$K(\omega, x) = \begin{cases} (\omega, \beta x - d_1(\omega, x)), & \text{if } x \in E_{a_j},\ j = 1, \ldots, m, \\ \\ (\sigma_q(\omega), \beta x - d_1(\omega, x)), & \text{if } x \in S_q,\ q = 1, \ldots, Q, \end{cases}$$

where the sequence of digits $\{d_n(\omega, x)\}_{n \geq 1}$ is given by

$$d_1(\omega, x) = \begin{cases} a_j, & \text{if } x \in E_{a_j},\ j = 1, \ldots, m, \\ \\ a_{\omega_1^{(q)}}, & \text{if } x \in S_q,\ q = 1, \ldots, Q, \end{cases}$$

and for $n \geq 2$, $d_n(\omega, x) = d_1(K^{n-1}(\omega, x))$.
To see how iterations of K generate β-expansions, we let $\pi : \Omega \times J_{a_1,a_m} \to J_{a_1,a_m}$ be the canonical projection onto the second coordinate. Then

$$\pi\left(K^n(\omega, x)\right) = \beta^n x - \beta^{n-1} d_1 - \cdots - \beta d_{n-1} - d_n,$$

and rewriting yields

$$x = \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_n}{\beta^n} + \frac{\pi\left(K^n(\omega,x)\right)}{\beta^n}.$$

Since $\pi\left(K^n(\omega,x)\right) \in J_{a_1,a_m}$, it follows that

$$x - \sum_{i=1}^{n} \frac{d_i}{\beta^i} = \frac{\pi_2\left(K^n(\omega,x)\right)}{\beta^n} \to 0 \qquad \text{as } n \to \infty.$$

This shows that for all $\omega \in \Omega$ and for all $x \in J_{a_1,a_m}$ one has that

$$x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} = \sum_{i=1}^{\infty} \frac{d_i(\omega,x)}{\beta^i}.$$

The random procedure just described shows that with each $\omega \in \Omega$ corresponds an algorithm that produces expansions in base β. Further, if we identify the point (ω, x) with $(\omega, (d_1(\omega,x), d_2(\omega,x), \ldots))$, then the action of K on the second coordinate corresponds to the left shift.

The following theorem states that all β-expansions of an element x can be generated, using a certain ω, except when $x = l_j = g_{j+k}$, then the lazy expansion of x is not generated by K. As stated in Remark (3.2), we disregard this case.

Theorem 3.1. *Suppose $x \in J_{a_1,a_m}$ can be written as $x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i}$, with $b_i \in A$ for all $i \geq 1$. Then there exists an $\omega \in \Omega$, such that $b_n = d_n(\omega,x)$ for all $n \geq 1$.*

Proof. This proof goes by induction on the number of digits of each $\omega^{(q)}$, that are determined. First, define the numbers $\{x_n\}_{n\geq 1}$ by $x_n = \sum_{i=1}^{\infty} \frac{b_{i+n-1}}{\beta^i}$ and for $q = 1, \ldots, Q$, let the set $\{l_n^{(q)}(x)\}_{n\geq 1}$ be given by

$$l_n^{(q)}(x) = \sum_{i=1}^{n} 1_{S_q}(x_i).$$

These numbers keep track of the number of times that the orbit of x hits the corresponding switch region.

- If $x \in E_{a_i}$, then by Lemma 3.1 we know that $b_1 = a_i$. Then for all $1 \leq j \leq Q$, $l_1^{(j)}(x) = 0$. Set $\Omega_1 = \Omega$.

- If $x \in S_q$, for some $1 \leq q \leq Q$, then by Lemma 3.1 we have $b_1 = a_i$ for some $a_i \in A_q$. We have $l_1^{(q)}(x) = 1$ and for all $j \neq q$, $l_1^{(j)}(x) = 0$. Set $\Omega_1^{(q)} = \{\omega^{(q)} \in \Omega^{(q)} : \omega_1^{(q)} = i\}$ and for all $j \neq q$, set $\Omega_1^{(j)} = \Omega^{(j)}$. Let $\Omega_1 = \prod_{j=1}^{Q} \Omega_1^{(j)}$.

Then in all cases, for each $1 \leq j \leq Q$, the set $\Omega_1^{(j)}$ is a cylinder set of length $l_1^{(j)}(x)$, where by a cylinder set of length 0 we mean the whole space $\Omega^{(j)}$. Furthermore, $b_1 = d_1(\omega,x)$ for all $\omega \in \Omega_1$. Now suppose we have obtained sets $\Omega_n \subseteq \ldots \subseteq \Omega_1$, so that for each $1 \leq j \leq Q$, $\Omega_n^{(j)}$ is a cylinder set of length $l_n^{(j)}(x)$ and that for all $1 \leq i \leq n$ and all $\omega \in \Omega_i$ we have $b_i = d_i(\omega,x)$.

- If $x_{n+1} \in E_{a_i}$, then for all $1 \le j \le Q$, $l_{n+1}^{(j)}(x) = l_n^{(j)}(x)$ and for all $\omega \in \Omega_n$, $b_{n+1} = a_i = d_{n+1}(\omega, x)$. Therefore, set $\Omega_{n+1} = \Omega_n$.

- If $x_{n+1} \in S_q$, then $l_{n+1}^{(q)}(x) = l_n^{(q)}(x) + 1$ and for all $j \neq q$, $l_{n+1}^{(j)}(x) = l_n^{(j)}(x)$. By Lemma 3.1, $b_{n+1} = a_i$ for some $a_i \in A_q$. Set $\Omega_{n+1}^{(q)} = \{\omega^{(q)} \in \Omega_n^{(q)} : \omega_{n+1}^{(q)} = i\}$ and for all $j \neq q$, let $\Omega_{n+1}^{(j)} = \Omega_n^{(j)}$. Set $\Omega_{n+1} = \prod_{j=1}^{Q} \Omega_n^{(j)}$. Then for each $\omega \in \Omega_{n+1}$ we have $b_{n+1} = d_{n+1}(\omega, x) = d_1(K^n(\omega, x))$.

The above shows that for each $1 \le j \le Q$, $\Omega_{n+1}^{(j)}$ is a cylinder set of length $l_{n+1}^{(j)}(x)$ and for all $\omega \in \Omega_{n+1}$ we have for all $1 \le i \le n+1$ that $b_i = d_i(\omega, x)$. If the map K hits one of the switch regions infinitely many times, then $l_n^{(q)}(x) \to \infty$ for the corresponding region and since all cylinder sets are compact, we know that $\bigcap_{n=1}^{\infty} \Omega_n^{(q)}$ in that case consists of one single point. In the other case the set $\{l_n^{(q)}(x) : n \in \mathbb{N}\}$ is finite and $\bigcap_{n=1}^{\infty} \Omega_n^{(q)}$ is exactly a cylinder set. In both cases, $\bigcap_{n=1}^{\infty} \Omega_n^{(q)}$ is non-empty and since this holds for each q, also the set $\bigcap_{n=1}^{\infty} \Omega_n$ consists of at least one element. Furthermore, the elements $\omega \in \bigcap_{n=1}^{\infty} \Omega_n$ satisfy the requirement of the theorem. $\square$

Remark 3.3. Notice that, if the sequence $\{x_n\}$ hits every switch region infinitely many times, the above procedure leads to a unique ω. Otherwise one gets a cylinder set or even the whole space in case x has a unique expansion, i.e. in case the orbit of x only visits the equality regions.

In the last part, we construct an isomorphism that links this transformation K to the uniform Bernoulli shift. Consider the probability space $(A^{\mathbb{N}}, \mathcal{A}, P)$, where $\mathcal{A}$ is the product σ-algebra on $A^{\mathbb{N}}$, and P is the uniform product measure. In case $l_j = g_{j+k}$ for some j, k, we remove from $A^{\mathbb{N}}$ all sequences that eventually end in $a_j, a_m, a_m, \ldots$, and we call the new set D. Let $(D, \mathcal{D}, \mathbb{P})$ be the probability space obtained from $(A^{\mathbb{N}}, \mathcal{A}, P)$, by restricting it to D and let σ' be the left shift on D. Let $\mathcal{F} \times \mathcal{B}$, be the product σ-algebra on $\Omega \times J_{a_1,a_m}$ where $\mathcal{F}$ is the product σ-algebra on Ω and $\mathcal{B}$ is the Borel σ-algebra on J_{a_1,a_m}. Define the function $\phi : \Omega \times J_{a_1,a_m} \to D$ by

$$\phi(\omega, x) = (d_1(\omega, x), d_2(\omega, x), \ldots).$$

Then ϕ is measurable, $\phi \circ K = \sigma' \circ \phi$ and Theorem 3.1 states that ϕ is surjective. We will indicate a subset of $\Omega \times J_{a_1,a_m}$ on which ϕ is invertible. For $1 \le q \le Q$, let

$$\begin{aligned} Z_q &= \{(\omega, x) \in \Omega \times J_{a_1,a_m} : \pi(K^n(\omega, x)) \in S_q \text{ i.o.}\}, \\ D_q &= \{(b_1, b_2, \ldots) \in D : \sum_{i=1}^{\infty} \frac{b_{j+i-1}}{\beta^i} \in S_q \text{ for infinitely many } j\text{'s}\}, \end{aligned}$$

and define the sets $Z = \bigcap_{q=1}^{Q} Z_q$ and $D^* = \bigcap_{q=1}^{Q} D_q$. It is clear that $\phi(Z) = D^*$, $K^{-1}(Z) = Z$ and $(\sigma')^{-1}(D^*) = D^*$. Let ϕ^* be the restriction of ϕ to Z. The next lemma states that ϕ^* is a bijection.

Lemma 3.2. *The map $\phi^* : Z \to D^*$ is a bimeasurable bijection.*

Proof. Let the sequence $(b_1, b_2, \ldots)$ be an element of D^* and define for each $1 \leq q \leq Q$ sequence $\{r_i^{(q)}\}_{i \geq 1}$ recursively:

$$\begin{aligned} r_1^{(q)} &= \min\{j \geq 1 : \sum_{n=1}^{\infty} \frac{b_{j+n-1}}{\beta^n} \in S_q\}, \\ r_i^{(q)} &= \min\{j > r_{i-1} : \sum_{n=1}^{\infty} \frac{b_{j+n-1}}{\beta^n} \in S_q\}. \end{aligned}$$

By Lemma 3.1, we know that $b_{r_i^{(q)}} = a_\ell \in A_q$. Set $\omega_i^{(q)} = \ell$. Take $\omega = (\omega^{(1)}, \ldots, \omega^{(Q)})$ and define $(\phi^*)^{-1} : D^* \to Z$ by

$$(\phi^*)^{-1}((b_1, b_2, \ldots)) = \left(\omega, \sum_{i=1}^{\infty} \frac{b_i}{\beta^i} \right).$$

We can easily check that $(\phi^*)^{-1}$ is measurable and is the inverse of ϕ^*. □

Lemma 3.3. $\mathbb{P}(D^*) = 1$.

Proof. Fix q. We know that if the q-th switch region is given by $S_q = S_{a_j, \ldots, a_{j+k}}$, then it is bounded on the left by $\max\{g_{j+k-1}, l_{j-1}\}$ and on the right by $\min\{g_{j+k}, l_j\}$. Let $x = \max\{g_{j+k-1}, l_{j-1}\}$ be the left endpoint of S_q and suppose that the greedy expansion of x is given by

$$x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i},$$

with $d_i \in A$ for all $i \geq 1$. Notice that $d_i \neq a_m$ for infinitely many i. Let

$$0 < \delta < \min\{g_{j+k} - x, l_j - x\}$$

and choose ℓ sufficiently large, such that

(i) $\displaystyle\sum_{i=\ell}^{\infty} \frac{a_m}{\beta^i} < \delta$ and

(ii) $d_\ell \neq a_m$.

Let $(b_1, b_2, \ldots) \in D$ be an arbitrary sequence of digits and set

$$x_\ell = \sum_{i=1}^{\ell-1} \frac{d_i}{\beta^i} + \frac{a_m}{\beta^\ell} + \sum_{i=\ell+1}^{\infty} \frac{b_{i-\ell}}{\beta^i}.$$

Since, by the definition of the greedy expansion, d_ℓ is the largest element of A such that

$$\sum_{i=1}^{\ell} \frac{d_i}{\beta^i} + \sum_{i=\ell+1}^{\infty} \frac{a_1}{\beta^i} \leq x,$$

we have

$$x_\ell \geq \sum_{i=1}^{\ell-1} \frac{d_i}{\beta^i} + \frac{a_m}{\beta^\ell} + \sum_{i=\ell+1}^{\infty} \frac{a_1}{\beta^i} > x.$$

Also,

$$x_\ell \leq \sum_{i=1}^{\ell-1} \frac{d_i}{\beta^i} + \frac{a_m}{\beta^\ell} + \sum_{i=\ell+1}^{\infty} \frac{a_m}{\beta^i} < x + \delta < \min\{g_{j+k}, l_j\}.$$

So, $x_\ell \in S_q$. Define the set

$D_q^* = \{(b_1, b_2, \ldots) \in A^{\mathbb{N}} : b_j, \ldots, b_{j+\ell-1} = d_1, \ldots, d_{\ell-1}, a_m$ for infinitely many j's$\}$.

By the second Borel-Cantelli Lemma, $\mathbb{P}(D_q^*) = 1$ and since obviously $D_q^* \subseteq D_q$, also $\mathbb{P}(D_q) = 1$. This holds for all $q \in \{1, \ldots, Q\}$, so we have $\mathbb{P}(D^*) = 1$. □

Using the product measure $\mathbb{P}$, we can define the K-invariant measure ν on $\mathcal{F} \times \mathcal{B}$, by setting

$$\nu(A) = \mathbb{P}(\phi(Z \cap A)).$$

As a direct consequence of Lemmas 3.2 and 3.3, we have the following theorem.

Theorem 3.2. *Let $\beta > 1$ and suppose $A = \{a_1, \ldots, a_m\}$ is an allowable digit set. The dynamical systems $(\Omega \times J_{a_1,a_m}, \mathcal{F} \times \mathcal{B}, \nu, K)$ and $(D, \mathcal{D}, \mathbb{P}, \sigma')$ are measurably isomorphic.*

As an immediate consequence of the isomorphism between the two dynamical systems, we have the following corollary about the entropy of K.

Corollary 3.1. *The entropy of the transformation K with respect to the measure ν is given by*

$$h_\nu(K) = \log m.$$

Remark 3.4. Let $\mathcal{M}$ be the family of measures μ defined on $(D, \mathcal{D})$, that are shift invariant and have $\mu(D^*) = 1$. Then for each $\mu \in \mathcal{M}$ we can define the measure ν_μ on $(\Omega \times J_{a_1,a_m}, \mathcal{F} \times \mathcal{B})$ by

$$\nu_\mu(A) = \mu(\phi(A \cap Z)).$$

It is clear that ν_μ is K-invariant and that if $\nu_\mu \neq \mathbb{P}$, then

$$h_{\nu_\mu}(K) < \log m.$$

We are interested in identifying the projection of the measure ν on the second coordinate, namely the measure $\nu \circ \pi^{-1}$ defined on J_{a_1,a_m} To do so, we consider the purely discrete measures $\{\delta_i\}_{i \geq 1}$ defined on $\mathbb{R}$ as follows:

$$\delta_i(\{a_1\beta^{-i}\}) = \frac{1}{m}, \ldots, \delta_i(\{a_m\beta^{-i}\}) = \frac{1}{m}.$$

Let δ be the corresponding infinite Bernoulli convolution,

$$\delta = \lim_{n\to\infty} \delta_1 * \ldots * \delta_n.$$

Proposition 3.2. $\nu \circ \pi^{-1} = \delta$.

Proof. Let $h : D \to J_{a_1,a_m}$ be given by $h(y) = \sum_{i=1}^{\infty} \frac{y_i}{\beta^i}$, where $y = (y_1, y_2, \ldots)$. Then, $\pi = h \circ \phi$, and $\delta = \mathbb{P} \circ h^{-1}$. Since $\mathbb{P} = \nu \circ \phi^{-1}$, it follows that $\nu_\beta \circ \pi^{-1} = \delta$. □

If $\beta \in (1, 2)$ and $A = \{0, 1\}$, then δ is an Erdös measure on $[0, \frac{1}{\beta-1}]$, and lots of things are already known. For example, if β is a Pisot number, then δ is singular with respect to Lebesgue measure; [E1, E2, S]. Further, for almost all $\beta \in (1, 2)$ the measure δ is equivalent to Lebesgue measure; [So, MS]. There are many generalizations of these results to the case of an arbitrary digit set (see [PSS] for more references and results).

Final Remarks.

(i) Instead of assigning a different die to each of the switch regions, we can also use the same m-sided die for each switch region and only consider the outcomes that are meaningful. We will define the dynamical system that generates the random β-expansions with deleted digits in this way. It can be shown that this system is isomorphic to the Bernoulli shift on A and hence also to the system described above.

Let the partition $\mathcal{P}$ be constructed as before. Let $\bar{\Omega} = \{1, \dots, m\}^{\mathbb{N}}$ and let σ be the left shift. Elements of $\bar{\Omega}$ now indicate a series of outcomes of our m-sided die and thus specify which transformation we choose. We first define the function $\bar{K} : \bar{\Omega} \times J_{a_1,a_m} \to \bar{\Omega} \times J_{a_1,a_m}$ by

$$\bar{K}(\bar{\omega}, x) = \begin{cases} (\bar{\omega}, \beta x - a_j) & \text{if } x \in E_{a_j}, \\ (\sigma(\bar{\omega}), \beta x - a_{\bar{\omega}_1}) & \text{if } x \in S_{a_j,\dots,a_{j+k}} \text{ and } \bar{\omega}_1 \in \{j, \dots, j+k\}, \\ (\sigma(\bar{\omega}), x) & \text{if } x \in S_{a_j,\dots,a_{j+k}} \text{ and } \bar{\omega}_1 \notin \{j, \dots, j+k\}. \end{cases}$$

Define the set S^* by

$$S^* = \bigcup_{j=1}^{m-1} \bigcup_{k=1}^{m-j} \{\{\bar{\omega}\} \times S_{a_j,\dots,a_{j+k}} : \bar{\omega}_1 \notin \{j, \dots, j+k\}\}.$$

Let $X = \bar{\Omega} \times J_{a_1,a_m} \setminus S^*$ and $X^* = \bigcap_{n=0}^{\infty} \bar{K}^{-n}(X)$. Consider the restriction of $\bar{K}$ to this set X^* and call it $R = R_\beta$, that is

$$R : X^* \to X^* : (\bar{\omega}, x) \mapsto \bar{K}(\bar{\omega}, x).$$

Notice that the set X^* only contains those combinations of $\bar{\omega}$'s and x's that yield 'valid' choices at every moment the die is thrown.

Let the dynamical system $(D, \mathcal{D}, \mathcal{P}, \sigma')$ be as before. Let $\bar{\mathcal{F}}$ be the product σ-algebra on $\bar{\Omega}$ and let $\mathcal{B}$ again be the Borel σ-algebra on J_{a_1,a_m}. Define

$$\mathcal{X} = \{A \cap X^* : A \in \bar{\mathcal{F}} \times \mathcal{B}\}.$$

Then, by defining the sequence of digits $\{\bar{d}_n\}_{n \geq 1}$ and the function $\bar{\phi}$ as we have done before, we can prove Theorem 3.1, Lemma 3.2 and Lemma 3.3 by only making slight adjustments to the proofs. If we also define the R-invariant measure $\bar{\nu}$ in a similar way, then we have shown that the system $(X^*, \mathcal{X}, \bar{\nu}, R)$ is isomorphic to $(D, \mathcal{D}, \mathbb{P}, \sigma')$.

(ii) We have established a measurable isomorphism between the dynamical systems $(\Omega \times J_{a_1,a_m}, \mathcal{F} \times \mathcal{B}, \nu, K)$, $(X^*, \mathcal{X}, \bar{\nu}, R)$ and $(D, \mathcal{D}, \mathbb{P}, \sigma')$, but still know very little about the measures ν and $\bar{\nu}$. An interesting question would be, whether or not these measures are measures of maximal entropy and if so, if they are the unique measures with this property. Another point of interest would be to find a measure, whose projection onto the second coordinate is equivalent to Lebesgue measure.

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