AN OPTIMAL ADAPTIVE FINITE ELEMENT METHOD FOR THE STOKES PROBLEM*

YAROSLAV KONDRATYUK, ROB STEVENSON†

Abstract. A new adaptive finite element method for solving the Stokes equations is developed, which is shown to converge with the best possible rate. The method consists of 3 nested loops. For a sequence of finite element spaces with respect to adaptively refined partitions, in the outmost loop the solution is approximated by that of the Stokes problem in which the divergence-free condition is reduced to orthogonality of the divergence to the finite element space in which the approximate pressure is sought. For solving each of these semi-discrete problems, the Uzawa iteration is applied. Finally, the elliptic system for the velocity that has to be solved in each Uzawa step is approximately solved by an adaptive finite element method.

Key words. Adaptive finite element method, convergence rates, computational complexity, a posteriori error estimators, Uzawa iteration, Stokes equations.

AMS subject classifications. 65N30, 65N50, 65N15, 65Y20, 41A25.

1. Introduction. Often the solution of a boundary value problem exhibits singularities, e.g., due to a non-smooth boundary. Then, because of the lacking (Sobolev) smoothness of the solution, finite element methods based on quasi-uniform partitions converge with a rate that is smaller than is allowed by the polynomial degree that is applied. This can be repaired when suitable refinements are made near those singularities. The optimal size of the elements as function of the distance to a singularity depends on the strength of the singularity, which is generally unknown.

With adaptive finite element methods (AFEMs), a sequence of nested partitions is created, where, when creating the next partition, the decision where to refine is made on basis of an a posteriori estimator of the error in the current finite element approximation. Although being successfully in use for more than 25 years, in more than one space dimension, even for second order *elliptic* equations, their convergence was not shown before the works of Dörfler ([Dör96]) and that of Morin, Nochetto and Siebert ([MNS00]). Convergence alone, however, does not show that the use of an adaptive method for a solution that has singularities improves upon, or even competes with that of a non-adaptive one. Recently, after the derivation of such a result by Binev, Dahmen and DeVore ([BDD04]) for an AFEM extended with a so-called coarsening routine, in [Ste05b] we could prove that standard AFEMs converge with the best possible rate in linear complexity. So this rate is equal to that of finite element approximations with respect to the sequence over $N \in \mathbf{N}$ of the best partitions with N elements.

In this paper, as a model *saddle point problem*, we consider the Stokes equations

$$\begin{cases} -\triangle \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{on } \Omega \subset \mathbf{R}^d, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

(although, in this introduction, we write equations in strong form, actually we always mean the corresponding variational formulations). In [DDU02], Dahlke, Dahmnen and Urban analyzed an adaptive wavelet method for solving these equations. The

*This work was supported by the Netherlands Organization for Scientific Research and by the European Community's Human Potential Programme under contract HPRN-CT-2002-00286.

†Department of Mathematics, Utrecht University, P.O. Box 80.010, NL-3508 TA Utrecht, The Netherlands ({kondratyuk, stevenson}@math.uu.nl)

starting point was the application of the Uzawa iteration on the continuous level, i.e., given some p_0, to compute for $j = 0, 1, \ldots$

$$\begin{cases} -\triangle \mathbf{u}_{j+1} &= \mathbf{f} - \nabla p_j, \\ \quad p_{j+1} &= p_j - \mathrm{div}\mathbf{u}_j, \end{cases}$$

Of course, this iteration cannot be performed exactly, and in each iteration the solution of the elliptic system was approximated using an adaptive wavelet method within decreasing tolerances as the iteration proceeds. Convergence was shown, and by the inclusion of coarsening steps, even optimal rates and linear complexity were demonstrated. Since nowhere Galerkin discretizations were formed of the mixed problem, the so-called LBB stability was not required.

In [BMN02], Bänsch, Morin and Nochetto studied above solution method with the adaptive wavelet method replaced by an AFEM. They proved convergence, and despite of the fact that they did not include coarsening, in numerical experiments they observed optimal rates, at least when the elliptic problems were solved not too accurately. When prescribing an a priori tolerance of the form γ^j for the jth iteration, it was needed to take γ in the range $[\approx .95, 1)$. By the addition of coarsening to this method, in [Kon06] optimal computational complexity was demonstrated.

When starting this work, our aim was to prove optimal computational complexity of basically the method from [BMN02]. For a reason that will be indicated in Remark 6.8, we didn't succeed to do this. Instead, for a somewhat more complicated algorithm involving an additional loop, we will prove optimal rates, and under some mild assumption (Assumption 6.4), also optimal computational complexity.

The pressure p can be found as the solution of the Schur complement equation that one obtains by eliminating the velocity $\mathbf{u}$ from the Stokes equations. This equation is *elliptic*, with corresponding energy norm equivalent to the $L_2(\Omega)$-norm. Given a finite element space $\mathbb{P}_{\sigma_i}$, where σ_i denotes the underlying finite element partition, the best approximation from this space to p with respect to this energy norm is the Galerkin solution $p_i \in \mathbb{P}_{\sigma_i}$. With Q_{σ_i} denoting the $L_2(\Omega)$-orthogonal projection onto $\mathbb{P}_{\sigma_i}$, this p_i can be shown to be the unique solution of

$$\begin{cases} -\triangle \mathbf{u}^{(i)} + \nabla p_i &= \mathbf{f} \quad \text{on } \Omega, \\ \quad Q_{\sigma_i}\mathrm{div}\mathbf{u}^{(i)} &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

i.e., the Stokes equations in which the divergence-free condition has been relaxed. We refer to this system as the reduced Stokes equations.

Concerning $\mathbf{u}^{(i)}$, this is still a problem posed over an infinite dimensional space. Assuming for the moment that we can solve it exactly, or more precisely, with a sufficient accuracy, the error in energy norm in p_i can be shown to be equivalent to the *a posteriori error estimator* $\|\mathrm{div}\mathbf{u}^{(i)}\|_{L_2(\Omega)}$. Furthermore, for any refined partition σ_{i+1}, the energy norm of $p_{i+1} - p_i$ is equivalent to $\|Q_{\sigma_{i+1}}\mathrm{div}\mathbf{u}^{(i)}\|_{L_2(\Omega)}$. Now following the lines of [Dör96, MNS00] for Poisson type problems, if, for some $\theta \in (0, 1]$, σ_{i+1} is selected such that $\|Q_{\sigma_{i+1}}\mathrm{div}\mathbf{u}^{(i)}\|_{L_2(\Omega)} \geq \theta\|\mathrm{div}\mathbf{u}^{(i)}\|_{L_2(\Omega)}$ ("bulk criterion"), then the so-called saturation property is guaranteed, and a linearly convergent sequence $(p_i)_i$ towards p is obtained. Moreover, if, depending on the efficiency index of this a posteriori error estimator, θ is small enough, and σ_{i+1} is selected with quasi minimal cardinality, then following the lines of [Ste05b], we can show convergence with the optimal rate. Compared to the adaptive methods for Poisson type problems, a complication is that to find such a σ_{i+1}, it is generally not sufficient to search it within the set of partitions that can be created by refining each element of σ_i only a small, fixed number of times. For our theoretical considerations, we studied the adaptive tree algorithms by Binev and DeVore from [BD04], whereas in our experiments we relied on the easy implementable greedy approach.

For solving the reduced Stokes problem for given i, we follow the approach from [BMN02] for the full Stokes problem. That is, we apply Uzawa, where the pressure update then reads as $p_{j+1}^{(i)} = p_j^{(i)} - Q_{\sigma_i}\mathrm{div}\mathbf{u}_j^{(i)}$, and where we solve the inner elliptic systems $-\triangle\mathbf{u}_{j+1}^{(i)} = \mathbf{f} - \nabla p_j^{(i)}$ with AFEM. Knowing that $p_j^{(i)} \in \mathbb{P}_{\sigma_i}$, and having control over $\#\sigma_i$, we are now able to prove also optimal rates of the velocity approximations towards $\mathbf{u}$. Note that other than in [BMN02], we have two different partitions underlying pressure and velocity approximations. Throughout the algorithm, both partitions become increasingly more refined, i.e., no derefinements are made.

This paper is organized as follows: In Section 2, we recall some properties of the Stokes problem. In Section 3, we define the finite element spaces that we will use. We give a procedure for refining partitions, which is a generalization to arbitrary space dimensions of the newest vertex bisection method in two space dimensions. An overview of the solution method will be given in Section 4. In Section 5, a posteriori error estimators are derived for the various problems that occur in our solution method. In Section 6, the adaptive refinement routines for pressure and velocity partitions are given. In Section 7, we give the detailed description of the adaptive method in the simplified situation that the right-hand side $\mathbf{f}$ is piecewise polynomial with respect to the initial partition. We prove convergence with the optimal rates. In this section, we assume that the arising finite dimensional linear systems are solved exactly, ignoring the question of computational complexity. In Section 8, we give the method for general right-hand sides, and replace the direct solvers by iterative solution methods, with which we end up with a method of optimal computational complexity. Finally, in Section 9, we present numerical results, and compare them with those obtained with the method from [BMN02]. As we will see, in this example both methods give similar results.

In this paper, by $C \lesssim D$ we will mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2. Stokes problem. Let Ω be a polygonal domain in $\mathbf{R}^d$. We consider the Stokes problem in variational form: With

$$\mathbb{V} := H_0^1(\Omega)^d, \quad \mathbb{P} := L_{2,0}(\Omega),$$

and given an $\mathbf{f} \in \mathbb{V}'$, *throughout this paper* $\mathbf{u} \in \mathbb{V}$ *(the velocity) and* $p \in \mathbb{P}$ *(the pressure) will denote the solutions of*

$$\boxed{a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) + b(\mathbf{u},q) = \mathbf{f}(\mathbf{v}), \quad (\mathbf{v} \in \mathbb{V},\, q \in \mathbb{P})}, \tag{2.1}$$

where $a : \mathbb{V} \times \mathbb{V} \to \mathbf{R}$, $b : \mathbb{V} \times \mathbb{P} \to \mathbf{R}$ are defined by

$$a(\mathbf{w},\mathbf{v}) := \int_\Omega \nabla\mathbf{w} : \nabla\mathbf{v}, \quad b(\mathbf{v},q) := -\int_\Omega q\,\mathrm{div}\mathbf{v}.$$

It is well-known that

$$\|\mathbf{v}\|_{\mathbb{V}} := a(\mathbf{v},\mathbf{v})^{\frac{1}{2}} \eqsim \|\mathbf{v}\|_{H^1(\Omega)^d}, \quad (\mathbf{v} \in \mathbb{V}),$$

b is bounded, and

$$\beta := \inf_{0 \neq q \in \mathbb{P}} \sup_{0 \neq \mathbf{v} \in \mathbb{V}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbb{V}} \|q\|_{L_2(\Omega)}} > 0. \tag{2.2}$$

As a consequence, the Stokes problem is well-posed, meaning that

$$\|\mathbf{w}\|_{\mathbb{V}} + \|r\|_{\mathbb{P}} \eqsim \sup_{0 \neq (\mathbf{v}, q) \in \mathbb{V} \times \mathbb{P}} \frac{a(\mathbf{w}, \mathbf{v}) + b(\mathbf{v}, r) + b(\mathbf{w}, q)}{\|\mathbf{v}\|_{\mathbb{V}} + \|q\|_{\mathbb{P}}}, \quad (\mathbf{w} \in \mathbb{V},\, r \in \mathbb{P}). \tag{2.3}$$

Remark 2.1. Clearly, for $r \in \tilde{\mathbb{P}} \subset \mathbb{P}$, (2.3) is also valid, uniformly over $\tilde{\mathbb{P}} \subset \mathbb{P}$, when the supremum over $\mathbb{V} \times \mathbb{P}$ is replaced by that over $\mathbb{V} \times \tilde{\mathbb{P}}$.
Using that $\|\mathrm{div}\mathbf{v}\|_{L_2(\Omega)} \leq \|\mathbf{v}\|_{\mathbb{V}}$ ($\mathbf{v} \in \mathbb{V}$) (see [NP04]), in particular we have $|b(\mathbf{v}, q)| \leq \|\mathbf{v}\|_{\mathbb{V}} \|q\|_{\mathbb{P}}$, and besides thus $\beta \leq 1$.

Defining $A : \mathbb{V} \to \mathbb{V}'$, $B : \mathbb{V} \to \mathbb{P}'$, and $B' : \mathbb{P} \to \mathbb{V}'$ by $(A\mathbf{v})(\mathbf{w}) = a(\mathbf{v}, \mathbf{w})$, $(B\mathbf{v})(q) = b(\mathbf{v}, q) = (B'q)(\mathbf{v})$, the problem (2.1) can be equivalently written as

$$\begin{bmatrix} A & B' \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix},$$

and, with the Schur complement $S := BA^{-1}B'$, p is also uniquely determined by

$$Sp = BA^{-1}f.$$

LEMMA 2.2. *We have*

$$(Sq)(q) = \sup_{0 \neq \mathbf{v} \in \mathbb{V}} \frac{b(\mathbf{v}, q)^2}{a(\mathbf{v}, \mathbf{v})},$$

so that, by the ellipticity of a*, the boundedness of* b*, and* (2.2)*,*

$$\|q\|_{\mathbb{P}} := (Sq)(q)^{\frac{1}{2}} \eqsim \|q\|_{L_2(\Omega)}, \quad (q \in \mathbb{P}).$$

Proof. With $\langle\, , \,\rangle = \langle\, , \,\rangle_{H^1(\Omega)^d}$ (or $\langle\, , \,\rangle = a(\, , \,)$), let $R : \mathbb{V} \to \mathbb{V}'$ be the mapping such that $\mathbf{g}(\mathbf{v}) = \langle \mathbf{v}, R\mathbf{g} \rangle$ ($\mathbf{v} \in \mathbb{V}$, $\mathbf{g} \in \mathbb{V}'$). Writing $\tilde{B}' = RB'$, $\tilde{A} = RA$, we have

$$\begin{aligned} \sup_{0 \neq \mathbf{v} \in \mathbb{V}} \frac{b(\mathbf{v}, q)^2}{a(\mathbf{v}, \mathbf{v})} &= \sup_{0 \neq \mathbf{v} \in \mathbb{V}} \frac{(B'q)(\mathbf{v})^2}{(A\mathbf{v})(\mathbf{v})} = \sup_{0 \neq \mathbf{v} \in \mathbb{V}} \frac{\langle \mathbf{v}, \tilde{B}'q \rangle^2}{\langle \mathbf{v}, \tilde{A}\mathbf{v} \rangle} = \sup_{0 \neq \mathbf{w} \in \mathbb{V}} \frac{\langle \mathbf{w}, \tilde{A}^{-\frac{1}{2}} \tilde{B}'q \rangle^2}{\langle \mathbf{w}, \mathbf{w} \rangle} \\ &= \langle \tilde{A}^{-\frac{1}{2}} \tilde{B}'q, \tilde{A}^{-\frac{1}{2}} \tilde{B}'q \rangle = \langle A^{-1}B'q, RB'q \rangle = (Sq)(q) \quad (q \in \mathbb{P}). \end{aligned}$$

□

For $\mathbf{g} \in \mathbb{V}'$, we set $\|\mathbf{g}\|_{\mathbb{V}'} = \sup_{0 \neq \mathbf{v} \in \mathbb{V}} \frac{|\mathbf{g}(\mathbf{v})|}{\|\mathbf{v}\|_{\mathbb{V}}}$. Equipped with norms $\|\ \|_{\mathbb{V}}$ and $\|\ \|_{\mathbb{V}'}$, respectively, $A : \mathbb{V} \to \mathbb{V}'$ is an isomorphism.

Functions $\mathbf{g} \in L_2(\Omega)^d$ will be interpreted as functionals by means of $\mathbf{g}(\mathbf{v}) := \int_\Omega \mathbf{g} \cdot \mathbf{v}$.

3. Finite element approximation. Given some fixed $k \in \mathbf{N}_{>0}$, and partitions τ and σ of $\bar{\Omega}$ into essentially disjoint (closed) d-simplices, we will search approximations for $\mathbf{u}$ and p from the finite element spaces

$$\mathbb{V}_\tau := \mathbb{V} \cap \prod_{T \in \tau} P_m(T)^d,$$

and

$$\mathbb{P}_\sigma := \mathbb{P} \cap \prod_{T\in\sigma} P_{m-1}(T),$$

respectively. For doing so, furthermore we will approximate the right-hand side $\mathbf{f}$ by functions from

$$\mathbb{V}_\tau^* := \prod_{T\in\tau} P_{m-1}(T)^d.$$

At any moment in our algorithm we will have that $\tau \supseteq \sigma$, meaning that τ is a refinement of σ, or is equal to σ.

Sometimes, we will view $\mathbb{V}$ and $\mathbb{P}$ formally as finite element spaces corresponding to the infinitely fine partition ∞, and denote them as $\mathbb{V}_\infty$ and $\mathbb{P}_\infty$, respectively.

Remark 3.1. The fact that the approximate pressure is a piecewise polynomial of degree not larger than $m-1$ will only be used in the forthcoming Proposition 5.3. It is most likely that also there higher degree polynomials can be permitted at the expense of having a more complicated refinement rule for the velocity partitions (it will be needed to create more interior vertices, cf. Figure 5.1). On the other hand, at least for our analysis, it will be essential that $\mathbb{P}_\tau \supseteq \mathrm{div}\mathbb{V}_\tau$ (cf. Remark 6.1).

Note that $(\mathbb{V}_\tau, \mathbb{P}_\tau)$ is *not* an LBB stable pair.

Below, we specify the type of partitions we will consider, and recall some results from [Ste05a], generalizing upon known results for *newest vertex bisection* in two dimensions. For $0 \leq i \leq d-1$, a simplex spanned by $i+1$ vertices of an d-simplex T is called a hyperface of T. For $i = d-1$, it will be called a *true hyperface*, and for $i \leq d-2$ it will called a *lower dimensional hyperface*. A partition τ is called *conforming* when the intersection of any two different $T, T' \in \tau$ is either empty, or a hyperface of both simplices. Different simplices T, T' that share a true hyperface will be called *neighbours*. (Actually, when $\Omega \neq \mathrm{int}(\overline{\Omega})$ above definition of a conforming partition can be unnecessarily restrictive. We refer to [Ste05a] for a discussion of this matter.)

Given a simplex T with vertices $\{x_0, \ldots, x_d\}$, we will identify $\frac{1}{2}(d+1)!$ *tagged* simplices given by all possible *ordered* sequences $(\{x_0, x_1\}, x_2, \ldots, x_d)$. So although, for convenience, we will write a tagged simplex as $(x_0, x_1, x_2, \ldots, x_d)$, the ordering of the first two coordinates is arbitrary. Given a tagged simplex $T = (x_0, x_1, x_2, \ldots, x_d)$, its children are the tagged simplices $(x_0, x_2, \ldots, x_d, \frac{x_0+x_1}{2})$ and $(x_1, x_2, \ldots, x_d, \frac{x_0+x_1}{2})$. So these children are generated by bisecting the edge $\overline{x_0x_1}$ of T, i.e., by connecting its midpoint with the other vertices $x_2, \ldots, x_d$, see Figure 3.1 for an illustration. The edge $\overline{x_0x_1}$ is called the *refinement edge* of T. In the $d = 2$ case, the vertex opposite to this edge is known as the *newest vertex*.

Given a fixed conforming initial partition τ_0 of tagged simplices, *we will exclusively consider partitions that can be created from τ_0 by recurrent bisections of tagged simplices*, for short *descendants of* τ_0. The simplices that can be created in this way are uniformly shape regular, only dependent on τ_0 and d. For the case that Ω might have slits, we assume that

$$\partial\Omega \textit{ is the union of true hyperfaces of } T \in \tau_0.$$

We will assume that the simplices from τ_0 are tagged in a way such that any two neighbours $T = (x_0, \ldots, x_d)$, $T' = (x_0', \ldots, x_d')$ in τ_0 *match* in the following sense: If $\overline{x_0x_1} \subset T'$, then $\overline{x_0'x_1'} = \overline{x_0x_1}$, and $x_i = x_i'$ for all but one $i \in \{2, \ldots, d\}$.

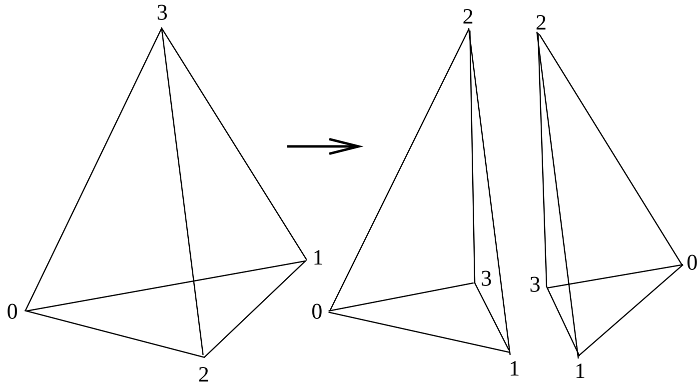

FIG. 3.1. *Bisection of a tagged tetrahedron*

It is known, see [BDD04] and the references therein, that for any conforming partition into triangles there exists a local numbering of the vertices so that the matching condition is satisfied. In more than two dimensions, this condition cannot be satisfied for each conforming partition. On the other hand, it can be shown that any conforming partition of d-simplices can be refined, inflating the number of simplices by not more than an absolute constant factor, into a conforming partition τ_0 that allows a local numbering of the vertices so that the matching condition is satisfied.

We note that any partition is given by the leaves of some subtree of the fixed infinite binary tree having as nodes all tagged simplices that can be created. The roots of this tree are the simplices of τ_0, and the children of any node are the simplices that result from its bisection.

For applying a posteriori error estimators, we will need that the partitions τ underlying the velocity approximations are conforming. So in the following

τ, τ', $\tilde{\tau}$ etc. will always denote conforming partitions.

Bisecting one or more simplices in a conforming partition τ generally results in a non-conforming partition ϱ. Conformity has to be restored by (recursively) bisecting any simplex $T \in \varrho$ that contains a vertex v of a $T' \in \varrho$ that does not coincides with any vertex of T (such a v is called a hanging vertex). This process, called completion, results into the smallest conforming refinement of ϱ.

Our adaptive method will be of the following form

`for` $j := 1$ `to` M
`do` *create some, possibly non-conforming refinement* ϱ_j *of* τ_{j-1}
    complete ϱ_j *to its smallest conforming refinement* τ_j
`endfor`

As we will see, we will be able to bound $\sum_{j=1}^M \#\varrho_j - \#\tau_{j-1}$. Because of the additional bisections made in the completion steps, however, generally $\#\tau_M - \#\tau_0$ will be larger. The following crucial result shows that these additional bisections inflate the total number of simplices by at most an absolute constant factor.

THEOREM 3.2 (generalizes upon [BDD04, Theorem 2.4] for $n = 2$).

$$\#\tau_M - \#\tau_0 \lesssim \sum_{j=1}^{M} \#\varrho_j - \#\tau_{j-1},$$

only dependent on τ_0 and d, and in particular thus independent of M.

We end this section by introducing two more notations. The smallest common refinement of partitions ϱ_1 and ϱ_2 will be denoted as $\varrho_1 \cup \varrho_2$. For a partition τ (thus a conforming one), E_τ will denote the set of internal true hyperfaces in τ, and $\mathcal{F}_\tau(T)$ denotes the set of T and its neighbours in τ.

4. Overview of the solution method. For a partition σ_i, we consider the Galerkin problem of finding $p^{(i)} \in \mathbb{P}_{\sigma_i}$ such that

$$(Sp^{(i)})(q) = (BA^{-1}\mathbf{f})(q), \quad (q \in \mathbb{P}_{\sigma_i}).$$

With $\mathbf{u}^{(i)} := A^{-1}(\mathbf{f} - B'p^{(i)})$, this problem is equivalent to the semi-discrete problem of finding $(\mathbf{u}^{(i)}, p^{(i)}) \in \mathbb{V} \times \mathbb{P}_{\sigma_i}$ such that

$$a(\mathbf{u}^{(i)}, \mathbf{v}) + b(\mathbf{v}, p^{(i)}) + b(\mathbf{u}^{(i)}, q) = \mathbf{f}(\mathbf{v}), \quad (\mathbf{v} \in \mathbb{V},\, q \in \mathbb{P}_{\sigma_i}). \tag{4.1}$$

Since this is just the Stokes problem in which the divergence-free constraint has been relaxed, we will refer to this problem as the *reduced Stokes problem*. The solution $p^{(i)}$ *is the best approximation to* p *from* $\mathbb{P}_{\sigma_i}$ *with respect to* $\|\;\|_{\mathbb{P}}$, and by creating a suitable adaptively refined sequence of partitions $\tau_0 =: \sigma_0 \subset \sigma_1 \subset \ldots$, a convergent sequence $(p^{(i)})_i$ towards p is obtained.

The reduced Stokes problem can however not be solved exactly. Defining $R_{\sigma_i} : \mathbb{P}' \to \mathbb{P}_i$ by $g(q) = \langle q, R_{\sigma_i} g\rangle_{L_2(\Omega)}$ $(q \in \mathbb{P}_{\sigma_i})$, it can be written as $R_{\sigma_i} S p^{(i)} = R_{\sigma_i} BA^{-1}\mathbf{f}$. Equipping $\mathbb{P}_{\sigma_i}$ with $\langle\;,\;\rangle_{L_2(\Omega)}$, the operator $R_{\sigma_i} S : \mathbb{P}_{\sigma_i} \to \mathbb{P}_{\sigma_i}$ is symmetric, bounded, and positive definite, with spectrum in $[\beta^2, 1]$ (for the lower bound, cf. Lemma 2.2 and (2.2); for the upper bound, cf. [NP04]). So to solve the reduced Stokes problem approximately, we may apply Richardson's iteration

$$p^{(i)}_{j+1} = p^{(i)}_j - (R_{\sigma_i} S p^{(i)}_j - R_{\sigma_i} BA^{-1}\mathbf{f}).$$

Writing $\mathbf{u}^{(i)}_j := A^{-1}(\mathbf{f} - B'p^{(i)}_j)$, and, with $Q_{\sigma_i} : \mathbb{P} \to \mathbb{P}_{\sigma_i}$ denoting the $L_2(\Omega)$-orthogonal projector, noting that $R_{\sigma_i} B = -Q_{\sigma_i}\mathrm{div}$, we arrive at the equivalent formulation

$$\left\{\begin{array}{rcl} a(\mathbf{u}^{(i)}_j, \mathbf{v}) &=& \mathbf{f}(\mathbf{v}) - b(\mathbf{v}, p^{(i)}_j), \quad (\mathbf{v} \in \mathbb{V}),\\ p^{(i)}_{j+1} &=& p^{(i)}_j - Q_{\sigma_i}\mathrm{div}\mathbf{u}^{(i)}_j, \end{array}\right. \tag{4.2}$$

known as the Uzawa iteration. The properties of $R_{\sigma_i} S$ show that

$$\|p^{(i)} - p^{(i)}_{j+1}\|_{L_2(\Omega)} \leq [1 - \beta^2]\|p^{(i)} - p^{(i)}_j\|_{L_2(\Omega)} \tag{4.3}$$

Also the Uzawa iteration for solving the reduced Stokes problem cannot be performed exactly since it involves solving an elliptic problem posed over $\mathbb{V}$. To solve this problem approximately, we will again consider Galerkin approximations: Given a partition $\tau^{(i)}_{j,k}$, let $\mathbf{u}^{(i)}_{j,k} \in \mathbb{V}_{\tau^{(i)}_{j,k}}$ being the solution of

$$a(\mathbf{u}^{(i)}_{j,k}, \mathbf{v}) = \mathbf{f}(\mathbf{v}) - b(\mathbf{v}, p^{(i)}_j), \quad (\mathbf{v} \in \mathbb{V}_{\tau^{(i)}_{j,k}}). \tag{4.4}$$

It is the best approximation to $\mathbf{u}^{(i)}_j$ from $\mathbb{V}_{\tau^{(i)}_{j,k}}$ with respect to $\|\;\|_{\mathbb{V}}$, and by creating a suitable adaptively refined sequence of partitions $\sigma_i \subseteq \tau^{(i)}_{j,0} \subset \tau^{(i)}_{j,1} \subset \ldots$, a convergent sequence $(\mathbf{u}^{(i)}_{j,k})_k$ towards $\mathbf{u}^{(i)}_j$ is obtained. To guarantee that $\mathbf{u}^{(i)}_{j,k+1}$ is indeed a better approximation than $\mathbf{u}_{j,k}^{(i)}$, we will need that $\mathbf{f} \in \mathbb{V}'$ can be sufficiently well approximated by a vector field from $\mathbb{V}^*_{\tau_{j,k}^{(i)}}$. To implement the latter requirement, instead of working with $\mathbf{f}$, we will replace it by suitable piecewise polynomial vector fields of degree $m-1$, that should become increasingly more accurate when the iterations proceeds.

Finally, instead of solving the finite dimensional linear systems (4.4) exactly, in order to obtain a method of (quasi-) optimal computational complexity, we will use approximate solutions obtained by employing optimal iterative solvers.

In order to stop each of the nested loops on time, as well as, for both Galerkin problems, to create adaptively refined partitions such that the corresponding approximations converge towards the solution and which partitions have quasi-optimal cardinalities, we need *a posteriori error estimators* that will be discussed in the next section. Stopping a loop on time on the one hand means that it should not stop too early in order to guarantee converge of the overall process, whereas on the other hand the iterations should not proceed too long in order to control the cardinalities of the partitions, that grow by the refinements, as well as the computational complexity.

In view of our solution method, we fix some notations. Throughout this paper, given some $r \in \mathbb{P}$, where we have in mind an approximation to p, and a partition τ, $\mathbf{u}_\tau^r \in \mathbb{V}_\tau$ will denote the solution of the (discretized) elliptic problem

$$\boxed{a(\mathbf{u}_\tau^r, \mathbf{v}_\tau) = \mathbf{f}(\mathbf{v}_\tau) - b(\mathbf{v}_\tau, r), \quad (\mathbf{v}_\tau \in \mathbb{V}_\tau)}. \tag{4.5}$$

Actually, we will consider this problem only for an $r \in \mathbb{P}_\tau$. As a special case of (4.5), $\mathbf{u}^r = \mathbf{u}_\infty^r \in \mathbb{V}$ denotes thus the solution of

$$a(\mathbf{u}^r, \mathbf{v}) = \mathbf{f}(\mathbf{v}) - b(\mathbf{v}, r), \quad (\mathbf{v} \in \mathbb{V}).$$

Given a partition σ, $(\mathbf{u}^\sigma, p_\sigma) \in \mathbb{V} \times \mathbb{P}_\sigma$ will denote the solution of the reduced Stokes problem

$$\boxed{a(\mathbf{u}^\sigma, \mathbf{v}) + b(\mathbf{v}, p_\sigma) + b(\mathbf{u}^\sigma, q_\sigma) = \mathbf{f}(\mathbf{v}), \quad (\mathbf{v} \in \mathbb{V},\, q_\sigma \in \mathbb{P}_\sigma)}.$$

5. A posteriori error estimators.

5.1. A posteriori error estimator for the inner elliptic problem. For a partition τ, $r_\tau \in \mathbb{P}_\tau$, $\mathbf{w}_\tau \in \mathbb{V}_\tau$, and $T \in \tau$, we set the local error indicator

$$\eta_T(\mathbf{f}, r_\tau, \mathbf{w}_\tau) := \operatorname{diam}(T)^2 \|\mathbf{f} - \nabla r_\tau + \triangle \mathbf{w}_\tau\|^2_{L_2(T)^d} + \operatorname{diam}(T) \|[\![r_\tau \mathbf{n} - \nabla \mathbf{w}_\tau \cdot \mathbf{n}]\!]_{\partial T}\|^2_{L_2(\partial T)^d},$$

where $[\![\,]\!]_{\partial T}$ denotes the jump of its argument over ∂T in the direction of $\mathbf{n}$, being a unit vector normal to ∂T. This jump is defined to be zero over $\partial\Omega$. We set the elliptic error estimator

$$\mathcal{E}^{\mathrm{E}}(\tau, \mathbf{f}, r_\tau, \mathbf{w}_\tau) := [\sum_{T \in \tau} \eta_T(\mathbf{f}, r_\tau, \mathbf{w}_\tau)]^{\frac{1}{2}}.$$

Note that the definition of the error estimator requires $\mathbf{f} \in L_2(\Omega)^d$, that we therefore assume here.

The following Proposition 5.2 is a generalization of [BMN02, Lemma 5.1(5.4)], see also [Ver96], in the sense that instead of $\|\mathbf{u}^{r_\tau} - \mathbf{u}_\tau^{r_\tau}\|_\mathbb{V}$, the difference $\|\mathbf{u}_{\tau'}^{r_\tau} - \mathbf{u}_\tau^{r_\tau}\|_\mathbb{V}$ for any $\tau' \supset \tau$ is estimated. It is shown that this difference can be bounded from above by the square root of the sum of the local error indicators corresponding to the simplices that were refined when creating τ' from τ, or those that have non-empty intersection with such simplices. By taking $\tau' = \infty$, this result yields the known bound for $\|\mathbf{u}^{r_\tau} - \mathbf{u}^{r_\tau}_\tau\|_{\mathbb{V}}$. Although our generalization is not so difficult to derive knowing the results from [BMN02, Ver96], for completeness we include a proof.

Remark 5.1. Although in this and the next subsection it will be allowed that $r_\tau \in \mathbb{P}_\tau$, actually we will think of it as being an approximation to p from $\mathbb{P}_\sigma$ for some "fixed" $\sigma \subseteq \tau$.

PROPOSITION 5.2. *Let $\tau' \supset \tau$ be partitions, $r_\tau \in \mathbb{P}_\tau$, $\mathbf{f} \in L_2(\Omega)^d$, and*

$$\overline{F} = \overline{F}(\tau, \tau') := \{T \in \tau : T \cap \tilde{T} \neq \emptyset \text{ for some } \tilde{T} \in \tau \text{ that has been refined in } \tau'\}.$$

Then we have

$$\|\mathbf{u}^{r_\tau}_{\tau'} - \mathbf{u}^{r_\tau}_\tau\|_{\mathbb{V}} \leq C_1 \Big[\sum_{T \in \overline{F}} \eta_T(\mathbf{f}, r_\tau, \mathbf{u}^{r_\tau}_\tau) \Big]^{\frac{1}{2}},$$

for some absolute constant $C_1 > 0$. Note that $\#\overline{F} \lesssim \#\tau' - \#\tau$.

In particular, by taking $\tau' = \infty$, we have

$$\|\mathbf{u}^{r_\tau} - \mathbf{u}^{r_\tau}_\tau\|_{\mathbb{V}} \leq C_1 \mathcal{E}^{\mathrm{E}}(\tau, \mathbf{f}, r_\tau, \mathbf{u}^{r_\tau}_\tau).$$

Proof. We have $\|\mathbf{u}^{r_\tau}_{\tau'} - \mathbf{u}^{r_\tau}_\tau\|_{\mathbb{V}} \eqsim \sup_{0 \neq \mathbf{v}_{\tau'} \in \mathbb{V}_{\tau'}} \frac{a(\mathbf{u}^{r_\tau}_{\tau'} - \mathbf{u}^{r_\tau}_\tau, \mathbf{v}_{\tau'})}{\|\mathbf{v}_{\tau'}\|_{\mathbb{V}}}$. For any $\mathbf{v}_{\tau'} \in \mathbb{V}_{\tau'}$, $\mathbf{v}_\tau \in \mathbb{V}_\tau$, from $\mathbf{u}^{r_\tau}_{\tau'} - \mathbf{u}^{r_\tau}_\tau \perp_{a(\,,\,)} \mathbb{V}_\tau$, the definition of $\mathbf{u}^{r_\tau}_{\tau'}$, and integration by parts, we have

$$a(\mathbf{u}^{r_\tau}_{\tau'} - \mathbf{u}^{r_\tau}_\tau, \mathbf{v}_{\tau'}) = a(\mathbf{u}^{r_\tau}_{\tau'} - \mathbf{u}^{r_\tau}_\tau, \mathbf{v}_{\tau'} - \mathbf{v}_\tau)$$

$$= \sum_{T \in \tau} \int_T \big[\mathbf{f} \cdot (\mathbf{v}_{\tau'} - \mathbf{v}_\tau) + r_\tau \mathrm{div}(\mathbf{v}_{\tau'} - \mathbf{v}_\tau) - \nabla \mathbf{u}^{r_\tau}_\tau : \nabla(\mathbf{v}_{\tau'} - \mathbf{v}_\tau)\big] \tag{5.1}$$

$$= \sum_{T \in \tau} \int_T (\mathbf{f} - \nabla r_\tau + \triangle \mathbf{u}^{r_\tau}_\tau) \cdot (\mathbf{v}_{\tau'} - \mathbf{v}_\tau) + \sum_{e \in E_\tau} \int_e [\![r_\tau \mathbf{n} - \nabla \mathbf{u}^{r_\tau}_\tau \cdot \mathbf{n}]\!]_e \cdot (\mathbf{v}_{\tau'} - \mathbf{v}_\tau). \tag{5.2}$$

We will select $\mathbf{v}_\tau$ as a quasi-interpolant of $\mathbf{v}_{\tau'}$. For any $T \in \tau$, let $N_T = \{x \in T : k\lambda_T(x) \in \mathbf{N}^d\}$, where $\lambda_T(x)$ denotes the barycentric coordinates of x with respect to T. Corresponding to the local nodal basis $\{\phi_{T,v} : v \in N_T\}$ of $P_m(T)$, defined by $\phi_{T,v}(w) = \delta_{vw}$ ($w \in N_T$), there exists a dual basis $\{\phi^*_{T,v} : v \in N_T\}$ of $P_m(T)$, defined by $\langle \phi_{T,v}, \phi^*_{T,w} \rangle_{L_2(T)} = \delta_{vw}$ ($w \in N_T$). A scaling argument shows that $\|\phi^*_{T,v}\|_{L_2(T)} \lesssim \mathrm{meas}(T)^{-1/2}$. For any nodal point $v \in \cup_{T \in \tau} N_T$, $v \notin \partial\Omega$, we now select a $T_v \in \tau$ with $v \in T_v$, and define $\mathbf{v}_\tau \in \mathbb{V}_\tau$ by $\mathbf{v}_\tau(v) = \int_{T_v} \mathbf{v}_{\tau'} \phi^*_{T,v}$. Its key properties are: For any $v \in \cup_{T \in \tau} N_T$, $\mathbf{v}_\tau(v) = \mathbf{v}_{\tau'}(v)$ when $T_v \in \tau'$; for any $T \in \tau$, $\|\mathbf{v}_\tau\|_{L_2(T)^d} \lesssim \|\mathbf{v}_{\tau'}\|_{L_2(\Omega_T)^d}$, where $\Omega_T := \cup\{\tilde{T} \in \tau : \tilde{T} \cap T \neq \emptyset\}$.

This first property shows that the sums in (5.2) vanish for any T or e for which all $\tilde{T} \in \tau$ with $\tilde{T} \cap T \neq \emptyset$ or $\tilde{T} \cap e \neq \emptyset$ are also in τ'. It also shows that the interpolator is actually a projector onto $\mathbb{V}_\tau$. From the second property, and either the fact that our interpolator reproduces any constant together with the Bramble-Hilbert lemma, or, in case $T \cap \partial\Omega \neq \emptyset$ so at least one of the $\tilde{T}$ that form Ω_T has a true hyperface on $\partial\Omega$, the Poincaré-Friedrichs inequality, we have

$$\operatorname{diam}(T)^{-1}\|\mathbf{v}_{\tau'} - \mathbf{v}_\tau\|_{L_2(T)^d} + |\mathbf{v}_{\tau'} - \mathbf{v}_\tau|_{H^1(T)^d} \lesssim |\mathbf{v}_{\tau'}|_{H^1(\Omega_T)^d}, \tag{5.3}$$

where also a homogeneity argument was used. For each $e \in E_\tau$ and either $T \in \tau$ on both sides of e, from the trace theorem and (4.2), we have

$$\begin{aligned} \|\mathbf{v}_{\tau'} - \mathbf{v}_\tau\|_{L_2(e)^d} &\lesssim \operatorname{diam}(e)^{-\frac{1}{2}}\|\mathbf{v}_{\tau'} - \mathbf{v}_\tau\|_{L_2(T)^d} + \operatorname{diam}(e)^{\frac{1}{2}}|\mathbf{v}_{\tau'} - \mathbf{v}_\tau|_{H^1(T)^d} \\ &\lesssim \operatorname{diam}(e)^{\frac{1}{2}}|\mathbf{v}_{\tau'}|_{H^1(\Omega_T)^d} \end{aligned} \tag{5.4}$$

By applying the Cauchy-Schwarz inequality to both sums from (5.2), and then substituting (5.3) or (5.4), the proof follows. $\square$

Next we study whether the error estimator also provides a lower bound for $\|\mathbf{u}^{r_\tau} - \mathbf{u}^{r_\tau}_\tau\|_{\mathbb{V}}$ and, when τ' is a sufficient refinement of τ, for $\|\mathbf{u}^{r_\tau}_{\tau'} - \mathbf{u}^{r_\tau}_\tau\|_{\mathbb{V}}$. In order to derive such estimates, for the time being we restrict further the type of right-hand sides to piecewise polynomials of degree $m-1$ with respect to τ. We will call $\tau' \supset \tau$ *a full refinement with respect to* $T \in \tau$, when

all $\tilde{T} \in \mathcal{F}_\tau(T)$, as well as all faces of T
contain a vertex of τ' in their interiors,

see Figure 5.1 for an illustration. The following proposition was shown in [BMN02,

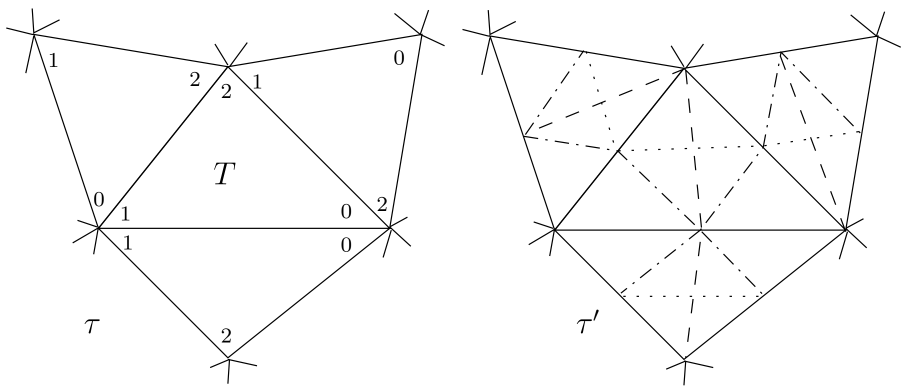

FIG. 5.1. *A refinement τ' of τ, which is a full refinement with respect to a triangle $T \in \tau$*

Lemma 5.3] [Actually, there a somewhat stronger condition on the refinement was imposed, but not used; a more general $\mathbf{f}$ was considered at the expense of an additional "oscillation" term in the expression; and, finally, there the general $\mathbf{w}_\tau \in \mathbb{V}_\tau$ reads as $\mathbf{u}^{r_\tau}_\tau$ whose additional property of being the solution of (4.5) was not used though].

PROPOSITION 5.3. *Let τ be a partition, $r_\tau \in \mathbb{P}_\tau$, and let us assume that $\mathbf{f} \in \mathbb{V}^*_\tau$. Let $\tau' \supset \tau$ be a full refinement of τ with respect to $T \in \tau$. Then for any $\mathbf{w}_\tau \in \mathbb{V}_\tau$, we have*

$$\eta_T(\mathbf{f}, r_\tau, \mathbf{w}_\tau) \lesssim \sum_{\tilde{T} \in \mathcal{F}_\tau(T)} |\mathbf{u}^{r_\tau}_{\tau'} - \mathbf{w}_\tau|^2_{H^1(\tilde{T})^d}.$$

As a straightforward consequence we have

COROLLARY 5.4. *In the situation of Proposition 5.3, let $\tau' \supset \tau$ be a full refinement of τ with respect to all T from some $\underline{F} \subset \tau$. Then*

$$c_2\Big[\sum_{T\in\underline{F}} \eta_T(\mathbf{f}, r_\tau, \mathbf{w}_\tau)\Big]^{\frac{1}{2}} \leq \|\mathbf{u}_{\tau'}^{r_\tau} - \mathbf{w}_\tau\|_{\mathbb{V}},$$

for some absolute constant $c_2 > 0$. In particular, we have

$$c_2\mathcal{E}^{\mathrm{E}}(\tau, \mathbf{f}, r_\tau, \mathbf{w}_\tau) \leq \|\mathbf{u}^{r_\tau} - \mathbf{w}_\tau\|_{\mathbb{V}}. \tag{5.5}$$

Finally in this subsection, we investigate the stability of the elliptic error estimator.

PROPOSITION 5.5. *Let τ be a partition, $r_\tau \in \mathbb{P}_\tau$, $\mathbf{f} \in L_2(\Omega)^d$, and $\mathbf{v}_\tau, \mathbf{w}_\tau \in \mathbb{V}_\tau$. Then*

$$c_2|\mathcal{E}^{\mathrm{E}}(\tau, \mathbf{f}, r_\tau, \mathbf{v}_\tau) - \mathcal{E}^{\mathrm{E}}(\tau, \mathbf{f}, r_\tau, \mathbf{w}_\tau)| \leq \|\mathbf{v}_\tau - \mathbf{w}_\tau\|_{\mathbb{V}}.$$

Proof. For $\mathbf{g} \in L_2(\Omega)^d$, $q_\tau \in \mathbb{P}_\tau$, by two applications of the triangle inequality in the form $\big|\|\cdot\| - \|\cdot\|\big|^2 \leq \|\cdot - \cdot\|^2$, first for vectors and then for functions, we have

$$|\mathcal{E}^{\mathrm{E}}(\tau, \mathbf{f}, r_\tau, \mathbf{v}_\tau) - \mathcal{E}^{\mathrm{E}}(\tau, \mathbf{g}, q_\tau, \mathbf{w}_\tau)| \leq \mathcal{E}^{\mathrm{E}}(\tau, \mathbf{f} - \mathbf{g}, r_\tau - q_\tau, \mathbf{v}_\tau - \mathbf{w}_\tau).$$

By substituting $\mathbf{g} = \mathbf{f}$ and $q_\tau = r_\tau$, and by applying (5.5) the proof is completed. ∎

5.2. A posteriori error estimator for the (reduced) Stokes problem. For partitions τ, ϱ, $r_\tau \in \mathbb{P}_\tau$, $\mathbf{f} \in L_2(\Omega)^d$, and $\mathbf{w}_\tau \in \mathbb{V}_\tau$, we consider the estimator

$$\mathcal{E}^{\mathrm{S}}(\varrho, \tau, \mathbf{f}, r_\tau, \mathbf{w}_\tau) := \mathcal{E}^{\mathrm{E}}(\tau, \mathbf{f}, r_\tau, \mathbf{w}_\tau) + \|Q_\varrho \mathrm{div} \mathbf{w}_\tau\|_{L_2(\Omega)}$$

We are going to apply the results from this subsection for $\varrho = \infty$ (the full, unreduced Stokes problem) as well as for $\varrho = \sigma \subseteq \tau$.

PROPOSITION 5.6. *For partitions ϱ, τ, $r_\tau \in \mathbb{P}_\tau$, and $\mathbf{f} \in L_2(\Omega)^d$, we have*

$$\|\mathbf{u}^\varrho - \mathbf{u}_\tau^{r_\tau}\|_{\mathbb{V}} + \|p_\varrho - r_\tau\|_{\mathbb{P}} \leq C_3\mathcal{E}^{\mathrm{S}}(\varrho, \tau, \mathbf{f}, r_\tau, \mathbf{u}_\tau^{r_\tau}),$$

for some absolute constant $C_3 > 0$.

Proof. The proof given in [Ver96] (cf. [BMN02, Lemma 4.1]) for $\varrho = \infty$ easily generalizes to $\varrho \subsetneq \infty$. Since it can be derived by a variation of the proof of Proposition 5.2, we only sketch the idea.

For any $(\mathbf{v}, q_\varrho) \in \mathbb{V} \times \mathbb{P}_\varrho$, $\mathbf{v}_\tau \in \mathbb{V}$, we have

$$\begin{aligned} a(\mathbf{u}^\varrho - \mathbf{u}_\tau^{r_\tau}, \mathbf{v}) &+ b(\mathbf{v}, p_\varrho - r_\tau) + b(\mathbf{u}^\varrho - \mathbf{u}_\tau^{r_\tau}, q_\varrho) \\ &= a(\mathbf{u}^\varrho - \mathbf{u}_\tau^{r_\tau}, \mathbf{v} - \mathbf{v}_\tau) + b(\mathbf{v} - \mathbf{v}_\tau, p_\varrho - r_\tau) - b(\mathbf{u}_\tau^{r_\tau}, q_\varrho). \end{aligned}$$

Together the first two terms in the second line are equal to (5.1) with $\mathbf{v}_{\tau'}$ reading as $\mathbf{v}$. By estimating them in the same way using that $r_\tau \in \mathbb{P}_\tau$, and applying the obvious estimate for $b(\mathbf{u}_\tau^{r_\tau}, q_\varrho)$, and, finally, by invoking (2.3) the proof follows. ∎

The following result was shown in [Ver96] (cf. [BMN02, Lemma 4.3]) for the case $\varrho = \infty$, but the proof generalizes immediately to general partitions ϱ.

Proposition 5.7. *Let τ, ϱ be partitions, $r_\tau \in \mathbb{P}_\tau$, $\mathbf{w}_\tau \in \mathbb{V}_\tau$, and let us assume that $\mathbf{f} \in \mathbb{V}_\tau^*$. Then for $T \in \tau$, we have*

$$\eta_T(\mathbf{f}, r_\tau, \mathbf{w}_\tau) \lesssim \sum_{\tilde{T} \in \mathcal{F}_\tau(T)} \left[|\mathbf{u}^\varrho - \mathbf{w}_\tau|^2_{H^1(\tilde{T})^d} + \|p_\varrho - r_\tau\|^2_{L_2(\tilde{T})^d} \right]$$

Since for $T \in \varrho$, with Q_T denoting the $L_2(T)$-orthogonal projector onto $P_{m-1}(T)$, $\|Q_T \mathrm{div} \mathbf{w}_\tau\|_{L_2(T)} \leq |\mathbf{u}^\varrho - \mathbf{w}_\tau|_{H^1(T)}$, we conclude that

$$c_4 \mathcal{E}^{\mathrm{S}}(\varrho, \tau, \mathbf{f}, r_\tau, \mathbf{w}_\tau) \leq \|\mathbf{u}^\varrho - \mathbf{w}_\tau\|_{\mathbb{V}} + \|p_\varrho - r_\tau\|_{\mathbb{P}}.$$

for some absolute constant $c_4 > 0$.

The last result in this subsection provides an a posteriori error error estimator for the outer elliptic problem.

Proposition 5.8. *For a partition ϱ, and $r \in \mathbb{P}$, we have*

$$c_6 \|Q_\varrho \mathrm{div} \mathbf{u}^r\|_{L_2(\Omega)} \leq \|p_\varrho - r\|_{\mathbb{P}} \leq C_5 \|Q_\varrho \mathrm{div} \mathbf{u}^r\|_{L_2(\Omega)},$$

for some absolute constants $C_5, c_6 > 0$.

Proof. Use $\sup_{0 \neq \mathbf{v} \in \mathbb{V}} \frac{b(\mathbf{v}, p_\varrho - r)}{\|\mathbf{v}\|_{\mathbb{V}}} = \sup_{0 \neq \mathbf{v} \in \mathbb{V}} \frac{a(\mathbf{u}^r - \mathbf{u}^\varrho, \mathbf{v})}{\|\mathbf{v}\|_{\mathbb{V}}} = \|\mathbf{u}^\varrho - \mathbf{u}^r\|_{\mathbb{V}}$, and thus

$$\beta \leq \frac{\|\mathbf{u}^\varrho - \mathbf{u}^r\|_{\mathbb{V}}}{\|p_\varrho - r\|_{\mathbb{P}}} \leq 1, \tag{5.6}$$

and

$$\begin{aligned} &\|p_\varrho - r\|_{\mathbb{P}} + \|\mathbf{u}^\varrho - \mathbf{u}^r\|_{\mathbb{V}} \\ &\eqsim \sup_{0 \neq (\mathbf{v}, q_\varrho) \in \mathbb{V} \times \mathbb{P}_\varrho} \frac{a(\mathbf{u}^\varrho - \mathbf{u}^r, \mathbf{v}) + b(\mathbf{v}, p_\varrho - r) + b(\mathbf{u}^\varrho - \mathbf{u}^r, q_\varrho)}{\|\mathbf{v}\|_{\mathbb{V}} + \|q_\varrho\|_{\mathbb{P}}} \\ &= \|Q_\varrho \mathrm{div} \mathbf{u}^r\|_{L_2(\Omega)}. \end{aligned}$$

□

6. Adaptive refinements resulting in error reduction . For both elliptic problems $Sp = BA^{-1}f$ and $a(\mathbf{u}^r, \mathbf{v}) = \mathbf{f}(\mathbf{v}) - b(\mathbf{v}, r)$ ($\mathbf{v} \in \mathbb{V}$), the latter for some given $r \in \mathbb{P}$, we construct adaptive refinement routines based on the a posteriori error estimators. Given (approximate) Galerkin solutions from $\mathbb{P}_\sigma$ or $\mathbb{V}_\tau$, respectively, they produce refinements $\tilde{\sigma} \supset \sigma$ or $\tilde{\tau} \supset \tau$ such that the Galerkin solutions with respect to these partitions have strictly smaller errors. Moreover, we will give bounds on the number of refined simplices that eventually will lead to the conclusion that our adaptive Stokes solver generates quasi-optimal partitions.

6.1. Adaptive pressure refinements. With C_5, c_6 being the constants from Proposition 5.8, for some absolute constants

$$d \in \Big(1 - \frac{c_6^2}{C_5^2}, 1\Big], \quad D \geq 1, \quad \theta \in \Big(0, \big[1 - \frac{1 - c_6^2/C_5^2}{d}\big]^{\frac{1}{2}}\Big), \tag{6.1}$$

we assume that we have the following routine available. We think of its arguments r_σ and $\mathbf{w}$ as being approximations to p and u^{r_σ}, respectively.

REFpres$[\sigma, r_\sigma, \mathbf{w}] \to \tilde{\sigma}$
% σ is a partition, $r_\sigma \in \mathbb{P}_\sigma$ and $\mathbf{w} \in \mathbb{V}$.
Select a partition $\tilde{\sigma} \supset \sigma$ with

$$\|Q_{\tilde{\sigma}} \mathrm{div}\mathbf{w}\|_{L_2(\Omega)} \geq \theta \|\mathrm{div}\mathbf{w}\|_{L_2(\Omega)}, \tag{6.2}$$

such that

$$\#\tilde{\sigma} - \#\sigma \leq D(\#\breve{\sigma} - \#\sigma)$$

for any $\breve{\sigma} \supset \sigma$ with $\|Q_{\breve{\sigma}} \mathrm{div}\mathbf{w}\|_{L_2(\Omega)} \geq \sqrt{1 - d(1-\theta^2)} \|\mathrm{div}\mathbf{w}\|_{L_2(\Omega)}$.

Remark 6.1. We will make calls $\tilde{\sigma} :=$ **REFpres**$[\sigma, r_\sigma, \mathbf{w}]$ only for the argument $\mathbf{w}$ from $\mathbb{V}_\tau$ for some $\tau \supset \sigma$, so that $\mathrm{div}\mathbf{w} \in \mathbb{P}_\tau$. It is therefore no restriction to assume that then $\tilde{\sigma} \subseteq \tau$. The fact that the partition underlying the pressure approximation is always contained in or equal to that underlying the velocity approximation will be essential for the adaptive refinement routine for reducing the error in the inner elliptic problem.

The benefit of **REFpres** appears from the following two lemmas:

LEMMA 6.2. *Let σ be a partition, and $r_\sigma \in \mathbb{P}_\sigma$. Then for $\tilde{\sigma} =$ **REFpres**$[\sigma, r_\sigma, \mathbf{u}^{r_\sigma}]$, we have*

$$\|p - p_{\tilde{\sigma}}\|_{\mathbb{P}} \leq \left[1 - \tfrac{c_6^2 \theta^2}{C_5^2}\right]^{\frac{1}{2}} \|p - r_\sigma\|_{\mathbb{P}}.$$

Moreover,

$$\#\tilde{\sigma} - \#\sigma \leq D[\#\bar{\sigma} - \#\tau_0]$$

for any *partition $\bar{\sigma}$ for which*

$$\inf_{q_{\bar{\sigma}} \in \mathbb{P}_{\bar{\sigma}}} \|p - q_{\bar{\sigma}}\|_{\mathbb{P}} \leq \left[1 - \tfrac{C_5^2}{c_6^2}(1 - d(1-\theta^2))\right]^{\frac{1}{2}} \|p - r_\sigma\|_{\mathbb{P}}. \tag{6.3}$$

(*Note that* (6.1) *implies that* $\frac{C_5^2}{c_6^2}(1 - d(1-\theta^2)) < 1$.)

Proof. The first statement follows from

$$\|p - r_\sigma\|_{\mathbb{P}}^2 = \|p - p_{\tilde{\sigma}}\|_{\mathbb{P}}^2 + \|p_{\tilde{\sigma}} - r_\sigma\|_{\mathbb{P}}^2$$

and $\|p_{\tilde{\sigma}} - r_\sigma\|_{\mathbb{P}} \geq c_6 \|Q_{\tilde{\sigma}} \mathrm{div}\mathbf{u}^{r_\sigma}\|_{L_2(\Omega)} \geq c_6 \theta \|\mathrm{div}\mathbf{u}^{r_\sigma}\|_{L_2(\Omega)} \geq \frac{c_6 \theta}{C_5} \|p - r_\sigma\|_{\mathbb{P}}$ by Proposition 5.8.

For a $\bar{\sigma}$ satisfying (6.3), let $\breve{\sigma} = \sigma \cup \bar{\sigma}$. Then from $\|p - p_{\breve{\sigma}}\|_{\mathbb{P}} \leq \inf_{q_{\bar{\sigma}} \in \mathbb{P}_{\bar{\sigma}}} \|p - q_{\bar{\sigma}}\|_{\mathbb{P}}$, with $\lambda := \frac{C_5^2}{c_6^2}(1 - d(1-\theta^2))$ we have

$$\begin{aligned} C_5^2 \|Q_{\breve{\sigma}} \mathrm{div}\mathbf{u}^{r_\sigma}\|_{L_2(\Omega)}^2 &\geq \|p_{\breve{\sigma}} - r_\sigma\|_{\mathbb{P}}^2 = \|p - r_\sigma\|_{\mathbb{P}}^2 - \|p - p_{\breve{\sigma}}\|_{\mathbb{P}}^2 \\ &\geq \lambda \|p - r_\sigma\|_{\mathbb{P}}^2 \geq \lambda c_6^2 \|\mathrm{div}\mathbf{u}^{r_\sigma}\|_{L_2(\Omega)}^2. \end{aligned}$$

Noting that $\frac{\lambda c_6^2}{C_5^2} = 1 - d(1-\theta^2)$, by construction of $\tilde{\sigma}$, we conclude that

$$\#\tilde{\sigma} - \#\sigma \leq D[\#\breve{\sigma} - \#\sigma] \leq D[\#\bar{\sigma} - \#\tau_0].$$

$\square$

Now we generalize Lemma 6.2 to the practical relevant situation that we have only an approximation to $\mathbf{u}^{r_\sigma}$ available:

LEMMA 6.3. *Let $\omega \in (0,\theta)$ be a constant, σ a partition, $r_\sigma \in \mathbb{P}_\sigma$, and $\mathbf{w} \in \mathbb{V}$ with*

$$\|\mathrm{div}\mathbf{u}^{r_\sigma} - \mathrm{div}\mathbf{w}\|_{L_2(\Omega)} \leq \omega \|\mathrm{div}\mathbf{w}\|_{L_2(\Omega)}.$$

Then for $\tilde{\sigma} = \mathbf{REFpres}[\sigma, r_\sigma, \mathbf{w}]$, we have

$$\|p - p_{\tilde{\sigma}}\|_{\mathbb{P}} \leq \left[1 - \tfrac{c_6^2(\theta-\omega)^2}{C_5^2(1+\omega)^2}\right]^{\frac{1}{2}} \|p - r_\sigma\|_{\mathbb{P}}.$$

Moreover, if ω is sufficiently small such that $\frac{\omega+\sqrt{1-d(1-\theta^2)}}{1-\omega} < \frac{c_6}{C_5}$, then

$$\#\tilde{\sigma} - \#\sigma \leq D[\#\bar{\sigma} - \#\tau_0]$$

for any *partition $\bar{\sigma}$ for which with $\xi := \left[1 - \left(\frac{\omega+\sqrt{1-d(1-\theta^2)}}{1-\omega}\frac{C_5}{c_6}\right)^2\right]^{\frac{1}{2}}$,*

$$\inf_{q_{\bar{\sigma}} \in \mathbb{P}_{\bar{\sigma}}} \|p - q_{\bar{\sigma}}\|_{\mathbb{P}} \leq \xi \|p - r_\sigma\|_{\mathbb{P}}. \tag{6.4}$$

Proof. Similar to the proof of Lemma 6.2. For the first part use that

$$\|Q_{\tilde{\sigma}}\mathrm{div}\mathbf{u}^{r_\sigma}\|_{L_2(\Omega)} \geq \|Q_{\tilde{\sigma}}\mathrm{div}\mathbf{w}\|_{L_2(\Omega)} - \omega\|\mathrm{div}\mathbf{w}\|_{L_2(\Omega)} \geq \tfrac{\theta-\omega}{1+\omega}\|\mathrm{div}\mathbf{u}^{r_\sigma}\|_{L_2(\Omega)},$$

and, for the second part, with any $\bar{\sigma}$ satisfying (6.4) and $\breve{\sigma} = \sigma \cup \bar{\sigma}$, that

$$\begin{aligned}
&C_5[\|Q_{\breve{\sigma}}\mathrm{div}\mathbf{w}\|_{L_2(\Omega)} + \omega\|\mathrm{div}\mathbf{w}\|_{L_2(\Omega)}] \geq C_5\|Q_{\breve{\sigma}}\mathrm{div}\mathbf{u}^{r_\sigma}\|_{L_2(\Omega)} \geq \|p_{\breve{\sigma}} - r_\sigma\|_{\mathbb{P}} \\
&= \left[\|p - r_\sigma\|_{\mathbb{P}}^2 - \|p - p_{\breve{\sigma}}\|_{\mathbb{P}}^2\right]^{\frac{1}{2}} \geq \sqrt{1-\xi^2}\|p - r_\sigma\|_{\mathbb{P}} \geq c_6\sqrt{1-\xi^2}\|\mathrm{div}\mathbf{u}^{r_\sigma}\|_{L_2(\Omega)} \\
&\geq (1-\omega)c_6\sqrt{1-\xi^2}\|\mathrm{div}\mathbf{w}\|_{L_2(\Omega)},
\end{aligned}$$

or equivalently, $\|Q_{\breve{\sigma}}\mathrm{div}\mathbf{w}\|_{L_2(\Omega)} \geq \sqrt{1-d(1-\theta^2)}\|\mathrm{div}\mathbf{w}\|_{L_2(\Omega)}$. ∎

As we said, we will make calls $\tilde{\sigma} := \mathbf{REFpres}[\sigma, r_\sigma, \mathbf{w}]$ only for the argument $\mathbf{w}$ from $\mathbb{V}_\tau$ for some $\tau \supset \sigma$, so that $\mathrm{div}\mathbf{w} \in \mathbb{P}_\tau$. Obviously, if, for some $\theta \in (0, c_6/C_5)$, **REFpres** is implemented as the selection of the *smallest* partition $\tilde{\sigma} \supset \sigma$ such that (6.2) is valid, it satisfies its requirements with $d = 1 = D$. Yet, in any case a naive implementation of this algorithm would require computing $\|Q_{\breve{\sigma}}\mathrm{div}\mathbf{w}\|_{L_2(\Omega)}$ for all partitions $\sigma \subsetneq \breve{\sigma} \subsetneq \tau$, which is prohibitive expensive.

Recalling that any partition corresponds to a subtree of the infinite binary tree that is determined by the initial partition τ_0 of tagged simplices, alternatively one may apply the adaptive tree approximation algorithms from [BD04]. Prescribing a $\theta \in (0,1)$, these algorithms are shown to fulfill the requirements on **REFpres** for *some* absolute constants $0 < d \leq 1 \leq D$. Assuming that for any simplex T from any partition $\sigma \subsetneq \breve{\sigma} \subsetneq \tau$ the values $\inf_{q \in P_{m-1}(T)} \|(\mathrm{div}\mathbf{w})|_T - q\|_{L_2(T)}$ are known, which values can be computed in $\mathcal{O}(\#\tau)$ operations, these algorithms produce $\tilde{\sigma}$ as in (6.2) in $\mathcal{O}(\#\tilde{\sigma})$ additional operations.

Unfortunately, it might be that the constant d derived in [BD04] is not larger than $1 - c_6^2/C_5^2$ as it should be in view of (6.1). So far, we have not verified whether the statements from [BD04] can be shown for any given $d \in (0,1)$ (likely at the expense of $D \to \infty$ when $d \uparrow 1$). Therefore, the statements in this paper concerning the *cost* of our adaptive algorithm are valid under the following assumption:

ASSUMPTION 6.4. *For $\mathbf{w} \in \mathbb{V}_\tau$, the call* $\mathbf{REFpres}[\cdot,\cdot,\mathbf{w}]$ *takes $\mathcal{O}(\#\tau)$ operations.*

Remark 6.5. Actually, so far for our experiments we used the easy implementable greedy algorithm: Starting from σ, we bisect that simplex T or those simplices T with maximum $\inf_{q \in P_{m-1}(T)} \|(\mathrm{div}\mathbf{w})|_T - q\|_{L_2(T)}$ until (6.2) is satisfied. Although there exist inputs $\mathbf{w}$ for which this greedy approach results in an output partition $\tilde{\sigma}$ that is not quasi-optimal, "usually" it works well (in any case when for all $T \in \sigma$, $(\mathrm{div}\mathbf{w})|_T$ is sufficiently smooth).

6.2. Adaptive velocity refinements. For some *fixed*

$$\zeta \in (0, \frac{c_2}{C_1}),$$

we will make use of the following routine to determine a suitable adaptive refinement for an update of the velocity:

$\mathbf{REFvel}[\tau, \mathbf{g}, r_\tau, \mathbf{w}_\tau] \to \tilde{\tau}$
% τ is a partition, $\mathbf{g} \in L_2(\Omega)^d$, $r_\tau \in \mathbb{P}_\tau$, and $\mathbf{w}_\tau \in \mathbb{V}_\tau$.
Select a set $\underline{F} \subset \tau$ with, up to some absolute factor, minimal cardinality *such that*

$$\sum_{T \in \underline{F}} \eta_T(\mathbf{g}, r_\tau, \mathbf{w}_\tau) \geq \zeta^2 \, \mathcal{E}^{\mathrm{E}}(\tau, \mathbf{g}, r_\tau, \mathbf{w}_\tau)^2. \tag{6.5}$$

Construct the smallest $\tilde{\tau} \supset \tau$ which is a full refinement with respect to all $T \in \underline{F}$.

The next lemma will show the benefit of $\mathbf{REFvel}$. It applies under the (unrealistic) assumptions that $\mathbf{f} \in \mathbb{V}_\tau^*$ and that the Galerkin problems are solved exactly. In Lemma 8.1 given in the next section, inexact Galerkin solutions will be allowed, and the given right-hand side $\mathbf{f} \in \mathbb{V}'$ will be replaced by an approximation from $\mathbb{V}_\tau^*$.

Note that when $\mathbf{f} \in \mathbb{V}_\tau^*$, the computation of all $\eta_T(\mathbf{f}, r_\tau, \mathbf{w}_\tau)$ $(T \in \tau)$ can be done in $\mathcal{O}(\#\tau)$ operations. By doing an *approximate* sorting of the $\eta_T(\mathbf{f}_\tau, r_\tau, \mathbf{w}_\tau)$ by their values, $\mathbf{REFvel}[\tau, \mathbf{f}, \cdot, \cdot]$ can be implemented in $\mathcal{O}(\#\tau)$ operations (cf. [Ste05b]).

LEMMA 6.6. *Let $\tau \supseteq \sigma$ be partitions, $\mathbf{f} \in \mathbb{V}_\tau^*$, and $r_\sigma \in \mathbb{P}_\sigma$. Then for $\tilde{\tau} =$ $\mathbf{REFvel}[\tau, \mathbf{f}, r_\sigma, \mathbf{u}_\tau^{r_\sigma}]$, we have*

$$\|\mathbf{u}^{r_\sigma} - \mathbf{u}_{\tilde{\tau}}^{r_\sigma}\|_{\mathbb{V}} \leq \left[1 - \tfrac{c_2^2 \zeta^2}{C_1^2}\right]^{\frac{1}{2}} \|\mathbf{u}^{r_\sigma} - \mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}}.$$

Moreover, if $\zeta < \frac{c_2}{C_1}$, and, for some absolute constant $\vartheta > 0$,

$$\|\mathbf{u}^{r_\sigma} - \mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}} \geq \vartheta \|\mathbf{u} - \mathbf{u}^{r_\sigma}\|_{\mathbb{V}}, \tag{6.6}$$

then for the set of marked simplices $\underline{F}$ inside $\mathbf{REFvel}$*, we have*

$$\#\underline{F} \lesssim \#\bar{\tau} + \#\bar{\sigma} + \#\sigma \tag{6.7}$$

for any partitions $\bar{\tau}$ and $\bar{\sigma}$ for which

$$\inf_{\mathbf{v}_{\bar{\tau}} \in \mathbb{V}_{\bar{\tau}}} \|\mathbf{u} - \mathbf{v}_{\bar{\tau}}\|_{\mathbb{V}} \leq \tfrac{1}{3}\left[1 - \tfrac{C_1^2 \zeta^2}{c_2^2}\right]^{\frac{1}{2}} \|\mathbf{u}^{r_\sigma} - \mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}} \tag{6.8}$$

$$\inf_{q_{\bar{\sigma}} \in \mathbb{P}_{\bar{\sigma}}} \|p - q_{\bar{\sigma}}\|_{\mathbb{V}} \leq \tfrac{1}{3}\left[1 - \tfrac{C_1^2 \zeta^2}{c_2^2}\right]^{\frac{1}{2}} \|\mathbf{u}^{r_\sigma} - \mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}} \tag{6.9}$$

Remark 6.7. Note that the bound on $\#\underline{F}$ in terms of $\mathbf{u}$, p (via $\bar{\tau}$ and $\bar{\sigma}$) and σ can only be shown when (the variational formulation of) $-\triangle \mathbf{u}^{r_\sigma} = \mathbf{f} - \nabla r^\sigma$ is *not solved too accurately*, which is enforced by (6.6). By assuming that $\mathbf{u}$ and p are in certain approximation classes, i.e., that these functions *can* be approximated with certain rates by finite element functions with respect to the *best partitions*, later we will derive quasi-optimal bounds for $\#\bar{\tau}$ and $\#\bar{\sigma}$, as well as for $\#\sigma$ via Lemma 6.3, and so in the end on $\#\underline{F}$. Without imposing (6.6), we would only arrive at a similar bound on $\#\underline{F}$ when we would assume that all approximations $\mathbf{u}^{r_\sigma}$ to $\mathbf{u}$ corresponding to all approximate pressures r_σ that are created in the adaptive method are similarly easy to approximate as $\mathbf{u}$, which is an unverifiable assumption.

Proof. The first statement follows from

$$\|\mathbf{u}^{r_\sigma} - \mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}}^2 = \|\mathbf{u}^{r_\sigma} - \mathbf{u}_{\tilde{\tau}}^{r_\sigma}\|_{\mathbb{V}}^2 + \|\mathbf{u}_{\tilde{\tau}}^{r_\sigma} - \mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}}^2,$$

and $\|\mathbf{u}_{\tilde{\tau}}^{r_\sigma} - \mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}} \geq c_2\zeta\mathcal{E}^{\mathrm{E}}(\tau, \mathbf{f}, r_\sigma, \mathbf{u}_\tau^{r_\sigma}) \geq \frac{c_2\zeta}{C_1}\|\mathbf{u}^{r_\sigma} - \mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}}$ by Corollary 5.4 and Proposition 5.2.

Let $\hat{\tau}$ be a partition for which

$$\inf_{\mathbf{v}_{\hat{\tau}} \in \mathbb{V}_{\hat{\tau}}} \|\mathbf{u}^{r_\sigma} - \mathbf{v}_{\hat{\tau}}\|_{\mathbb{V}} \leq \big[1 - \tfrac{C_1^2\zeta^2}{c_2^2}\big]^{\frac{1}{2}} \|\mathbf{u}^{r_\sigma} - \mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}}, \tag{6.10}$$

and let $\check{\tau} = \tau \cup \hat{\tau}$. Then with $\overline{F} = \overline{F}(\tau, \check{\tau})$ from Proposition 5.2, we have

$$\begin{aligned} C_1^2 \sum_{T \in \overline{F}} \eta_T(\mathbf{f}, r_\sigma, \mathbf{u}_\tau^{r_\sigma}) &\geq \|\mathbf{u}_{\check{\tau}}^{r_\sigma} - \mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}}^2 = \|\mathbf{u}^{r_\sigma} - \mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}}^2 - \|\mathbf{u}^{r_\sigma} - \mathbf{u}_{\check{\tau}}^{r_\sigma}\|_{\mathbb{V}}^2 \\ &\geq \tfrac{C_1^2\zeta^2}{c_2^2}\|\mathbf{u}^{r_\sigma} - \mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}}^2 \geq C_1^2\zeta^2\mathcal{E}^{\mathrm{E}}(\tau, \mathbf{f}, r_\sigma, \mathbf{u}_\tau^{r_\sigma})^2. \end{aligned}$$

By construction of $\underline{F}$, we infer that

$$\#\underline{F} \lesssim \overline{F} \lesssim \#\check{\tau} - \#\tau \leq \#\hat{\tau} - \#\tau_0. \tag{6.11}$$

It remains to bound $\#\hat{\tau} - \#\tau_0$ for a $\hat{\tau}$ as in (6.10). With $\bar{\sigma}$ as in (6.9), we write $\mathbf{u}^{r_\sigma} = (\mathbf{u}^{r_\sigma} - \mathbf{u}^{\bar{\sigma}}) + (\mathbf{u}^{\bar{\sigma}} - \mathbf{u}) + \mathbf{u}$, and approximate each of the three terms within tolerance $\frac{1}{3}\big[1 - \frac{C_1^2\zeta^2}{c_2^2}\big]^{\frac{1}{2}}$ with finite element functions (the second one with zero). From (5.6), we have

$$\|\mathbf{u} - \mathbf{u}^{\bar{\sigma}}\|_{\mathbb{V}} \leq \|p - p_{\bar{\sigma}}\|_{\mathbb{P}} \leq \tfrac{1}{3}\big[1 - \tfrac{C_1^2\zeta^2}{c_2^2}\big]^{\frac{1}{2}} \|\mathbf{u}^{r_\sigma} - \mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}}.$$

The vector field $\mathbf{w} = \mathbf{u}^{r_\sigma} - \mathbf{u}^{\bar{\sigma}}$ solves

$$a(\mathbf{w}, \mathbf{v}) = -b(\mathbf{v}, r_\sigma - p_{\bar{\sigma}}), \quad (\mathbf{v} \in \mathbb{V}).$$

With $\hat{\sigma} = \sigma \cup \bar{\sigma}$, the error in its best approximation $\mathbf{w}_{\hat{\sigma}}$ from $\mathbb{V}_{\hat{\sigma}}$ can be bounded by

$$\begin{aligned} \|\mathbf{w} - \mathbf{w}_{\hat{\sigma}}\|_{\mathbb{V}} \leq \|\mathbf{w}\|_{\mathbb{V}} &\leq \|\mathbf{u}^{r_\sigma} - \mathbf{u}\|_{\mathbb{V}} + \|\mathbf{u} - \mathbf{u}^{\bar{\sigma}}\|_{\mathbb{V}} \\ &\leq \big(\vartheta^{-1} + \frac{1}{3}\big[1 - \frac{C_1^2\zeta^2}{c_2^2}\big]^{\frac{1}{2}}\big)\|\mathbf{u}^{r_\sigma} - \mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}}. \end{aligned}$$

With $\hat{\sigma}_0 = \hat{\sigma}$, for $k = 1, 2, \ldots$ let $\hat{\sigma}_k \supset \hat{\sigma}_{k-1}$ be the smallest partition that is a full refinement with respect to all $T \in \hat{\sigma}_{k-1}$. Then using the fact that $r_\sigma - p_{\bar{\sigma}} \in \mathbb{P}_{\hat{\sigma}_0}$, as in the first part of this lemma with now $\zeta = 1$ we have $\|\mathbf{w}-\mathbf{w}_{\hat{\sigma}_k}\|_{\mathbb{V}} \leq \big[1-\frac{C_1^2}{c_2^2}\big]^{\frac{k}{2}}\|\mathbf{w}-\mathbf{w}_{\hat{\sigma}}\|_{\mathbb{V}}$, where $\mathbf{w}_{\hat{\sigma}_k}$ denotes the best approximation to $\mathbf{w}$ from $\mathbb{V}_{\sigma_k}$. With k being the smallest integer with $\big[1-\frac{C_1^2}{c_2^2}\big]^{\frac{k}{2}}\big(\vartheta^{-1}+\frac{1}{3}\big[1-\frac{C_1^2\zeta^2}{c_2^2}\big]^{\frac{1}{2}}\big) \leq \frac{1}{3}\big[1-\frac{C_1^2\zeta^2}{c_2^2}\big]^{\frac{1}{2}}$, we conclude that

$$\begin{aligned}\inf_{\mathbf{v}_{\bar{\tau}}\in\mathbb{V}_{\bar{\tau}}}\|\mathbf{u}^{r_\sigma}-(\mathbf{v}_{\bar{\tau}}+\mathbf{w}_{\hat{\sigma}_k})\|_{\mathbb{V}} &\leq \|\mathbf{u}^{r_\sigma}-\mathbf{u}^{\bar{\sigma}}-\mathbf{w}_{\hat{\sigma}_k}\|_{\mathbb{V}}+\|\mathbf{u}^{\bar{\sigma}}-\mathbf{u}\|_{\mathbb{V}}+\inf_{\mathbf{v}_{\bar{\tau}}\in\mathbb{V}_{\bar{\tau}}}\|\mathbf{u}-\mathbf{v}_{\bar{\tau}}\|_{\mathbb{V}}\\ &\leq \big[1-\tfrac{C_1^2\zeta^2}{c_2^2}\big]^{\frac{1}{2}}\|\mathbf{u}^{r_\sigma}-\mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}}.\end{aligned}$$

Since $\mathbf{v}_{\bar{\tau}}+\mathbf{w}_{\hat{\sigma}_k}\in\mathbb{V}_{\bar{\tau}\cup\hat{\sigma}_k}$, and $\#(\bar{\tau}\cup\hat{\sigma}_k)-\#\tau_0 \lesssim \#\bar{\tau}+\#\bar{\sigma}+\#\sigma$ (dependent on k and thus on ϑ), in view of (6.10) and (6.11) the proof is completed. ◻

Remark 6.8. If in (6.6), $\vartheta > \big[1-\frac{C_1^2\zeta^2}{c_2^2}\big]^{-\frac{1}{2}}$, then by a simplification of above proof, instead of (6.7), one obtains that

$$\#\underline{F} \lesssim \#\bar{\tau}-\#\tau_0$$

for any $\bar{\tau}$ with

$$\inf_{\mathbf{v}_{\bar{\tau}}\in\mathbb{V}_{\bar{\tau}}}\|\mathbf{u}-\mathbf{v}_{\bar{\tau}}\|_{\mathbb{V}} \leq \big(\big[1-\tfrac{C_1^2\zeta^2}{c_2^2}\big]^{\frac{1}{2}}-\vartheta^{-1}\big)\|\mathbf{u}^{r_\sigma}-\mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}},$$

which bound on $\#\underline{F}$ is in particular independent of the pressure p. It is, however, not clear whether under the condition $\|\mathbf{u}^{r_\sigma}-\mathbf{u}_\tau^{r_\sigma}\|_{\mathbb{V}} \geq \vartheta\|\mathbf{u}-\mathbf{u}^{r_\sigma}\|_{\mathbb{V}}$ for such ϑ, the inner elliptic problem is sufficiently accurately solved to obtain a convergent inexact Uzawa algorithm for solving the reduced Stokes problem of finding $(\mathbf{u}^\sigma, p_\sigma)$. Knowing that we can control $\#\sigma$ because our outmost loop producing Galerkin approximations to p, Lemma 6.6 provides a way to avoid the condition that $\vartheta > \big[1-\frac{C_1^2\zeta^2}{c_2^2}\big]^{-\frac{1}{2}}$. This point is the exact reason why we did not succeed to prove quasi-optimality of the Uzawa iteration for solving the full Stokes problem, so without our outmost loop. Indeed, with that method there is no separate control over the partitions that underly the pressure approximations.

7. An adaptive method for the Stokes problem in an idealized setting.

For $s > 0$, we define the approximation class

$$\mathcal{A}_{\mathbb{V}}^s = \{\mathbf{v}\in\mathbb{V} : |\mathbf{v}|_{\mathcal{A}_{\mathbb{V}}^s} := \sup_{\varepsilon>0}\varepsilon \inf_{\{\tau:\inf_{\mathbf{v}_\tau\in\mathbb{V}_\tau}\|\mathbf{v}-\mathbf{v}_\tau\|_{\mathbb{V}}\leq\varepsilon\}}[\#\tau-\#\tau_0]^s < \infty\},$$

and equip it with norm $\|\mathbf{v}\|_{\mathcal{A}_{\mathbb{V}}^s} := \|\mathbf{v}\|_{\mathbb{V}}+|\mathbf{v}|_{\mathcal{A}_{\mathbb{V}}^s}$. So $\mathcal{A}_{\mathbb{V}}^s$ is the class of vector fields that *can* be approximated within any given tolerance $\varepsilon > 0$ by a $\mathbf{v}_\tau\in\mathbb{V}_\tau$ for *some* partition τ with $\#\tau-\#\tau_0 \lesssim \varepsilon^{-1/s}|\mathbf{u}|_{\mathcal{A}_{\mathbb{V}}^s}^{1/s}$. Similarly, we define

$$\mathcal{A}_{\mathbb{P}}^s = \{q\in\mathbb{P} : |q|_{\mathcal{A}_{\mathbb{P}}^s} := \sup_{\varepsilon>0}\varepsilon \inf_{\{\sigma:\inf_{q_\sigma\in\mathbb{P}_\sigma}\|q-q_\sigma\|_{\mathbb{P}}\leq\varepsilon\}}[\#\sigma-\#\tau_0]^s < \infty\},$$

and equip it with norm $\|q\|_{\mathcal{A}_{\mathbb{P}}^s} := \|q\|_{\mathbb{P}}+|q|_{\mathcal{A}_{\mathbb{P}}^s}$.

Because of the polynomial degrees of our approximations, only for $s \leq m/d$ membership of $\mathbf{u}\in\mathcal{A}_{\mathbb{V}}^s$ or $p\in\mathcal{A}_{\mathbb{P}}^s$ can be enforced by imposing suitable smoothness conditions on $\mathbf{u}$ or p, respectively. These smoothness conditions, however, are much milder than requiring that $\mathbf{u}\in H^{1+sd}(\Omega)^d$ or $p\in H^{sd}(\Omega)$, that would be needed when only uniformly refined partitions were considered. The approximations classes can be (nearly) characterized as certain Besov spaces (see [BDDP02] for details). In any case for $d = 2$ and sufficiently smooth $\mathbf{f}$, it is known (see [Dah99]) that $\mathbf{u}$ and p have sufficient Besov smoothness so that they are in $\mathcal{A}^s_{\mathbb{V}}$ or $\mathcal{A}^s_{\mathbb{P}}$ for any $s < m/d$.

The results derived in Section 6.2 concerning adaptive velocity refinements were valid under the assumption that $\mathbf{f}$ was piecewise polynomial of degree $m-1$ with respect to the current partition. In order to make our exposition not too complicated, *in this section we will assume that* $\mathbf{f}$ *is piecewise polynomial of degree* $m-1$ *with respect to any partition that we encounter, i.e., that is in* $\mathbb{V}^*_{\tau_0}$. In the next section, we will remove this restriction. Furthermore, for the moment we assume that the arising finite dimensional linear systems are solved exactly, i.e., we do not care about the computational cost. In the next section, by applying iterative solvers, we will show quasi-optimal computational complexity.

The following algorithm is an implementation of the solution method that was announced in Section 4 with the simplifications mentioned above. Note that the `do`-loop in the algorithm actually consists of 3 nested loops over i, j and k. The loop over i concerns an adaptive method for solving p from the Schur complement equation. The loop over j concerns the Uzawa method for solving p^{σ_i} from the reduced Stokes problem. Finally, the loop over k concerns an adaptive method for solving $\mathbf{u}^{p_j^{(i)}} \in \mathbb{V}$ from $a(\mathbf{u}^{p_j^{(i)}}, \mathbf{v}) = \mathbf{f}(\mathbf{v}) - b(\mathbf{v}, p_j^{(i)})$ $(\mathbf{v} \in \mathbb{V})$. We have formulated these loops as one loop to deal efficiently with the complicated stopping criteria. E.g., the innermost one stops when either $\mathcal{E}^{\mathrm{E}}(\cdots) \le \alpha\mathcal{E}^{\mathrm{S}}(\sigma_i,\cdots)$ or $\mathcal{E}^{\mathrm{S}}(\sigma_i,\cdots) \le \kappa\mathcal{E}^{\mathrm{S}}(\infty,\cdots)$ or $\mathcal{E}^{\mathrm{S}}(\infty,\cdots) \le \varepsilon$, i.e., when either

$$\begin{aligned}
\|\mathbf{u}^{p_j^{(i)}} - \mathbf{u}^{(i)}_{j,k}\|_{\mathbb{V}} &\le C_1\alpha c_4^{-1}[\|p_{\sigma_i} - p_j^{(i)}\|_{\mathbb{P}} + \|\mathbf{u}^{p_{\sigma_i}} - \mathbf{u}^{(i)}_{j,k}\|_{\mathbb{V}}] \quad \text{or}\\
\|p^{\sigma_i} - p_j^{(i)}\|_{\mathbb{P}} + \|\mathbf{u}^{p_{\sigma_i}} - \mathbf{u}^{(i)}_{j,k}\|_{\mathbb{V}} &\le C_3\kappa c_4^{-1}[\|p - p_j^{(i)}\|_{\mathbb{P}} + \|\mathbf{u} - \mathbf{u}^{(i)}_{j,k}\|_{\mathbb{V}}] \quad \text{or}\\
\|p - p_j^{(i)}\|_{\mathbb{P}} + \|\mathbf{u} - \mathbf{u}^{(i)}_{j,k}\|_{\mathbb{V}} &\le \varepsilon.
\end{aligned}$$

STOKESSOLVE$_0[f,\varepsilon] \to [\sigma_j^{(i)}, p_j^{(i)}, \tau^{(i)}_{j,k}, \mathbf{u}^{(i)}_{j,k}]$
% For this preliminary version of the adaptive solver it is assumed
% that $f \in \mathbb{V}^*_{\tau_0}$.
% Let the parameter ζ *from* **REFvel** *satisfy* $\zeta \in (0, \frac{c_2}{C_1})$, *and* θ *from* **REFpres**
% satisfy $\theta \in (0, [1 - \frac{1-c_6^2/C_5^2}{d}]^{\frac{1}{2}})$. *For some* $\omega \in (0,\theta)$ *small enough such that*
% $\frac{\omega + \sqrt{1-d(1-\theta^2)}}{1-\omega} < \frac{c_6}{C_5}$, *fix some sufficiently small constants* $\kappa, \alpha > 0$ *such that*
% $\kappa < 1$, $C_1\frac{\kappa}{1-\kappa} \le \omega$, $\kappa C_1 < c_4$, $[1 - \frac{2\kappa C_3}{c_4 - \kappa C_1}]^{-1}[1 - \frac{c_6^2(\theta-\omega)^2}{c_5^2(1+\omega)^2}]^{\frac{1}{2}} < 1$, $\alpha C_1 < c_4$,
% and $1 - \beta^2 + \frac{2\alpha C_1}{c_4 - \alpha C_1} < 1$.
$p_0^{(0)} := 0$, $\sigma_0 := \tau^{(0)}_{0,0} := \tau_0$
$i := j := k := 0$
`do` $\mathbf{u}^{(i)}_{j,k} := \mathbf{u}^{p_j^{(i)}}_{\tau^{(i)}_{j,k}}$ *% i.e.,* $\mathbf{u}^{(i)}_{j,k} \in \mathbb{V}_{\tau^{(i)}_{j,k}}$, $a(\mathbf{u}^{(i)}_{j,k}, \mathbf{v}) = \mathbf{f}(\mathbf{v}) - b(\mathbf{v}, p_j^{(i)})$, $(\mathbf{v} \in \mathbb{V}_{\tau^{(i)}_{j,k}})$
  `if` $C_3\mathcal{E}^{\mathrm{S}}(\infty, \tau^{(i)}_{j,k}, \mathbf{f}, p_j^{(i)}, \mathbf{u}^{(i)}_{j,k}) \le \varepsilon$ `then stop`
  `elsif` $\mathcal{E}^{\mathrm{S}}(\sigma_i, \tau^{(i)}_{j,k}, \mathbf{f}, p_j^{(i)}, \mathbf{u}^{(i)}_{j,k}) \le \kappa\mathcal{E}^{\mathrm{S}}(\infty, \tau^{(i)}_{j,k}, \mathbf{f}, p_j^{(i)}, \mathbf{u}^{(i)}_{j,k})$ `then`
    $\sigma_{i+1} :=$ **REFpres**$[\sigma_i, p_j^{(i)}, \mathbf{u}^{(i)}_{j,k}]$
    $p_0^{(i+1)} := p_j^{(i)}$, $\tau^{(i+1)}_{0,0} := \tau^{(i)}_{j,k}$
    $i{+}{+}$, $j := k := 0$
  `elsif` $\mathcal{E}^{\mathrm{E}}(\tau^{(i)}_{j,k}, \mathbf{f}, p_j^{(i)}, \mathbf{u}^{(i)}_{j,k}) \le \alpha\mathcal{E}^{\mathrm{S}}(\sigma_i, \tau^{(i)}_{j,k}, \mathbf{f}, p_j^{(i)}, \mathbf{u}^{(i)}_{j,k})$ `then`
    $p^{(i)}_{j+1} := p_j^{(i)} - Q_{\sigma_i}\mathrm{div}\,\mathbf{u}^{(i)}_{j,k}$

```
        τ^{(i)}_{j+1,0} := τ^{(i)}_{j,k}
        j++, k := 0
  else  τ^{(i)}_{j,k+1} := REFvel[τ^{(i)}_{j,k}, f, p^{(i)}_j, u^{(i)}_{j,k}]
        k++
  endif
enddo
```

THEOREM 7.1. **(I)** *Let* $\mathbf{f} \in \mathbb{V}^*_{\tau_0}$, *then* $[\sigma^{(i)}_j, p^{(i)}_j, \tau^{(i)}_{j,k}, \mathbf{u}^{(i)}_{j,k}] := \mathbf{STOKESSOLVE}_0[f, \varepsilon]$ *terminates, and* $\|\mathbf{u} - \mathbf{u}^{(i)}_{j,k}\|_{\mathbb{V}} + \|p - p^{(i)}_j\|_{\mathbb{P}} \leq \varepsilon$. **(II)** *If, for some* $s > 0$, $p \in \mathcal{A}^s_{\mathbb{P}}$, *then* $\#\sigma^{(i)}_j - \#\tau_0 \lesssim \varepsilon^{-1/s}|p|^{1/s}_{\mathcal{A}^s_{\mathbb{P}}}$, *only dependent on* τ_0 *and on* s *when it tends to* 0 *or infinity. If, in addition, for some* $\tilde{s} > 0$, $\mathbf{u} \in \mathcal{A}^{\tilde{s}}_{\mathbb{V}}$, *then with* $\bar{s} = \min(s, \tilde{s})$, $\#\tau^{(i)}_{j,k} - \#\tau_0 \lesssim \varepsilon^{-1/\bar{s}}(\|p\|^{1/\bar{s}}_{\mathcal{A}^{\bar{s}}_{\mathbb{P}}} + \|\mathbf{u}\|^{1/\bar{s}}_{\mathcal{A}^{\bar{s}}_{\mathbb{V}}})$, *only dependent on* τ_0, *and on* $\bar{s}$ *when it tends to* 0 *or infinity.*

Remark 7.2. Note that in view of the assumptions, $\#\sigma^{(i)}_j - \#\tau_0$ is at most a constant multiple larger than this expression for the *best partition* $\sigma^{(i)}_j$ giving rise to such an error in the pressure. Similarly, $\#\tau^{(i)}_{j,k} - \#\tau_0$ is at most a constant multiple larger than this expression for the *best partition* $\tau^{(i)}_{j,k}$ on which p and $\mathbf{u}$ can be approximated by piecewise polynomials of degree $m-1$, or continuous piecewise polynomials of degree m with errors less than or equal to ε in $\|\,\|_{\mathbb{P}}$ or $\|\,\|_{\mathbb{V}}$, respectively.

Proof. **(I)** Given i and j, $\underline{k} = \underline{k}(i,j)$ will denote the maximum value of k for those i and j. Given an i, $\underline{j} = \underline{j}(i)$ will denote the maximum value of j for that i, and $\underline{\underline{k}} = \underline{\underline{k}}(i) := \underline{k}(i, \underline{j}(i))$ is the the maximum value of k for that i. Finally, $\underline{i}$ will denote the maximum value of i.

If $C_3\mathcal{E}^{\mathrm{S}}(\infty, \tau^{(i)}_{j,k}, \mathbf{f}, p^{(i)}_j, \mathbf{u}^{(i)}_{j,k}) \leq \varepsilon$ *is passed*, i.e., $(i,j,k) = (\underline{i}, \underline{j}(\underline{i}), \underline{\underline{k}}(\underline{i}))$, then $\|\mathbf{u} - \mathbf{u}^{(\underline{i})}_{\underline{j},\underline{\underline{k}}}\|_{\mathbb{V}} + \|p - p^{(\underline{i})}_{\underline{j}}\|_{\mathbb{P}} \leq \varepsilon$ by Proposition 5.6.

The inequality $\mathcal{E}^{\mathrm{S}}(\sigma_i, \tau^{(i)}_{j,k}, \mathbf{f}, p^{(i)}_j, \mathbf{u}^{(i)}_{j,k}) \leq \kappa\mathcal{E}^{\mathrm{S}}(\infty, \tau^{(i)}_{j,k}, \mathbf{f}, p^{(i)}_j, \mathbf{u}^{(i)}_{j,k})$ is equivalent to $\mathcal{E}^{\mathrm{E}}(\tau^{(i)}_{j,k}, \mathbf{f}, p^{(i)}_j, \mathbf{u}^{(i)}_{j,k}) \leq (1-\kappa)^{-1}(\kappa\|\mathrm{div}\mathbf{u}^{(i)}_{j,k}\|_{L_2(\Omega)} - \|Q_{\sigma_i}\mathrm{div}\mathbf{u}^{(i)}_{j,k}\|_{L_2(\Omega)})$. So *if this test is passed*, i.e., $(j,k) = (\underline{j}(i), \underline{\underline{k}}(i))$, then by Propositions 5.2, 5.7 and 5.6,

$$C_1^{-1}\|\mathbf{u}^{p^{(i)}_{\underline{j}}} - \mathbf{u}^{(i)}_{\underline{j},\underline{\underline{k}}}\|_{\mathbb{V}} \leq \mathcal{E}^{\mathrm{E}}(\tau^{(i)}_{\underline{j},\underline{\underline{k}}}, \mathbf{f}, p^{(i)}_{\underline{j}}, \mathbf{u}^{(i)}_{\underline{j},\underline{\underline{k}}}) \leq \tfrac{\kappa}{1-\kappa}\|\mathrm{div}\mathbf{u}^{(i)}_{\underline{j},\underline{\underline{k}}}\|_{L_2(\Omega)}, \tag{7.1}$$

$$C_1^{-1}\|\mathbf{u}^{p^{(i)}_{\underline{j}}} - \mathbf{u}^{(i)}_{\underline{j},\underline{\underline{k}}}\|_{\mathbb{V}} \leq \kappa c_4^{-1}(\|\mathbf{u} - \mathbf{u}^{(i)}_{\underline{j},\underline{\underline{k}}}\|_{\mathbb{V}} + \|p - p^{(i)}_{\underline{j}}\|_{\mathbb{P}}), \tag{7.2}$$

$$C_3^{-1}\|p_{\sigma_i} - p^{(i)}_{\underline{j}}\|_{\mathbb{P}} \leq \kappa c_4^{-1}(\|\mathbf{u} - \mathbf{u}^{(i)}_{\underline{j},\underline{\underline{k}}}\|_{\mathbb{V}} + \|p - p^{(i)}_{\underline{j}}\|_{\mathbb{P}}). \tag{7.3}$$

By (7.1), $\|\mathrm{div}\cdot\|_{L_2(\Omega)} \leq \|\cdot\|_{\mathbb{V}}$ on $\mathbb{V}$, and $C_1\frac{\kappa}{1-\kappa} \leq \omega$, Lemma 6.3 shows that

$$\|p - p_{\sigma_{i+1}}\|_{\mathbb{P}} \leq \left[1 - \tfrac{c_6^2(\theta-\omega)^2}{C_5^2(1+\omega)^2}\right]^{\frac{1}{2}}\|p - p^{(i)}_{\underline{j}}\|_{\mathbb{P}}, \tag{7.4}$$

and furthermore, since $\frac{\omega+\sqrt{1-d(1-\theta^2)}}{1-\omega} < \frac{c_6}{C_5}$ and in view of the definition of $\mathcal{A}^s_{\mathbb{P}}$, that

$$\#\sigma_{i+1} - \#\sigma_i \lesssim \|p - p^{(i)}_{\underline{j}}\|_{\mathbb{P}}^{-1/s}|p|^{1/s}_{\mathcal{A}^s_{\mathbb{P}}}. \tag{7.5}$$

By (7.2), $\kappa \leq c_4 C_1^{-1}$, and (5.6), we have

$$\boxed{\|\mathbf{u}-\mathbf{u}^{(i)}_{\underline{j},\underline{\underline{k}}}\|_{\mathbb{V}}} \leq \|\mathbf{u}-\mathbf{u}^{p^{(i)}_{\underline{j}}}\|_{\mathbb{V}} + \tfrac{\kappa C_1}{c_4-\kappa C_1}(\|\mathbf{u}-\mathbf{u}^{p^{(i)}_{\underline{j}}}\|_{\mathbb{V}} + \|p-p^{(i)}_{\underline{j}}\|_{\mathbb{P}})$$

$$\leq \boxed{\tfrac{c_4+\kappa C_1}{c_4-\kappa C_1}\|p-p^{(i)}_{\underline{j}}\|_{\mathbb{P}}}, \tag{7.6}$$

and so by (7.3),

$$\|p_{\sigma_i}-p^{(i)}_{\underline{j}}\|_{\mathbb{P}} \leq \tfrac{2\kappa C_3}{c_4-\kappa C_1}\|p-p^{(i)}_{\underline{j}}\|_{\mathbb{P}}. \tag{7.7}$$

With $\rho_1 := [1-\frac{2\kappa C_3}{c_4-\kappa C_1}]^{-1}[1-\frac{c_6^2(\theta-\omega)^2}{c_5^2(1+\omega)^2}]^{\frac{1}{2}} < 1$, combining (7.7), with i reading as $i+1$, and (7.4) shows that

$$\boxed{\|p-p^{(i+1)}_{\underline{j}(i+1)}\|_{\mathbb{P}} \leq \rho_1\|p-p^{(i)}_{\underline{j}(i)}\|_{\mathbb{P}}}. \tag{7.8}$$

Since $c_4\mathcal{E}^{\mathrm{S}}(\infty,\tau^{(i)}_{\underline{j},\underline{\underline{k}}},\mathbf{f},p^{(i)}_{\underline{j}},\mathbf{u}^{(i)}_{\underline{j},\underline{\underline{k}}}) \leq \|\mathbf{u}-\mathbf{u}^{(i)}_{\underline{j},\underline{\underline{k}}}\|_{\mathbb{V}} + \|p-p^{(i)}_{\underline{j}}\|_{\mathbb{P}}$, together (7.8) and (7.6) show that *the algorithm terminates, assuming each loop over j does,* which we show next.

If $\mathcal{E}^{\mathrm{E}}(\tau^{(i)}_{j,k},\mathbf{f},p^{(i)}_j,\mathbf{u}^{(i)}_{j,k}) \leq \alpha\mathcal{E}^{\mathrm{S}}(\sigma_i,\tau^{(i)}_{j,k},\mathbf{f},p^{(i)}_j,\mathbf{u}^{(i)}_{j,k})$ *is passed*, i.e., $k=\underline{k}(i,j)$, then

$$\|\mathbf{u}^{p^{(i)}_j}-\mathbf{u}^{(i)}_{j,\underline{k}}\|_{\mathbb{V}} \leq \alpha C_1 c_4^{-1}(\|p_{\sigma_i}-p^{(i)}_j\|_{\mathbb{P}} + \|\mathbf{u}^{\sigma_i}-\mathbf{u}^{(i)}_{j,\underline{k}}\|_{\mathbb{V}}). \tag{7.9}$$

Estimating $\|\mathbf{u}^{\sigma_i}-\mathbf{u}^{(i)}_{j,\underline{k}}\|_{\mathbb{V}} \leq \|\mathbf{u}^{\sigma_i}-\mathbf{u}^{p^{(i)}_j}\|_{\mathbb{V}} + \|\mathbf{u}^{p^{(i)}_j}-\mathbf{u}^{(i)}_{j,\underline{k}}\|_{\mathbb{V}}$, applying (7.9) and $\|\mathbf{u}^{\sigma_i}-\mathbf{u}^{p^{(i)}_j}\|_{\mathbb{V}} \leq \|p_{\sigma_i}-p^{(i)}_j\|_{\mathbb{P}}$ ((5.6)), we obtain that

$$\|\mathbf{u}^{\sigma_i}-\mathbf{u}^{(i)}_{j,\underline{k}}\|_{\mathbb{V}} \leq \tfrac{c_4+\alpha C_1}{c_4-\alpha C_1}\|p_{\sigma_i}-p^{(i)}_j\|_{\mathbb{P}}, \tag{7.10}$$

and by substituting this in (7.9), that

$$\boxed{\|\mathbf{u}^{p^{(i)}_j}-\mathbf{u}^{(i)}_{j,\underline{k}}\|_{\mathbb{V}} \leq \tfrac{2\alpha C_1}{c_4-\alpha C_1}\|p_{\sigma_i}-p^{(i)}_j\|_{\mathbb{P}}}. \tag{7.11}$$

With $\rho_2 := 1-\beta^2+\frac{2\alpha C_1}{c_4-\alpha C_1} < 1$, by the statement (4.3) concerning convergence of the exact Uzawa iteration, we infer that

$$\boxed{\|p_{\sigma_i}-p^{(i)}_{j+1}\|_{\mathbb{P}} \leq \rho_2\|p_{\sigma_i}-p^{(i)}_j\|_{\mathbb{P}}}. \tag{7.12}$$

Since $c_4\mathcal{E}^{\mathrm{S}}(\sigma_i,\tau^{(i)}_{j,\underline{k}},\mathbf{f},p^{(i)}_j,\mathbf{u}^{(i)}_{j,\underline{k}}) \leq \|\mathbf{u}^{\sigma_i}-\mathbf{u}^{(i)}_{j,\underline{k}}\|_{\mathbb{V}} + \|p_{\sigma_i}-p^{(i)}_j\|_{\mathbb{P}}$, together (7.11) and (7.12) show that *each loop over j terminates* by either $\mathcal{E}^{\mathrm{S}}(\sigma_i,\tau^{(i)}_{j,k},\mathbf{f},p^{(i)}_j,\mathbf{u}^{(i)}_{j,k}) \leq \kappa\mathcal{E}^{\mathrm{S}}(\infty,\tau^{(i)}_{j,k},\mathbf{f},p^{(i)}_j,\mathbf{u}^{(i)}_{j,k})$ or $C_3\mathcal{E}^{\mathrm{S}}(\infty,\tau^{(i)}_{j,k},\mathbf{f},p^{(i)}_j,\mathbf{u}^{(i)}_{j,k}) \leq \varepsilon$, *assuming each loop over k terminates*, which we show next.

With $\rho_3 := [1-\frac{c_2^2\zeta^2}{C_1^2}]^{\frac{1}{2}} < 1$, Lemma 6.6 shows that

$$\boxed{\|\mathbf{u}^{p^{(i)}_j}-\mathbf{u}^{(i)}_{j,k+1}\|_{\mathbb{V}} \leq \rho_3\|\mathbf{u}^{p^{(i)}_j}-\mathbf{u}^{(i)}_{j,k}\|_{\mathbb{V}}}. \tag{7.13}$$

Since $c_2\mathcal{E}^{\mathrm{E}}(\tau_{j,k}^{(i)},\mathbf{f},p_j^{(i)},\mathbf{u}_{j,k}^{(i)}) \leq \|\mathbf{u}^{p_j^{(i)}}-\mathbf{u}_{j,k}^{(i)}\|_{\mathbb{V}}$, from (7.13) we infer that *each loop over k terminates* by either $\mathcal{E}^{\mathrm{E}}(\tau_{j,k}^{(i)},\mathbf{f},p_j^{(i)},\mathbf{u}_{j,k}^{(i)}) \leq \alpha\mathcal{E}^{\mathrm{S}}(\sigma_i,\tau_{j,k}^{(i)},\mathbf{f},p_j^{(i)},\mathbf{u}_{j,k}^{(i)})$ or $\mathcal{E}^{\mathrm{S}}(\sigma_i,\tau_{j,k}^{(i)},\mathbf{f},p_j^{(i)},\mathbf{u}_{j,k}^{(i)}) \leq \kappa\mathcal{E}^{\mathrm{S}}(\infty,\tau_{j,k}^{(i)},\mathbf{f},p_j^{(i)},\mathbf{u}_{j,k}^{(i)})$ or $C_3\mathcal{E}^{\mathrm{S}}(\infty,\tau_{j,k}^{(i)},\mathbf{f},p_j^{(i)},\mathbf{u}_{j,k}^{(i)}) \leq \varepsilon$. With this, part **(I)** of the theorem is proven.

Before starting with the second part, we collect some estimates for $\|p-p_{\underline{j}(0)}^{(0)}\|_{\mathbb{P}}$, $\|p_{\sigma_i}-p_0^{(i)}\|_{\mathbb{P}}$ and $\|\mathbf{u}^{p_j^{(i)}}-\mathbf{u}_{j,0}^{(i)}\|_{\mathbb{V}}$, i.e., the initial values for the recursions (7.8), (7.12), and (7.13) over i, j and k, respectively.

From (7.7) we infer that

$$\|p-p_{\underline{j}(0)}^{(0)}\|_{\mathbb{P}} \leq [1-\tfrac{2\kappa C_3}{c_4-\kappa C_1}]^{-1}\|p-p_{\tau_0}\|_{\mathbb{P}} \leq [1-\tfrac{2\kappa C_3}{c_4-\kappa C_1}]^{-1}\|p\|_{\mathbb{P}}. \tag{7.14}$$

For $j=0$ and $i=0$, we have that

$$\|p_{\sigma_0}-p_0^{(0)}\|_{\mathbb{P}} \leq \|p_{\sigma_0}-p\|_{\mathbb{P}}+\|p\|_{\mathbb{P}} \leq 2\|p\|_{\mathbb{P}}, \tag{7.15}$$

and for $j=0$ and $i>0$, that

$$\begin{aligned}\|p_{\sigma_i}-p_0^{(i)}\|_{\mathbb{P}} &= \|p_{\sigma_i}-p_{\underline{j}(i-1)}^{(i-1)}\|_{\mathbb{P}} \leq \|p_{\sigma_i}-p\|_{\mathbb{P}}+\|p-p_{\underline{j}(i-1)}^{(i-1)}\|_{\mathbb{P}}\\ &\leq \big([1-\tfrac{c_6^2(\theta-\omega)^2}{C_5^2(1+\omega)^2}]^{\frac{1}{2}}+1\big)\|p-p_{\underline{j}(i-1)}^{(i-1)}\|_{\mathbb{P}}\end{aligned} \tag{7.16}$$

by (7.4).

For $k=j=i=0$, we have

$$\|\mathbf{u}^{p_0^{(0)}}-\mathbf{u}_{0,0}^{(0)}\|_{\mathbb{V}} \leq \|\mathbf{u}^{p_0^{(0)}}\|_{\mathbb{V}} = \|\mathbf{f}\|_{\mathbb{V}'}. \tag{7.17}$$

For $k=j=0$ and $i>0$, we have $p_0^{(i)}=p_{\underline{j}(i-1)}^{(i-1)}$ and $\tau_{0,0}^{(i)}=\tau_{\underline{j}(i-1),\underline{k}(i-1)}^{(i-1)}$, and so

$$\|\mathbf{u}^{p_0^{(i)}}-\mathbf{u}_{0,0}^{(i)}\|_{\mathbb{V}} = \|\mathbf{u}^{p_{\underline{j}(i-1)}^{(i-1)}}-\mathbf{u}_{\underline{j}(i-1),\underline{k}(i-1)}^{(i-1)}\|_{\mathbb{V}} \leq \tfrac{2C_1\kappa}{c_4-\kappa C_1}\|p-p_{\underline{j}(i-1)}^{(i-1)}\|_{\mathbb{P}}, \tag{7.18}$$

by (7.2) and (7.6). For $k=0$ and $j>0$, we have

$$\begin{aligned}\|\mathbf{u}^{p_j^{(i)}}-\mathbf{u}_{j,0}^{(i)}\|_{\mathbb{V}} &\leq \|\mathbf{u}^{p_j^{(i)}}-\mathbf{u}_{j-1,\underline{k}(j-1)}^{(i)}\|_{\mathbb{V}}\\ &\leq \|\mathbf{u}^{p_j^{(i)}}-\mathbf{u}^{p_{j-1}^{(i)}}\|_{\mathbb{V}}+\|\mathbf{u}^{p_{j-1}^{(i)}}-\mathbf{u}_{j-1,\underline{k}(j-1)}^{(i)}\|_{\mathbb{V}}.\end{aligned}$$

Now using that by (5.6)

$$\begin{aligned}&\|\mathbf{u}^{p_j^{(i)}}-\mathbf{u}^{p_{j-1}^{(i)}}\|_{\mathbb{V}} \leq \|p_j^{(i)}-p_{j-1}^{(i)}\|_{\mathbb{P}} \leq \|\mathbf{u}^{\sigma_i}-\mathbf{u}_{j-1,\underline{k}(j-1)}^{(i)}\|_{\mathbb{V}}\\ &\leq \|\mathbf{u}^{\sigma_i}-\mathbf{u}^{p_{j-1}^{(i)}}\|_{\mathbb{V}}+\|\mathbf{u}^{p_{j-1}^{(i)}}-\mathbf{u}_{j-1,\underline{k}(j-1)}^{(i)}\|_{\mathbb{V}} \leq \|p_{\sigma_i}-p_{j-1}^{(i)}\|_{\mathbb{P}}+\|\mathbf{u}^{p_{j-1}^{(i)}}-\mathbf{u}_{j-1,\underline{k}(j-1)}^{(i)}\|_{\mathbb{V}},\end{aligned}$$

and

$$\|\mathbf{u}^{p_{j-1}^{(i)}}-\mathbf{u}_{j-1,\underline{k}(j-1)}^{(i)}\|_{\mathbb{V}} \leq \tfrac{2\alpha C_1}{c_4-\alpha C_1}\|p_{\sigma_i}-p_{j-1}^{(i)}\|_{\mathbb{P}}$$

by (7.11), we find that

$$\|\mathbf{u}^{p_j^{(i)}}-\mathbf{u}_{j,0}^{(i)}\|_{\mathbb{V}} \leq \big(1+\tfrac{4\alpha C_1}{c_4-\alpha C_1}\big)\|p_{\sigma_i}-p_{j-1}^{(i)}\|_{\mathbb{P}}. \tag{7.19}$$

(II) At the moment of a call $\sigma_{i+1} := \mathbf{REFpres}[\sigma_i, p^{(i)}_{\underline{j}}, \mathbf{u}^{(i)}_{\underline{j},\underline{\underline{k}}}]$, we have

$$\|p - p^{(i)}_{\underline{j}}\|_{\mathbb{P}} \geq c_4 \mathcal{E}^{\mathrm{S}}(\infty, \tau^{(i)}_{\underline{j},\underline{\underline{k}}}, \mathbf{f}, p^{(i)}_{\underline{j}}, \mathbf{u}^{(i)}_{\underline{j},\underline{\underline{k}}}) > c_4 C_3^{-1} \varepsilon,$$

so that in view of (7.5) and (7.8), we have

$$\#\sigma_{\underline{i}} - \#\tau_0 \leq \sum_{i=0}^{\underline{i}} \#\sigma_i - \#\sigma_{i-1} \lesssim \varepsilon^{-1/s} |p|^{1/s}_{\mathcal{A}^s_{\mathbb{P}}}.$$

At the moment of a call $\tau^{(i)}_{j,k+1} := \mathbf{REFvel}[\tau^{(i)}_{j,k}, \mathbf{f}, p^{(i)}_j, \mathbf{u}^{(i)}_{j,k}]$, we have

$$\begin{aligned}\|\mathbf{u}^{p^{(i)}_j} - \mathbf{u}^{(i)}_{j,k}\|_{\mathbb{V}} &> c_2 \mathcal{E}^{\mathrm{E}}(\tau^{(i)}_{j,k}, \mathbf{f}, p^{(i)}_j, \mathbf{u}^{(i)}_{j,k}) > c_2 \alpha \mathcal{E}^{\mathrm{S}}(\sigma_i, \tau^{(i)}_{j,k}, \mathbf{f}, p^{(i)}_j, \mathbf{u}^{(i)}_{j,k}) \\ &> c_2 \alpha \kappa \mathcal{E}^{\mathrm{S}}(\infty, \tau^{(i)}_{j,k}, \mathbf{f}, p^{(i)}_j, \mathbf{u}^{(i)}_{j,k}) \geq c_2 \alpha \kappa C_3^{-1} \|\mathbf{u} - \mathbf{u}^{(i)}_{j,k}\|_{\mathbb{V}},\end{aligned}$$

where for the first time in this proof we used the fact that innermost iteration is stopped in time. In view of the definitions of $\mathcal{A}^{\bar{s}}_{\mathbb{P}}$ and $\mathcal{A}^{\bar{s}}_{\mathbb{V}}$, Lemma 6.6 now shows that the set of marked simplices $\underline{F}_{i,j,k} = \underline{F}$ inside $\mathbf{REFvel}[\tau^{(i)}_{j,k}, \mathbf{f}, p^{(i)}_j, \mathbf{u}^{(i)}_{j,k}]$ satisfies

$$\#\underline{F}_{i,j,k} \lesssim \|\mathbf{u}^{p^{(i)}_j} - \mathbf{u}^{(i)}_{j,k}\|^{-1/\bar{s}}_{\mathbb{V}} (|\mathbf{u}|^{1/\bar{s}}_{\mathcal{A}^{\bar{s}}_{\mathbb{V}}} + |p|^{1/\bar{s}}_{\mathcal{A}^{\bar{s}}_{\mathbb{P}}}) + \#\sigma_i - \#\tau_0 + \#\tau_0. \tag{7.20}$$

From $\|\mathbf{u}^{p^{(i)}_j} - \mathbf{u}^{(i)}_{j,k}\|_{\mathbb{V}} \leq \|\mathbf{u}^{p^{(i)}_j}\|_{\mathbb{V}} \lesssim \|\mathbf{u}\|_{\mathbb{V}} + \|p\|_{\mathbb{P}}$, we have

$$\#\tau_0 \lesssim 1 \lesssim \|\mathbf{u}^{p^{(i)}_j} - \mathbf{u}^{(i)}_{j,k}\|^{-1/\bar{s}}_{\mathbb{V}} (\|\mathbf{u}\|^{1/\bar{s}}_{\mathbb{V}} + \|p\|^{1/\bar{s}}_{\mathbb{P}}).$$

For $i > 0$, we have $\#\sigma_i - \#\tau_0 \lesssim \|p - p^{(i-1)}_{\underline{j}(i-1)}\|^{-1/\bar{s}}_{\mathbb{P}} |p|^{1/\bar{s}}_{\mathcal{A}^{\bar{s}}_{\mathbb{P}}}$ by (7.5). By (7.18), (7.16) and (7.19), and the decrease of $\|p_{\sigma_i} - p^{(i)}_j\|_{\mathbb{P}}$ and $\|\mathbf{u}^{p^{(i)}_j} - \mathbf{u}^{(i)}_{j,k}\|_{\mathbb{V}}$ as function of j or k, respectively, we have

$$\begin{aligned}\|p - p^{(i-1)}_{\underline{j}(i-1)}\|_{\mathbb{P}} &\gtrsim \begin{cases} \|\mathbf{u}^{p^{(i)}_0} - \mathbf{u}^{(i)}_{0,0}\|_{\mathbb{V}} \\ \|p_{\sigma_i} - p^{(i)}_0\|_{\mathbb{P}} \geq \|p_{\sigma_i} - p^{(i)}_{j-1}\|_{\mathbb{P}} \gtrsim \|\mathbf{u}^{p^{(i)}_j} - \mathbf{u}^{(i)}_{j,0}\|_{\mathbb{V}} \quad (j > 0) \end{cases} \\ &\gtrsim \|\mathbf{u}^{p^{(i)}_j} - \mathbf{u}^{(i)}_{j,k}\|_{\mathbb{V}}\end{aligned}$$

uniformly in i, j, k. We conclude that

$$\#\underline{F}_{i,j,k} \lesssim \|\mathbf{u}^{p^{(i)}_j} - \mathbf{u}^{(i)}_{j,k}\|^{-1/\bar{s}}_{\mathbb{V}} (\|\mathbf{u}\|^{1/\bar{s}}_{\mathcal{A}^{\bar{s}}_{\mathbb{V}}} + \|p\|^{1/\bar{s}}_{\mathcal{A}^{\bar{s}}_{\mathbb{P}}}),$$

and so, by Theorem 3.2 and (7.13), that

$$\begin{aligned}\#\tau^{(\underline{i})}_{\underline{j}(\underline{i}),\underline{\underline{k}}(\underline{i})} - \tau_0 &\lesssim (\|\mathbf{u}\|^{1/\bar{s}}_{\mathcal{A}^{\bar{s}}_{\mathbb{V}}} + \|p\|^{1/\bar{s}}_{\mathcal{A}^{\bar{s}}_{\mathbb{P}}}) \sum_{i=0}^{\underline{i}} \sum_{j=0}^{\underline{j}(i)} \sum_{k=0}^{\underline{k}(j)-1} \|\mathbf{u}^{p^{(i)}_j} - \mathbf{u}^{(i)}_{j,k}\|^{-1/\bar{s}}_{\mathbb{V}} \\ &\lesssim (\|\mathbf{u}\|^{1/\bar{s}}_{\mathcal{A}^{\bar{s}}_{\mathbb{V}}} + |p|^{1/\bar{s}}_{\mathcal{A}^{\bar{s}}_{\mathbb{P}}}) \sum_{i=0}^{\underline{i}} \sum_{j=0}^{\underline{j}(i)} \|\mathbf{u}^{p^{(i)}_j} - \mathbf{u}^{(i)}_{j,\underline{k}(j)-1}\|^{-1/\bar{s}}_{\mathbb{V}}.\end{aligned}$$

We will bound this expression by using that each of the three nested iterations is stopped in time.

For any i and j, by definition of $\underline{k}$, we have

$$\begin{aligned}\mathcal{E}^{\mathrm{E}}(\tau^{(i)}_{j,\underline{k}-1},\mathbf{f},p^{(i)}_j,\mathbf{u}^{(i)}_{j,\underline{k}-1}) &> \alpha\mathcal{E}^{\mathrm{S}}(\sigma_i,\tau^{(i)}_{j,\underline{k}-1},\mathbf{f},p^{(i)}_j,\mathbf{u}^{(i)}_{j,\underline{k}-1})\\ &> \alpha\kappa\mathcal{E}^{\mathrm{S}}(\infty,\tau^{(i)}_{j,\underline{k}-1},\mathbf{f},p^{(i)}_j,\mathbf{u}^{(i)}_{j,\underline{k}-1}) > \alpha\kappa C_3^{-1}\varepsilon,\end{aligned}$$

or

$$\begin{aligned}\|\mathbf{u}^{p^{(i)}_j}-\mathbf{u}^{(i)}_{j,\underline{k}-1}\|_{\mathbb{V}} &\gtrsim \|\mathbf{u}^{\sigma_i}-\mathbf{u}^{(i)}_{j,\underline{k}-1}\|_{\mathbb{V}}+\|p_{\sigma_i}-p^{(i)}_j\|_{\mathbb{P}}\\ &\gtrsim \|\mathbf{u}-\mathbf{u}^{(i)}_{j,\underline{k}-1}\|_{\mathbb{V}}+\|p-p^{(i)}_j\|_{\mathbb{P}} \gtrsim \varepsilon.\end{aligned}$$

Similarly, for any i, by definition of $\underline{j}$ we have

$$\mathcal{E}^{\mathrm{S}}(\sigma_i,\tau^{(i)}_{\underline{j}-1,\underline{k}},\mathbf{f},p^{(i)}_{\underline{j}-1},\mathbf{u}^{(i)}_{\underline{j}-1,\underline{k}}) > \kappa\mathcal{E}^{\mathrm{S}}(\infty,\tau^{(i)}_{\underline{j}-1,\underline{k}},\mathbf{f},p^{(i)}_{\underline{j}-1},\mathbf{u}^{(i)}_{\underline{j}-1,\underline{k}}) > \kappa C_3^{-1}\varepsilon,$$

or

$$\|p_{\sigma_i}-p^{(i)}_{\underline{j}-1}\|_{\mathbb{P}} \eqsim \|\mathbf{u}^{\sigma_i}-\mathbf{u}^{(i)}_{\underline{j}-1,\underline{k}}\|_{\mathbb{V}}+\|p_{\sigma_i}-p^{(i)}_{\underline{j}-1}\|_{\mathbb{P}} \gtrsim \|\mathbf{u}-\mathbf{u}^{(i)}_{\underline{j}-1,\underline{k}}\|_{\mathbb{V}}+\|p-p^{(i)}_{\underline{j}-1}\|_{\mathbb{P}} \gtrsim \varepsilon,$$

where for "$\eqsim$" we used (7.10). Finally, by definition of $\underline{i}$, we have

$$\mathcal{E}^{\mathrm{S}}(\infty,\tau^{(\underline{i}-1)}_{\underline{j},\underline{k}},\mathbf{f},p^{(\underline{i}-1)}_{\underline{j}},\mathbf{u}^{(\underline{i}-1)}_{\underline{j},\underline{k}}) > C_3^{-1}\varepsilon,$$

or

$$\|p-p^{(\underline{i}-1)}_{\underline{j}}\|_{\mathbb{P}} \eqsim \|\mathbf{u}-\mathbf{u}^{(\underline{i}-1)}_{\underline{j},\underline{k}}\|_{\mathbb{V}}+\|p-p^{(\underline{i}-1)}_{\underline{j}}\|_{\mathbb{P}} \gtrsim \varepsilon,$$

where for "$\eqsim$" we used (7.6). By using in addition (7.12) and (7.8), we find that

$$\begin{aligned}&\sum_{i=0}^{\underline{i}}\sum_{j=0}^{\underline{j}(i)}\|\mathbf{u}^{p^{(i)}_j}-\mathbf{u}^{(i)}_{j,\underline{k}(j)-1}\|_{\mathbb{V}}^{-1/\bar{s}}\\ &\lesssim \sum_{i=0}^{\underline{i}}\sum_{j=0}^{\underline{j}(i)-1}\|p_{\sigma_i}-p^{(i)}_j\|_{\mathbb{P}}^{-1/\bar{s}}+\sum_{i=0}^{\underline{i}-1}\|p-p^{(i)}_{\underline{j}(i)}\|_{\mathbb{P}}^{-1/\bar{s}}+\varepsilon^{-1/\bar{s}}\\ &\lesssim \sum_{i=0}^{\underline{i}}\|p_{\sigma_i}-p^{(i)}_{\underline{j}(i)-1}\|_{\mathbb{P}}^{-1/\bar{s}}+\|p-p^{(\underline{i}-1)}_{\underline{j}(\underline{i}-1)}\|_{\mathbb{P}}^{-1/\bar{s}}+\varepsilon^{-1/\bar{s}}\\ &\lesssim \sum_{i=0}^{\underline{i}-1}\|p-p^{(i)}_{\underline{j}(i)-1}\|_{\mathbb{P}}^{-1/\bar{s}}+\|p_{\sigma_{\underline{i}}}-p^{(\underline{i})}_{\underline{j}(\underline{i})-1}\|_{\mathbb{P}}^{-1/\bar{s}}+\varepsilon^{-1/\bar{s}}\\ &\lesssim \sum_{i=0}^{\underline{i}-1}\|p-p^{(i)}_{\underline{j}(i)}\|_{\mathbb{P}}^{-1/\bar{s}}+\varepsilon^{-1/\bar{s}} \lesssim \varepsilon^{-1/\bar{s}},\end{aligned}$$

which completes the proof of the theorem. ∎

8. A practical adaptive method for the Stokes problem. The following lemma generalizes upon Lemma 6.6, relaxing both the condition that $\mathbf{f}\in\mathbb{V}^*_\tau$, and the assumption that we have the exact Galerkin solutions for the inner elliptic problems available, assuming that the deviations from that ideal situation are sufficiently small in a relative sense. The idea is that we *replace* $\mathbf{f}$ by a sufficiently accurate piecewise polynomial approximation, and that we solve the arising linear system only approximately using an iterative solver.

Since we are going to consider Galerkin systems with modified right-hand sides, we introduce the following notation: Given $r \in \mathbb{P}$, $\mathbf{g} \in \mathbb{V}'$, and a partition τ, $\mathbf{u}_\tau^{r,\mathbf{g}} \in \mathbb{V}_\tau$ will denote the solution of the Galerkin problem

$$\boxed{a(\mathbf{u}_\tau^{r,\mathbf{g}}, \mathbf{v}_\tau) = \mathbf{g}(\mathbf{v}_\tau) - b(\mathbf{v}_\tau, r), \quad (\mathbf{v}_\tau \in \mathbb{V}_\tau)}. \tag{8.1}$$

So in view of the notation introduced in (4.5), we have $\mathbf{u}_\tau^{r,\mathbf{f}} = \mathbf{u}_\tau^r$. Note that $\|\mathbf{u}_\tau^r - \mathbf{u}_\tau^{r,\mathbf{g}}\|_\mathbb{V} \le \|\mathbf{f} - \mathbf{g}\|_\mathbb{V}$.

LEMMA 8.1. *There exist constants* $\chi_1 = \chi_1(\zeta, C_1, c_2) > 0$, *and* $\lambda = \lambda(\chi_1, C_1, c_2) \in (0, \frac{1}{3}\big[1 - \frac{C_1^2\zeta^2}{c_2^2}\big]^{\frac{1}{2}}]$ *such that if for any* $\mathbf{f} \in \mathbb{V}'$, *partitions* $\tau \supseteq \sigma$, $r_\sigma \in \mathbb{P}_\sigma$, $\mathbf{f}_\tau \in \mathbb{V}_\tau^*$, $\mathbf{w}_\tau \in \mathbb{V}_\tau$ *with*

$$\|\mathbf{f} - \mathbf{f}_\tau\|_{\mathbb{V}'} + \|\mathbf{u}_\tau^{r_\sigma,\mathbf{f}_\tau} - \mathbf{w}_\tau\|_\mathbb{V} \le \chi_1 \mathcal{E}(\tau, \mathbf{f}_\tau, r_\sigma, \mathbf{w}_\tau), \tag{8.2}$$

and, for some absolute constant $\vartheta > 0$,

$$\|\mathbf{u}^{r_\sigma} - \mathbf{w}_\tau\|_\mathbb{V} \ge \vartheta \|\mathbf{u} - \mathbf{u}^{r_\sigma}\|_\mathbb{V},$$

then the set of marked simplices $\underline{F}$ *inside the call* $\tilde{\tau} := \mathbf{REFvel}[\tau, \mathbf{f}_\tau, r_\sigma, \mathbf{w}_\tau]$ *satisfies*

$$\#\underline{F} \lesssim \#\bar{\tau} + \#\bar{\sigma} + \#\sigma$$

for any partitions $\bar{\tau}$ *and* $\bar{\sigma}$ *for which*

$$\inf_{\mathbf{v}_{\bar{\tau}} \in \mathbb{V}_{\bar{\tau}}} \|\mathbf{u} - \mathbf{v}_{\bar{\tau}}\|_\mathbb{V} \le \lambda \|\mathbf{u}^{r_\sigma} - \mathbf{w}_\tau\|_\mathbb{V}, \quad \inf_{q_{\bar{\sigma}} \in \mathbb{P}_{\bar{\sigma}}} \|p - q_{\bar{\sigma}}\|_\mathbb{V} \le \lambda \|\mathbf{u}^{r_\sigma} - \mathbf{w}_\tau\|_\mathbb{V}. \tag{8.3}$$

Furthermore, given a

$$\mu \in \big(\big[1 - \tfrac{c_2^2\zeta^2}{C_1^2}\big]^{\frac{1}{2}}, 1\big),$$

there exists an $\chi_2 = \chi_2(\mu, \zeta, C_1, c_2) > 0$, *such that if* (5.6) *is valid with* χ_1 *reading as* χ_2, *and for* $\tau' \supseteq \tilde{\tau}$, $\mathbf{f}_{\tau'} \in \mathbb{V}'$ *and* $\mathbf{w}_{\tau'} \in \mathbb{V}_{\tau'}$,

$$\|\mathbf{f} - \mathbf{f}_{\tau'}\|_{\mathbb{V}'} + \|\mathbf{u}_{\tau'}^{r_\sigma,\mathbf{f}_{\tau'}} - \mathbf{w}_{\tau'}\|_\mathbb{V} \le \chi_2 \mathcal{E}^{\mathrm{E}}(\tau, \mathbf{f}_\tau, r_\sigma, \mathbf{w}_\tau),$$

then

$$\|\mathbf{u}^{r_\sigma} - \mathbf{w}_{\tau'}\|_\mathbb{V} \le \mu \|\mathbf{u}^{r_\sigma} - \mathbf{w}_\tau\|_\mathbb{V}.$$

Proof. Following the lines of [Ste05b, proof of Lemma 6.1], for suitable constants χ_1 and λ, one can show that $\#\underline{F} \lesssim \#\hat{\tau} - \#\tau_0$ for any partition $\hat{\tau}$ with $\inf_{\mathbf{v}_{\hat{\tau}} \in \mathbb{V}_{\hat{\tau}}} \|\mathbf{u}^{r_\sigma} - \mathbf{v}_{\hat{\tau}}\|_\mathbb{V} \le \lambda \|\mathbf{u}^{r_\sigma} - \mathbf{w}_\tau\|_\mathbb{V}$. Then, following the proof of Lemma 6.6, we infer that $\#\hat{\tau} - \#\tau_0 \le \#\bar{\tau} + \#\bar{\sigma} + \#\sigma$, with $\bar{\tau}$ and $\#\bar{\sigma}$ from (8.3).

The second statement can be proven as in [Ste05b, Lemma 6.2]. ☐

For solving the Galerkin systems approximately, we assume that we have an iterative solver of optimal type available:

$\mathbf{GALSOLVE}[\tau, \mathbf{f}_\tau, r_\tau, \mathbf{w}_\tau^{(0)}, \eta] \to \bar{\mathbf{w}}_\tau$
% τ is a partition, $f_\tau \in \mathbb{V}_\tau^*$, $r_\tau \in \mathbb{P}_\tau$, $\mathbf{w}_\tau^{(0)} \in \mathbb{V}_\tau$, the latter being an initial

% *approximation for an iterative solver, and* $\eta > 0$.
% *The output* $\bar{\mathbf{w}}_\tau \in \mathbb{V}_\tau$ *satisfies*

$$\|\mathbf{u}_\tau^{r_\tau,\mathbf{f}_\tau} - \bar{\mathbf{w}}_\tau\|_{\mathbb{V}} \leq \eta.$$

% *The call requires* $\lesssim \max\{1, \log(\eta^{-1}\|\mathbf{u}_\tau^{r_\tau,\mathbf{f}_\tau} - \mathbf{w}_\tau^{(0)}\|_{\mathbb{V}})\}\#\tau$ *arithmetic operations.*

Additive or multiplicative multigrid methods with local smoothing are known to be of this type.

A routine called **RHS** will be needed to find a sufficiently accurate piecewise polynomial approximation of degree $m-1$ to the right-hand side $\mathbf{f}$. Since this might not be possible with respect to the current partition τ, a call of **RHS** may result in a further refinement.

$\mathbf{RHS}[\tau, \eta] \to [\tau', \mathbf{f}_{\tau'}]$
% $\eta > 0$. *The output consists of an* $\mathbf{f}_{\tau'} \in \mathbb{V}_{\tau'}^*$, *where* $\tau' = \tau$, *or, if necessary,*
% $\tau' \supset \tau$, *such that* $\|\mathbf{f} - \mathbf{f}_{\tau'}\|_{\mathbb{V}'} \leq \eta$.

Assuming that $p \in \mathcal{A}_{\mathbb{P}}^s$ and $\mathbf{u} \in \mathcal{A}_{\mathbb{V}}^{\tilde{s}}$ for some $s, \tilde{s} > 0$, the cost of approximating the right-hand side $\mathbf{f}$ using **RHS** will generally not dominate the other costs of our adaptive method only if there is some constant $c_{\mathbf{f}}$ such that, with $\bar{s} = \max(s, \tilde{s})$, for any $\eta > 0$ and any partition τ, for $[\tau', \cdot] := \mathbf{RHS}[\tau, \eta]$, it holds that

$$\#\tau' - \#\tau \leq c_{\mathbf{f}}^{1/\bar{s}} \eta^{-1/\bar{s}},$$

and the number of arithmetic operations required by the call is $\lesssim \#\tau'$. We will call such a **RHS** to be $\bar{s}$-*optimal* with constant $c_{\mathbf{f}}$. Obviously, given $\bar{s}$, such a routine can only exist when $\mathbf{f} \in \mathcal{A}_{\mathbb{V}'}^{\bar{s}}$, defined by

$$\mathcal{A}_{\mathbb{V}'}^{\bar{s}} = \{\mathbf{g} \in \mathbb{V}' : \sup_{\varepsilon > 0} \varepsilon \inf_{\{\tau : \inf_{\mathbf{g}_\tau \in \mathbb{V}_\tau^*} \|\mathbf{g} - \mathbf{g}_\tau\|_{\mathbb{V}'} \leq \varepsilon\}} [\#\tau - \#\tau_0]^{\bar{s}} < \infty\}.$$

Knowing that $\mathbf{f} \in \mathcal{A}_{\mathbb{V}'}^{\bar{s}}$ is a different thing than knowing how to *construct* suitable approximations. If $s \in [1/d, (m+1)/d]$ and $\mathbf{f} \in H^{sd-1}(\Omega)^d$, then $\mathbf{f} \in \mathcal{A}_{\mathbb{V}'}^{\bar{s}}$ and $\mathbf{f}_{\tau'}$ constructed as the best approximation from $\mathbb{V}_{\tau'}^*$ to $\mathbf{f}$ with respect to $L_2(\Omega)^d$ using (the smallest common refinement of the input partition and) uniform refined partitions τ' are known to converge with the required rate. For general $\mathbf{f} \in \bar{\mathcal{A}}_{\mathbb{V}'}^{s}$, however, a realization of a suitable routine **RHS** has to depend on $\mathbf{f}$ at hand.

Remark 8.2. As we have said, in our adaptive method, for both computing the error estimator and setting up the Galerkin system for the inner elliptic problem, we will replace $\mathbf{f}$ by a piecewise polynomial approximation. This has the advantages that we can consider $\mathbf{f} \notin L_2(\Omega)^d$, for which thus the error estimator is not defined, and that it simplifies the analysis, since we don't have to take into quadrature errors on various places.

Thinking of smooth p, $\mathbf{u}$ and thus $\mathbf{f}$, we have $p \in \mathcal{A}_{\mathbb{P}}^{m/d}$, $\mathbf{u} \in \mathcal{A}_{\mathbb{V}}^{m/d}$ (and generally $p \notin \mathcal{A}_{\mathbb{P}}^s$, $\mathbf{u} \notin \mathcal{A}_{\mathbb{V}}^s$ for $s > m/d$), whereas $\mathbf{f} \in \mathcal{A}_{\mathbb{V}'}^{(m+1)/d}$. Moreover, these rates are realized using quasi-uniform partitions. In this case, an adaptive method will create such partitions, and we see that at least asymptotically calls of **RHS** will not give rise to refinements. So in this case, we can simply skip these calls, and work with the exact $\mathbf{f}$, where one can view the generally now necessary application of quadrature for the evaluation of the error estimator and for setting up the right-hand sides of the Galerkin systems as an implicit replacement of $\mathbf{f}$ by a piecewise polynomial. Also for less smooth p and/or $\mathbf{u}$, and an $\mathbf{f}$ that is at least in $L_2(\Omega)^d$, one can expect that usually refinements are not needed for obtaining a sufficiently accurate approximation of $\mathbf{f}$ by a piecewise polynomial of degree $m-1$.

Now we are ready to formulate our practical adaptive Stokes solver **STOKESSOLVE**.

$\mathbf{STOKESSOLVE}[f,\varepsilon] \to [\sigma_j^{(i)}, p_j^{(i)}, \tau_{j,k}^{(i)}, \mathbf{w}_{j,k}^{(i)}]$
% Let the parameter ζ from **REFvel** *satisfy $\zeta \in (0, \frac{c_2}{C_1})$, and θ from* **REFpres**
% satisfy $\theta \in (0, [1 - \frac{1-c_6^2/C_5^2}{d}]^{\frac{1}{2}})$. Let $\kappa, \beta, \alpha, \chi > 0$ be sufficiently small constants.
$\sigma_0 := \tau_{0,0}^{(0)} := \tau_0$, $p_0^{(0)} := 0$, $\mathbf{w}_{0,0}^{(0)} := 0$, $\delta \eqsim \|\mathbf{f}\|_{\mathbb{V}'}$, $i := j := k := 0$
`do` $[\mathbf{f}_{j,k}^{(i)}, \tau_{j,k}^{(i)}] := \mathbf{RHS}[\delta/2, \tau_{j,k}^{(i)}]$
  $\mathbf{w}_{j,k}^{(i)} := \mathbf{GALSOLVE}[\tau_{j,k}^{(i)}, \mathbf{f}_{j,k}^{(i)}, p_j^{(i)}, \mathbf{w}_{j,k}^{(i)}, \delta/2]$
  `if` $C_3\mathcal{E}^{\mathrm{S}}(\infty, \tau_{j,k}^{(i)}, \mathbf{f}_{j,k}^{(i)}, p_j^{(i)}, \mathbf{w}_{j,k}^{(i)}) + (1 + C_3(2c_2^{-1}+1))\delta/2 \le \varepsilon$ `then stop`
  `elsif` $\mathcal{E}^{\mathrm{S}}(\sigma_i, \tau_{j,k}^{(i)}, \mathbf{f}_{j,k}^{(i)}, p_j^{(i)}, \mathbf{w}_{j,k}^{(i)}) + \delta \le \kappa\mathcal{E}^{\mathrm{S}}(\infty, \tau_{j,k}^{(i)}, \mathbf{f}_{j,k}^{(i)}, p_j^{(i)}, \mathbf{w}_{j,k}^{(i)})$ `then`
    $\sigma_{i+1} := \mathbf{REFpres}[\sigma_i, p_j^{(i)}, \mathbf{w}_{j,k}^{(i)}]$
    $p_0^{(i+1)} := p_j^{(i)}$, $\tau_{0,0}^{(i+1)} := \tau_{j,k}^{(i)}$, $\mathbf{w}_{0,0}^{(i+1)} := \mathbf{w}_{j,k}^{(i)}$
    $\delta := \beta(\mathcal{E}^{\mathrm{S}}(\infty, \tau_{j,k}^{(i)}, \mathbf{f}_{j,k}^{(i)}, p_j^{(i)}, \mathbf{w}_{j,k}^{(i)}) + \delta)$
    $i{+}{+}$, $j := k := 0$
  `elsif` $\mathcal{E}^{\mathrm{E}}(\tau_{j,k}^{(i)}, \mathbf{f}_{j,k}^{(i)}, p_j^{(i)}, \mathbf{w}_{j,k}^{(i)}) + \delta \le \alpha\mathcal{E}^{\mathrm{S}}(\sigma_i, \tau_{j,k}^{(i)}, \mathbf{f}_{j,k}^{(i)}, p_j^{(i)}, \mathbf{w}_{j,k}^{(i)})$ `then`
    $p_{j+1}^{(i)} := p_j^{(i)} - Q_{\sigma_i}\mathrm{div}\mathbf{w}_{j,k}^{(i)}$
    $\tau_{j+1,0}^{(i)} := \tau_{j,k}^{(i)}$, $\mathbf{w}_{j+1,0}^{(i)} := \mathbf{w}_{j,k}^{(i)}$
    $\delta := \beta(\mathcal{E}^{\mathrm{S}}(\sigma_i, \tau_{j,k}^{(i)}, \mathbf{f}_{j,k}^{(i)}, p_j^{(i)}, \mathbf{w}_{j,k}^{(i)}) + \delta)$
    $j{+}{+}$, $k := 0$
  `elsif` $\delta \le \chi\mathcal{E}^{\mathrm{E}}(\tau_{j,k}^{(i)}, \mathbf{f}_{j,k,}^{(i)}, p_j^{(i)}, \mathbf{w}_{j,k}^{(i)})$
    $\tau_{j,k+1}^{(i)} := \mathbf{REFvel}[\tau_{j,k}^{(i)}, \mathbf{f}_{j,k}^{(i)}, p_j^{(i)}, \mathbf{w}_{j,k}^{(i)}]$, $\mathbf{w}_{j,k+1}^{(i)} := \mathbf{w}_{j,k}^{(i)}$
    $\delta := \beta(\mathcal{E}^{\mathrm{E}}(\tau_{j,k}^{(i)}, \mathbf{f}_{j,k}^{(i)}, p_j^{(i)}, \mathbf{w}_{j,k}^{(i)}) + \delta)$
    $k{+}{+}$
  `else` $\delta := \delta/2$
  `endif`
`enddo`

Compared to the preliminary version $\mathbf{STOKESSOLVE}_0$, in **STOKESSOLVE** the exact solution $\mathbf{u}_{j,k}^{(i)}$ of the inner elliptic problem is replaced by an approximate one $\mathbf{w}_{j,k}^{(i)}$, which is moreover computed using an approximation $\mathbf{f}_{j,k}^{(i)} \in \mathbb{V}^*_{\tau_{j,k}^{(i)}}$ of the right-hand side $\mathbf{f} \in \mathbb{V}'$. As a consequence, the results on the a posteriori error estimators from Section 5 cannot be applied directly. Using the stability of the problems defining $\mathbf{u}$ and p with respect to perturbations in $\mathbf{f}$, and the stability of the error estimator $\mathcal{E}^{\mathrm{E}}$ demonstrated in Proposition 5.5, for sufficiently small α and κ, stopping by either $\mathcal{E}^{\mathrm{E}}(\cdots)+\delta \le \alpha\mathcal{E}^{\mathrm{S}}(\sigma_i,\cdots)$, $\mathcal{E}^{\mathrm{S}}(\sigma_i,\cdots)+\delta \le \kappa\mathcal{E}^{\mathrm{S}}(\infty,\cdots)$ or $\mathcal{E}^{\mathrm{S}}(\infty,\cdots)+(1+C_3(2c_2^{-1}+1))\delta/2 \le \varepsilon$, means that

$$\|\mathbf{u}^{p_j^{(i)}} - \mathbf{w}_{j,k}^{(i)}\|_{\mathbb{V}} \le D_1(\alpha)[\|p_{\sigma_i} - p_j^{(i)}\|_{\mathbb{P}} + \|\mathbf{u}^{p_{\sigma_i}} - \mathbf{w}_{j,k}^{(i)}\|_{\mathbb{V}}],$$
$$\|p^{\sigma_i} - p_j^{(i)}\|_{\mathbb{P}} + \|\mathbf{u}^{p_{\sigma_i}} - \mathbf{w}_{j,k}^{(i)}\|_{\mathbb{V}} \le D_2(\kappa)[\|p - p_j^{(i)}\|_{\mathbb{P}} + \|\mathbf{u} - \mathbf{w}_{j,k}^{(i)}\|_{\mathbb{V}}] \quad \text{or}$$
$$\|p - p_j^{(i)}\|_{\mathbb{P}} + \|\mathbf{u} - \mathbf{w}_{j,k}^{(i)}\|_{\mathbb{V}} \le \varepsilon,$$

respectively, with $D_1(\alpha), D_2(\kappa) > 0$ being some constants that tend to zero when α

or κ tend to zero.

Since the tolerance $\delta/2$ for the error in the approximations for the right-hand side and in that for the solution of the inner Galerkin problem decreases until $\delta \leq \chi\mathcal{E}^{\mathrm{E}}(\cdots)$, where, for β small enough, the next δ respects the same bound, the second part of Lemma 8.1 shows that for each i and j, $\mathbf{w}_{j,k}^{(i)}$ converges linearly towards $\mathbf{u}^{p_j^{(i)}}$. Here we silently assumed that by halving δ, at some point $\delta \leq \chi\mathcal{E}^{\mathrm{E}}(\cdots)$ is valid. This, however, is not necessarily true since $\mathcal{E}^{\mathrm{E}}(\cdots)$ changes as well. E.g. think of the (unlikely) situation that we have reached a partition on which $\mathbf{u}^{p_j^{(i)}}$ can be represented exactly. Yet, in that case we also have linear convergence of $\mathbf{w}_{j,k}^{(i)}$ towards $\mathbf{u}^{p_j^{(i)}}$. Indeed, if during the process of halving δ it remains larger than $\chi\mathcal{E}^{\mathrm{E}}(\cdots)$, then $\chi\mathcal{E}^{\mathrm{E}}(\cdots)+\delta$, up to some constant factor being an upper bound for $\|\mathbf{u}^{p_j^{(i)}} - \mathbf{w}_{j,k}^{(i)}\|_{\mathbb{V}}$, decreases linearly.

Based on these observations, similarly as in the proof of Part **(I)** of Theorem 8.3, for sufficiently small κ and α, one shows convergence of **STOKESSOLVE**, with $\|p - p_j^{(i)}\|_{\mathbb{P}} + \|\mathbf{u} - \mathbf{w}_{j,k}^{(i)}\|_{\mathbb{V}} \leq \varepsilon$ at termination.

When **REFvel** is called, from $\delta \leq \chi\mathcal{E}^{\mathrm{E}}(\cdots)$ for sufficiently small χ, $\mathcal{E}^{\mathrm{E}}(\cdots)+\delta > \alpha\mathcal{E}^{\mathrm{S}}(\sigma_i, \cdots)$ and $\mathcal{E}^{\mathrm{S}}(\sigma_i, \cdots) + \delta > \kappa\mathcal{E}^{\mathrm{S}}(\infty, \cdots)$, we have

$$\begin{aligned}\|\mathbf{u}^{p_j^{(i)}} - \mathbf{w}_{j,k}^{(i)}\|_{\mathbb{V}} \gtrsim \mathcal{E}^{\mathrm{E}}(\cdots) - \delta \gtrsim \mathcal{E}^{\mathrm{E}}(\cdots) + \delta \gtrsim \mathcal{E}^{\mathrm{S}}(\sigma_i, \cdots) + \delta \gtrsim \mathcal{E}^{\mathrm{S}}(\infty, \cdots) + \delta \\ \gtrsim \|\mathbf{u} - \mathbf{w}_{j,k}^{(i)}\|_{\mathbb{V}},\end{aligned}$$

meaning that we may apply the first part of Lemma 8.1 to bound the cardinality of the set $\underline{F}_{i,j,k}$ of marked simplices. Assuming that $p \in \mathcal{A}_{\mathbb{P}}^{\bar{s}}$ and $\mathbf{u} \in \mathcal{A}_{\mathbb{V}}^{\bar{s}}$, as in (7.20) we find

$$\#\underline{F}_{i,j,k} \lesssim \|\mathbf{u}^{p_j^{(i)}} - \mathbf{w}_{j,k}^{(i)}\|_{\mathbb{V}}^{-1/\bar{s}}(|\mathbf{u}|_{\mathcal{A}_{\mathbb{V}}^{\bar{s}}}^{1/\bar{s}} + |p|_{\mathcal{A}_{\mathbb{P}}^{\bar{s}}}^{1/\bar{s}}) + \#\sigma_i - \#\tau_0 + \#\tau_0.$$

Other than in **STOKESSOLVE**$_0$, in **STOKESSOLVE** refinements can also be made by calls $[\mathbf{f}_{j,k}^{(i)}, \tau_{j,k}^{(i)}] :=$ **RHS**$[\delta/2, \tilde{\tau}_{j,k}^{(i)}]$. Assuming that **RHS** $\bar{s}$-optimal with constant $c_{\mathbf{f}}$, then $\#\tau_{j,k}^{(i)} - \#\tilde{\tau}_{j,k}^{(i)} \leq c_{\mathbf{f}}^{-1/\bar{s}}(\delta/2)^{-1/\bar{s}}$ Using that such a call can only be made when

$$\delta \gtrsim \mathcal{E}^{\mathrm{E}}(\cdots) + \delta \gtrsim \mathcal{E}^{\mathrm{S}}(\sigma_i, \cdots) + \delta \gtrsim \mathcal{E}^{\mathrm{S}}(\infty, \cdots) + \delta \gtrsim \varepsilon,$$

applying similar techniques as in the proof of Theorem 7.1 one can prove that **STOKESSOLVE** outputs quasi-optimal partitions.

Finally, one can prove that **STOKESSOLVE** is of quasi-optimal computational complexity. The main point is that with a call **GALSOLVE**$[\tau_{j,k}^{(i)}, p_j^{(i)}, \mathbf{f}_{j,k}^{(i)}, \mathbf{w}_{j,k}^{(i)}, \delta/2]$, it holds that $\|\mathbf{u}^{p_j^{(i)}, \mathbf{f}_{j,k}^{(i)}} - \mathbf{w}_{j,k}^{(i)}\|_{\mathbb{V}} \lesssim \delta/2$, so that the error has to be reduced by only a constant factor.

Along the lines indicated above, we end up with the following theorem. We have chosen not to include a full proof, since this would require another level of technicalities on top of those from the proof of Theorem 7.1.

Theorem 8.3. **(I)** $[\sigma_j^{(i)}, p_j^{(i)}, \tau_{j,k}^{(i)}, \mathbf{w}_{j,k}^{(i)}] :=$ **STOKESSOLVE**$[f, \varepsilon]$ *terminates, and* $\|\mathbf{u} - \mathbf{w}_{j,k}^{(i)}\|_{\mathbb{V}} + \|p - p_j^{(i)}\|_{\mathbb{P}} \leq \varepsilon$. **(II)** *If, for some* $s > 0$, $p \in \mathcal{A}_{\mathbb{P}}^{s}$, *then* $\#\sigma_j^{(i)} - \#\tau_0 \lesssim \varepsilon^{-1/s}|p|_{\mathcal{A}_{\mathbb{P}}^{s}}^{1/s}$, *only dependent on* τ_0 *and on* s *when it tends to* 0 *or infinity. If, in addition, for some* $\tilde{s} > 0$, $\mathbf{u} \in \mathcal{A}_{\mathbb{V}}^{\tilde{s}}$ *and, with* $\bar{s} = \min(s, \tilde{s})$, **RHS** *$\bar{s}$-optimal with* *constant $c_{\mathbf{f}}$, then $\#\tau_{j,k}^{(i)} - \#\tau_0 \lesssim \varepsilon^{-1/\bar{s}}(\|p\|_{\mathcal{A}_{\mathbb{P}}^{\bar{s}}}^{1/\bar{s}} + \|\mathbf{u}\|_{\mathcal{A}_{\mathbb{V}}^{\bar{s}}}^{1/\bar{s}} + c_{\mathbf{f}}^{1/\bar{s}})$, only dependent on τ_0, and on $\bar{s}$ when it tends to 0 or infinity. Under Assumption 6.4, and when $\varepsilon \lesssim \|\mathbf{f}\|_{\mathbb{V}'}$, the number of arithmetic operations and storage locations required by the call is also bounded by a multiple of the same expression as $\varepsilon^{-1/\bar{s}}(\|p\|_{\mathcal{A}_{\mathbb{P}}^{\bar{s}}}^{1/\bar{s}} + \|\mathbf{u}\|_{\mathcal{A}_{\mathbb{V}}^{\bar{s}}}^{1/\bar{s}} + c_{\mathbf{f}}^{1/\bar{s}})$.*

9. Numerical experiment. We consider $d = 2$, the L-shaped domain $\Omega = (0,1)^2 \backslash (\frac{1}{2}, 1)^2$, $\mathbf{f} : \mathbf{x} \mapsto 25(4\mathbf{x}_2 - 1, 1 - 4\mathbf{x}_1)$, and $m = 1$, i.e., continuous piecewise linear approximation for the velocity, and piecewise constant approximation for the pressure. The initial partition τ_0 is illustrated in Figure 9.1.

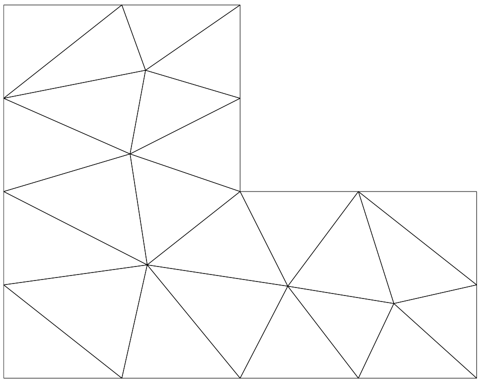

FIG. 9.1. *Initial partition τ_0*

The "bulk chasing" parameters θ and ζ inside **REFpres** or **REFvel** were chosen to be 0.7 and $\sqrt{0.3}$, repectively. Since $\mathbf{f}$ is smooth, we followed the approach discussed in Remark 8.2, and skipped the calls of **RHS**. Furthermore, instead of solving each arising finite dimensional linear system within tolerance $\delta/2$ for the first value of $\delta/2$ obtained by successively halving that is less than $\chi\mathcal{E}^{\mathrm{E}}(\tau_{j,k}^{(i)}, \mathbf{f}_{j,k}^{(i)}, p_j^{(i)}, \mathbf{w}_{j,k}^{(i)})$, we always approximately solved it by 3 multigrid iterations with local smoothing (cf. [WC03]) starting with the previously computed approximate velocity. The parameters κ and α are chosen to be 0.88 and 0.9, respectively.

For comparison, we also implemented the adaptive Uzawa method for solving the full Stokes problem, i.e., **STOKESSOLVE** without the outmost loop over i, i.e., without the part starting from the first `elsif`-statement until the second one, and all remaining occurrences of σ_i replaced by ∞. In particular the pressure update $p_{j+1}^{(i)} := p_j^{(i)} + Q_{\sigma_i}\mathrm{div}\mathbf{w}_{j,k}^{(i)}$ then reads as $p_{j+1} := p_j + \mathrm{div}\mathbf{w}_{j,k}$. This is the algorithm studied in [BMN02], apart from the replacement of a priori prescribed tolerances by a posteriori ones.

In Figure 9.2, we plotted the full Stokes error estimator $\mathcal{E}^{\mathrm{S}}(\infty, \tau_{\underline{j},\underline{\underline{k}}}^{(i)}, \mathbf{f}, p_{\underline{j}}^{(i)}, \mathbf{w}_{\underline{j},\underline{\underline{k}}}^{(i)})$ vs. $\#\tau_{\underline{j},\underline{\underline{k}}}^{(i)}$ (or $\mathcal{E}^{\mathrm{S}}(\infty, \tau_{j,\underline{k}}, \mathbf{f}, p_j, \mathbf{w}_{j,\underline{k}})$ vs. $\#\tau_{j,\underline{k}}$). Ignoring the fact that generally $\mathbf{w}_{\underline{j},\underline{\underline{k}}}^{(i)} \neq \mathbf{u}_{\tau_{\underline{j},\underline{\underline{k}}}^{(i)}}^{p_{\underline{j}}^{(i)}}$ (or $\mathbf{w}_{j,\underline{k}} \neq \mathbf{u}_{\tau_{j,\underline{k}}}^{p_j}$), modulo a constant factor this estimator is an upper bound for $\|\mathbf{u} - \mathbf{w}_{\underline{j},\underline{\underline{k}}}^{(i)}\|_{\mathbb{V}} + \|p - p_{\underline{j}}^{(i)}\|_{\mathbb{P}}$ (or $\|\mathbf{u} - \mathbf{w}_{j,\underline{k}}\|_{\mathbb{V}} + \|p - p_j\|_{\mathbb{P}}$) (For $\mathbf{f} \in \mathbb{V}_{\tau_{\underline{j},\underline{\underline{k}}}^{(i)}}^*$ (or $\mathbf{f} \in \mathbb{V}_{\tau_{j,\underline{k}}}^*$), it would also be a lower bound). In Figure 9.3, we plotted the pressure error estimator $\|\mathrm{div}\mathbf{w}_{\underline{j},\underline{\underline{k}}}^{(i)}\|_{L_2(\Omega)}$ vs. $\#\sigma_i$ (or $\|\mathrm{div}\mathbf{w}_{j,\underline{k}}\|_{L_2(\Omega)}$ vs. $\#\tau_{j,0}$). Ignoring that generally $\mathbf{w}_{\underline{j},\underline{\underline{k}}}^{(i)} \neq$

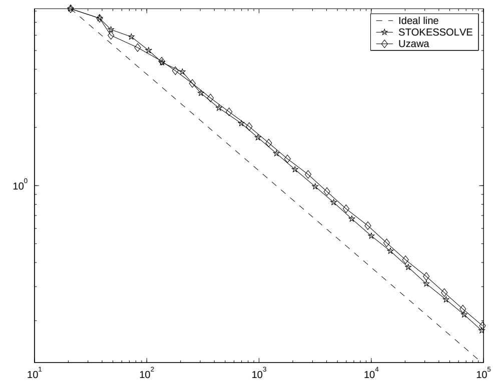

FIG. 9.2. *Estimated Stokes error vs. cardinality underlying partition*

$\mathbf{u}^{p_{\underline{j}}^{(i)}}$ (or $\mathbf{w}_{j,\underline{k}} \neq \mathbf{u}^{p_j}$), this estimator is equivalent to $\|p - p_{\underline{j}}^{(i)}\|_{\mathbb{P}}$ (or $\|p - p_j\|_{\mathbb{P}}$). As predicted by the theory, the approximations produced by **STOKESSOLVE** converge with the best possible rates. In this example, we observe that the same is true for the adaptive Uzawa method.

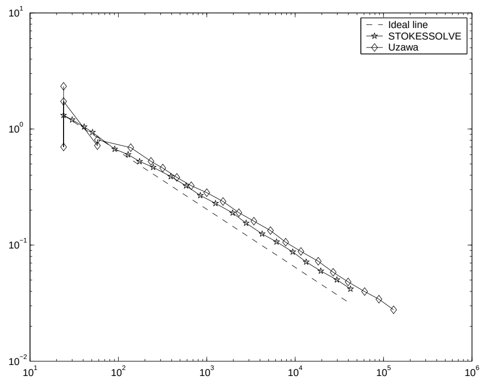

FIG. 9.3. *Estimated pressure error vs. cardinality underlying partition*

Our implementation is partly written in C, and, for our convenience, partly in MATLAB making use of the PDE toolbox. Due to the datastructures used in the MATLAB part, our code is not of optimal computational complexity; the time needed for a call of **REFvel** is not proportional to the cardinality of its output partition. Subtracting the times spent for this routine, we observed computing times that are proportional to the cardinality of the final partition.

In Figure 9.4 we show two partitions σ_i that were produced by **STOKESSOLVE**. Note that these partitions are nonconforming. In Figure 9.5 we give two partitions $\tau_{j,k}^{(i)}$. Finally, in Figure 9.6 plots of p and $\mathbf{u}$ are given.

Acknowledgements. The authors are indebted to Dr. Haijun Wu (Nanjing University, China) for making available to us his code implementing multigrid on locally refined partitions.

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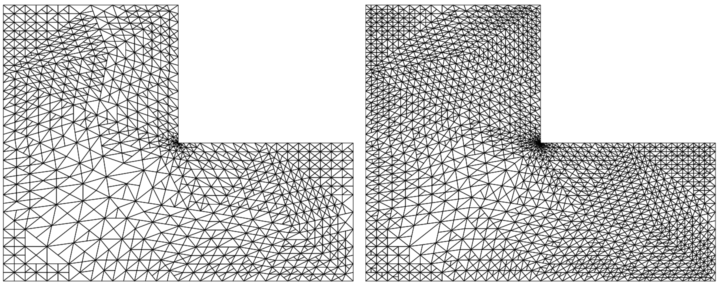

FIG. 9.4. *Pressure partitions with* 1965 *or* 4235 *triangles*

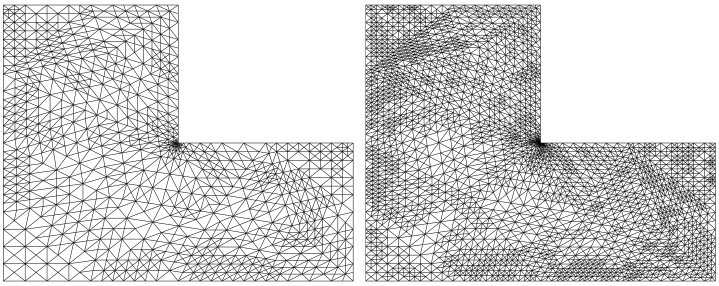

FIG. 9.5. *Velocity partitions with* 1842 *or* 6126 *triangles*

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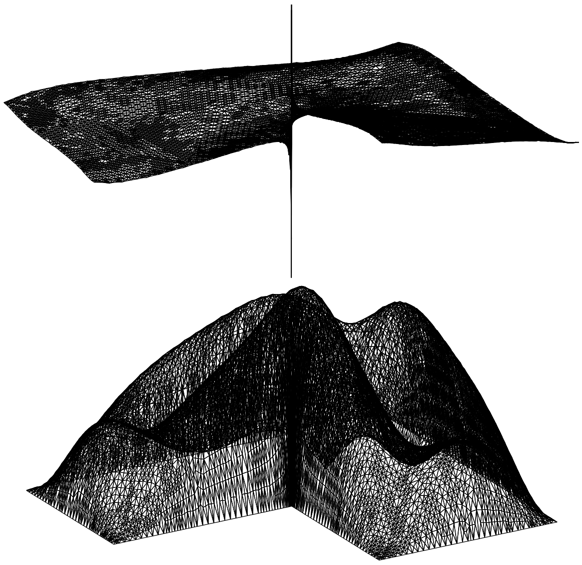

FIG. 9.6. *Pressure p and the norm $|\mathbf{u}|$ of the velocity*

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