Dynamics of Hamiltonian Systems

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Abstract

Hamiltonian systems are a class of dynamical systems which can be characterised by preservation of a symplectic form. This allows to write down the equations of motion in terms of a single function, the Hamiltonian function. They were conceived in the 19th century to study physical systems varying from optics to frictionless mechanics in a unified way. This description turned out to be particularly efficient for symmetry reduction and perturbation analysis.

1 Introduction

The best-known Hamiltonian system is the harmonic oscillator. The second order differential equation

$$\ddot{x} \;+\; \varpi^2 x \;=\; 0$$

models the motion of a point mass attached to a massless spring (Hooke's law) and has the general solution

$$x(t) \;=\; x_0 \cos \varpi t \;+\; \frac{y_0}{\varpi} \sin \varpi t$$

with initial conditions $(x(0), \dot{x}(0)) = (x_0, y_0)$. Choosing co-ordinates

$$q \;=\; \sqrt{\varpi} x \quad \text{and} \quad p \;=\; \frac{y}{\sqrt{\varpi}} \tag{1}$$

yields the first order system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} q \\ p \end{pmatrix} \;=\; \begin{pmatrix} 0 & \varpi \\ -\varpi & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \;,$$

the solutions of which move along the circles $p^2 + q^2 = const.$ This last observation lies at the basis of the Hamiltonian formalism, defining

$$H(q,p) \;=\; \varpi \frac{p^2 + q^2}{2}$$

to be the Hamiltonian function of the system. The Hamiltonian vector field is then defined to be perpendicular to the gradient ∇H, ensuring that H is a conserved quantity. Passing back through (1) yields the energy

$$H(x, y) \;=\; \frac{\dot{x}^2}{2} \;+\; \varpi^2 \frac{x^2}{2}$$

of the spring system (the point mass being scaled to $m = 1$).

The conserved quantity H makes two-dimensional systems, said to have one *degree of freedom*, easy to study and many efforts are made to reduce more complex systems to this setting, often using symmetries. In such a case the system is *integrable* and can in principle be explicitly solved. Where the symmetries are only approximately preserving the system, a more geometric understanding allows to analyse how the perturbation from the integrable approximation to the original system alters the dynamics.

2 The Phase Space

The simplest type of example of a Hamiltonian system is that of a 1–dimensional particle with kinetic energy

$$T \;=\; \frac{1}{2} m v^2 \;=\; \frac{p^2}{2m} \tag{2}$$

and potential energy $V = V(q)$. The canonical equations derived from the Hamiltonian function $H = T + V$ are

$$\begin{aligned} \dot{q} \;&=\; \frac{\partial H}{\partial p} \;=\; \frac{p}{m} \\ \dot{p} \;&=\; -\frac{\partial H}{\partial q} \;=\; -V'(q) \end{aligned}$$

and are equivalent with Newton's laws, where $F = -V'$ is the force field with potential V. An immediate consequence is

$$\dot{H} \;=\; \frac{\partial H}{\partial q} \dot{q} \;+\; \frac{\partial H}{\partial p} \dot{p} \;=\; 0 \;,$$

the conservation of energy.

The same conclusion holds in $\mathbb{R}^{2n}$, with *canonical co-ordinates* $q_1, \ldots, q_n, \; p_1, \ldots, p_n$, where a Hamiltonian function $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ defines the canonical equations

$$\dot{q}_i \;=\; \frac{\partial H}{\partial p_i} \;, \quad i = 1, \ldots, n \tag{3a}$$

$$\dot{p}_i \;=\; -\frac{\partial H}{\partial q_i} \;, \quad i = 1, \ldots, n \;. \tag{3b}$$

The Hamiltonian vector field X_H defined by these equations satisfies

$$i_{X_H} \lrcorner\, \omega \;:=\; \omega(X_H, \_) \;=\; \mathrm{d}H \tag{4}$$

where

$$\omega \;=\; \mathrm{d}q_1 \wedge \mathrm{d}p_1 \;+\; \ldots \;+\; \mathrm{d}q_n \wedge \mathrm{d}p_n \tag{5}$$

is the canonical *symplectic form* on $\mathbb{R}^{2n}$.

Hamiltonian mechanics becomes conceptually easier if one abstracts from well-chosen canonical co-ordinates and considers the phase space to be a *symplectic manifold* $(\mathcal{P}, \omega)$. A Hamiltonian function $H : \mathcal{P} \longrightarrow \mathbb{R}$ then defines by means of (4) the Hamiltonian vector field X_H on $\mathcal{P}$ in a co-ordinate free way. With Darboux's Theorem [1, 2, 24, 28] one can always return to canonical co-ordinates, locally around any given point of $\mathcal{P}$.

The flow $\varphi : \mathbb{R} \times \mathcal{P} \longrightarrow \mathcal{P}$ of a Hamiltonian vector field preserves the symplectic structure, *i.e.* $\varphi_t^* \omega = \omega$ [2, 24, 12, 28]. An immediate consequence is Liouville's Theorem that the phase space volume $\omega^n = \omega \wedge \ldots \wedge \omega$ is preserved as well. As a corollary one obtains Poincaré's Reccurrence Theorem.

Theorem (Poincaré). *Let $\Omega \subseteq \mathcal{P}$ be compact and invariant under φ_t. Then every neighbourhood U of every point $a \in \Omega$ has a trajectory that returns to U.*
The proof consists of "a walk in the park" – however small my shoes are, I am bound to step on my own trail if I walk forever in a park of finite size.

Hamiltonian systems have special properties that are untypical for general dissipative systems. An important aspect is that volume preservation excludes the existence of attractors. In dissipative systems the restriction of the flow to an attractor often allows to dramatically lower the dimension of the system; more generally one restricts with the same aim to the non-wandering set. In the present conservative context, if *e.g.* the energy level sets are compact then the non-wandering set consists of the whole phase space.

One speaks of a *simple mechanical system* [1, 2, 24] if the phase space $\mathcal{P} = T^*M$ is the cotangent bundle of a Riemannian manifold M, the configuration space, and the Hamiltonian $H = T + V$ is the sum of kinetic and potential energy. Furthermore, the kinetic energy

$$T(\alpha_q) \;=\; \langle \alpha_q \mid \alpha_q \rangle_q$$

is given by the Riemannian metric (evaluated at the base point $q \in M$ of $\alpha_q \in T_q^* M$) and the potential energy

$$V(\alpha_q) \;=\; V(q)$$

depends only on the configuration $q \in M$. The cotangent bundle has a *canonical* 1*–form* ϑ defined by

$$\vartheta(v_{\alpha_q}) \;=\; \alpha_q(T\pi(v_{\alpha_q})) \quad \text{for all } v_{\alpha_q} \in T_{\alpha_q} T^* M$$

where $\pi : T^*M \longrightarrow M$ is the bundle projection to the configuration space and $T\pi : TT^*M \longrightarrow TM$ its derivative, see figure 1. From ϑ the symplectic form is obtained as the exterior derivative $\omega = -\mathrm{d}\vartheta$.

Choosing any co-ordinates $q_1, \ldots, q_n$ on the configuration space M and completing them with the conjugate momenta $p_1, \ldots, p_n$ one automatically has a canonical co-ordinate

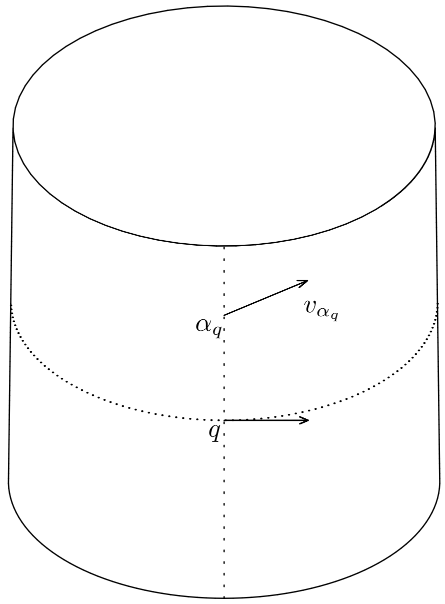

Figure 1: Projection of a vector v_{α_q} tangent to T^*M to the tangent space of M at $q = \pi(\alpha_q)$. The resulting tangent vector can be fed into $\alpha_q : T_qM \longrightarrow \mathbb{R}$.

system on the phase space $\mathcal{P} = T^*M$ in which the canonical 1–form reads

$$\vartheta \;=\; \sum_{i=1}^{n} p_i \mathrm{d}q_i$$

and the symplectic form is given by (5). This freedom of choosing any co-ordinate system is lost on the whole phase space if one insists on the equations of motion to be in canonical form (3).

A significant part of the classical literature [20, 35, 16, 2] is devoted to *generating functions*, a means to generate transformations that turn canonical co-ordinates into canonical co-ordinates, therefore called *canonical transformations*. A contemporary means to obtain canonical transformations is to use the time–1–flow φ_1 of a well-chosen Hamiltonian function as these are better suited for implementation on a computer. Since $\varphi_1^{-1} = \varphi_{-1}$ it is as simple (or complicated) to compute the inverse of such a transformation as it is to compute the transformation itself.

In a (not necessarily canonical) co-ordinate system $z_1, \ldots, z_{2n}$ one can use the *Poisson bracket*, defined by

$$\{F, H\} \;:=\; \omega(X_F, X_H) \ ,$$

to write down the equations of motion

$$\dot{z}_j \;=\; \{z_j, H\} \;=\; \sum_{k=1}^{2n} \frac{\partial H}{\partial z_k} \{z_j, z_k\}$$

which have the canonical form precisely when it is possible to write $(z_1, \ldots, z_n) = (q_1, \ldots, q_n)$ and $(z_{n+1}, \ldots, z_{2n}) = (p_1, \ldots, p_n)$ with Poisson bracket relations

$$\{q_i, q_j\} \;=\; 0 \;, \quad \{p_i, p_j\} \;=\; 0 \;, \quad \{q_i, p_j\} \;=\; \delta_{ij} \;.$$

While a symplectic manifold has necessarily even dimension, one can turn also manifolds of odd dimension into a Poisson space. An important example is the Poisson structure

$$\{F, H\}(z) \;:=\; \langle \nabla F(z) \times \nabla H(z) \mid z \rangle \tag{6}$$

on $\mathbb{R}^3$ (with its inner product $\langle x \mid y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$) for which the equations of motion read

$$\dot{z} \;=\; \nabla H(z) \times z \;. \tag{7}$$

The function $R(z) = \frac{1}{2}\langle z \mid z \rangle$ is a *Casimir function* of the Poisson structure as it is invariant under every Hamiltonian flow on $\mathbb{R}^3$ since $\{R, H\} = 0$ for all (Hamiltonian) functions $H : \mathbb{R}^3 \longrightarrow \mathbb{R}$. This fibrates $\mathbb{R}^3$ into invariant spheres $\{R = \frac{1}{2}\rho^2\}$ of radius ρ, with a singular sphere reduced to a point at the origin. The area element σ makes each sphere S^2_ρ a symplectic manifold, and the restriction of the Poisson bracket (6) to S^2_ρ satisfies

$$\{F, H\} \;=\; \frac{1}{\rho^2}\sigma(X_F, X_H) \;.$$

A similar (though in general slightly more complicated) decomposition into symplectic leaves exists for every Poisson manifold [2, 24, 30].

A first measure for the complexity of a Hamiltonian system is the number of degrees of freedom. For a simple mechanical system this is the dimension of the configuration space, and accordingly one defines this number as $\frac{1}{2}\dim\mathcal{P}$ for a symplectic manifold $\mathcal{P}$. On a Poisson space this number is related to the rank of the Poisson bracket, given by $\operatorname{rank}(\{z_j, z_k\})_{j,k=1,\ldots,m}$ in local co-ordinates $z_1, \ldots, z_m$. This even number is upper semi-continuous and coincides at each point with the dimension of the symplectic leaf passing through that point. Hence, the number of degrees of freedom is defined as one half of the maximal rank of the Poisson structure.

3 Systems in One Degree Of Freedom

For the simple mechanical systems consisting of a 1–dimensional particle with kinetic energy (2) moving in a potential $V = V(q)$ the trajectories coincide with the level curves $\{H = h\}$ of $H = T + V$. Finding the time parametrisations of the trajectories is thereby reduced to the quadrature

$$\int \frac{\mathrm{d}q}{\sqrt{2m(h - V(q))}} \;,$$

and correspondingly one speaks of an integrable system. Where this time parametrisation is not important one can always multiply the Hamiltonian by a strictly positive function, in the present one-degree-of-freedom situation one has then the extra choice [8, 17] of performing any co-ordinate transformation and still writing the equations in canonical form (with respect to the transformed Hamiltonian function).

The phase portraits in a Poisson space $\mathcal{P}$ with one degree of freedom can be obtained by intersecting the level sets of the energy with the symplectic leaves. In particular, a point where the rank of the Poisson structure drops from 2 to 0 is an equilibrium for every Hamiltonian system on $\mathcal{P}$. Equilibria on regular symplectic leaves are called regular equilibria.

There are four types of linearizations of regular equilibria in one degree of freedom. In canonical co-ordinates these are given by quadratic Hamiltonian functions H_2.

Elliptic. $H_2(q,p) = \frac{\varpi}{2}(p^2+q^2)$, the nearby motion is periodic with frequency close to ϖ. This is the harmonic oscillator.

Hyperbolic. $H_2(q,p) = \frac{a}{2}(p^2-q^2)$, the equilibrium is a *saddle*.

Parabolic. $H_2(q,p) = \frac{a}{2}p^2$, the higher order terms of the Hamiltonian around the equilibrium determine the character of the flow.

Vanishing. $H_2(q,p) \equiv 0$, the linearization contains no information and the flow is given by the higher order terms.

Generic Hamiltonian systems on a symplectic surface have only elliptic and hyperbolic equilibria, see figure 2. Where the system depends on external parameters or is defined on a family of symplectic leaves one may also encounter parabolic equilibria. The phenomenon of vanishing linearization is of co-dimension three.

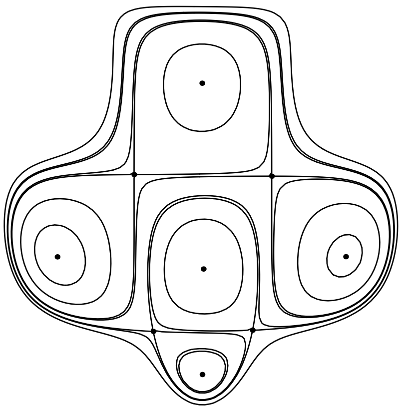

Figure 2: Typical recurrent flow in one degree of freedom

As an example consider the phase space $\mathbb{R}^3$ with Poisson structure (6) and Hamiltonian energy function

$$H(z) \;=\; \sum_{i=1}^{3} \frac{a_i}{2} z_i^2 + b_i z_i$$

depending on the external paremeters $(a, b) \in \mathbb{R}^3 \times \mathbb{R}^3$. On each sphere S^2_ρ the points with minimal and maximal energy are elliptic equilibria. All equilibria occur where S^2_ρ touches the quadric $\{H = h\}$ of constant energy. Where this happens with coinciding curvature along a line the equilibrium is parabolic. For $a, b \in \mathbb{R}^3$ in general position a *centre-saddle bifurcation* occurs as ρ passes through such a value. If *e.g.* the three conditions $a_1 = a_2$, $b_1 = b_2 = 0$ hold, then the curvatures at

$$z \;=\; \begin{pmatrix} 0 \\ 0 \\ -\frac{b_i}{a_i} \end{pmatrix}$$

coincide along all lines and the linearization at this equilibrium vanishes. The origin $z = 0$ is for all values of $a, b \in \mathbb{R}^3$ a (singular) equilibrium.

4 Systems in Two Degrees Of Freedom

While all recurrent motion is periodic in one degree of freedom, the flow can have a stochastic (or chaotic) character already in two degrees of freedom. The level sets $\{H = h\}$ of the Hamiltonian are now 3–dimensional invariant hypersurfaces, and complicated dynamics is known to be possible from dimension three on. Still, being Hamiltonian imposes certain restrictions.

Leaving aside equilibria with vanishing eigenvalues, there are the following types of linearizations of regular equilibria in two degrees of freedom, with quadratic Hamiltonian H_2 in canonical co-ordinates.

Elliptic. The quadratic part

$$H_2(q, p) \;=\; \alpha \frac{p_1^2 + q_1^2}{2} \;+\; \varpi \frac{p_2^2 + q_2^2}{2} \;, \quad \text{with } |\alpha| \le |\varpi|, \tag{8}$$

is the superposition of two harmonic oscillators. For $\alpha/\varpi \in \mathbb{Q}$ the motion is periodic, but for irrational frequency ratio the trajectories spin densely around invariant tori. In the case $\varpi = -\alpha$ of 1:−1 resonance one can add a nilpotent part

$$\frac{p_1^2 + q_1^2 \;+\; p_2^2 + q_2^2}{4} \;-\; \frac{p_1 q_2 + p_2 q_1}{2}$$

whence the linear flow becomes unbounded.

Hypo-elliptic. The linear vector field with quadratic Hamiltonian

$$H_2(q,p) \;=\; a\,\frac{p_1^2-q_1^2}{2} \;+\; \varpi\,\frac{p_2^2+q_2^2}{2}$$

has one pair of eigenvalues on the real axis and one pair of eigenvalues on the imaginary axis. One also speaks of a saddle-centre equilibrium.

Hyperbolic. The linearization has no eigenvalues on the imaginary axis. In case the spectrum consists of two pairs of real eigenvalues the standard form of the Hamiltonian is

$$H_2(q,p) \;=\; a\,\frac{p_1^2-q_1^2}{2} \;+\; b\,\frac{p_2^2-q_2^2}{2}$$

and one speaks of a saddle-saddle equilibrium (or real saddle). In the alternative case of a complex quartet $\pm\alpha \pm \mathrm{i}\varpi$ one has

$$H_2(q,p) \;=\; \alpha(p_1q_1+p_2q_2) \;+\; \varpi(p_1q_2-p_2q_1)$$

and speaks of a *focus-focus* equilibrium (or complex saddle), since the flow on the stable manifold spirals into the equilibrium and the flow on the unstable manifold spirals away from the equilibrium.

The 1:−1 resonance is special in that it typically triggers off a *Hamiltonian Hopf bifurcation* during which an elliptic equilibrium becomes hyperbolic, of focus-focus type. Other bifurcations occur where eigenvalues vanish under variation of an external parameter (possibly parametrising 4–dimensional symplectic leaves in a higher-dimensional Poisson space).

In generic Hamiltonian systems all equilibria are isolated and the periodic orbits form 1–parameter families. In one degree of freedom these families extend between (elliptic) equilibria and/or the *(un)stable manifolds* of (hyperbolic) equilibria, see figure 2. In two degrees of freedom the 1–parameter families of periodic orbits are special solutions and one may hope that they similarly organize the dynamics. The normal linear behaviour of a periodic orbit with period τ is determined by its Floquet multipliers, the eigenvalues of the linearization $D\varphi_\tau$ of the flow.

The simplest way to find periodic orbits in a Hamiltonian system, *e.g.* as a starting point for continuation, is to look for the *normal modes* of an equilibrium.

Theorem (Liapunov). *Let $\lambda_\pm = \pm\mathrm{i}\alpha$ be a purely imaginary pair of eigenvalues of the linearization $A = DX_H(z)$ at an equilibrium $z \in \mathcal{P}$ of a Hamiltonian system X_H on a symplectic manifold $\mathcal{P}$ for which no integer multiple $k\lambda_\pm$, $k \in \mathbb{N}$, is an(other) eigenvalue of A. Then X_H admits a 1–parameter familiy $(\gamma_\varepsilon)_{0<\varepsilon\leq\varepsilon_0}$ of periodic orbits that approach z as $\varepsilon \longrightarrow 0$ with periods $T_\varepsilon \xrightarrow{\varepsilon\to 0} 2\pi/\alpha$. The union*

$$\{z\} \;\cup \bigcup_{0<\varepsilon\leq\varepsilon_0} \gamma_\varepsilon$$

forms a smooth 2–dimensional submanifold of $\mathcal{P}$ with boundary γ_{ε_0} that is diffeomorphic to the closed disk in $\mathbb{R}^2$.

For a proof see [1, 25, 28]; this result immediately generalizes to n degrees of freedom.

If the Hessean $D^2H(z)$ is positive (or negative) definite, the non-resonance condition $\lambda \neq k\lambda_\pm$ for the remaining eigenvalues of A can be dropped, but the resulting families of periodic orbits may no longer form manifolds through z and only form cones with z as vertex instead. In two degrees of freedom the 1:-1 and 1:-2 resonances provide examples where the normal mode is lacking, but for 1:$-k$ resonant equilibria with $k \geq 3$ it turns out to be generic for the normal modes to exist.

The normal mode of a hypo-elliptic equilibrium in two degrees of freedom is a hyperbolic periodic orbit with a real pair $a, \frac{1}{a} \neq \pm 1$ of Floquet multipliers. The periodic orbits born at an elliptic equilibrium also inherit their normal behaviour from the "second" pair of eigenvalues and have a pair $\mathrm{e}^{\pm \mathrm{i}\varpi} \neq \pm 1$ of Floquet multipliers on the unit circle. Coupling two generic systems with one degree of freedom with a sufficiently weak interaction yields periodic orbits that are globally organized in this fashion, each periodic orbit being the superposition of a centre or a saddle in one subsystem and a periodic orbit in the other subsystem.

The periodic orbits that are neither elliptic nor hyperbolic are isolated within the 1–parameter families of periodic orbits. The arcs in between consist either completely of elliptic or completely of hyperbolic periodic orbits and may be parametrised by the values of the energy. The following types of bifurcations are triggered off by parabolic periodic orbits, for which the *normal linear* behaviour is governed by a Floquet matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}-1&1\\0&-1\end{smallmatrix}\right)$.

The periodic centre-saddle bifurcation. Under variation of the energy an elliptic and a hyperbolic periodic orbit meet at a parabolic periodic orbit with all Floquet multipliers equal to 1. No periodic orbit remains when further in(or de)creasing the energy. In a suitable projection

$$(H, I): \ \mathcal{P} \ \longrightarrow \ \mathbb{R}^2 \ \xrightarrow{H} \ \mathbb{R}$$

the family of periodic orbits forms a fold. See also figure 4 below.

The Hamiltonian period-doubling bifurcation. Under variation of the energy an elliptic periodic orbit turns hyperbolic (or *vice versa*) when passing through a parabolic periodic orbit with Floquet multipliers -1. Furthermore a family of periodic orbits with twice the period emerges from the parabolic periodic orbit, inheriting the normal linear behaviour from the initial periodic orbit. See also figure 5 below.

For proofs of this and the following see [25, 22].

In generic systems with two degrees of freedom these are the only occuring bifurcations of periodic orbits. In three and more degrees of freedom there is "more space" in the normal direction and also periodic Hamiltonian Hopf bifurcations may occur. A prominent example is the gyroscopic stabilization of the rotating Lagrange top "standing up", cf. [2, 24, 12].

Along arcs of elliptic periodic orbits the pair $e^{\pm i\varpi}$ of Floquet multipliers passes regularly through roots of unity. Generically this happens on a dense set of parametrising energy values, but for fixed denominator ℓ in $e^{\pm i\varpi} = e^{\pm 2\pi i k/\ell}$ the corresponding energy values are again isolated. The cases $\ell = 1$ and $\ell = 2$ correspond to the above bifurcations so the "first" case is $\ell = 3$.

Two arcs of hyperbolic periodic orbits emerge at elliptic orbits with Floquet multipliers $e^{\pm 2\pi i k/3}$, both with three times the period. One extends for lower and the other for higher energy values. The periodic orbit with Floquet multipliers $e^{\pm 2\pi i k/3}$ momentarily loses its (normal) stability due to these approaching unstable orbits.

In the case of Floquet multipliers $e^{\pm 2\pi i k/\ell}$ with $\ell \geq 5$ again two arcs of periodic orbits with ℓ times the period emerge, but now one is elliptic and the other hyperbolic, and they both extend to the same side of energy values (either lower or higher). Furthermore there is no momentary loss of (normal) stability as these families detach. For $\ell = 4$ both the $\ell = 3$ and the $\ell \geq 5$ scenarios can occur.

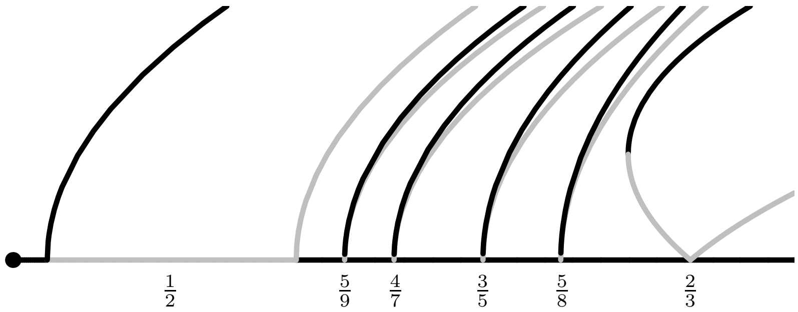

Figure 3: Subharmonic branching along a normal mode of an elliptic equilibrium in a response diagram with axes along normal frequency and amplitude. Where normal and internal frequency of the periodic orbit emanating from the equilibrium have a ratio $\varpi = 2\pi\frac{k}{\ell}$ with integers $k, \ell \in \mathbb{Z}$ two periodic orbits with ℓ times the period branch off, one elliptic and one hyperbolic (shown in grey). For $\ell = 3$ the distance between these widens as the two periodic orbits come into existence in a 3–periodic centre-saddle bifurcation "before" the resonance after which the hyperbolic 3–periodic orbit passes through the normal mode at the $\frac{2}{3}$–resonance (shown reflected as the vertical depicts the amplitude). For $\ell = 2$ the resonance leads to a gap within the elliptic normal mode, filled by hyperbolic periodic orbits and with boundaries marked by two Hamiltonian period-doubling bifurcations.

It is at isolated values of the energy that the two arcs of $(\ell \geq 5)$–times periodic orbits (or $(\ell = 4)$–times where appropriate) change from extending along lower energy values to extending along higher energy values. For definiteness let us consider a sub-arc between such values where all these exist for higher energy values. Where this sub-arc contains periodic orbits with Floquet multipliers $e^{\pm 2\pi i k/3}$ (or $e^{\pm 2\pi i k/4}$ where appropriate) the family of 3–times periodic orbits that extends for lower energy values typically vanishes in a nearby periodic centre–saddle bifurcation. The family of elliptic periodic orbits born in this bifurcation then extends along higher energy values as well. A possible arc of elliptic periodic orbits is sketched in figure 3. For more details on periodic solutions of Hamiltonian systems see [33].

To visualize the flow on the 3–dimensional *energy shells* $\{H = h\}$ one uses iso-energetic Poincaré-sections, *i.e.* surfaces $\Sigma_h \subset \{H = h\}$ that are everywhere transverse to the vector field X_H. For recurrent points $z \in \Sigma_h$ the (first) return time

$$\tau(z) \quad := \quad \min \{ \, T > 0 \mid \varphi_T(z) \in \Sigma_h \, \}$$

allows to define the *iso-energetic Poincaré-mapping*

$$\begin{array}{rccc} P_h : & \Sigma_h & \longrightarrow & \Sigma_h \\ & z & \mapsto & \varphi_{\tau(z)}(z) \end{array} .$$

The phase space volume $\omega^2 = \omega \wedge \omega$ coming from the symplectic structure induces an area element σ on Σ_h that is preserved by P_h. On the other hand, every area-preserving mapping can be realized as an iso-energetic Poincaré-mapping of a Hamiltonian system.

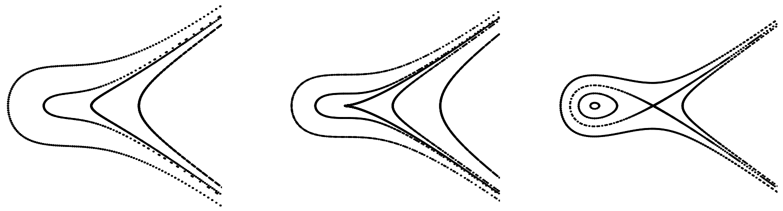

Figure 4: Centre-saddle bifurcation in an area-preserving mapping.

A periodic orbit γ of X_H with energy h corresponds to a fixed point $x \in \Sigma_h$ (or to a periodic point of period $k \in \mathbb{N}$ if γ has $k - 1$ intersections with Σ_h before returning to x). Two of the Floquet multipliers of γ are equal to 1, reflecting that periodic orbits form 1–parameter families in Hamiltonian systems and that moving the initial condition within γ yields that same periodic orbit with a translated time parametrisation. The remaining two Floquet multipliers are the eigenvalues of the linearization $DP_h(z)$ of the iso-energetic Poincaré-mapping at a fixed point z.

Because of area-preservation $\det DP_h(z) = 1$, so one eigenvalue of $DP_h(z)$ has to be the inverse of the other eigenvalue. For an elliptic periodic orbit both eigenvalues lie on the unit circle where the inverse equals the complex conjugate. For hyperbolic γ both eigenvalues are real and one sometimes makes the distinction between the direct hyperbolic case of positive eigenvalues and the inverse hyperbolic case of negative eigenvalues. See figure 4 for the centre-saddle bifurcation triggered off by the double eigenvalue $+1$ and figure 5 for the period-doubling bifurcation triggered off by the double eigenvalue -1.

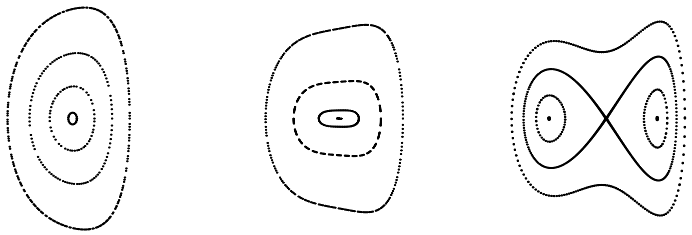

Figure 5: Period-doubling bifurcation in an area-preserving mapping.

5 Symmetry Reduction

When a dynamical system admits a symmetry group it is possible to simplify the dynamics. This reduction process is especially rewarding in Hamiltonian systems, where Noether's theorem yields for every continuous symmetry a conserved quantity. One even has the choice between first fixing the values of the conserved quantities and then reducing what is left of the symmetry, or first reducing the symmetry and then fixing the remaining conserved quantities.

Let $H \in C^\infty(\mathcal{P})$ be a Hamiltonian function on the symplectic manifold $(\mathcal{P}, \omega)$ and G a compact Lie group (*i.e.* the group operation is smooth) with Lie algebra $\mathfrak{g}$ (the tangent space T_eG at the neutral element $e \in G$, provided with the Lie bracket that measures the non-commutativity of the group operation). Results for more general groups do exist, but some kind of compactness, *e.g.* that the *group action* be proper, is always needed. Assume that the group action

$$\begin{array}{ccc} G \times \mathcal{P} & \longrightarrow & \mathcal{P} \\ (g, z) & \mapsto & gz \end{array} \tag{9}$$

preserves both the Hamiltonian function H and the symplectic form ω. Then G also preserves the Hamiltonian vector field X_H and the resulting flow commutes with the group action (9), *i.e.* $\varphi_t \circ g = g \circ \varphi_t$ for all $(t, g) \in \mathbb{R} \times G$. In fact, this allows to combine the flow $\varphi : \mathbb{R} \times \mathcal{P} \longrightarrow \mathcal{P}$ of X_H and the action (9) to the action

$$\begin{array}{ccc} (\mathbb{R} \times G) \times \mathcal{P} & \longrightarrow & \mathcal{P} \\ ((t, g), z) & \mapsto & \varphi_t(gz) \end{array} \tag{10}$$

of the Lie group $\mathbb{R} \times G$ on $\mathcal{P}$.

Reduction aims to find a phase space of smaller dimension on which the dynamics can be studied. Identifying points $z \sim gz$ that are transformed into each other by group elements leads to the quotient space $\mathcal{P}/_G$ with canonical projection

$$\mathcal{P} \longrightarrow \mathcal{P}/_G \; .$$

As $H(z) = H(gz)$ for all $g \in G$ the Hamiltonian induces a function on $\mathcal{P}/_G$, again denoted by H. However, the symplectic structure on $\mathcal{P}$ does not induce a symplectic structure on the quotient space. What can be transferred from $\mathcal{P}$ to $\mathcal{P}/_G$ is the Poisson structure.

The symplectic form is preserved by G, whence the mappings $z \mapsto gz$ are canonical transformations, satisfying

$$\{F \circ g, H \circ g\} \;=\; \{F, H\} \circ g \qquad \text{for all } F, H \in C^\infty(\mathcal{P}).$$

Thus, the Poisson bracket of G–invariant functions is again G–invariant, defining a Poisson bracket on $\mathcal{P}/_G$. The Casimir functions on this Poisson space correspond to those G–invariant functions on $\mathcal{P}$ that are conserved quantities for every G–invariant Hamiltonian function.

As an example consider the free rigid body with a fixed point, subject only to its own inertia. The configuration space is $SO(3)$, all (rigid) rotations about the fixed point, with a Riemannian metric provided by the mass distribution. On the phase space $T^*SO(3)$ the Hamiltonian $H = T$ is given by the resulting kinetic energy.

The mass distribution and hence the Hamiltonian are invariant under rotations, making $SO(3)$ a symmetry group. Using the left trivialization

$$\begin{array}{rccc} \lambda: & T^*SO(3) & \longrightarrow & SO(3) \times T_e^*SO(3) \\ & \alpha_g & \mapsto & (g, T_e^*L_g(\alpha_g)) \end{array}$$

defined by means of the left translation

$$\begin{array}{rccc} L_g: & SO(3) & \longrightarrow & SO(3) \\ & h & \mapsto & gh \end{array}$$

the group action reads

$$\begin{array}{ccc} SO(3) \times (SO(3) \times \mathbb{R}^3) & \longrightarrow & SO(3) \times \mathbb{R}^3 \\ (g, (h, \ell)) & \mapsto & (gh, \ell) \end{array}$$

and reveals $T^*SO(3)/_{SO(3)} \cong \mathbb{R}^3$. Here $\ell \in \mathbb{R}^3 \cong \mathfrak{so}(3)^* = T_e^*SO(3)$ consists of the three components of the angular momentum with respect to a set of axes fixed in the body, whence one calls $\lambda(\alpha_g) = (g, \ell)$ body co-ordinates. Choosing the body set of axes along the principal axes of the rigid body the Hamiltonian takes the form

$$H(g, \ell) \;=\; H(\ell) \;=\; \frac{\ell_1^2}{2I_1} + \frac{\ell_2^2}{2I_2} + \frac{\ell_3^2}{2I_3}$$

where I_1, I_2, I_3 are the principal moments of inertia. The Poisson bracket relations inherited from $T^*SO(3)$ are

$$\{\ell_1, \ell_2\} \;=\; -\ell_3 \;, \quad \{\ell_2, \ell_3\} \;=\; -\ell_1 \;, \quad \{\ell_3, \ell_1\} \;=\; -\ell_2 \tag{11}$$

whence the Poisson structure on $\mathbb{R}^3$ is almost (6) considered in Section 2, differing only by a minus sign. The Casimir function $R(\ell) = \frac{1}{2}(\ell_1^2 + \ell_2^2 + \ell_3^2)$ measures (half of the square of) the length of the angular momentum, fixing this conserved quantity yields the symplectic leaves of the quotient space $\mathbb{R}^3$.

The alternative approach to symmetric Hamiltonian systems first fixes the conserved quantities. For $\xi \in \mathfrak{g}$ the 1–prameter subgroup $\{\exp(s\xi) \mid s \in \mathbb{R}\}$ of G yields a conserved quantity

$$J^\xi : \mathcal{P} \longrightarrow \mathbb{R} \tag{12}$$

by Noether's theorem, cf. [2, 8, 25]. The Hamiltonian vector field X_{J^ξ} has the flow $(s, z) \mapsto \exp(s\xi) \cdot z$ provided by the 1–parameter subgroup.

Here and from now on the assumption is made that the (mild) conditions for Noether's theorem are fulfilled, making (9) a Hamiltonian group action. What has to be avoided is that the flow provided by the 1–parameter subgroup is only locally Hamiltonian, see [1, 30, 31] for more details.

In the example of the free rigid body a conserved quantity (12) is the component of the angular momentum along an axis fixed in space. Collecting these conserved quantities by means of

$$\begin{array}{rcccccc} J: & \mathcal{P} & \longrightarrow & \mathfrak{g}^* & & & \\ & z & \mapsto & J(z): & \mathfrak{g} & \longrightarrow & \mathbb{R} \\ & & & & \xi & \mapsto & J^\xi(z) \end{array} \tag{13}$$

yields the *momentum mapping*. For the rigid body this amounts to fixing a set of axes in space and assigning to $\alpha_g \in T_g^*SO(3)$ the three components $\mu \in \mathbb{R}^3 \cong \mathfrak{so}(3)^*$ of the angular momentum with respect to these axes. Correspondingly, the right trivialization reads

$$\begin{array}{rccc} \varrho: & T^*SO(3) & \longrightarrow & SO(3) \times \mathbb{R}^3 \\ & \alpha_g & \mapsto & (g, \mu) \end{array}$$

where $\mu = T_e^* R_g(\alpha_g)$ is obtained differentiating the right translation $R_g(h) = hg$. In space co-ordinates $\varrho(\alpha_g) = (g, \mu)$ the group action takes the form

$$\begin{array}{ccc} SO(3) \times (SO(3) \times \mathbb{R}^3) & \longrightarrow & SO(3) \times \mathbb{R}^3 \\ (g, (h, \mu)) & \mapsto & (hg, g(\mu)) \end{array}$$

and one sees that the momentum mapping

$$SO(3) \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \tag{14}$$

intertwines between the $SO(3)$–actions on the phase space and on $\mathbb{R}^3$.

To formulate the corresponding equivariance property of the momentum mapping in the general case the co-adjoint action

$$\begin{array}{ccc} G \times \mathfrak{g}^* & \longrightarrow & \mathfrak{g}^* \\ (g, \mu) & \mapsto & \mathrm{Ad}^*_{g^{-1}}(\mu) = \mu \circ \mathrm{Ad}_g^{-1} \end{array} \tag{15}$$

is needed, here $\mathrm{Ad}_g : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the derivative $T_e(L_g R_{g^{-1}})$ of the inner automorphism $h \mapsto ghg^{-1} = L_g R_{g^{-1}} h$ at the neutral element. The momentum mapping J of the Hamiltonian group action (9) is now called equivariant if it intertwines between (9) and (15), *i.e.*

$$J(gz) \;=\; \mathrm{Ad}^*_{g^{-1}} J(z) \qquad \text{for all } (g, z) \in G \times \mathcal{P}.$$

In this case the assignment $\xi \mapsto J^\xi \in C^\infty(\mathcal{P})$ of conserved quantities to Lie algebra elements $\xi \in \mathfrak{g}$ turns out to be a Lie algebra homomorphism, cf. [1, 24, 31]. Providing $\mathfrak{g}^*$ with the *Lie–Poisson bracket* (defined in terms of the commutator on $\mathfrak{g}$) makes (13) a canonical mapping, see [30, 31] for more details. The identification $\mathfrak{so}(3)^* \cong \mathbb{R}^3$ turns the Lie–Poisson bracket into (6) and indeed $\{\mu_1, \mu_2\} = \mu_3$ etc. for the spatial representation of the angular momentum. Working with right group actions (instead of acting from the left) would yield a minus sign here and a plus sign in (11).

Let $\mu \in \mathfrak{g}^*$ be a regular value of the equivariant momentum mapping (13). Then $J^{-1}(\mu)$ is a submanifold of $\mathcal{P}$ and the (compact) isotropy subgroup

$$G_\mu \quad = \quad \left\{ \, g \in G \mid \mathrm{Ad}^*_{g^{-1}}(\mu) = \mu \, \right\}$$

of the co-adjoint action leaves this manifold invariant as

$$J(gz) \quad = \quad \mathrm{Ad}^*_{g^{-1}} J(z) \quad = \quad \mu \qquad \text{for all } (g, z) \in G_\mu \times J^{-1}(\mu)$$

whence the restriction

$$\begin{array}{ccc} G_\mu \times J^{-1}(\mu) & \longrightarrow & J^{-1}(\mu) \\ (g, z) & \mapsto & gz \end{array} \tag{16}$$

defines a Lie group action on the manifold $J^{-1}(\mu)$. Passing to the quotient $J^{-1}(\mu)/_{G_\mu}$ is called symplectic point reduction.

Theorem (Marsden and Weinstein). *Let μ be a regular value of J and assume that the action* (16) *is free. Then the reduced phase space $\mathcal{P}_\mu = J^{-1}(\mu)/_{G_\mu}$ is a symplectic manifold, with ω_μ uniquely determined by $\omega_{|J^{-1}(\mu)} = \omega_\mu \circ \pi_\mu$ where π_μ is the quotient projection. A G–invariant Hamiltonian function $H \in C^\infty(\mathcal{P})$ induces $H_\mu \in C^\infty(\mathcal{P}_\mu)$ and π_μ intertwines between the flows of X_H and X_{H_μ}.*

For a proof see [1, 2, 12, 31].

In the example of the free rigid body the momentum mapping $SO(3) \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is surjective and all values are regular, with inverse images $SO(3) \times \{\mu\}$. For $\mu \neq 0$ the isotropy subgroup $G_\mu \cong S^1$ consists of all rotations in $\mathbb{R}^3$ about the axis along μ and the reduced phase space

$$SO(3) \times \{\mu\}/_{G_\mu} \quad \cong \quad SO(3)/_{S^1} \quad \cong \quad S^2$$

can be identified with the sphere of radius $|\mu|$ as

$$\begin{array}{ccc} SO(3) \times \{\mu\} & \longrightarrow & S^2_{|\mu|} \\ (g, \mu) & \mapsto & g(\mu) \end{array}$$

performs the reduction. The symplectic form on $S^2_{|\mu|}$ is a multiple

$$\frac{-1}{|\mu|^2} \cdot \sigma$$

of the area element σ (the minus sign is the "same" as in (11)). The isotropy subgroup of $\mu = 0$ is the whole $SO(3)$ and the quotient space consists of the origin $\ell = g(0)$ in $\mathbb{R}^3$.

As illustrated in this example, symplectic point reduction provides the symplectic leaves of the Poisson space $\mathcal{P}/_G$ obtained by directly reducing the action (9) on $\mathcal{P}$. A link between these two procedures of symmetry reduction is symplectic orbit reduction. By equivariance of (13) the inverse image $J^{-1}(G(\mu))$ of the co-adjoint orbit

$$G(\mu) \quad = \quad \left\{ \operatorname{Ad}^*_{g^{-1}}\mu \mid g \in G \right\} \tag{17}$$

is invariant under the whole group G as

$$J(gz) \quad = \quad \operatorname{Ad}^*_{g^{-1}} J(z) \quad \in \quad G(\mu) \qquad \text{for all } (g, z) \in G \times J^{-1}(G(\mu)) \ .$$

Hence, the restriction of (9) to $J^{-1}(G(\mu))$ defines a group action and the quotient spaces $J^{-1}(G(\mu))/_G$ form a partition of $\mathcal{P}/_G$ (since

$$G \quad = \quad \bigcup_{\mu \in \mathfrak{g}^*} G(\mu)$$

and two orbits are either disjoint or equal). On the other hand, the two quotients $J^{-1}(G(\mu))/_G$ and $J^{-1}(\mu)/_{G_\mu}$ are symplectomorphic, see [30, 31] for more details. In the example of the free rigid body, the inverse image of the co-adjoint orbit $G(\mu)$ is the sphere bundle in $T^*SO(3)$ of radius $|\mu|$ which is given by $SO(3) \times S^2_{|\mu|}$ both in body and space co-ordinates.

The group action (9) is free if every $z \in \mathcal{P}$ is moved away from z by every group element, *i.e.* all isotropy subgroups

$$G_z \quad = \quad \{ \, g \in G \mid gz = z \, \}$$

of (9) (not of the co-adjoint action) are trivial. For free actions the Poisson space $\mathcal{P}/_G$ is a Poisson manifold. Non-trivial isotropy groups typically lead to singularities. This can already be seen in the example of S^1 acting on $\mathbb{R}^4 = T^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ by simultaneous rotation in both planes, for the general theory see [12, 30] and references therein.

The group $S^1 = SO(2)$ of planar rotations is commutative, whence the isotropy subgroups of the co-adjoint action coincide with the whole group and the generator $l = q_1 p_2 - q_2 p_1$ of the action

$$\begin{pmatrix} q \\ p \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \cos\rho & -\sin\rho & 0 & 0 \\ \sin\rho & \cos\rho & 0 & 0 \\ 0 & 0 & \cos\rho & -\sin\rho \\ 0 & 0 & \sin\rho & \cos\rho \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \tag{18}$$

may be fixed both before and after passing to the quotient. The ring of S^1–invariant functions is generated by the polynomials

$$x \;=\; \frac{q_1^2 + q_2^2}{2} \;, \quad y \;=\; \frac{p_1^2 + p_2^2}{2} \;, \quad z \;=\; p_1 q_1 \,+\, p_2 q_2 \quad \text{and } \ l$$

which are restricted by the relations

$$x \geq 0 \;, \quad y \geq 0 \;, \quad \text{and} \quad R_l(x,y,z) \;=\; 0$$

where

$$R_l(x,y,z) \;=\; \frac{1}{2}z^2 \;-\; 2xy \;+\; \frac{1}{2}l^2$$

defines the *syzygy* between these invariants. In particular, the S^1–invariant Hamiltonian function H may be written as a function $H = H_l(x,y,z)$.

This allows to refrain from local co-ordinates and use (x,y,z) as (global) variables on the reduced phase spaces

$$\mathcal{P}_l \;=\; \left\{ \, (x,y,z) \in \mathbb{R}^3 \mid x \geq 0 \,,\; y \geq 0 \,,\; R_l(x,y,z) = 0 \, \right\} \tag{19}$$

which foliate the Poisson space

$$\mathbb{R}^4/_{S^1} \;=\; \bigcup_{l \in \mathbb{R}} \mathcal{P}_l \times \{l\} \;\subseteq\; \mathbb{R}^4$$

with Poisson brackets given in table 1. Fixing the Casimir function l, the resulting Poisson structure on $\mathbb{R}^3$ can also be written as

$$\{F,H\} \;=\; \langle \nabla F \times \nabla H \mid \nabla R_l \rangle \tag{20}$$

(generalizing (6)), revealing R_l to be the second Casimir function.

Table 1: Poisson structure on $\mathbb{R}^4$.

$\{\downarrow,\rightarrow\}$	x	y	z	l
x	0	z	$2x$	0
y	$-z$	0	$-2y$	0
z	$-2x$	$2y$	0	0
l	0	0	0	0

The fixed point $(q,p) = 0$ of (18) has the whole group S^1 as isotropy group, all other isotropy groups of (18) are trivial. Thus, for $l \neq 0$ the positive sheets $\mathcal{P}_l$ of the two-sheeted hyperboloids $R_l^{-1}(0)$ yield symplectic leaves of the Poisson space $\mathbb{R}^4/_{S^1}$. For $l = 0$ the reduced phase space $\mathcal{P}_0$ is *stratified* into two symplectic strata of dimensions 2 and 0, the positive part $x+y > 0$ of the double cone $R_0^{-1}(0)$ and the vertex of this cone, respectively. Hence, the vertex is automatically an equilibrium of the reduced Hamiltonian system.

Where an action (9) of a discrete group G preserves Hamiltonian H and symplectic form ω there is no resulting conserved quantity, but one still can pass to the quotient $\mathcal{P}/_G$. For instance, on $\mathbb{R}^2$ the Hamiltonians

$$H(q,p) \;=\; \frac{p^2}{2} \pm \frac{q^4}{24} \;+\; \frac{\lambda q^2}{2}$$

admit the symmetry group $G = \{\pm\mathrm{id}\}$ that also preserves the symplectic form $\mathrm{d}q \wedge \mathrm{d}p$. The reduced Poisson space is the cone (19) with $l = 0$.

A symmetry may preserve only the equations of motion, but neither the Hamiltonian function nor the symplectic form. An example on $\mathbb{R}^2$ is given by

$$H(q,p) \;=\; \frac{p^3}{6} + \frac{pq^3}{6} \;+\; \lambda p \;+\; \mu pq$$

satisfying $H(q,-p) = -H(q,p)$ and $\mathrm{d}q\wedge\mathrm{d}(-p) = -\mathrm{d}q\wedge\mathrm{d}p$. The quotient may be realized as

$$\mathbb{R}^2/_{\{p \mapsto -p\}} \;=\; \left\{\; (q,p) \in \mathbb{R}^2 \mid p \geq 0 \;\right\}$$

and inherits outside the boundary the symplectic structure $\mathrm{d}q\wedge\mathrm{d}p$. The q–axis is invariant under the flow, with equation of motion $\dot{q} = \partial H/\partial p$.

6 Integrable Systems

Let $(\mathcal{P},\omega)$ be a symplectic manifold of dimension $\dim \mathcal{P} = 2n$ and $H \in C^\infty(\mathcal{P})$ a Hamiltonian function. The Hamiltonian vector field X_H is completely (or Liouville) integrable if there are n functions $F_1,\ldots,F_n \in C^\infty(\mathcal{P})$ with $\{F_i, H\} = 0$, $i = 1,\ldots,n$ and $\{F_i, F_j\} = 0$, $i,j = 1,\ldots,n$ (the F_i are *integrals* in involution) that are functionally independent outside a set of measure zero [1] or on an open and dense set [12], *i.e.* the closed set

$$\left\{\; z \in \mathcal{P} \mid \det\big(X_{F_1}(z),\ldots,X_{F_n}(z)\big) = 0 \;\right\} \tag{21}$$

is small, *e.g.* of non-zero co-dimension. Examples abound, among them are all linear systems, all systems with one degree of freedom and uncoupled superpositions of completely integrable systems. Since one may choose $F_1 := H$ a system in two degrees of freedom is integrable as soon as it has a conserved quantity F_2. Where there is an S^1–symmetry, as in the (planar) two body problem and the geodesic flow on a surface of revolution, the momentum mapping $J =: F_2$ provides this conserved quantity.

The simplifying assumptions usually made when modelling *e.g.* a mechanical system often introduce extra symmetries. Consequently, some of the problems from classical mechanics, like the Lagrange top, turned out to be integrable. The continuous efforts of the 19th century lead to more integrable systems, like the geodesic flow on a triaxial ellipsoid and the Kovalevskaya top. Eventually it became clear that integrable systems are the exception and non-integrable systems are the rule, with as most prominent example the three body problem. However, the discovery of the Toda lattice renewed interest and the list of known integrable systems is still growing.

Theorem (Liouville). *Let the components of $F : \mathcal{P} \longrightarrow \mathbb{R}^n$ be n integrals in involution of the Hamiltonian $H = F_1$. For a regular value $c \in \operatorname{im} F \subseteq \mathbb{R}^n$ a compact connected component $\mathcal{R}_c$ of $F^{-1}(c)$ is an X_H–invariant manifold that is diffeomorphic to the n–torus $\mathbb{T}^n = \mathbb{R}^n/_{\mathbb{Z}^n}$. The subset $\mathcal{R}_c \subseteq \mathcal{P}$ has an open neighbourhood $\mathcal{U}$ on which X_H admits* *action angle variables* $(x, y) \in \mathbb{T}^n \times \mathbb{Y}$, $\mathbb{Y} \subseteq \mathbb{R}^n$ *open, i.e. the diffeomorphism*

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathbb{T}^n \times \mathbb{Y} \\ z & \mapsto & (x(z), y(z)) \end{array}$$

turns the symplectic structure ω *into* $\sum \mathrm{d}x_i \wedge \mathrm{d}y_i$ *and* $y : \mathcal{U} \longrightarrow \mathbb{Y}$ *factors through* $F : \mathcal{U} \longrightarrow \mathbb{R}^n$.

For a proof see [1, 2, 12, 28].

The last statement makes F independent of the angle variable x. In particular, $H = H(y)$ for the Hamiltonian function (though not $H = H(y_1)$ as in Hamilton–Jacobi theory) and the equations of motion read

$$\begin{aligned} \dot{x} &= \varpi(y) := DH(y) \\ \dot{y} &= 0 \end{aligned}$$

whence the flow $\varphi_t(x, y) = (x + t \cdot \varpi(y), y)$ is easily computed. Thus, constructing action angle variables of an integrable system is equivalent to explicitly solving the equations of motion. The term "completely integrable" indicates that this can be achieved by solving algebraic equations and indefinite integrals.

The invariant torus $\mathcal{R}_c$ is Lagrangean, *i.e.* an *isotropic* submanifold (the symplectic structure vanishes on $\mathcal{R}_c$) and maximal with that property. The dimension of *Lagrangean submanifolds* is always equal to the number of degrees of freedom. The flow on $\mathcal{R}_c$ is *conditionally periodic*, in the angle variable x it is parallel with frequency vector $\varpi(0)$, assuming that $y = 0 \in \mathbb{Y}$ is the action value of $\mathcal{R}_c$.

Denote by $C \subseteq \operatorname{im} F \subseteq \mathbb{R}^n$ the regular values of F, the complement of the image of (21) under F in $\operatorname{im} F$. Assume from now on that all level sets $F^{-1}(c)$, $c \in C$ are compact, *e.g.* because $F : \mathcal{P} \longrightarrow \mathbb{R}^n$ is proper or all energy level sets are compact. Putting $\mathcal{R} := F^{-1}(C)$ yields an n–torus bundle with fibres $\mathcal{R}_c$, and assuming that all $F^{-1}(c)$ are connected, the restriction

$$F : \mathcal{R} \longrightarrow C \tag{22}$$

can be used as projection mapping of this bundle. In case $\mathcal{P} = T^*M$ is the cotangent bundle of a configuration space M the $\mathbb{T}^n$–bundle is trivial if and only if there are global action angle variables. The first obstruction for a torus bundle to be trivial is *monodromy*. For instance, if the inverse image $\mathcal{S} = \mathcal{P} \setminus \mathcal{R}$ of the singular values of F contains an $(n-2)$–parameter family of invariant $(n-2)$–tori with normal linear behaviour of focus-focus type, then the bundle (22) has non-trivial monodromy.

Associated to (22) is the homology bundle $H_1(\mathcal{R}/C, \mathbb{Z})$ of $\mathcal{R}$ over C with fibre $H_1(\mathcal{R}_c, \mathbb{Z}) \cong \mathbb{Z}^n$, supplying the lattice that has to be divided out of $H_1(\mathcal{R}_c, \mathbb{R}) \cong \mathbb{R}^n$ to obtain the particular torus $\mathcal{R}_c$ at $c \in C$. For each path $\gamma : [0, 1] \longrightarrow C$ the lift to $H_1(\mathcal{R}, \mathbb{Z})$ yields a bijective orientation-preserving $\mathbb{Z}$–linear mapping between the lattices at $\gamma(0)$ and $\gamma(1)$. For a closed loop the two lattices coincide and the mapping is represented by a matrix

$\mathcal{M}(\gamma) \in SL_n(\mathbb{Z})$ with integer coefficients and determinant 1. This discrete object remains invariant under homotopies, and the resulting homomorphism

$$\mathcal{M} : \ \pi_1(C) \ \longrightarrow \ SL_n(\mathbb{Z})$$

from the first homotopy group of C into $SL_n(\mathbb{Z})$ is the monodromy of the n–torus bundle $\mathcal{R}$. In case $\mathcal{M} \equiv \mathrm{id}$ one can uniquely move a chosen basis $Y_1(c), \ldots, Y_n(c)$ of $H_1(\mathcal{R}_c, \mathbb{Z})$ at some chosen point $c \in C$ to all other fibres of the bundle $H_1(\mathcal{R}, \mathbb{Z})$, using paths with $\gamma(0) = c$. This yields Hamiltonian vector fields

$$X_{y_1} \ = \ Y_1 \circ F \ , \quad \ldots \ , \quad X_{y_n} \ = \ Y_n \circ F$$

on $\mathcal{R}$ for which the Hamiltonian functions $y_1, \ldots, y_n \in C^\infty(\mathcal{R})$ are global action variables. The remaining task is to also find global angle variables.

The global actions y_i define Hamiltonian vector fields with periodic flows. This yields a free Hamiltonian group action

$$\mathbb{T}^n \times \mathcal{R} \ \longrightarrow \ \mathcal{R}$$

and makes (22) a *principal* torus bundle.

Now principal torus bundles are classified by their *Chern class* in $H^2(C, \mathbb{Z}^n)$, a discrete invariant measuring the obstruction to the existence of a global section

$$\sigma : \ C \ \longrightarrow \ \mathcal{R}$$

(*i.e.* a mapping satisfying $F \circ \sigma = \mathrm{id}$). Such a global section yields in every fibre $\mathcal{R}_c$ the desired "origin" $x = 0$ of the angle variables and then one can let the group $\mathbb{T}^n$ act. For the resulting globally defined x, y to be action angle variables the equation

$$\mathrm{d}x \wedge \mathrm{d}y \ = \ \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i \ = \ \omega$$

has to be fulfilled, if necessary adapting the section σ accordingly. The cohomology class $[\sigma^* \omega] \in H^2(C, \mathbb{R})$ is a continuous invariant that vanishes if and only if this can be achieved. In the particular case that ω is exact, being defined by means of a canonical 1–form, one has $[\sigma^* \omega] = \sigma^* [\omega] = 0$. See [14] for more details.

A special but important situation occurs if the globally defined Hamiltonian vector fields

$$X_{F_2}, \ldots, X_{F_n} \tag{23}$$

already have periodic flows themselves (and only $F_1 = H$ has to be replaced by a local action y_1 in the construction of action angle variables). This defines an action

$$\mathbb{T}^{n-1} \times \mathcal{P} \ \longrightarrow \ \mathcal{P} \tag{24}$$

globally on the phase space (the restriction of which to the regular part $\mathcal{R}$ is free, giving $\mathcal{R}$ the structure of a principal $(n-1)$–torus bundle). Reducing this symmetry yields a one-degree-of-freedom problem on (the symplectic leaves of) the base space in

$$\sigma : \; \mathcal{P} \;\longrightarrow\; \mathcal{P}/_{\mathbb{T}^{n-1}}$$

with set Σ of singular values. Constructing action angle variables amounts to finding the time parametrisations of the (relative) periodic trajectories together with the areas encircled by these. Let $\Xi \subseteq \mathcal{P}/_{\mathbb{T}^{n-1}}$ denote the set of (relative) regular equilibria.

The singular part $\mathcal{S} = \mathcal{P}^{\setminus \mathcal{R}}$ is the union of the energy level sets containing points of $\sigma^{-1}(\Sigma)$ – here the $n-1$ vector fields (23) are linearly dependent – and those containing points of $\sigma^{-1}(\Xi)$ – where $X_H(z)$ is a linear combination of the linear independent vector fields (23). This makes $\mathcal{P}$ a *ramified torus bundle*, with regular fibres in $\mathcal{R}$ forming n–parameter families of Lagrangean n–tori, the distribution of which is determined by the collection $\mathcal{S}$ of singular fibres.

In case the action (24) is free, the set Σ is empty and σ makes the whole phase space $\mathcal{P}$ a principal $\mathbb{T}^{n-1}$–bundle. The isotropic $(n-1)$–tori reconstructed from Ξ behave similar to the periodic orbits in a two-degrees-of-freedom system described in Section 4. Thus, the families of Lagrangean tori shrink down to elliptic $(n-1)$–tori and are separated by the (un)stable manifolds of ($(n-1)$–parameter families of) hyperbolic $(n-1)$–tori, and the $(n-1)$–tori may undergo bifurcations. However, these bifurcations are more involved than those of periodic orbits for three reasons.

- Normal-internal resonances $\langle k \mid \varpi \rangle = \ell\alpha$ between the internal frequencies $\varpi_1, \ldots, \varpi_{n-1}$ and the *normal frequency* α of elliptic $(n-1)$–tori with $k \in \mathbb{Z}^{n-1}$ and $\ell = 1, 2$ are dense, triggering off quasi-periodic centre-saddle bifurcations and *frequency-halving bifurcations*.

- The occurring bifurcations may be degenerate and typically have co-dimensions up to $n-1$. This is a genericity condition on H, within the "universe" of integrable Hamiltonian systems on $\mathcal{P}$.

- It is also generic for heteroclinic bifurcations re-connecting the (un)stable manifolds of $(n-1)$–tori to involve parabolic $(n-1)$–tori.

In case the action (24) has non-trivial isotropy groups, the invariant tori reconstructed from Σ of dimensions $n-2, n-3, \ldots, 2, 1, 0$ (the latter two being periodic orbits and equilibria) and their (un)stable manifolds form $\sigma^{-1}(\Sigma)$.

The description of $\mathcal{P}$ as a ramified n–torus bundle still applies when some (or all) of the vector fields (23) do not have periodic flows and some (or all) of the action variables $y_2, \ldots, y_n$ are only locally defined. The Lagrangean tori in $\mathcal{R}$ form n–parameter families and the singular fibres in $\mathcal{S}$ determine how these families fit together. At the $(n-1)$–parameter families of elliptic $(n-1)$–tori the Lagrangean tori shrink down in the same way as periodic orbits shrink down to centres in one degree of freedom. Different families of Lagrangean tori are separated by $(n-1)$–parameter families of hyperbolic $(n-1)$–tori and their (un)stable manifolds.

This picture is repeated in how the $(n-1)$–tori shrink down to $(n-2)$–parameter families of (partially) elliptic $(n-2)$–tori and are separated by $(n-2)$–parameter families of (partially) hyperbolic $(n-2)$–tori and (part of) their (un)stable manifolds. Furthermore there are $(n-2)$–parameter families of hyperbolic $(n-2)$–tori of focus-focus type, together with their (un)stable manifolds these form *pinched n–tori*. These three ways lead to invariant tori of smaller and smaller dimension until ending up with 1–parameter families of periodic orbits and isolated equilibria.

Within the family of all $(n-1)$–tori one encounters quasi-periodic centre–saddle and frequency halving bifurcations along $(n-2)$–parameter subfamilies and more generally bifurcations of co–dimension $k \leq n-1$ along $(n-k-1)$–parameter subfamilies. Similarly, invariant $(n-2)$–tori undergoing a quasi-periodic Hamiltonian Hopf bifurcation form $(n-3)$–parameter families and the m–parameter families of invariant m–tori have $(m-k)$–parameter subfamilies where bifurcations of co-dimension $k \leq m$ occur. Such bifurcations are not restricted to those of *semi-local* type, but may also involve coinciding stable and unstable manifolds of different invariant tori. For instance, heteroclinic orbits between hyperbolic $(n-1)$–tori form $(2n-2)$–dimensional submanifolds of the phase space.

Let $F_1, \ldots, F_r \in C^\infty(\mathcal{P})$ be functionally independent integrals in involution. Fixing a point $z_0 \in \mathcal{P}$, the orbit $\mathbb{T}^r(z_0)$ of the $\mathbb{T}^r$–action generated by

$$X_{F_1}, \ldots, X_{F_r} \tag{25}$$

is an r–torus to which the vector fields (25) are tangent. Hence, the symplectic structure ω vanishes on the manifold $\mathbb{T}^r(z_0) \subseteq \mathcal{P}$ whence this torus is isotropic, implying $r \leq n$. Consequently, if a Hamiltonian system X_H on $\mathcal{P}$ is *superintegrable*, having more functionally independent integrals of motion than degrees of freedom, these cannot be all in involution. One therefore also speaks of non-commutative integrability in this context.

The "extra" integral makes superintegrable systems even more exceptional than integrable systems, although it usually can be attributed to a non-commutative symmetry group. An example is the Euler case of a free rigid body for which the four integrals are the energy and the three components of the angular momentum mapping (14) induced by the symmetry group $SO(3)$. For the planar and spatial two body problem (the Kepler system) the symmetry groups $SO(3)$ and $SO(4)$ lead to 3 and 5 independent integrals of motion, respectively. The latter is exceptional *even within the class of superintegrable systems*; replacing the inverse square attraction by another central force (not the harmonic oscillator with its 5 independent integrals of motion due to an $SU(3)$–symmetry) breaks the symmetry down to $SO(3)$ with still 4 independent integrals of motions.

Theorem (Nekhoroshev, Mishchenko and Fomenko). *On the subset $\mathcal{R} \subseteq \mathcal{P}$ let $F : \mathcal{R} \longrightarrow \mathbb{R}^{2n-r}$ be a submersion with compact and connected fibres (hence, a fibration). Assume that $\{F_i, F_j\} = P_{ij} \circ F$, $i, j = 1, \ldots, 2n-r$ and that the matrix P with entries $P_{ij} : \mathcal{P} \longrightarrow \mathbb{R}$ has rank $2(n-r)$ at all points of $F(\mathcal{P})$. Then every fibre of F is* *diffeomorphic to $\mathbb{T}^r$ and the fibration F has local trivialisations which are symplectic.*

This formulation is taken from [15], where the geometric contents of this theorem is thoroughly described in terms of this fibration and an associated co-fibration with fibres of dimension $2n - r$.

Thus, every fibre of F has a neighbourhood $\mathcal{U}$ with co–ordinates

$$(x, y, q, p) \; : \; \mathcal{U} \;\; \longrightarrow \;\; \mathbb{T}^r \times \mathbb{R}^r \times \mathbb{R}^{n-r} \times \mathbb{R}^{n-r}$$

such that the level sets of F coincide with the level sets of (y, q, p) and

$$\sigma_{|\mathcal{U}} \;\; = \;\; \sum_{i=1}^{r} \mathrm{d}x_i \wedge \mathrm{d}y_i \; + \; \sum_{j=1}^{n-r} \mathrm{d}q_j \wedge \mathrm{d}p_j \quad .$$

These co–ordinates are Nekhoroshev's generalized action–angle variables. Where superintegrability is due to a non-commutative symmetry group G, the $2(n - r)$ parameters "live" in the co-adjoint orbits (17).

7 Perturbation Analysis

An important property of integrable Hamiltonian systems is their behaviour under small perturbations. For a satisfactory description of *e.g.* mechanical systems the simplifying assumptions used to derive the model should not completely change the dynamics, some kind of "robustness" is desirable. An instance where the underlying approximation by a "simpler" system is part of the mathematical treatment is normal form theory.

Using a series expansion of the Hamiltonian function, the aim of normalization is to find co-ordinates in which the terms of the expansion look particularly simple (whence the Hamiltonian vector field takes a particularly simple form as well). This is an algorithmic procedure that inductively pushes a torus symmetry through the series. While the resulting series are typically divergent, a well-chosen truncation yields a normalized approximation for which good estimates are available. Already existing symmetries are preserved and the choices to be made when normalizing a given Hamiltonian are largely dictated ensuring that the combination of acquired and inherited symmetries render the system integrable. This does not always work, the integrability of normal forms is automatic only in two degrees of freedom, cf. [32]. For more details on normal forms in perturbation theory see [6].

Normalization often results in a torus action (24) on the phase space $\mathcal{P}$, for which the components $F_2, \ldots, F_n$ of the momentum mapping

$$J : \;\; \mathcal{P} \;\; \longrightarrow \mathbb{R}^{n-1}$$

define Hamiltonian vector fields (23) with periodic flows. This allows for the detailed description given in the previous section, and the question is what remains of this description when the symmetry (24) is broken.

The most prominent part of the ramified torus bundle defined by an integrable Hamiltonian system are the families of Lagrangean tori. Persistence of invariant tori can only be expected if these have a dynamical meaning. For instance, an invariant 1–torus that consists of a union of equilibria instead of being a periodic orbit is highly unlikely to remain present in a perturbed system. Similarly, an invariant torus with conditionally periodic motion has dynamical meaning if it is the closure of a dense orbit. This excludes resonances

$$\langle k \mid \varpi \rangle \;=\; 0 \;, \quad k \in \mathbb{Z}^m \backslash \{0\}$$

between the frequencies $\varpi_1, \ldots, \varpi_n$ of the parallel flow on the invariant torus T. The parallel nature of the flow implies that for a non-resonant frequency vector ϖ the time average

$$\int_{-\infty}^{+\infty} f(x(t)) \mathrm{d}t \;=\; \int_{\mathsf{T}} f(x) \mathrm{d}x$$

of some function $f : \mathsf{T} \longrightarrow \mathbb{R}$ along the quasi-periodic motion $x(t) = x + t\varpi$ is equal to the space average over the torus.

This space average is approximated "quickly" – taking the time average over finite intervals of time – if ϖ is *Diophantine*, satisfying the strong non-resonance condition

$$|\, \langle k \mid \varpi \rangle \,| \;\geq\; \frac{\gamma}{|k|^\tau} \qquad \text{for all } k \in \mathbb{Z}^m \backslash \{0\} \tag{26}$$

with constants $\gamma > 0$ and $\tau > n - 1$. The set $\mathbb{R}^n_{\tau,\gamma}$ of (τ, γ)–Diophantine $\varpi \in \mathbb{R}^n$ has the local structure

$$\mathbb{R} \;\times\; \textit{Cantor dust} \;; \tag{27}$$

for $\varpi \in \mathbb{R}^n_{\tau,\gamma}$ also $s\varpi \in \mathbb{R}^n_{\tau,\gamma}$ for all $s \geq 1$, and the intersection $S^{n-1} \cap \mathbb{R}^n_{\tau,\gamma}$ is perfect and totally disconnected. While (Lebesgue)-almost all frequency vectors are non-resonant, the complement of Diophantine frequency vectors is an open and dense set. Still, the relative measure of $\mathbb{R}^n_{\tau,\gamma}$ goes to 1 as $\gamma \to 0$. The celebrated KAM theorem yields persistence of Lagrangean tori with Diophantine frequency vector.

Theorem (Kolmogorov, Arnol'd and Moser). *Let $\mathbb{Y} \subseteq \mathbb{R}^n$ be an open neighbourhood of the origin and consider the phase space $\mathcal{P} = \mathbb{T}^n \times \mathbb{Y}$ with symplectic structure $\mathrm{d}x \wedge \mathrm{d}y$. Let the Hamiltonian*

$$H_\varepsilon(x, y) \;=\; H_0(y) \;+\; \varepsilon H_1(x, y; \varepsilon)$$

be real analytic and non-degenerate, satisfying

$$\det D^2 H_0(y) \;\neq\; 0 \qquad \textit{for all } y \in \mathbb{Y}. \tag{28}$$

Then there exists $\varepsilon_0 > 0$ such that for all $|\varepsilon| < \varepsilon_0$ there is a canonical transformation ϕ_ε near the identity and a measure-theoretically large set $\mathbb{Y}'_\varepsilon \subseteq \mathbb{Y}$ with the property that for $p \in \mathbb{Y}'_\varepsilon$ the transformed Hamiltonian $H_\varepsilon \circ \phi_\varepsilon$ does not depend on $q \in \mathbb{T}^n$.

For a proof see [3, 27, 10].

The Kolmogorov non-degeneracy condition (28) expresses that the frequency mapping

$$\begin{array}{rccc} \varpi : & \mathbb{Y} & \longrightarrow & \mathbb{R}^n \\ & y & \mapsto & DH_0(y) \end{array} \tag{29}$$

is locally a diffeomorphism. This allows to pull back the whole geometry defined by (26) into phase space to obtain

$$\mathbb{Y}_\varepsilon \;=\; \varpi^{-1}(\mathbb{R}^n_{\tau,\gamma})$$

(and from this the subset $\mathbb{Y}'_\varepsilon \subseteq \mathbb{Y}_\varepsilon$ by omitting points γ–close to the boundary $\partial\mathbb{Y}$). The constant γ is chosen as a function of ε – of order $\mathcal{O}(\sqrt{\varepsilon})$ – to find an optimal balance between the (relative) measure of $\mathbb{Y}'_\varepsilon$ and the deviation of ϕ_ε from the identity.

The subset $\mathbb{T}^n \times \mathbb{Y}'_\varepsilon \subseteq \mathcal{P}$ consists (in the transformed variables) of quasi-periodic tori since $\dot{p}$ vanishes for $p \in \mathbb{Y}'_\varepsilon$. The theorem makes no statement for $p \in \mathbb{Y} \backslash \mathbb{Y}'_\varepsilon$. For $n \geq 3$ this leaves the possibility of Arnol'd diffusion, trajectories that venture off to distant parts of the phase space; for more details see [37].

While the KAM theorem concerns the fate of "most" trajectories and for all times, the complementary Nekhoroshev theory concerns all trajectories and states that they stay close to the unperturbed tori for times of the order

$$\exp\left[\left(\frac{\varepsilon_0}{\varepsilon}\right)^{\frac{1}{2n}}\right] \quad .$$

Here analyticity of the Hamiltonian is a necessary ingredient, for finitely differentiable Hamiltonians one only obtains polynomial times. In the above formulation of the KAM theorem the assumption of analyticity of the Hamiltonian can be weakend without essential changes for the result, during the proof one merely has to intersperse an analytic approximation at each iteration step. The diffusion is even superexponentially slow for trajectories starting close to surviving tori, see [34] for more details on this phenomenon of exponential condensation, and see [29] for more details on Nekhoroshev theory.

Where the energy level sets are transversal to the continuous direction in $\mathbb{Y}'_\varepsilon$ one has persistence of most Lagrangean tori on each energy shell, parametrised by *Cantor dust*. The same result is obtained under the condition of iso-energetic non-degeneracy

$$\det \begin{pmatrix} D^2H_0(y) & \nabla H_0(y) \\ DH_0(y) & 0 \end{pmatrix} \;\neq\; 0 \;, \tag{30}$$

which is independent of Kolmogorov's condition. It is generic for an integrable system to satisfy both conditions almost everywhere. However, in applications it is a non-trivial task to actually check this and to determine the hypersurfaces in action space where these determinants vanish.

In two degrees of freedom the energy shells $\{H = h\}$ are 3–dimensional invariant manifolds for regular values h of the Hamiltonian and are separated by each Lagrangean torus. Thus, for initial conditions (q_0, p_0) in the co-ordinates provided by the KAM theorem with $p_0 \notin \mathbb{Y}'_\varepsilon$ the persistent tori parametrised by $\mathbb{Y}'_\varepsilon$ still have dynamical consequences as the trajectory is confined between two such tori, so $|p(t) - p_0|$ admits a bound of order $\mathcal{O}(\sqrt{\varepsilon})$. As a consequence, one obtains (dynamical) stability for all elliptic equilibria in generic Hamiltonian systems with two degrees of freedom.

Indeed, if the Hessian of (8) is (positive or negative) definite, then the Hamiltonian serves as a Liapunov function. In the indefinite case one also includes third and fourth order terms in the analysis and passes to a Birkhoff normal form

$$\alpha I_1 + \varphi I_2 + \frac{\beta}{2} I_1^2 + \delta I_1 I_2 + \frac{\chi}{2} I_2^2 \tag{31}$$

with $I_1 = \frac{1}{2}(p_1^2 + q_1^2)$ and $I_2 = \frac{1}{2}(p_2^2 + q_2^2)$. This can be achieved if there are no low order resonances $\varpi = -k\alpha$, $k = 1, 2, 3$ between the two (normal) frequencies of the equilibrium. A second genericity condition on the Hamiltonian is that the linear part of (31) does not divide the quadratic part (in the I_i), ensuring that (30) holds in a whole neighbourhood of the elliptic equilibrium.

The Cantor set structure defined by the Diophantine conditions (26) can be used to weaken the necessary non-degeneracy condition. Since the gaps are defined by linear inequalities, the conditions on the first derivatives of the frequency mapping (29) can be replaced by conditions on the curvature or even higher derivatives. Such Rüssmann-like conditions still guarantee that the relative measure of surviving tori tends to 1 as the perturbation strength tends to zero, but at a price. For instance, the highest derivative $L \in \mathbb{N}$ needed in

$$< \frac{\partial^{|\ell|}\omega}{\partial y} \;\mid\; |\ell| \leq L > \;=\; \mathbb{R}^n \tag{32}$$

enters the Diophantine conditions on the frequency vector by means of the inequality $\tau > nL - 1$ on the Diophantine constant τ. For more details see [11].

The KAM theorem is a semi-local result, valid in the neighbourhood of an initial torus that admits action angle variables. A global version is obtained in [7]. The global conjugacy is glued together from convex combinations of local conjugacies using a partition of unity, the key ingredient being a unicity result on the tori obtained in the KAM theorem.

In the integrable approximation the distribution of the n–parameter families of Lagrangean tori is determined by the singular part $\mathcal{S}$ of the ramified torus bundle $\mathcal{P}$. Since $\mathcal{S}$ consists of families of lower dimensional tori together with their (un)stable manifolds, the persistence of (isotropic) m–tori, $m < n$, becomes important. For $m = 0$ the persistence of equilibria, together with their linear behaviour (a superposition of what is possible in one and two degrees of freedom) follows from the implicit mapping theorem as it is generic for the Hamiltonian function that no equilibrium has vanishing (and neither multiple) eigenvalues. In the periodic case $m = 1$ the Diophantine condition (26) ensures $\varpi \neq 0$ and the 1–parameter families of periodic orbits persist as well, together with occurring bifurcations. In 1–parameter families all bifurcations are generically of co-dimension 1 – a genericity condition on the Hamiltonian. Bifurcations of higher co-dimension would not be expected to persist.

For hyperbolic tori the criteria remain valid almost verbatim; the key step is to pass to a centre manifold. A technical difficulty is that even for analytic Hamiltonians centre manifolds may only be of finite differentiability. While KAM theorems remain true in this context, the analytic context has its advantages – for instance (32) is satisfied for some $L \in \mathbb{N}$ for an analytic frequency mapping ϖ if and only if $\operatorname{im} \varpi$ does not lie within a linear hyperplane. An alternative is therefore to prove persistence of hyperbolic tori directly, this also gives a more direct hold on their stable and unstable manifolds.

Elliptic $(n-1)$–tori need one extra parameter to control the normal frequency as well. Similar to the iso-energetic case one can use time re-parametrisation and obtain Cantor families of persistent elliptic $(n-1)$–tori parametrised by Cantor dust. Where there are more than one normal frequency to control this can no longer be done in a linear way; a problem solved by Rüssmann-like conditions on the higher derivatives of the frequency vector, see [10] and references therein. In case the mapping of internal frequencies satisfies Kolmogorov's condition, the higher order derivatives are only needed of normal frequencies. Now normal frequencies α_j enter the Diophantine conditions

$$| \, 2\pi \langle k \mid \varpi \rangle \; + \; \langle \ell \mid \alpha \rangle \, | \quad \geq \quad \frac{\gamma}{|k|^{\tau}} \tag{33}$$

only as combinations $\langle \ell \mid \alpha \rangle$ with $|\ell| \leq 2$. This allows to extend the result to finite-dimensional elliptic tori in infinitely many degrees of freedom, cf. [19, 18, 4]. For hypo-elliptic tori one may deal with the hyperbolic part by means of a centre manifold or use a direct approach. Such m–tori have k additional pairs of purely imaginary Floquet exponents and excitation of normal modes leads for $l = 1, \ldots, k$ to $(m+l)$–parameter families of $(n+l)$–tori inheriting the "remaining" normal linear behaviour, see [34] and references therein.

Where (lower-dimensional) m–tori undergo a semi-local bifurcation the m actions y *conjugate* to the toral angles x first of all have to versally unfold the bifurcation scenario. It is generic for the integrable Hamiltonian H that the m–parameter families of m–tori, $1 \leq m \leq n-1$, do not encounter bifurcations of co-dimension higher than m, so this is possible. The curvature of the frequency mapping is then used to ensure Diophanticity of most bifurcating tori, *i.e.* a Rüssmann-like condition with $L = 2$ is sufficient, cf. [17]. This curvature requirement is not necessary for 2–tori; these may undergo the quasi-periodic analogues of the co-dimension one bifurcations of periodic orbits as co-dimension two bifurcations are isolated within these 2–parameter families and cannot be prevented to disappear in resonance gaps.

Small perturbations of an integrable Hamiltonian thus lead to a Cantorification of the ramified n–torus bundle as sketched in figure 6, the stratification of the action space into various subfamilies parametrising the tori is replaced by a Cantor stratification. Of equal importance are those changes that make sure that the non-integrable perturbed dynamics is indeed qualitatively different from the integrable unperturbed dynamics. While the former persistence results are obtained upon genericity conditions on the unperturbed system, such changes require the perturbation to be generic.

Disintegrating Lagrangean tori lead to invariant m–tori, where $n-m$ is the number of independent resonances $\langle k \mid \varpi \rangle = 0$ of the (internal) frequencies. Most of these lower dimensional tori will be elliptic or hyperbolic. The new hyperbolic tori lie at the basis of

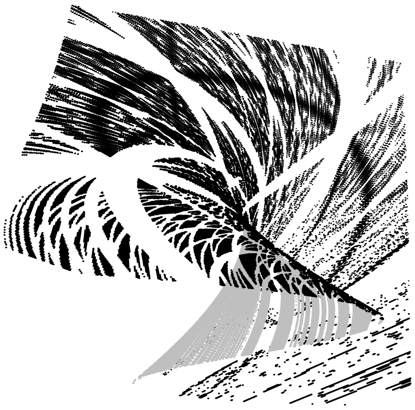

Figure 6: A typical decomposition of the action space of a nearly integrable system in three degrees of freedom. The 2–dimensional Cantor dust parametrises elliptic 2–tori that vanish in quasi-periodic centre-saddle bifurcations along the fold lines. Between these extend arcs of hyperbolic 2–tori, parametrised by a 2–dimensional set of the form (27). The 1–dimensional Cantor dust along the folds consists of Lebesgue density points of these 2–dimensional parameter sets in the same way that these consist of Lebesgue density points of 3–dimensional sets of the form (27) above and below the surface which parametrise Lagrangean 3–tori.

a possible scenario for Arnol'd diffusion. One of the effects of a small generic perturbation is that stable and unstable manifolds of hyperbolic periodic orbits no longer coincide, but split and intersect transversely, cf. [1]. This caries over to hyperbolic tori. The splitting of separatrices also leads to transverse intersections of stable and unstable manifolds of neighbouring hyperbolic tori in the same energy shell. These hyperbolic tori form a Cantor family, and one of the main problems is to make sure that the transition chain of hyperbolic tori and their heteroclinic connections bridges the occuring gaps, cf. [13]. The dynamics in the gaps of Cantor families of hyperbolic tori can already be studied in the perturbation near resonant singular fibres of the ramified n–torus bundle. On the centre manifold these become again (resonant) regular fibres, but the full perturbed motion is superposed by the hyperbolic dynamics in the symplectic normal directions.

The nature of the gaps opened by violations of (33) in families of elliptic tori is twofold. Internal resonances $\langle k \mid \varpi \rangle = 0$ lead to the destruction of the torus. Boundary points of the gaps resulting from normal-internal resonances are related to quasi-periodic bifurcations. The resonance $\alpha = 2\pi \langle k \mid \varpi \rangle$ triggers off a (quasi-periodic) centre-saddle bifurcation and resonance gaps

$$| \, 2\pi \langle k \mid \varpi \rangle \; + \; 2\alpha \, | \;\; < \;\; \frac{\gamma}{|k|^\tau}$$

are completely filled by hyperbolic tori that terminate in frequency halving bifurcations.

The maximal tori of superintegrable systems are m–tori in $n > m$ degrees of freedom and their normal behaviour vanishes, both linearly and non-linearly. The strategy when studying a perturbation of such a *properly degenerate* system is to find an intermediate system that is also integrable, but non-degenerately so. The perturbation H_1 of a superintegrable Hamiltonian H_0 *removes the degeneracy* if the perturbed Hamiltonian $H_\varepsilon = H_0 + \varepsilon H_1$ can be written in the form

$$H \;=\; H_0 \;+\; \varepsilon \bar{H}_1 \;+\; \varepsilon^2 H_2$$

where $H_0 + \varepsilon \bar{H}_1$ is a non-degenerate integrable Hamiltonian. Since $\bar{H}_1$ is defined in terms of H_1 (*e.g.* as its average along the unperturbed flow defined by H_0) every genericity condition on the intermediate system puts genericity conditions on the perturbation H_1. This first step of a normal form procedure lies also at the basis of Nekhoroshev theory for superintegrable systems, see [15] for more details.

8 Future Directions

Many Hamiltonian systems modelling real phenomena have symmetries, and the conditions of the regular reduction theorem of Marsden and Weinstein are often not fulfilled. The regularity assumptions could be successfully removed, cf. [12, 30, 26], and progress is still made in weakening the compactness conditions. Since the flow $\varphi : \mathbb{R} \times \mathcal{P} \longrightarrow \mathcal{P}$ itself is also a group action some condition has to exclude too general situations. It is for this reason that the symmetry (9) is studied in Section 5 and not the larger symmetry (10).

In an integrable system, action angle variables define a $\mathbb{T}^n$–action in the neighbourhood of a given Lagrangean torus. Globally one only has the $\mathbb{R}^n$–action defined by the commuting flows of the integrals $F_1, \ldots, F_n$. The flow of $X_H = X_{F_1}$ is the actual object of study, and in general the flows of the vector fields (23) may be as complicated.

On the topological level, the $\mathbb{T}^n$–bundle $\mathcal{R}$ has a monodromy mapping $\mathcal{M}$ and only if $\mathcal{M}$ is trivial can one uniquely characterise $\mathcal{R}$ by its Chern class, such bundles are isomorphic if and only if their Chern classes coincide. An extension of this characterization to the case of non-trivial monodromy does not yet exist. Such a classification of all torus bundles on a given base space C might need further invariants.

The global version of the KAM theorem provides a Cantorification of the global bundle $\mathcal{R}$. This makes also non-local properties subject to the perturbation analysis, for instance showing a discrete invariant like monodromy to persist. Globally on the phase space, perturbation not only of the Hamiltonian H but also of the symplectic structure ω becomes a well-defined problem. For instance, if $\omega = -\mathrm{d}\vartheta$ is exact one may add a small non-exact closed 2–form $\varepsilon\sigma$ whence $\omega + \varepsilon\sigma$ is a non-exact symplectic form on the phase space.

A Lagrangean torus with $n - 1$ independent resonances consists of periodic orbits. When the torus breaks up under the perturbation, only finitely many of these are expected to survive. At the same time the trivial normal behaviour of these periodic orbits changes, resulting in hyperbolic and elliptic periodic orbits. The latter can serve as starting points for the construction of *solenoids*, cf. [5, 23]. This construction should carry over to elliptic tori, where the "encircling" tori emerge from normal-internal resonances and might also result in solenoids that are limits of tori with varying dimension.

The results in [17] address persistence of Diophantine tori involved in a bifurcation and the corresponding gaps trigger off new phenomena, cf. [21]. Internally resonant tori involved in a quasi-periodic bifurcation may result in large dynamical instabilities, especially where multiple parabolic resonances are encountered. The effect is further amplified for tangent (or flat) parabolic resonances, which fail to satisfy the iso-energetic non-degeneracy condition.

The high co-dimensions of bifurcations that may be encountered within families of isotropic tori makes it necessary to study Hamiltonian bifurcations that have been left aside since they generically do not occur for periodic orbits. The coupling of the three types of co-dimension one bifurcations is unavoidable where the resonance gaps defined by (33) with $\ell \neq 0$ intersect. Already for an equilibrium with linearization

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

the nearby dynamics is extremely complicated, with all possible resonances of equilibria in two degrees of freedom occurring in a versal unfolding.

A perturbed superintegrable system can lead to the combination of two bifurcations in both the fast and the slow dynamics. With two different time scales *e.g.* the dynamics triggered off by two simultaneous violations of (33) appears to be of $(1+1)$-degree-of-freedom rather than having truly 2 degrees of freedom. This might render this problem more accessible.

For more details on Hamiltonian perturbation theory see [9] and references therein. An important subject are variational methods, which can be used to obtain periodic solutions (but also more complicated dynamics); see [33] on this subject and also [36] for a striking application to celestial mechanics.

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