Hamiltonian Perturbation Theory (and Transition to Chaos)

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1 Abstract

The fundamental problem of mechanics is to study Hamiltonian systems that are small perturbations of integrable systems. But also perturbations that destroy the Hamiltonian character are important, be it to study the effect of a small amount of friction, or to further the theory of dissipative systems themselves which surprisingly often revolves around certain well-chosen Hamiltonian systems. Furthermore there are approaches like KAM theory that historically were first applied to Hamiltonian systems. Typically perturbation theory explains only part of the dynamics, and in the resulting 'gaps' the orderly unperturbed motion is replaced by random or chaotic motion.

2 Introduction

We outline perturbation theory from a general point of view, illustrated by a few examples.

2.1 The Perturbation Problem

The aim of perturbation theory is to approximate a given dynamical system by a more familiar one, regarding the former as a perturbation of the latter. The problem then is to deduce certain dynamical properties from the 'unperturbed' to the 'perturbed' case.

What is familiar may or may not be a matter of taste, at least it depends a lot on the dynamical properties of one's interest. Still the most frequently used 'unperturbed' systems are

- Linear systems;
- Integrable Hamiltonian systems, compare with [71] and references therein;

- Normal form truncations, compare with [19] and references therein;

- Etc.

To some extent the second category can be seen as a special case of the third. To avoid technicalities in this section we assume all systems to be 'sufficiently' smooth, say of class C^∞ or real analytic. Moreover in our considerations ε will be a real parameter. The 'unperturbed' case always corresponds to $\varepsilon = 0$ and the 'perturbed' one to $\varepsilon \neq 0$ or $\varepsilon > 0$.

2.1.1 Examples of perturbation problems

To begin with consider the autonomous differential equation

$$\ddot{x} + \varepsilon\dot{x} + \frac{dV}{dx}(x) = 0,$$

modelling an oscillator with small damping. Rewriting this equation of motion as a planar vector field

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\varepsilon y - \frac{dV}{dx}(x), \end{aligned}$$

we consider the energy $H(x, y) = \frac{1}{2}y^2 + V(x)$. For $\varepsilon = 0$ the system is Hamiltonian with Hamiltonian function H. Indeed, generally we have $\dot{H}(x, y) = -\varepsilon y^2$, implying that for $\varepsilon > 0$ there is dissipation of energy. Evidently for $\varepsilon \neq 0$ the system is no longer Hamiltonian.

The reader is invited to compare the phase portraits of the cases $\varepsilon = 0$ and $\varepsilon > 0$ for $V(x) = -\cos x$ (the pendulum) or $V(x) = \frac{1}{2}\lambda x^2 + \frac{1}{24}bx^4$ (Duffing).

Another type of example is provided by the non-autonomous equation

$$\ddot{x} + \frac{dV}{dx}(x) = \varepsilon f(x, \dot{x}, t),$$

which can be regarded as the equation of motion of an oscillator with small external forcing. Again rewriting as a vector field, we obtain

$$\begin{aligned} \dot{t} &= 1 \\ \dot{x} &= y \\ \dot{y} &= -\frac{dV}{dx}(x) + \varepsilon f(x, y, t), \end{aligned}$$

now on the generalized phase space $\mathbb{R}^3 = \{t, x, y\}$. In the case where the t-dependence is periodic, we can take $\mathbb{S}^1 \times \mathbb{R}^2$ for (generalized) phase space.

Remarks.

- A small variation of concerns a parametrically forced oscillator like

$$\ddot{x} + (\omega^2 + \varepsilon \cos t) \sin x = 0,$$

which happens to be entirely in the world of Hamiltonian systems.

- It may be useful to study the Poincaré or period mapping of such time periodic systems, which happens to be a mapping of the plane. We recall that in the Hamiltonian cases this mapping preserves area.

There are lots of variations and generalizations. One example is the solar system, where the unperturbed case consists of a number of uncoupled 2-body problems concerning the Sun and each of the planets, and where the interaction between the planets is considered as small [9, 6, 110, 111].

Remarks.

- One variation is a restriction to fewer bodies, for example only three. Examples of this are systems like Sun–Jupiter–Saturn, Earth–Moon–Sun or Earth–Moon–Satellite.
- Often Sun, Moon and planets are considered as point masses, in which case the dynamics usually are modelled as a Hamiltonian system. It is also possible to extend this approach taking tidal effects into account, which have a non-conservative nature.
- The Solar system is close to resonance, which makes application of KAM theory problematic. There exist however other integrable approximations that takes resonance into account. [2, 64].

Quite another perturbation setting is local, e.g., near an equilibrium point.To fix thoughts consider

$$\dot{x} = Ax + f(x), \; x \in \mathbb{R}^n$$

with $A \in \mathrm{gl}(n, \mathbb{R})$, $f(0) = 0$ and $\mathrm{D}_x f(0) = 0$. By the scaling $x = \varepsilon \bar{x}$ we rewrite the system to

$$\dot{\bar{x}} = A\bar{x} + \varepsilon g(\bar{x}).$$

So, here we take the linear part as unperturbed system. Observe that for small ε the perturbation is small on a compact neighbourhood of $\bar{x} = 0$.

This setting also has many variations. In fact, any normal form approximation may be treated in this way [19]. Then the normalized truncation forms the 'unperturbed' part and the higher order terms the perturbation.

Remark. In the above we took the classical viewpoint which involves a perturbation parameter controlling the size of the perturbation. Often one can generalize this by considering a suitable topology (like the Whitney topologies) on the corresponding class of systems [74].

2.2 Questions of Persistence

What are the kind of questions Perturbation Theory asks? A large class of questions concerns the *persistence* of certain dynamical properties as known for the unperturbed case. To fix thoughts we give a few examples.

To begin with consider equilibria and periodic orbits. So we put

$$\dot{x} = f(x, \varepsilon), \; x \in \mathbb{R}^n, \; \varepsilon \in \mathbb{R}, \tag{1}$$

for a map $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$. Recall that equilibria are given by the equation $f(x, \varepsilon) = 0$. The following theorem that continuates equilibria in the unperturbed system for $\varepsilon \neq 0$, is a direct consequence of the Implicit Function Theorem.

Theorem 1. (Persistence of equilibria) *Suppose that* $f(x_0, 0) = 0$ *and that*

$$f(x_0, 0) = 0 \textit{ and that } \mathrm{D}_x f(x_0, 0) \textit{ has maximal rank.}$$

Then there exists a local arc $\varepsilon \mapsto x(\varepsilon)$ *with* $x(0) = x_0$ *such that*

$$f(x(\varepsilon), \varepsilon) \equiv 0.$$

Periodic orbits can be approximated in a similar way. Indeed, let the system (1) for $\epsilon = 0$ have a periodic orbit γ_0. Let Σ be a local transversal section of γ_0 and $P_0 : \Sigma \to \Sigma$ the corresponding Poincaré map. Then P_0 has a fixed point $x_0 \in \Sigma \cap \gamma_0$. By transversality, for $|\varepsilon|$ small, a local Poincaré map $P_\varepsilon : \Sigma \to \Sigma$ is well-defined for (1). Observe that fixed points x_ε of P_ε correspond to periodic orbits γ_ε of (1). We now have, again as another direct consequence of the Implicit Function Theorem

Theorem 2. (Persistence of periodic orbits) *In the above assume that*

$$P_0(x_0) = x_0 \textit{ and } \mathrm{D}_x P_0(x_0) \textit{ has no eigenvalue } 1.$$

Then there exists a local arc $\varepsilon \mapsto x(\varepsilon)$ *with* $x(0) = x_0$ *such that*

$$P_\varepsilon(x(\varepsilon)) \equiv x_\varepsilon.$$

Remarks.

- Often the conditions of Theorem 2 are not easy to verify. Sometimes it is useful here to use Floquet Theory, see [100]. In fact, if T_0 is the period of γ_0 and Ω_0 its Floquet matrix, then $\mathrm{D}_x P_0(x_0) = \exp(T_0 \Omega_0)$.

- The format of the Theorems 1 and 2 with the perturbation parameter ε directly allows for algorithmic approaches. One way to proceed is by perturbation series, leading to asympotic formulæ that in the real analytic setting have positive radius of convergence. In the latter case the names of Poincaré and Lindstedt are associated to the method, cf. [10].

  Also numerical continuation programmes exist based on the Newton method.

- The Theorems 1 and 2 can be seen as special cases of a a general theorem for *normally hyperbolic invariant manifolds* [75], Theorem 4.1. In all cases a contraction principle on a suitable Banach space of graphs leads to persistence of the invariant dynamical object.

  This method in particular yields existence and persistence of stable and unstable manifolds [55, 56].

Another type of dynamics subject to Perturbation Theory is quasi-periodic. We emphasize that persistence of (Diophantine) quasi-periodic invariant tori occurs both in the conservative setting and in many others, like in the reversible and the general (dissipative) setting. In the latter case this leads to persistent occurrence of families of quasi-periodic attractors [128]. These results are in the domain of Kolmogorov-Arnold-Moser (KAM) Theory. For details we refer to §6 below or to [41, 54], the first reference containing more than 400 references in this area.

Remarks.

- Concerning the Solar system, KAM Theory always has aimed at proving that it contains many quasi-periodic motions, in the sense of positive Liouville measure. This would imply that there is positive probability that a given initial condition lies on such a 'stable' quasi-periodic motion [2, 64], however, also see [88].

- Another type of result in this direction compares the distance of certain individual solutions of the perturbed and the unperturbed system, with coinciding initial conditions over time scales that are long in terms of ε. Compare with [41].

Apart from persistence properties related to invariant manifolds or individual solutions, the aim can also be to obtain a more global persistence result. As an example of this we mention the Hartman-Grobman Theorem, e.g., [7, 119, 126]. Here the setting once more is

$$\dot{x} = Ax + f(x), \; x \in \mathbb{R}^n,$$

with $A \in \mathrm{gl}(n, \mathbb{R})$, $f(0) = 0$ and $\mathrm{D}_x f(0) = 0$. Now we assume A to be hyperbolic (i.e., with no purely imaginary eigenvalues). In that case the full system, near the origin, is topologically conjugated to the linear system $\dot{x} = Ax$. Therefore all global, *qualitative* properties of the unperturbed (linear) system are persistent under perturbation to the full system.

It is said that the hyperbolic linear system $\dot{x} = Ax$ is (locally) *structurally stable*. This kind of thinking was introduced to the dynamical systems area by Thom [136], with a first, succesful application to catastrophe theory. For further details, see [7, 70, 119].

2.3 General dynamics

We give a few remarks on the general dynamics in a neighborhood of Hamiltonian KAM tori. In particular this concerns so-called 'superexponential stickiness' of the KAM tori and adiabatic stability of the action variables, involving the so-called Nekhoroshev estimate.

To begin with, emphasize the following difference between the cases $n = 2$ and $n \geq 3$ in the classical KAM Theorem of §6.1. For $n = 2$ the level surfaces of the Hamiltonian are three-dimensional, while the Lagrangean tori have dimension two and hence codimension one in the energy hypersurfaces. This means that for open sets of initial conditions, the evolution curves are forever trapped in between KAM tori, as these tori foliate over nowhere dense sets of positive measure. This implies perpetual adiabatic stability of the action variables. In contrast, for $n \geq 3$ the Lagrangean tori have codimension $n-1>1$ in the energy hypersurfaces and evolution curves may escape.

This actually occurs in the case of so-called *Arnold diffusion*. The literature on this subject is immense, and we here just quote [5, 9, 112, 96], for many more references see [41].

Next we consider the motion in a neighborhood of the KAM tori, in the case where the systems are real analytic or at least Gevrey smooth. First we mention that, measured in terms of the distance to the KAM torus, nearby evolution curves generically stay nearby over a *superexponentially* long time [105, 106]. This property often is referred to as 'superexponential stickiness' of the KAM tori, see [41] for more references.

Second, nearly integrable Hamiltonian systems, in terms of the perturbation size, generically exhibit *exponentially* long adiabatic stability of the action variables, see e.g. [112, 113, 96, 106, 91, 92, 93, 123] and many others, for more references see [41]. This property is referred to as the *Nekhoroshev estimate* or the *Nekhoroshev theorem*. For related work on perturbations of so-called superintegrable systems, also see [41] and references therein.

2.4 Chaos

In the previous subsection we discussed persistent and some non-persistent features of dynamicals systems under small perturbations. Here we discuss properties related to splitting of separatrices, caused by generic perturbations.

A first example was met earlier, when comparing the pendulum with and without (small) damping. The unperturbed system is the undamped one and this is a Hamiltonian system. The perturbation however no longer is Hamiltonian. We see that the equilibria are persistent, as should be according to Theorem 1, but that none of the periodic orbits survives the perturbation. Such qualitative changes go with perturbing away from the Hamiltonian setting.

Similar examples concern the breaking of a certian symmetry by the perturbation. The latter often occurs in the case of normal form approximations. Then the normalized truncation is viewed as the unperturbed system, which is perturbed by the higher order terms. The truncation often displays a reasonable amount of symmetry (e.g., toroidal symmetry), which *generically* is forbidden for the class of systems under consideration, e.g. see [47].

To fix thoughts we reconsider the conservative example

$$\ddot{x} + (\omega^2 + \varepsilon \cos t) \sin x = 0$$

of the previous section. The corresponding (time dependent, Hamiltonian [6]) vector field reads

$$\begin{aligned} \dot{t} &= 1 \\ \dot{x} &= y \\ \dot{y} &= -(\omega^2 + \varepsilon \cos t) \sin x. \end{aligned}$$

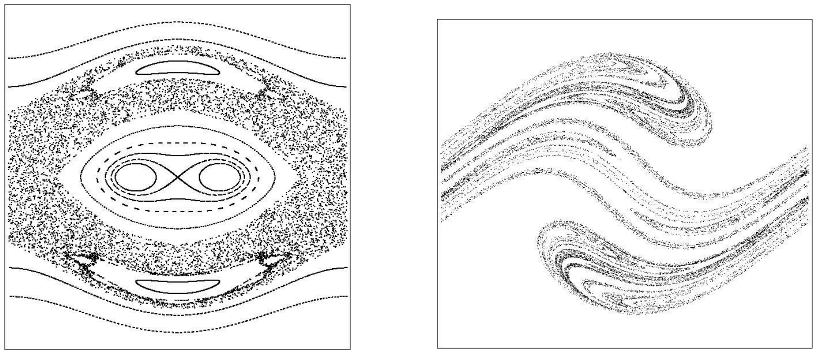

Figure 1: Chaos. Left: Poincaré map $P_{\omega,\varepsilon}$ near the $1:2$ resonance $\omega = \frac{1}{2}$ and for $\varepsilon > 0$ 'not too small'. Right: A dissipative analogue.

Let $P_{\omega,\varepsilon} : \mathbb{R}^2 \to \mathbb{R}^2$ be the corresponding (area-preserving) Poincaré map. Let us consider the unperturbed map $P_{\omega,0}$ which is just the flow over time 2π of the free pendulum $\ddot{x} + \omega^2 \sin x = 0$. Such a map is called *integrable*, since it is the stroboscopic map of a 2-dimensional vector field, hence displaying the $\mathbb{R}$-symmetry of a flow. When perturbed to the *nearly integrable* case $\varepsilon \neq 0$, this symmetry generically is broken. We list a few of the generic properties for such maps [126]:

- The homoclinic and heteroclinic points occur at transversal intersections of the corresponding stable and unstable manifolds.
- The periodic points of period less than a given bound are isolated.

This means generically that the separatrices 'split' and that the 'resonant' invariant circles filled with periodic points with the same (rational) rotation number fall apart. In any concrete example the issue remains whether or not it satisfies appropriate genericity conditions. One method to chek this is due to Melnikov, compare [67, 140], for more sophicated tools see [66]. Often this leads to elliptic (Abelian) integrals.

In nearly integrable systems chaos can occur. This fact is at the heart of the celebrated non-integrability of the 3-body problem as addressed by Poincaré [121, 110, 111, 12, 61]. A long standing open conjecture is that the clouds of points as visible in Figure 1, Left, densely fill sets of positive area, thereby leading to ergodicity [9].

In the case of dissipation, see Figure 1, Right we conjecture the occurrence of a Hénon like strange attractor [14, 129, 38].

Remarks.

- The persistent occurrence of periodic points of a given rotation number follows from the Poincaré-Birkhoff fixed point theorem [110, 76, 99], i.e., on topological grounds.
- The above arguments are not restricted to the conservative setting, although quite a number of 'unperturbed' systems come from this world. Again see Figure 1.

3 One Degree Of Freedom

Planar Hamiltonian systems are always integrable and the orbits are given by the level sets of the Hamiltonian function. This still leaves room for a perturbation theory. The recurrent dynamics consists of periodic orbits, equilibria and asymptotic trajectories forming the (un)stable manifolds of unstable equilibria. The equilibria organize the phase portrait, and generically all equilibria are elliptic (purely imaginary eigenvalues) or hyperbolic (real eigenvalues), i.e. there is no equilibrium with a vanishing eigenvalue. If the system depends on a parameter such vanishing eigenvalues may be unavoidable and it becomes possible that the coresponding dynamics persist under perturbations.

Perturbations may also destroy the Hamiltonian character of the flow. This happens especially where the starting point is a dissipative planar system and e.g. a scaling leads for $\varepsilon = 0$ to a limiting Hamiltonian flow. The perturbation problem then becomes twofold. Equilibria still persist by Theorem 1 and hyperbolic equilibria moreover persist as such, with sum of eigenvalues of order $\mathcal{O}(\varepsilon)$. Also for elliptic eigenvalues the sum of eigenvalues is of order $\mathcal{O}(\varepsilon)$ after the perturbation, but here this number measures the dissipation whence the equilibrium becomes (weakly) attractive for negative values and (weakly) unstable for positive values. The 1–parameter families of periodic orbits of a Hamiltonian system do not persist under dissipative perturbations, the very fact that they form families imposes the corresponding fixed point of the Poincaré mapping to have an eigenvalue 1 and Theorem 2 does not apply. Typically only finitely many periodic orbits survive a dissipative perturbation and it is already a difficult task to determine their number.

3.1 Hamiltonian Perturbations

The Duffing oscillator has the Hamiltonian function

$$H(x,y) \;=\; \tfrac{1}{2}y^2 \;+\; \tfrac{1}{24}bx^4 \;+\; \tfrac{1}{2}\lambda x^2 \tag{2}$$

where b is a constant distinguishing the two cases $b = \pm 1$ and λ is a parameter. Under variation of the parameter the equations of motion

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\tfrac{1}{6}bx^3 - \lambda x \end{aligned}$$

display a Hamiltonian pitchfork bifurcation, supercritical for positive b and subcritical in case b is negative. Correspondingly, the linearization at the equilibrium $x = 0$ of the anharmonic oscillator $\lambda = 0$ is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

whence this equilibrium is parabolic.

The typical way in which a parabolic equilibrium bifurcates is the centre-saddle bifurcation. Here the Hamiltonian reads

$$H(x,y) \;=\; \tfrac{1}{2}ay^2 \;+\; \tfrac{1}{6}bx^3 \;+\; c\lambda x \tag{3}$$

where $a, b, c \in \mathbb{R}$ are nonzero constants, for instance $a = b = c = 1$. Note that this is a completely different unfolding of the parabolic equilibrium at the origin. A closer look at the phase portraits and in particular at the Hamiltonian function of the Hamiltonian pitchfork bifurcation reveals the symmetry $x \mapsto -x$ of the Duffing oscillator. This suggests to add the non-symmetric term μx. The resulting 2–paramter family

$$H_{\lambda,\mu}(x, y) \;=\; \tfrac{1}{2}y^2 \;+\; \tfrac{1}{24}bx^4 \;+\; \tfrac{1}{2}\lambda x^2 \;+\; \mu x$$

of Hamiltonian systems is indeed structurally stable. This implies not only that all equilibria of a Hamiltonian perturbation of the Duffing oscillator have a local flow equivalent to the local flow near a suitable equilibrium in this 2–paramter family, but that every 1–parameter family of $\mathbb{Z}_2$–symmetric Hamiltonian systems that is a perturbation of (2) has equivalent dynamics. For more details see [34] and references therein.

This approach applies *mutatis mutandi* to every non-degenerate planar singularity, cf. [133, 70]. At an equilibrium all partial derivatives of the Hamiltonian vanish and the resulting singularity is called non-degenerate if it has finite multiplicity, which implies that it admits a versal unfolding H_λ with finitely many parameters. The family of Hamiltonian systems defined by this versal unfolding contains all possible (local) dynamics that the initial equilibrium may be perturbed to. Imposing additional discrete symmetries is immediate, the necessary symmetric versal unfolding is obtained by averaging

$$H_\lambda^G \;=\; \frac{1}{|G|} \sum_{g \in G} H_\lambda \circ g$$

along the orbits of the symmetry group G.

3.2 Dissipative Perturbations

In a generic dissipative system all equilibria are hyperbolic. Qualitatively, i.e. up to topological equivalence, the local dynamics is completely determined by the number of eigenvalues with positive real part. Those hyperbolic equilibria that can appear in Hamiltonian systems (the eigenvalues forming pairs $\pm\nu$) do not play an important rôle. Rather, planar Hamiltonian systems become important as a tool to understand certain bifurcations triggered off by non-hyperbolic equilibria. Again this requires the system to depend on external parameters.

The simplest example is the Hopf bifurcation, a co-dimension one bifurcation where an equilibrium loses stability as the pair of eigenvalues crosses the imaginary axis, say at $\pm\mathrm{i}$. At the bifurcation the linearization is a Hamiltonian system with an elliptic equilibrium (the co-dimension one bifurcations where a single eigenvalues crosses the imaginary axis through 0 do not have a Hamiltonian linearization). This limiting Hamiltonian system has a 1–parameter family of periodic orbits around the equilibrium, and the non-linear terms determine the fate of these periodic orbits. The normal form of order 3 reads

$$\begin{aligned}
\dot{x} &= -y\left(1 + b(x^2 + y^2)\right) + x\left(\lambda + a(x^2 + y^2)\right) \\
\dot{x} &= x\left(1 + b(x^2 + y^2)\right) + y\left(\lambda + a(x^2 + y^2)\right)
\end{aligned}$$

and is Hamiltonian if and only if $(\lambda, a) = (0, 0)$. The sign of the coefficient distinguishes between the supercritical case $a > 0$, in which there are no periodic orbits coexisting with the attractive equilibria (i.e. when $\lambda < 0$) and one attracting periodic orbit for each $\lambda > 0$ (coexisting with the unstable equilibrium), and the subcritical case $a < 0$, in which the family of periodic orbits is unstable and coexists with the attractive equilibria (with no periodic orbits for parameters $\lambda > 0$). As $\lambda \to 0$ the family of periodic orbits shrinks down to the origin, so also this 'Hamiltonian feature' is preserved.

Equilibria with a double eigenvalue 0 need two parameters to persistently occur in families of dissipative systems. The generic case is the Takens–Bogdanov bifurcation. Here the linear part is too degenerate to be helpful, but the nonlinear Hamiltonian system defined by (3) with $a = 1 = c\lambda$ and $b = -3$ provides the periodic and heteroclinic orbit(s) that constitute the nontrivial part of the bifurcation diagram. Where discrete symmetries are present, e.g. for equilibria in dissipative systems originating from other generic bifurcations, the limiting Hamiltonian system exhibits that same discrete symmetry. For more details see [56, 67, 85] and references therein.

The continuation of certain periodic orbits from an unperturbed Hamiltonian system under dissipative perturbation can be based on Melnikov like methods, again see [67, 140]. As above, this often leads to Abelian integrals, for instance to count the number of periodic orbits that branch off.

3.3 Reversible Perturbations

A dynamical system that admits a reflection symmetry R mapping trajectories $\varphi(t, z_0)$ to trajectories $\varphi(-t, R(z_0))$ is called reversible. In the planar case we may restrict to the reversing reflection

$$R : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \mapsto & (x, -y) \end{array} . \tag{4}$$

All Hamiltonian functions $H = \frac{1}{2}y^2 + V(x)$ which have an interpretation 'kinetic + potential energy' are reversible, and in general the class of reversible systems is positioned 'between' the class of Hamiltonian systems and the class of dissipative systems. A guiding example is the perturbed Duffing oscillator (with the rôles of x and y exchanged so that (4) remains the reversing symmetry)

$$\begin{aligned} \dot{x} &= -\tfrac{1}{6}y^3 - y + \varepsilon xy \\ \dot{y} &= x \end{aligned}$$

that combines the Hamiltonian character of the equilibrium at the origin with the dissipative character of the two other equilibria. Note that all orbits 'outside' the homoclinic loop are periodic.

There are two ways in which the reversing symmetry (4) imposes a Hamiltonian character on the dynamics. An equilibrium that lies on the symmetry line $\{y = 0\}$ has a linearization that is itself a reversible system and consequently the eigenvalues are subject to the same constraints as in the Hamiltonian case. (For equilibria z_0 that do not lie on the symmetry line the reflection $R(z_0)$ is also an equilibrium, and it is to the union of their eigenvalues that these constraints still apply.) Furthermore, every orbit that crosses $\{y = 0\}$ more than once is automatically periodic, and these periodic orbits form 1–parameter families. In particular, elliptic equilibria are still surrounded by periodic orbits.

The dissipative character of a reversible system is most obvious for orbits that do not cross the symmetry line. Here R merely maps the orbit to a reflected counterpart. The above perturbed Duffing oscillator exemplifies that the character of an orbit crossing $\{y = 0\}$ exactly once is undetermined. While the homoclinic orbit of the saddle at the origin has a Hamiltonian character, the heteroclinic orbits between the other two equilibria behave like in a dissipative system.

4 Perturbations of Periodic Orbits

The perturbation of a one-degree-of-freedom system by a periodic forcing is a perturbation that changes the phase space. Treating the time variable t as a phase space variable leads to the extended phase space $\mathbb{S}^1 \times \mathbb{R}^2$ and equilibria of the unperturbed system become periodic orbits, inheriting the normal behaviour. Furthermore introducing an action conjugate to the 'angle' t yields a Hamiltonian system in two degrees of freedom.

While the 1–parameter families of periodic orbits merely provide the 'typical' recurrent motion in one degree of freedom, they form special solutions in two or more degrees of freedom. Arcs of elliptic periodic orbits are particularly instructive. Note that these occur generically in both the Hamiltonian and the reversible context.

4.1 Conservative Perturbations

Along the family of elliptic periodic orbits a pair $\mathrm{e}^{\pm \mathrm{i}\Omega}$ of Floquet multipliers passes regularly through roots of unity. Generically this happens on a dense set of parameters values, but for fixed denominator q in $\mathrm{e}^{\pm \mathrm{i}\Omega} = \mathrm{e}^{\pm 2\pi \mathrm{i} p/q}$ the corresponding energy values are isolated. The most important of such resonances are those with small denominators q.

For $q = 1$ generically a periodic centre-saddle bifurcation takes place where an elliptic and a hyperbolic periodic orbit meet at a parabolic periodic orbit. No periodic orbit remains under further variation of a suitable parameter.

The generic bifurcation for $q = 2$ is the period-doubling bifurcation where an elliptic periodic orbit turns hyperbolic (or *vice versa*) when passing through a parabolic periodic orbit with Floquet multipliers -1. Furthermore a family of periodic orbits with twice the period emerges from the parabolic periodic orbit, inheriting the normal linear behaviour from the initial periodic orbit.

In case $q = 3$, and possibly also for $q = 4$, generically two arcs of hyperbolic periodic orbits emerge, both with three (resp. four) times the period. One of these extends for lower and the other for higher parameter values. The initial elliptic periodic orbit momentarily loses its stability due to these approaching unstable orbits.

Denominators $q \geq 5$ (and also the second possibility for $q = 4$) lead to a pair of subharmonic periodic orbits of q times the period emerging either for lower or for higher parameter values. This is (especially for large q) comparable to the behaviour at Diophantine $\mathrm{e}^{\pm \mathrm{i}\Omega}$ where a family of invariant tori emerges, cf. § 5 below.

For a single pair $\mathrm{e}^{\pm \mathrm{i}\Omega}$ of Floquet multipliers this behaviour is traditionally studied for the (iso-energetic) Poincaré-mapping, cf. [95] and references therein. However, the above description remains true in higher dimensions, where additionally multiple pairs of Floquet multipliers may interact. An instructive example is the Lagrange top, the 'sleeping' motion of which is gyroscopically stabilized after a periodic Hamiltonian Hopf bifurcation; see [58] for more details.

4.2 Dissipative Perturbations

There exists a large class of local bifurcations in the dissipative setting, that can be arranged in a perturbation theory setting, where the unperturbed system is Hamiltonian. The arrangement consists of changes of variables and rescalings. An early example of this is the Bogdanov-Takens bifurcation [134, 135]. For other examples regarding nilpotent singularities, see [39, 40] and references therein.

To fix thoughts, consider families of planar maps and let the unperturbed Hamiltonian part contain a center (possibly surrounded by by a homoclinic loop). The question then is which of these persist when adding the dissipative perturbation.

Usually only a definite finite number persists. As in §2.4, a Melnikov function can be invoked here, possibly again leading to elliptic (Abelian) integrals, Picard Fuchs equations, etc. For details see [63, 127] and references therein.

5 Invariant Curves of Planar Diffeomorphisms

This section starts with general considerations on circle diffeomorphisms, in particular focussing on persistence properties of quasi-periodic dynamics. Our main references are [4, 72, 73, 142, 37, 36, 41]. For a definiton of rotation number see [60]. After this we turn to area preserving maps of an annulus where we discuss Moser's Twist Map Theorem [107], also see [37, 36, 41]. The section is concluded by a description of the holomorphic linearization of a fixed point in a planar map [7, 104, 144, 145].

Our main perspective will be perturbative, where we consider circle maps near a rigid rotation. It turns out that generally parameters are needed for persistence of quasi-periodicity under perturbations. In the area preserving setting we consider perturbations of a pure twist map.

5.1 Circle Maps

We start with the following general problem. Given a 2-parameter family

$$P_{\alpha,\varepsilon} : \mathbb{T}^1 \to \mathbb{T}^1,\ x \mapsto x + 2\pi\alpha + \varepsilon a(x, \alpha, \varepsilon)$$

of circle maps of class C^∞. It turns out to be convenient to view this 2-parameter family as a 1-parameter family of maps

$$P_\varepsilon : \mathbb{T}^1 \times [0, 1] \to \mathbb{T}^1 \times [0, 1],\ (x, \alpha) \mapsto (x + 2\pi\alpha + \varepsilon a(x, \alpha, \varepsilon), \alpha)$$

of the cylinder. Note that the unperturbed system P_0 is a family of rigid circle rotations, viewed as a cylinder map, where the individual map $P_{\alpha,0}$ has rotation number α. The question now is what will be the fate of this rigid dynamics for $0 \neq |\varepsilon| \ll 1$.

The classical way to address this question is to look for a conjugation Φ_ε, that makes the following diagram commute

$$\begin{array}{ccc} \mathbb{T}^1 \times [0,1] & \xrightarrow{P_\varepsilon} & \mathbb{T}^1 \times [0,1] \\ \uparrow \Phi_\varepsilon & & \uparrow \Phi_\varepsilon \\ \mathbb{T}^1 \times [0,1] & \xrightarrow{P_0} & \mathbb{T}^1 \times [0,1], \end{array}$$

i.e., such that

$$P_\varepsilon \circ \Phi_\varepsilon = \Phi_\varepsilon \circ P_0.$$

Due to the format of P_ε we take Φ_ε as a skew map

$$\Phi_\varepsilon(x, \alpha) = (x + \varepsilon U(x, \alpha, \varepsilon), \alpha + \varepsilon\sigma(\alpha, \varepsilon)),$$

which leads to the *nonlinear* equation

$$U(x + 2\pi\alpha, \alpha, \varepsilon) - U(x, \alpha, \varepsilon) = 2\pi\sigma(\alpha, \varepsilon) + a\,(x + \varepsilon U(x, \alpha, \varepsilon), \alpha + \varepsilon\sigma(\alpha, \varepsilon), \varepsilon)$$

in the unknown maps U and σ. Expanding in powers of ε and comparing at lowest order yields the linear equation

$$U_0(x + 2\pi\alpha, \alpha) - U_0(x, \alpha) = 2\pi\sigma_0(\alpha) + a_0(x, \alpha)$$

which can be directly solved by Fourier-series. Indeed, writing

$$a_0(x, \alpha) = \sum_{k \in \mathbf{Z}} a_{0k}(\alpha) \mathrm{e}^{\mathrm{i}kx}, U_0(x, \alpha) = \sum_{k \in \mathbf{Z}} U_{0k}(\alpha) \mathrm{e}^{\mathrm{i}kx}$$

we find $\sigma_0 = -\frac{1}{2\pi} a_{00}$ and

$$U_{0k}(\alpha) = \frac{a_{0k}(\alpha)}{\mathrm{e}^{2\pi\mathrm{i}k\alpha} - 1}.$$

It follows that in general a formal solution exists if and only if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Still, the accumulation of $\mathrm{e}^{2\pi\mathrm{i}k\alpha} - 1$ on 0 leads to the celebrated *small divisors* [9, 111], also see [37, 36, 57, 41].

The classical solution considers the following Diophantine nonresonance conditions. Fixing $\tau > 2$ and $\gamma > 0$ consider $\alpha \in [0, 1]$ such that for all rationals p/q

$$\left| \alpha - \frac{p}{q} \right| \geq \gamma q^{-\tau}. \tag{5}$$

This subset of such α's is denoted by $[0, 1]_{\tau,\gamma}$ and is well-known to be nowhere dense but of large measure as $\gamma > 0$ gets small [118]. Note that Diophantine numbers are irrational.

Theorem 3. (Circle Map Theorem) *For γ sufficiently small and for the perturbation εa sufficiently small in the C^∞-topology, there exists a C^∞ transformation $\Phi_\varepsilon : \mathbb{T}^1 \times [0, 1] \to \mathbb{T}^1 \times [0, 1]$, conjugating the restriction $P_0|_{[0,1]_{\tau,\gamma}}$ to a subsystem of P_ε.*

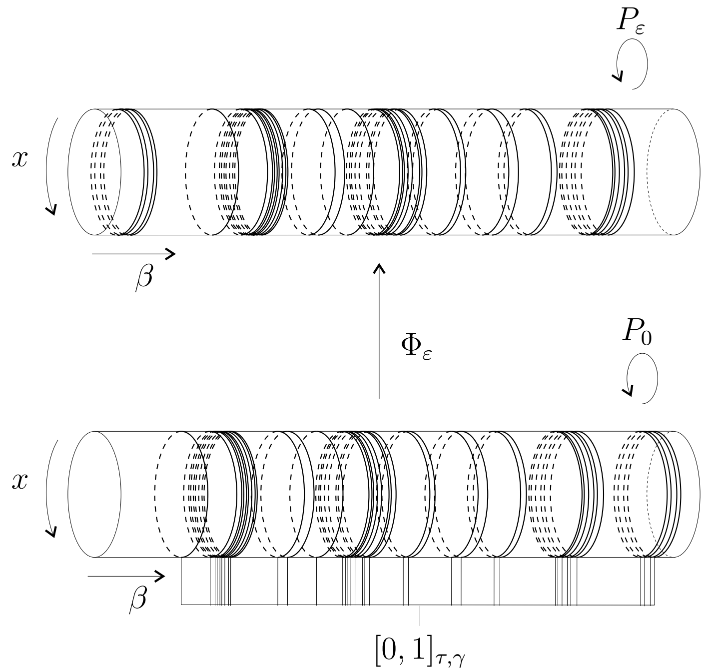

Figure 2: Skew cylinder map, conjugating (Diophantine) quasi-periodic invariant circles of P_0 and P_ε.

Theorem 3 in the present structural stability formulation (compare with Figure 2) is a special case of the results in [37, 36]. We here speak of *quasi-periodic stability*. For earlier versions see [4, 9].

Remarks.

- Rotation numbers are preserved by the map Φ_ε and irrational rotation numbers correspond to quasi-periodicity. Theorem 3 thus ensures that *typically* quasi-periodicity occurs with *positive* measure in the parameter space. Note that since Cantor sets are perfect, quasi-periodicity typically has a non-isolated occurrence.

- The map Φ_ε has no dynamical meaning inside the gaps. The gap dynamics in the case of circle maps can be illustrated by the Arnold family of circle maps [4, 7, 60], given by

$$P_{\alpha,\varepsilon}(x) = x + 2\pi\alpha + \varepsilon \sin x$$

  which exhibits a countable union of open resonance tongues where the dynamics is periodic, see Figure 3. Note that this map only is a diffeomorphism for $|\varepsilon| < 1$.

- We like to mention that non-perturbative versions of Theorem 3 have been proven in [72, 73, 142].

- For simplicity we formulated Theorem 3 under C^∞-regularity, noting that there exist many ways to generalize this. On the one hand there exist C^k-versions for finite k and on the other hand there exist fine tunings in terms of real-analytic and

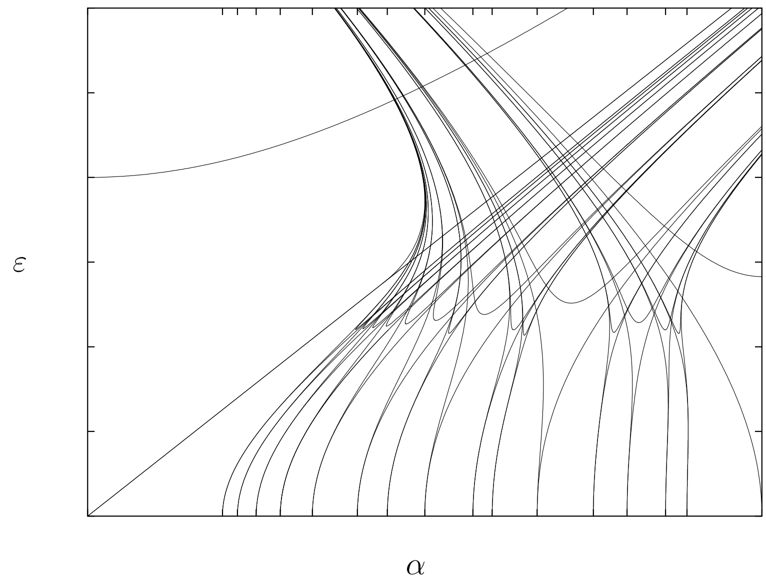

Figure 3: Arnold resonance tongues; for $\varepsilon \geq 1$ the maps are endomorphic.

Gevrey regularity. For details wer refer to [36, 41] and references therein. This same remarks applies to other results in this section and in §6 on KAM Theory.

A possible application of Theorem 3 runs as follows. Consider a system of weakly coupled Van der Pol oscillators

$$\begin{aligned} \ddot{y}_1 + c_1\dot{y}_1 + a_1 y_1 + f_1(y_1, \dot{y}_1) &= \varepsilon g_1(y_1, y_2, \dot{y}_1, \dot{y}_2) \\ \ddot{y}_2 + c_2\dot{y}_2 + a_2 y_2 + f_2(y_2, \dot{y}_2) &= \varepsilon g_2(y_1, y_2, \dot{y}_1, \dot{y}_2). \end{aligned}$$

Writing $\dot{y}_j = z_j, j = 1, 2$, one obtains a vector field in the 4-dimensional phase space $\mathbb{R}^2 \times \mathbb{R}^2 = \{(y_1, z_1), (y_2, z_2)\}$. For $\varepsilon = 0$ this vector field has an invariant 2-torus, which is the product of the periodic motions of the individual Van der Pol oscillations. This 2-torus is normally hyperbolic and therefore persistent for $|\varepsilon| \ll 1$ [75]. In fact the torus is an attractor and we can define a Poincaré return map within this torus attractor. If we include some of the coefficients of the equations as parameters, Theorem 3 is directly applicable. The above statements on quasi-periodic circle maps then directly translate to the case of quasi-periodic invariant 2-tori. Concerning the resonant cases, generically a tongue structure like in Figure 3 occurs; for the dynamics corresponding to parameter values inside such a tongue one speaks of *phase lock*.

Remarks.

- The celebrated synchronisation of Huygens's clocks [79] is related to a 1:1 resonance, meaning that the corresponding Poincaré map would have its parameters in the 'main' tongue with rotation number 0. Compare with Figure 3.

- There exist direct generalizations to cases with n-oscillators ($n \in \mathbb{N}$), leading to families of invariant n-tori carrying quasi-periodic flow, forming a nowhere dense set of positive measure. An alteration with resonance occurs as roughly sketched in Figure 3. In higher dimension the gap dynamics, apart from periodicity, also can contain strange attractors [115, 129]. We shall come back to this subject in a later section.

5.2 Area-Preserving Maps

The above setting historically was preceeded by an area preserving analogue [107] that has its origin in the Hamiltonian dynamics of frictionless mechanics.

Let $\Delta \subseteq \mathbb{R}^2 \setminus \{(0,0)\}$ be an annulus, with sympectic polar coordinates $(\varphi, I) \in \mathbb{T}^1 \times \mathbf{K}$, where $\mathbf{K}$ is an interval. Moreover, let $\sigma = d\varphi \wedge dI$ be the area form on Δ.

We consider a σ-preserving smooth map $P_\varepsilon : \Delta \to \Delta$ of the form

$$P_\varepsilon(\varphi, I) = (\varphi + 2\pi\alpha(I), I) + O(\varepsilon),$$

where we assume that the map $I \mapsto \alpha(I)$ is a (local) diffeomorphism. This assumption is known as the *twist condition* and P_ε is called a *twist map*. For the unperturbed case $\varepsilon = 0$ we are dealing with a pure twist map and its dynamics is comparable to the unperturbed family of cylinder maps as met in §5.1. Indeed it is again a family of rigid rotations, parametrized by I and where $P_0(., I)$ has rotation number $\alpha(I)$. In this case the question is what will be the fate of this family of invariant circles, as well as with the corresponding rigidly rotational dynamics.

Regarding the rotation number we again introduce Diophantine conditions. Indeed, for $\tau > 2$ and $\gamma > 0$ the subset $[0,1]_{\tau,\gamma}$ is defined as in (5), i.e., it contains all $\alpha \in [0,1]$, such that for all rationals p/q

$$\left| \alpha - \frac{p}{q} \right| \geq \gamma q^{-\tau}.$$

Pulling back $[0,1]_{\tau,\gamma}$ along the map α we obtain a subset $\Delta_{\tau,\gamma} \subseteq \Delta$.

Theorem 4. [107] (Twist Map Theorem) *For γ sufficiently small and for the perturbation $O(\varepsilon)$ sufficiently small in C^∞-topology, there exists a C^∞ transformation $\Phi_\varepsilon : \Delta \to \Delta$, conjugating the restriction $P_0|_{\Delta_{\tau,\gamma}}$ to a subsystem of P_ε.*

As in the case of Theorem 3 again we chose the formulation of [37, 36]. Largely the remarks following Theorem 3 also apply here.

Remarks.

- Compare the format of the Theorems 3 and 4 and observe that in the latter case the role of the parameter α has been taken by the 'action' variable I. Theorem 4 implies that *typically* quasi-periodicity occurs with positive measure in phase space.

- In the gaps typically we have coexistence of periodicity, quasi-periodicity and chaos [9, 6, 110, 111, 29, 126, 140]. The latter follows from transversality of homo- and heteroclinic connections that give rise to positive topological entropy. Open problems are whether the corresponding Lyapunov exponents also are positive, compare with the discussion at the end of §2.

Similar to the applications of Theorem 3 given at the end of §5.1, here direct applications are possible in the conservative setting. Indeed, consider a system of weakly coupled pendula

$$\begin{aligned}\ddot{y}_1 + \alpha_1^2 \sin y_1 &= \varepsilon \frac{\partial U}{\partial y_1}(y_1, y_2) \\ \ddot{y}_2 + \alpha_2^2 \sin y_2 &= \varepsilon \frac{\partial U}{\partial y_2}(y_1, y_2).\end{aligned}$$

Writing $\dot{y}_j = z_j$, $j = 1, 2$ as before, we again get a vector field in the 4-dimensional phase space $\mathbb{R}^2 \times \mathbb{R}^2 = \{(y_1, y_2), (z_1, z_2)\}$. In this case the energy

$$H_\varepsilon(y_1, y_2, z_1, z_2) = \tfrac{1}{2}z_1^2 + \tfrac{1}{2}z_2^2 - \alpha_1^2 \cos y_1 - \alpha_2^2 \cos y_2 + \varepsilon U(y_1, y_2)$$

is a constant of motion. Restricting to a 3-dimensional energy surface $H_\varepsilon^{-1} =$ const., the iso-energetic Poincaré map P_ε is a Twist Map and application of Theorem 4 yields the conclusion of quasi-periodicity (on invariant 2-tori) occurring with positive measure in the energy surfaces of H_ε.

Remark. As in the dissipative case this example directly generalizes to cases with n oscillators ($n \in \mathbb{N}$), again leading to invariant n-tori with quasi-periodic flow. We shall return to this subject in a later section.

5.3 Linearization of Complex Maps

The §§5.1 and 5.2 both deal with smooth circle maps that are conjugated to rigid rotations. Presently the concern is with planar holomorphic maps that are conjugated to a rigid rotation on an open subset of the plane. Historically this is the first time that a small divisor problem was solved [7, 104, 144, 145].

5.3.1 Complex linearization

Given is a holomorphic germ $F : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ of the form $F(z) = \lambda z + f(z)$, with $f(0) = f'(0) = 0$. The problem is to find a biholomorphic germ $\Phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that

$$\Phi \circ F = \lambda \cdot \Phi.$$

Such a diffeomorphism Φ is called a *linearization* of F near 0.

We begin with the formal approach. Given the series $f(z) = \sum_{j \geq 2} f_j z^j$, we look for $\Phi(z) = z + \sum_{j \geq 2} \phi_j z^j$. It turns out that a solution always exists whenever $\lambda \neq 0$ is not a root of unity. Indeed, direct computation reveals the following set of equations that can be solved recursively:

For $j = 2:$ get the equation $\lambda(1 - \lambda)\phi_2 = f_2$

For $j = 3:$ get the equation $\lambda(1 - \lambda^2)\phi_3 = f_3 + 2\lambda f_2 \phi_2$

For $j = n:$ get the equation$\lambda(1 - \lambda^{n-1})\phi_n = f_n +$ known.

The question now reduces to whether this formal solution has positive radius of convergence.

The hyperbolic case $0 < |\lambda| \neq 1$ was already solved by Poincaré, for a description see [7]. The elliptic case $|\lambda| = 1$ again has small divisors and was solved by Siegel when for some $\gamma > 0$ and $\tau > 2$ we have the Diophantine nonresonance condition

$$|\lambda - \mathrm{e}^{2\pi \mathrm{i}\frac{p}{q}}| \geq \gamma |q|^{-\tau}.$$

The corresponding set of λ constitutes a set of full measure in $\mathbb{T}^1 = \{\lambda\}$.

Yoccoz [144] completely solved the elliptic case using the Bruno-condition. If

$$\lambda = e^{2\pi i\alpha} \text{ and when } \frac{p_n}{q_n}$$

is the nth convergent in the continued fraction expansion of α then the Bruno-condition reads

$$\sum_n \frac{\log(q_{n+1})}{q_n} < \infty.$$

This condition turns out to be necessary and sufficient for Φ having positive radius of convergence [144, 145].

5.3.2 Cremer's example in Herman's version

As an example consider the map

$$F(z) = \lambda z + z^2,$$

where $\lambda \in \mathbb{T}^1$ is not a root of unity.

Observe that a point $z \in \mathbb{C}$ is a periodic point of F with period q if and only if $F^q(z) = z$, where obviously

$$F^q(z) = \lambda^q z + \cdots + z^{2^q}.$$

Writing

$$F^q(z) - z = z(\lambda^q - 1 + \cdots + z^{2^q-1}),$$

the period q periodic points exactly are the roots of the right hand side polynomial. Abbreviating $N = 2^q - 1$, it directly follows that, if $z_1, z_2, \ldots, z_N$ are the nontrivial roots, then for their product we have

$$z_1 \cdot z_2 \cdot \cdots \cdot z_N = \lambda^q - 1.$$

It follows that there exists a nontrivial root within radius

$$|\lambda^q - 1|^{1/N}$$

of $z = 0$.

Now consider the set of $\Lambda \subset \mathbb{T}^1$ defined as follows: $\lambda \in \Lambda$ whenever

$$\operatorname{liminf}_{q\to\infty} |\lambda^q - 1|^{1/N} = 0.$$

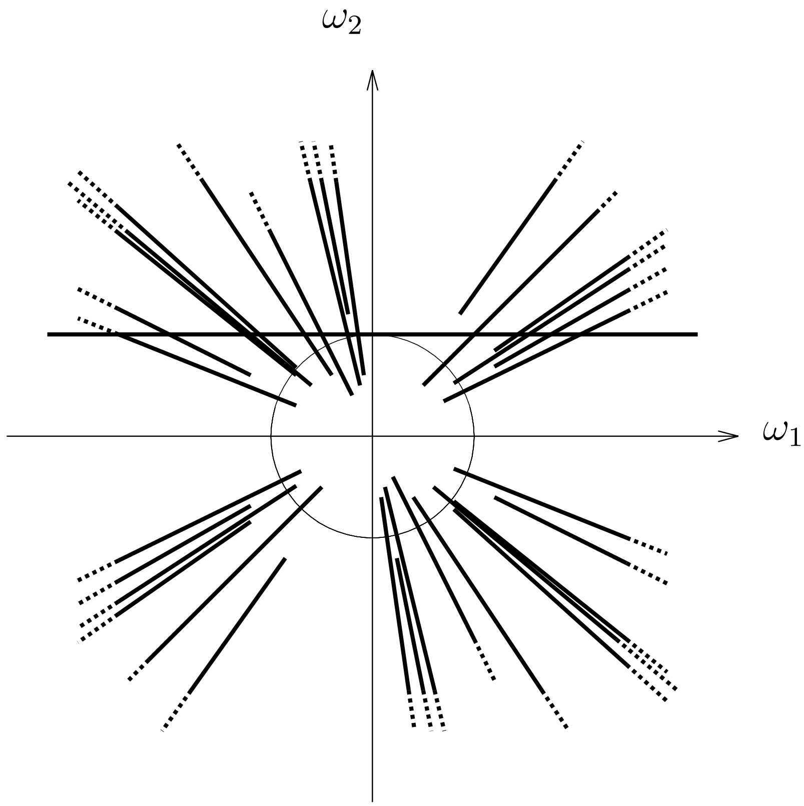

Figure 4: The Diophantine set $\mathbb{R}^n_{\tau,\gamma}$ has the closed half line geometry and the intersection $\mathbb{S}^{n-1} \cap \mathbb{R}^n_{\tau,\gamma}$ is a Cantor set of measure $\mathbb{S}^{n-1} \setminus \mathbb{R}^n_{\tau,\gamma} = O(\gamma)$ as $\gamma \downarrow 0$.

It can be directly shown that Λ is *residual*, again compare with [118]. It also follows that for $\lambda \in \Lambda$ linearization is impossible. Indeed, since the rotation is irrational, the existence of periodic points in any neighbourhood of $z = 0$ implies zero radius of convergence.

Remarks.

- Notice that the residual set Λ is in the complement of the full measure set of all Diophantine numbers, again see [118].

- Considering $\lambda \in \mathbb{T}^1$ as a parameter, we see a certain analogy of these results on complex linearization with the Theorems 3 and 4. Indeed, in this case for a full measure set of λ's on a neighbourhood of $z = 0$ the map $F = F_\lambda$ is conjugated to a rigid irrational rotation.

  Such a domain in the z-plane often is referred to as a Siegel disc. For a more general discussion of these and of Herman rings, see [104].

6 KAM Theory: an overview

In §5 we described the persistent occurrence of quasi-periodicity in the setting of diffeomorphisms of the circle or the plane. The general Perturbation Theory of quasi-periodic motions is known under the name Kolmogorov-Arnold-Moser (or KAM) Theory and discussed extensively in [54]. Presently we briefly summarize parts of this KAM Theory in broad terms, as this fits in our considerations, thereby largely referring to [82, 83, 3, 122, 124, 146, 147], also see [21, 41, 57].

In general quasi-periodicity is defined by a smooth conjugation. First on the n-torus $\mathbb{T}^n = \mathbb{R}^n/(2\pi Z)^n$ consider the vector field

$$\mathbb{X}_\omega = \sum_{j=1}^{n} \omega_j \frac{\partial}{\partial \varphi_j},$$

where $\omega_1, \omega_2, \ldots, \omega_n$ are called frequencies [109, 24]. Now, given a smooth (say, of class C^∞) vector field X on a manifold M, with $T \subseteq M$ an invariant n-torus, we say that the restriction $X|_T$ is *parallel* if there exists $\omega \in \mathbb{R}^n$ and a smooth diffeomorphism $\Phi : T \to \mathbb{T}^n$, such that $\Phi_*(X|_T) = \mathbb{X}_\omega$. We say that $X|_T$ is *quasi-periodic* if the frequencies $\omega_1, \omega_2, \ldots, \omega_n$ are independent over $\mathbb{Q}$.

A quasi-periodic vector field $X|_T$ leads to an integer affine structure on the torus T. In fact, since each orbit is dense, it follows that the self conjugations of $\mathbb{X}_\omega$ exactly are the translations of $\mathbb{T}^n$, which completely determine the affine structure of $\mathbb{T}^n$. Then, given $\Phi : T \to \mathbb{T}^n$ with $\Phi_*(X|_T) = \mathbb{X}_\omega$, it follows that the self conjugations of $X|_T$ determines a natural affine structure on the torus T. Note that the conjugation Φ is unique modulo translations in T and $\mathbb{T}^n$.

Note that the composition of Φ by a translation of $\mathbb{T}^n$ does not change the frequency vector ω. However, the compostion by a linear invertible map $S \in \mathrm{GL}(n, \mathbb{Z})$ yields $S_*\mathbb{X}_\omega = \mathbb{X}_{S\omega}$. We here speak of an *integer affine* structure [24].

Remarks.

- The transition maps of an integer affine structure are translations and elements of $\mathrm{GL}(n, \mathbb{Z})$.
- The current construction is compatible with the integrable affine structure on the the Liouville tori of an integrable Hamiltonian systems [6]. Note that in that case the structure extends to all parallel tori.

6.1 Classical KAM Theory

The classical KAM Theory deals with smooth, nearly integrable Hamiltonian systems of the form

$$\begin{aligned} \dot{\varphi} &= \omega(I) + \varepsilon f(I, \varphi, \varepsilon) \\ \dot{I} &= \varepsilon g(I, \varphi, \varepsilon), \end{aligned} \tag{6}$$

where I varies over an open subset of $\mathbb{R}^n$ and φ over the standard torus $\mathbb{T}^n$. Note that for $\varepsilon = 0$ the phase space as an open subset of $\mathbb{R}^n \times \mathbb{T}^n$, is foliated by invariant tori, parametrized by I. Each of the tori is parametrized by φ and the corresponding motion is parallel (or multi- periodic or conditionally periodic) with frequency vector $\omega(I)$.

Perturbation Theory asks for persistence of the invariant n-tori and the parallelity of their motion for small values of $|\varepsilon|$. The answer that KAM Theory gives needs two essential ingredients. The first ingredient is that of *Kolmogorov non-degeneracy* which states that the map $I \in \mathbb{R}^n \mapsto \omega(I) \in \mathbb{R}^n$ is a (local) diffeomorphism. Compare with the twist condition of §5. The second ingredient generalizes the Diophantine conditions (5) of §5 as follows: for $\tau > n - 1$ and $\gamma > 0$ consider the set

$$\mathbb{R}^n_{\tau,\gamma} = \{\omega \in \mathbb{R}^n \mid |\langle \omega, k\rangle| \geq \gamma |k|^{-\tau}, k \in \mathbb{Z}^n \setminus \{0\}\}. \tag{7}$$

The following properties are more or less direct. First $\mathbb{R}^n_{\tau,\gamma}$ has a closed half line geometry in the sense that if $\omega \in \mathbb{R}^n_{\tau,\gamma}$ and $s \geq 1$ then also $s\omega \in \mathbb{R}^n_{\tau,\gamma}$. Moreover, the intersection $\mathbb{S}^{n-1} \cap \mathbb{R}^n_{\tau,\gamma}$ is a Cantor set of measure $\mathbb{S}^{n-1} \setminus \mathbb{R}^n_{\tau,\gamma} = O(\gamma)$ as $\gamma \downarrow 0$, see Figure 4.

Completely in the spirit of Theorem 4, the classical KAM Theorem roughly states that a Kolmogorov non-degenerate nearly integrable system $(6)_\varepsilon$, for $|\varepsilon| \ll 1$ is smoothly conjugated to the unperturbed version $(6)_0$, provided that the frequency map ω is co-restricted to the Diophantine set $\mathbb{R}^n_{\tau,\gamma}$. In this formulation smoothness has to be taken in the sense of Whitney [122, 139], also compare with [124, 37, 36, 21, 41, 57].

As a consequence we may say that in Hamiltonian systems of n degrees of freedom typically quasi-periodic invariant (Lagrangian) n-tori occur with positive measure in phase space. It should be said that also an iso-energetic version of this classical result exists, implying a similar conclusion restricted to energy hypersurfaces [9, 6, 35, 41]. The Twist Map Theorem 4 is closely related to the iso-energetic KAM Theorem.

Remarks.

- We chose the quasi-periodic stability format as in §5. For regularity issues compare with a remark following Theorem 3.
- For applications we largely refer to §2 and to [36, 41] and references therein.
- Continuing the discussion on affine structures at the beginning of this section, we mention that by means of the symplectic form, the domain of the I–variables in $\mathbb{R}^n$ inherits an affine structure [62], also see [94] and references therein.

Statistical Mechanics deals with particle systems that are large, often infinitely large. The *Ergodic Hypothesis* roughly says that in a bounded energy hypersurface, the dynamics are ergodic, meaning that any evolution in the energy level set comes near every point of this set.

The taking of limits as the number of particles tends to infinity is a notoriously difficult subject. Here we discuss a few direct consequences of classical KAM Theory for many degrees of freedom. This discussion starts with Kolmogorov's papers [83, 82], which we now present in a slightly rephrased form. First, we recall that for Hamiltonian systems (say, with n degrees of freedom), typically the union of Diophantine quasi-periodic Lagrangean invariant n-tori fills up positive measure in the phase space and also in the energy hypersurfaces. Second, such a collection of KAM tori immediately gives rise to non-ergodicity, since it clearly implies the existence of distinct invariant sets of positive measure. For background on Ergodic Theory, see e.g. [9, 49] and [41] for more references. Apparently the KAM tori form an 'obstruction' to ergodicity, and a question is how bad this obstruction is as $n \to \infty$. Results in [5, 80] indicate that this KAM Theory obstruction is not too bad as the size of the system tends to infinity. In general the role of the Ergodic Hypothesis in Statistical Mechanics has turned out to be much more subtle than was expected, see e.g. [17, 65].

6.2 Dissipative KAM theory

As already noted by Moser [108, 109], KAM Theory extends outside the world of Hamiltonian systems, like to volume preserving systems, or to equivariant or reversible systems. This also holds for the class of general smooth systems, often called 'dissipative'. In fact, the KAM Theorem allows for a Lie algebra proof, that can be used to cover all these special cases [37, 36, 41, 33]. It turns out that in many cases parameters are needed for persistent occurrence of (Diophantine) quasi-periodic tori.

As an example we now consider the dissipative setting, where we discuss a parametrized system with normally hyperbolic invariant n-tori carrying quasi-periodic motion. From [75] it follows that this is a persistent situation and that, up to a smooth (in this case of class C^k for large k) diffeomorphism, we can retrict to the case where $\mathbb{T}^n$ is the phase space. To fix thoughts we consider the smooth system

$$\begin{aligned} \dot{\varphi} &= \omega(\mu) + \varepsilon f(\varphi, \mu, \varepsilon) \\ \dot{\mu} &= 0, \end{aligned} \tag{8}$$

where $\mu \in \mathbb{R}^n$ is a multi-parameter. The results of the classical KAM Theorem regarding $(6)_\varepsilon$ largely carry over to $(8)_{\mu,\varepsilon}$.

Now, for $\varepsilon = 0$ the product of phase space and parameter space as an open subset of $\mathbb{T}^n \times \mathbb{R}^n$ is completely foliated by invariant n-tori and since the perturbation does not concern the $\dot{\mu}$–equation, this foliation is persistent. The interest is with the dynamics on the resulting invariant tori that remains parallel after the perturbation, compare with the setting of Theorem 3. As just stated, KAM Theory here gives a solution similar to the Hamiltonian case. The analogue of the Kolmogorov non-degeneracy condition here is that the frequency map $\mu \mapsto \omega(\mu)$ is a (local) diffeomorphism. Then, in the spirit of Theorem 3, we state that the system $(8)_{\mu,\varepsilon}$ is smoothly conjugated to $(8)_{\mu,0}$, as before, provided that the map ω is co-restricted to the Diophantine set $\mathbb{R}^n_{\tau,\gamma}$. Again the smoothness has to be taken in the sense of Whitney [122, 146, 147, 37, 139], also see [36, 21, 41, 57].

It follows that the occurrence of normally hyperbolic invariant tori carrying (Diophantine) quasi-periodic flow is typical for families of systems with sufficiently many parameters, where this occurrence has positive measure in parameter space. In fact, if the number of parameters equals the dimension of the tori, the geometry as sketched in Figure 4 carries over in a diffeomorphic way.

Remarks.

- Many remarks following §6.1 and Theorem 3 also hold here.

- In cases where the system is degenerate, for instance because there is 'lack of parameters', a path formalism can be invoked, where the parameter 'path' is required to be a generic subfamily of the Diophantine set $\mathbb{R}^n_{\tau,\gamma}$, see Figure 4. This amounts to the Rüssmann nondegeneracy, that still gives positive measure of quasi-periodicity in the parameter space, compare with [36, 41] and references therein.

- In the dissipative case the KAM Theorem gives rise to families of quasi-periodic attractors in a typical way. This is of importance in center manifold reductions of infinite dimensional dynamics as, e.g., in fluid mechanics [128, 129]. In §8 we shall return to this subject.

6.3 Lower dimensional tori

We extend the above approach to the case of lower dimensional tori, i.e., where the dynamics transversal to the tori is also taken into account. We largely follow the set-up of [37, 33] that follows Moser [109]. Also see [36, 41] and references therein. Changing notation a little, we now consider the phase space $\mathbb{T}^n \times \mathbb{R}^m = \{x(\bmod 2\pi), y\}$, as well a parameter space $\{\mu\} = P \subset \mathbb{R}^s$. We consider a C^∞-family of vector fields $X(x, y, \mu)$ as before, having $\mathbb{T}^n \times \{0\} \subset \mathbb{T}^n \times \mathbb{R}^m$ as an invariant n-torus for $\mu = \mu_0 \in P$.

$$\begin{aligned} \dot{x} &= \omega(\mu) + f(y, \mu) \\ \dot{y} &= \Omega(\mu)\, y + g(y, \mu) \\ \dot{\mu} &= 0, \end{aligned} \tag{9}$$

with $f(y, \mu_0) = O(|y|)$ and $g(y, \mu_0) = O(|y|^2)$, so we assume the invariant torus to be of Floquet type.

The system $X = X(x, y, \mu)$ is integrable in the sense that it is $\mathbb{T}^n$-symmetric, i.e., x-independent [37]. The interest is with the fate of the invariant torus $\mathbb{T}^n \times \{0\}$ and its parallel dynamics under small perturbation to a system $\tilde{X} = \tilde{X}(x, y, \mu)$ that no longer needs to be integrable.

Consider the smooth mappings $\omega : P \to \mathbb{R}^n$ and $\Omega : P \to \mathrm{gl}\,(m, \mathbb{R})$. To begin with we restrict to the case where all eigenvalues of $\Omega(\mu_0)$ are simple and nonzero. In general for such a matrix $\Omega \in \mathrm{gl}\,(m, \mathbb{R})$, let the eigenvalues be given by $\alpha_1 \pm \mathrm{i}\beta_1, \ldots, \alpha_{N_1} \pm \mathrm{i}\beta_{N_1}$ and $\delta_1, \ldots, \delta_{N_2}$, where all α_j, β_j and δ_j are real and hence $m = 2N_1 + N_2$. Also consider the map $\mathrm{spec} : \mathrm{gl}\,(m, \mathbb{R}) \to \mathbb{R}^{2N_1+N_2}$, given by $\Omega \mapsto (\alpha, \beta, \delta)$. Next to the internal frequency vector $\omega \in \mathbb{R}^n$, we also have the vector $\beta \in \mathbb{R}^{N_1}$ of normal frequencies.

The present analogue of Kolmogorov non-degeneracy is the Broer-Huitema-Takens (BHT) non-degeneracy condition [37, 130], which requires that the product map $\omega \times (\mathrm{spec}\,) \circ \Omega : P \to \mathbb{R}^n \times \mathrm{gl}\,(m, \mathbb{R})$ at $\mu = \mu_0$ has a surjective derivative and hence is a local submersion [74].

Furthermore, we need Diophantine conditions on both the internal and the normal frequencies, generalizing (7). Given $\tau > n-1$ and $\gamma > 0$, it is required for all $k \in \mathbb{Z}^n \setminus \{0\}$ and all $\ell \in \mathbb{Z}^{N_1}$ with $|\ell| \leq 2$ that

$$|\langle k, \omega \rangle + \langle \ell, \beta \rangle| \geq \gamma |k|^{-\tau}. \tag{10}$$

Inside $\mathbb{R}^n \times \mathbb{R}^{N_1} = \{\omega, \beta\}$ this yields a 'Cantor set' as before (compare Figure 4). This set has to be 'pulled back' along the submersion $\omega \times (\text{spec}\,) \circ \Omega$, for examples see §§7.2 and 8.1 below.

The KAM Theorem for this setting is quasi-periodic stability of the n-tori under consideration, as in §6.2, yielding typical examples where quasi-periodicity has positive measure in parameter space. In fact, we get a little more here, since the normal linear behaviour of the n-tori is preserved by the Whitney smooth conjugations. This is expressed as normal linear stability, which is of importance for quasi-periodic bifurcations, see §8.1 below.

Remarks.

- A more general set-up of the normal stability theory [33] adapts the above to the case of non-simple (multiple) eigenvalues. Here the BHT non-degeneracy condition is formulated in terms of versal unfoldings of the matrix $\Omega(\mu_0)$ [7]. For possible conditions under which vanishing eigenvalues are admissible see [37, 70, 23] and references therein.

- This general set-up allows for a stucture preserving formulation as mentioned earlier, thereby including the Hamiltonian and volume preserving case, as well as equivariant and reversible cases. This allows, for example, to deal with quasi-periodic versions of the Hamiltonian and the reversible Hopf bifurcation [22, 23, 27, 28].

- The Parametrized KAM Theory discussed here *a priori* needs many parameters. In many cases the parameters are 'distinguished' in the sense that they are given by action variables, etc. For an example see §7.2 on Hamiltonian $(n-1)$–tori Also see [130] and [36, 41] where the case of Rüssmann non-degeneracy is included. This generalizes a remark at the end of §6.2.

6.4 Global KAM Theory

We stay in the Hamiltonian setting, considering Lagrangian invariant n-tori as these occur in a Liouville integrable system with n degrees of freedom. The union of these tori forms a smooth $\mathbb{T}^n$-bundle $f : M \to B$ (where we leave out all singular fibres). It is known that this bundle can be non-trivial [62, 58] as can be measured by monodromy and Chern class. In this case global action angle variables are not defined. This non-triviality, among other things, is of importance for semi-classical versions of the classical system at hand, in particular for certain spectrum defects [59, 137, 138], for more references also see [41].

Restricting to the classical case, the problem is what happens to the (non-trivial) $\mathbb{T}^n$-bundle f under small, non-integrable perturbation. From the classical KAM Theory, see §6.1 we already know that on trivializing charts of f Diophantine quasi-periodic n-tori

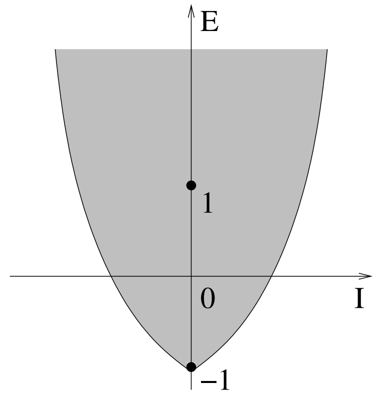

Figure 5: Range of the energy-momentum map of the spherical pendulum.

persist. In fact, at this level, a Whitney smooth conjugation exists between the integrable system and its perturbation, which is even Gevrey regular [139]. It turns out that these 'local' KAM conjugations can be glued together so to obtain a global conjugation at the level of quasi-periodic tori, thereby implying global quasi-periodic stability [24]. Here we need unicity of KAM tori, i.e., independence of the action-angle chart used in the classical KAM Theorem [48]. The proof uses the integer affine structure on the quasi-periodic tori, which enables taking convex combinations of the local conjugations subjected to a suitable partition of unity [74, 132]. In this way the geometry of the integrable bundle can be carried over to the nearly-integrable one.

The classical example of a Liouville integrable system with non-trivial monodromy [62, 58] is the spherical pendulum, which we now briefly revisit. The configuration space is $\mathbb{S}^2 = \{q \in \mathbb{R}^3 \mid \langle q, q \rangle = 1\}$ and the phase space $T^*\mathbb{S}^2 \cong \{(q,p) \in \mathbb{R}^6 \mid \langle q, q \rangle = 1 \text{ and } \langle q, p \rangle = 0\}$. The two integrals $I = q_1p_2 - q_2p_1$ (angular momentum) and $E = \frac{1}{2}\langle p, p \rangle + q_3$ (energy) lead to an energy momentum map $\mathcal{EM} : T^*\mathbb{S}^2 \to \mathbb{R}^2$, given by $(q,p) \mapsto (I, E) = \left(q_1p_2 - q_2p_1, \frac{1}{2}\langle p, p \rangle + q_3\right)$. In Figure 5 we show the image of the map $\mathcal{EM}$. The shaded area B consists of regular values, the fibre above which is a Lagrangian 2-torus; the union of these gives rise to a bundle $f : M \to B$ as described before, where $f = \mathcal{EM}|_M$. The motion in the 2-tori is a superposition of Huygens's rotations and pendulum-like swinging, and the non-existence of global action angle variables reflects that the three interpretations of 'rotating oscillation', 'oscillating rotation' and 'rotating rotation' cannot be reconciled in a consistent way. The singularities of the fibration include the equilibria $(q, p) = ((0, 0, \pm 1), (0, 0, 0)) \mapsto (I, E) = (0, \pm 1)$. The boundary of this image also consists of singular points, where the fibre is a circle that corresponds to Huygens's horizontal rotations of the pendulum. The fibre above the upper equilibrium point $(I, E) = (0, 1)$ is a pinched torus [58], leading to non-trival monodromy, in a suitable bases of the period lattices, given by

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}).$$

The question here is what remains of the bundle f when the system is perturbed. Here we observe that locally Kolmogorov non-degeneracy is implied by the non-trivial monodromy

[117, 125]. From [24, 125] it follows that the non-trival monodromy can be extended in the perturbed case.

Remarks.

- The case where this perturbation remains integrable is covered in [98], but presently the interest is with the nearly integrable case, so where the axial symmetry is broken. Also compare [41] and many of its references.
- The global conjugations of [24] are Whitney smooth (even Gevrey regular [139]) and near the identity map in the C^∞-topology [74]. Geometrically speaking these diffeomorphisms also are $\mathbb{T}^n$-bundle isormorphisms between the unperturbed and the perturbed bundle, the basis of which is a 'Cantor' set of positive measure.

7 Splitting of Separatrices

KAM theory does not predict the fate of close-to-resonant tori under perturbations. For fully resonant tori the phenomenon of frequency locking leads to the destruction of the torus under (sufficiently rich) perturbations, and other resonant tori disintegrate as well. In case of a single resonance between otherwise Diophantine frequencies the perturbation leads to quasi-periodic bifurcations, cf. § 8.

While KAM theory concerns the fate of 'most' trajectories and for all times, a complementary theorem has been obtained in [112, 113, 96, 116]. It concerns all trajectories and states that they stay close to the unperturbed tori for *long* times that are exponential in the inverse of the perturbation strength. For trajectories starting close to surviving tori the diffusion is even superexponentially slow, cf. [105, 106]. Here a form of smoothness exceeding the mere existence of ∞ many derivatives of the Hamiltonian is a necessary ingredient, for finitely differentiable Hamiltonians one only obtains polynomial times.

Solenoids, which cannot be present in integrable systems, are constructed for generic Hamiltonian systems in [15, 97, 101], yielding the simultaneous existence of representatives of all homeomorphy-classes of solenoids. Hyperbolic tori form the core of a construction proposed in [5] of trajectories that venture off to distant points of the phase space. In the unperturbed system the union of a family of hyperbolic tori, parametrised by the actions conjugate to the toral angles, form a normally hyperbolic manifold. The latter is persistent under perturbations, cf. [75, 103], and carries a Hamiltonian flow with fewer degrees of freedom. The main difference between integrable and non-integrable systems already occurs for periodic orbits.

7.1 Periodic Orbits

A sharp difference to dissipative systems is that it is generic for hyperbolic periodic orbits on compact energy shells in Hamiltonian systems to have homoclinic orbits, cf. [1] and references therein. For integrable systems these form together a pinched torus, but under generic perturbations the stable and unstable manifold of a hyperbolic periodic orbit intersect transversely. It is a nontrivial task to actually check this genericity condition for a given non-integrable perturbation, a first-order condition going back to Poincaré requires the computation of the so-called Mel'nikov integral, see [67, 140] for more details. In two degrees of freedom normalization leads to approximations that are integrable to all orders, which implies that the Melnikov integral is a flat function. In the real analytic case the Melnikov criterion is still decisive in many examples [66].

Genericity conditions are traditionally formulated in the universe of smooth vector fields, and this makes the whole class of analytic vector fields appear to be non-generic. This is an overly pessimistic view as the conditions defining a certain class of 'generic' vector fields may certainly be satisfied by a given analytic system. In this respect it is interesting that the generic properties may also be formulated in the universe of analytic vector fields, see [46] for more details.

7.2 $(n-1)$–Tori

The $(n-1)$–parameter families of invariant $(n-1)$–tori organize the dynamics of an integrable Hamiltonian system in n degrees of freedom, and under small perturbations the parameter space of persisting analytic tori is Cantorised. This still allows for a global understanding of a substantial part of the dynamics, but also leads to additional questions.

A hyperbolic invariant torus $\mathbb{T}^{n-1}$ has its Floquet exponents off the imaginary axis. Note that $\mathbb{T}^{n-1}$ is not a normally hyperbolic manifold. Indeed, the normal linear behaviour involves the $n-1$ zero eigenvalues in the direction of the parametrising actions as well; similar to (9) the format

$$\begin{aligned} \dot{x} &= \omega(y) + O(y) + O(z^2) \\ \dot{y} &= O(y) + O(z^3) \\ \dot{z} &= \Omega(y)z + O(z^2) \end{aligned}$$

in Floquet co-ordinates yields an x–independent matrix Ω that describes the symplectic normal linear behaviour, cf. [37]. The union $\{ z = 0 \}$ over the family of $(n-1)$–tori is a normally hyperbolic manifold and constitutes the centre manifold of $\mathbb{T}^{n-1}$. Separatrices splitting yields the dividing surfaces in the sense of Wiggins et al. [141].

The persistence of elliptic tori under perturbation from an integrable system involves not only the internal frequencies of $\mathbb{T}^{n-1}$, but also the normal frequencies. Next to the internal resonances the necessary Diophantine conditions (10) exclude the normal-internal resonances

$$\langle k, \omega \rangle = \alpha_j \tag{11}$$
$$\langle k, \omega \rangle = 2\alpha_j \tag{12}$$
$$\langle k, \omega \rangle = \alpha_i + \alpha_j \tag{13}$$
$$\langle k, \omega \rangle = \alpha_i - \alpha_j \; . \tag{14}$$

The first three resonances lead to the quasi-periodic center-saddle bifurcation studied in §8, the frequency-halving (or quasi-periodic period doubling) bifurcation and the quasi-periodic Hamiltonian Hopf bifurcation, respectively. The resonance (14) generalizes an equilibrium in 1:1 resonance whence $\mathbb{T}^{n-1}$ persists and remains elliptic, cf. [80]. When passing through resonances (12) and (13) the lower-dimensional tori lose ellipticity and aquire hyperbolic Floquet exponents. Elliptic $(n-1)$–tori have a single normal frequency whence (11) and (12) are the only normal-internal resonances. See [29] for a thorough treatment of the ensuing possibilities.

The restriction to a single normal–internal resonance is dictated by our present possibilities. Indeed, already the bifurcation of equilibria with a fourfold zero eigenvalue leads to unfoldings that simultaneously contain all possible normal resonances. Thus, a satisfactory study of such tori which already may form one–parameter families in integrable Hamiltonian systems with five degrees of freedom has to await further progress in local bifurcation theory.

8 Transitions to chaos

One of the main interests over the second half of the twentieth century has been the transition between orderly and complicated forms of dynamics upon variation of either intial states or of system parameters. By 'orderly' we here mean equilibrium and periodic dynamics and by 'complicated' quasi-periodic and chaotic dynamics, although we note that only chaotic dynamics is associated to unpredictability, e.g. see [49]. As already discussed in §2 systems like a forced nonlinear oscillator or the planar 3-body problem exhibit coexistence of periodic, quasi-periodic and chaotic dynamics, also compare with Figure 1.

Similar remarks go for the onset of turbulence in fluid dynamics. Around 1950 this led to the scenario of Hopf-Landau-Lifschitz [78, 86, 87], which roughly amounts to the following. Stationary fluid motion corresponds to an equilibrium point in an ∞-dimensional state space of velocity fields. The first transition is a Hopf bifurcation [77, 67, 85], where a periodic solution branches off. In a second transition of similar nature a quasi-periodic 2-torus branches off, then a quasi-periodic 3-torus, etc. The idea is that the motion picks up more and more frequencies and thus obtains an increasingly complicated power spectrum. In the early 1970's this idea was modified in the Ruelle-Takens route to turbulence, based on the observation that, for flows, a 3-torus can carry chaotic (or 'strange') attractors [129, 115], giving rise to a broad band power spectrum. By the quasi-periodic bifurcation theory [37, 36, 41] as sketched below these two approaches are unified in a generic way, keeping track of measure theoretic aspects. For general background in dynamical systems theory we refer to [49, 81].

Another transition to chaos was detected in the quadratic family of interval maps

$$f_\mu(x) = \mu x(1-x),$$

see [60, 102, 104], also for a holomorphic version. This transition consists of an infinite sequence of period doubling bifurcations ending up in chaos; it has several universal aspects and occurs persistently in families of dynamical systems. In many of these cases also homoclinic bifurcations show up, where sometimes the transition to chaos is immediate when parameters cross a certain boundary, for general theory see [13, 14, 25, 120]. There

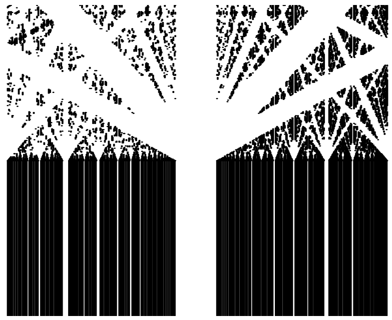

Figure 6: Sketch of the Cantorized Fold, as the bifurcation set of the quasi-periodic center-saddle bifurcation for $n = 2$ [68, 70], where the horizontal axis indicates the frequency ratio $\omega_2 : \omega_1$, cf. (15). The lower part of the figure corresponds to hyperbolic tori and the upper part to elliptic ones. See the text for further interpretations.

exist quite a number of case studies where all three of the above scenario's plays a role, e.g., see [42, 43, 44] and many of their references.

8.1 Quasi-periodic bifurcations

For the classical bifurcations of equilibria and periodic orbits, the bifurcation sets and diagrams are generally determined by a classical geometry in the product of phase space and parameter space as already established by, e.g., [8, 136], often using Singularity Theory. Quasi-periodic bifurcation theory concerns the extension of these bifurcations to invariant tori in nearly-integrable systems, e.g., when the tori lose their normal hyperbolicity or when certain (strong) resonances occur. In that case the dense set of resonances, also responsable for the small divisors, leads to a 'Cantorisation' of the classical geometries obtained from Singularity Theory [37, 33, 50, 71, 70, 29, 30, 31, 27, 22, 28], also see[36, 41, 57]. Broadly speaking one could say that in these cases the Preparation Theorem [136] is partly replaced by KAM Theory. Since the KAM Theory has been developed in several settings with or without preservation of structure, see §6, for the ensuing quasi-periodic bifurcation theory the same holds.

8.1.1 Hamiltonian cases

To fix thoughts we start with an example in the Hamiltonian setting, where a robust model for the quasi-periodic center-saddle bifurcation is given by

$$H_{\omega_1,\omega_2,\mu,\varepsilon}(I,\varphi,p,q) = \omega_1 I_1 + \omega_2 I_2 + \tfrac{1}{2}p^2 + V_\mu(q) + \varepsilon f(I,\varphi,p,q) \tag{15}$$

with $V_\mu(q) = \frac{1}{3}q^3 - \mu q$, compare with [68, 70]. The unperturbed (or integrable) case $\varepsilon = 0$, by factoring out the $\mathbb{T}^2$-symmetry, boils down to a standard center-saddle bifurcation, involving the Fold catastrophe [136] in the potential function $V = V_\mu(q)$. This results in the existence of two invariant 2-tori, one elliptic and the other hyperbolic. For $0 \neq |\varepsilon| \ll 1$

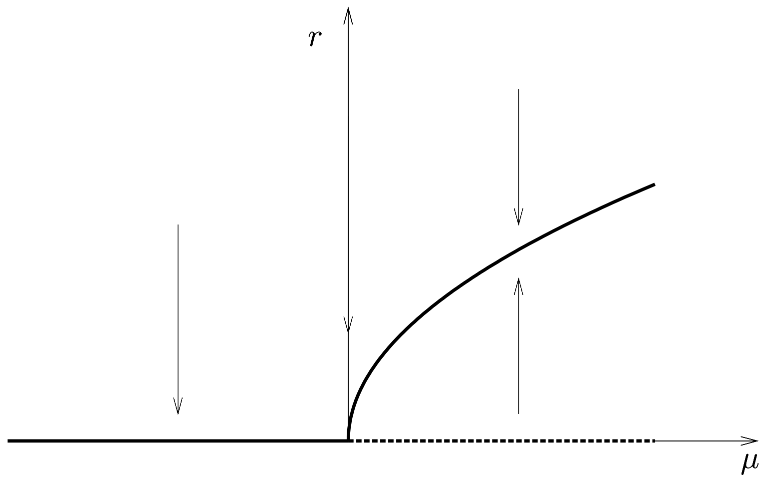

Figure 7: Bifurcation diagram of the Hopf bifurcation.

the dense set of resonances complicates this scenario, as sketched in Figure 6, determined by the Diophantine conditions

$$\begin{aligned} |\langle k, \omega \rangle| &\geq \gamma |k|^{-\tau}, \text{ for } q < 0, \\ |\langle k, \omega \rangle + \ell\, \beta(q)| &\geq \gamma |k|^{-\tau}, \text{ for } q > 0 \end{aligned} \tag{16}$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$ and for all $\ell \in \mathbb{Z}$ with $|\ell| \leq 2$. Here $\beta(q) = \sqrt{2q}$ is the normal frequency of the elliptic torus given by $q = \sqrt{\mu}$ for $\mu > 0$. As before, (cf. §§5, 6), this gives a 'Cantor set' of positive measure [109, 108, 37, 36, 33, 41, 68, 70].

For $0 < |\varepsilon| \ll 1$ Figure 6 will be distorted by a near-identity diffeomorphism, compare with the formulations of the Theorems 3 and 4. On the Diophantine 'Cantor set' the dynamics is quasi-periodic, while in the gaps generically there is coexistence of periodicity and chaos, roughly comparable with Figure 1, Left. The gaps at the border furthermore lead to the phenomenon of parabolic resonance, cf. [89].

Similar programs exist for all cuspoid and umbilic catastrophes [30, 31, 69] as well as for the Hamiltonian Hopf bifurcation [27, 28]. For applications of this approach see [29]. For a reversible analogue see [22]. As so often within the gaps generically there is an infinite regress of smaller gaps [11, 29]. For theoretical background we refer to [109, 37, 33], for more references also see [41].

8.1.2 Dissipative cases

In the general 'dissipative' case we basically follow the same strategy. Given the standard bifurcations of equilibria and periodic orbits, we get more complex situations when invariant tori are involved as well. The simplest examples are the quasi-periodic saddle-node and quasi-periodic period doubling [37] also see [36, 41].

To illustrate the whole approach let us start from the Hopf bifurcation of an equilibrium point of a vector field [77, 67, 119, 85] where a hyperbolic point attractor loses stability and branches off a periodic solution, cf. §3.2. A topological normal form is given by

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - (y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{17}$$

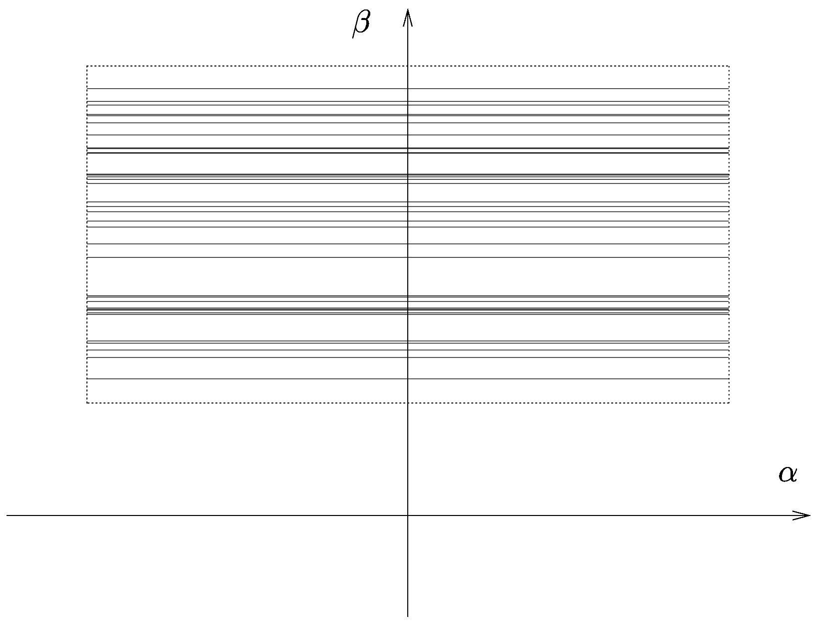

Figure 8: Planar section of the 'Cantor set' $\Gamma^{(2)}_{\tau,\gamma}$.

where $y = (y_1, y_2) \in \mathbb{R}^2$, ranging near $(0,0)$. In this representation usually one fixes $\beta = 1$ and lets $\alpha = \mu$ (near 0) serve as a (bifurcation) parameter, classifying modulo topological equivalence. In polar coordinates (17) so gets the form

$$\begin{aligned} \dot{\varphi} &= 1, \\ \dot{r} &= \mu r - r^3. \end{aligned}$$

Figure 7 shows an amplitude response diagram (often called bifurcation diagram). Observe the occurrence of the attracting periodic solution for $\mu > 0$ of amplitude $\sqrt{\mu}$.

Let us briefly consider the Hopf bifurcation for fixed points of diffeomorphisms. A simple example has the form

$$P(y) = \mathrm{e}^{2\pi(\alpha + i\beta)} y + O(|y|^2), \tag{18}$$

$y \in \mathbb{C} \cong \mathbb{R}^2$, near 0. To start with β is considered a constant, such that β is not rational with denominator less than 5, see [7, 135], and where $O(|y|^2)$ should contain generic third order terms. As before, we let $\alpha = \mu$ serve as a bifurcation parameter, varying near 0. On one side of the bifurcation value $\mu = 0$ this system by normal hyperbolicity and [75] has an invariant circle. Here, due to the invariance of the rotation numbers of the invariant circles, no topological stability can be obtained [114]. Still this bifurcation can be characterized by many persistent properties. Indeed, in a generic 2-parameter family (18), say with both α and β as parameters, the periodicity in the parameter plane is organized in resonance tongues [7, 26, 85]. (The tongue structure is hardly visible when only one parameter, like α, is used.) If the diffeomorphism is the return map of a periodic orbit for flows, this bifurcation produces an invariant 2-torus. Usually this counterpart for flows is called Neĭmark-Sacker bifurcation. The periodicity as it occurs in the resonance tongues, for the vector field is related to phase lock. The tongues are contained in gaps of a 'Cantor set' of quasi-periodic tori with Diophantine frequencies. Compare the discussion in §5.1, in particular also regarding the Arnold family and Figure 3. Also see §6 and again compare with [118].

Quasi-periodic versions exist of the saddle-node, the period doubling and the Hopf bifurcation. Returning to the setting with $\mathbb{T}^n \times \mathbb{R}^m$ as the phase space, we remark that the quasi-periodic saddle-node and period doubling already occur for $m = 1$, or in an analogous center manifold. The quasi-periodic Hopf bifurcation needs $m \geq 2$. We shall illustrate our results on the latter of these cases, compare with [36, 20]. For earlier results in this direction see [53]. Our phase space is $\mathbb{T}^n \times \mathbb{R}^2 = \{x \pmod{2\pi}, y\}$, where we are dealing with the parallel invariant torus $\mathbb{T}^n \times \{0\}$. In the integrable case, by $\mathbb{T}^n$-symmetry we can reduce to $\mathbb{R}^2 = \{y\}$ and consider the bifurcations of relative equilibria. The present interest is with small non-integrable perturbations of such integrable models.

We now discuss the *quasi-periodic Hopf bifurcation* [16, 37], largely following [57]. The unperturbed, integrable family $X = X_\mu(x, y)$ on $\mathbb{T}^n \times \mathbb{R}^2$ has the form

$$X_\mu(x, y) = [\omega(\mu) + f(y, \mu)]\partial_x + [\Omega(\mu)y + g(y, \mu)]\partial_y, \tag{19}$$

were $f = O(|y|)$ and $g = O(|y|^2)$ as before. Moreover $\mu \in P$ is a multi-parameter and $\omega : P \to \mathbb{R}^n$ and $\Omega : P \to \mathrm{gl}(2, \mathbb{R})$ are smooth maps. Here we take

$$\Omega(\mu) = \begin{pmatrix} \alpha(\mu) & -\beta(\mu) \\ \beta(\mu) & \alpha(\mu) \end{pmatrix},$$

which makes the ∂_y component of (19) compatible with the planar Hopf family (17). The present form of Kolmogorov non-degeneracy is Broer-Huitema-Takens stability [37, 33, 23], requiring that there is a subset $\Gamma \subseteq P$ on which the map

$$\mu \in P \mapsto (\omega(\mu), \Omega(\mu)) \in \mathbb{R}^n \times \mathrm{gl}(2, \mathbb{R})$$

is a submersion. For simplicity we even assume that μ is replaced by

$$(\omega, (\alpha, \beta)) \in \mathbb{R}^n \times \mathbb{R}^2.$$

Observe that if the nonlinearity g satisfies the well-known Hopf nondegeneracy conditions, e.g., compare [67, 85], then the relative equilibrium $y = 0$ undergoes a standard planar Hopf bifurcation as described before. Here α again plays the role of bifurcation parameter and a closed orbit branches off at $\alpha = 0$. To fix thoughts we assume that $y = 0$ is attracting for $\alpha < 0$, and that the closed orbit occurs for $\alpha > 0$, and is attracting as well. For the integrable family X, qualitatively we have to multiply this planar scenario with $\mathbb{T}^n$, by which all equilibria turn into invariant attracting or repelling n-tori and the periodic attractor into an attracting invariant $(n + 1)$-torus. Presently the question is what happens to both the n- and the $(n+1)$-tori, when we apply a small near-integrable perturbation.

The story runs much like before. Apart from the BHT non-degeneracy condition we require Diophantine conditions (10), defining the 'Cantor set'

$$\Gamma^{(2)}_{\tau,\gamma} = \{(\omega, (\alpha, \beta)) \in \Gamma \mid |\langle k, \omega\rangle + \ell\beta| \geq \gamma|k|^{-\tau}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}, \forall \ell \in \mathbb{Z} \text{ with } |\ell| \leq 2\}, \tag{20}$$

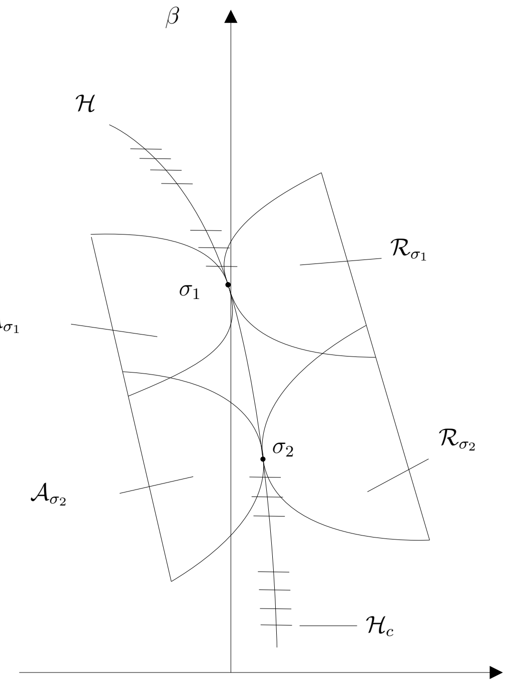

Figure 9: Fattening by normal hyperbolicity of a nowhere dense parameter set with invariant n-tori in the perturbed system. The curve $\mathcal{H}$ is the Whitney smooth (even Gevrey regular [139]) image of the β-axis in Figure 8. $\mathcal{H}$ interpolates the Cantor set $\mathcal{H}_c$ that contains the non-hyperbolic Diophantine quasi-periodic invariant n-tori, corresponding to $\Gamma^{(2)}_{\tau,\gamma}$, see (20). To the points $\sigma_{1,2} \in \mathcal{H}_c$ discs $\mathcal{A}_{\sigma_{1,2}}$ are attached where we find attracting normally hyperbolic n-tori and similarly in the discs $\mathcal{R}_{\sigma_{1,2}}$ repelling ones. The contact between the disc boundaries and $\mathcal{H}$ is infinitely flat [16, 37].

In Figure 8 we sketch the intersection of $\Gamma^{(2)}_{\tau,\gamma} \subset \mathbb{R}^n \times \mathbb{R}^2$ with a plane $\{\omega\} \times \mathbb{R}^2$ for a Diophantine (internal) frequency vector ω, cf. (7).

From [16, 37] it now follows that for any family $\tilde{X}$ on $\mathbb{T}^n \times \mathbb{R}^2 \times P$, sufficiently near X in the C^∞-topology a near-identity C^∞-diffeomorphism $\Phi : \mathbb{T}^n \times \mathbb{R}^2 \times \Gamma \to \mathbb{T}^n \times \mathbb{R}^2 \times \Gamma$ exists, defined near $\mathbb{T}^n \times \{0\} \times \Gamma$, that conjugates X to $\tilde{X}$ when further restricting to $\mathbb{T}^n \times \{0\} \times \Gamma^{(2)}_{\tau,\gamma}$. So this means that the Diophantine quasi-periodic invariant n-tori are persistent on a diffeomorphic image of the 'Cantor set' $\Gamma^{(2)}_{\tau,\gamma}$, compare with the formulations of the Theorems 3 and 4.

Similarly we can find invariant $(n+1)$-tori. We first have to develop a $\mathbb{T}^{n+1}$ symmetric normal form approximation [16, 37, 19]. For this purpose we extend the Diophantine conditions (20) by requiring that the inequality holds for all $|\ell| \leq N$ for $N = 7$. We thus find another large 'Cantor' set, again see Figure 8, where Diophantine quasi-periodic invariant $(n+1)$-tori are persistent. Here we have to restrict to $\alpha > 0$ for our choice of the sign of the normal form coefficient, compare with Figure 7.

In both the cases of n-tori and of $(n+1)$-tori, the nowhere dense subset of the parameter space containing the tori can be fattened by normal hyperbolicity to open subsets. Indeed, the quasi-periodic n- and $(n+1)$-tori are ∞-ly normally hyperbolic [75]. Exploiting the normal form theory [16, 37, 19] to the utmost and using a more or less standard contraction argument [55, 16], a fattening of the parameter domain with invariant tori can be obtained that leaves out only small 'bubbles' around the resonances, as sketched and explained in Figure 9 for the n-tori. For earlier results in the same spirit in a case study of the quasi-periodic saddle-node bifurcation see [50, 51, 52], also compare with [11].

8.2 A scenario for the onset of turbulence

Generally speaking, in may settings quasi-periodicity constitutes the order in between chaos [36]. In the Hopf-Landau-Lifschitz-Ruelle-Takens scenario [78, 86, 87, 129] we may consider a sequence of typical transitions as given by quasi-periodic Hopf bifurcations, starting with the standard Hopf or Hopf-Neĭmark-Sacker bifurcation as described before. In the gaps of the Diophantine 'Cantor' sets generically there will be coexistence of periodicity, quasi-periodicity and chaos in infinite regress. As said earlier, period doubling sequences and homoclinic bifurcations may accompany this.

As an example consider a family of maps that undergoes a generic quasi-periodic Hopf bifurcation from circle to 2-torus. It turns out that here the Cantorized fold of Figure 6 is relevant, where now the vertical coordinate is a bifurcation parameter. Moreover compare with Figure 3, where also variation of ε is taken into account. The 'Cantor set' contains the quasi-periodic dynamics, while in the gaps we can have chaos, e.g., in the form of Hénon like strange attractors [115, 44]. A fattening process as explained above, also can be carried out here.

9 Future Directions

One important general issue is the mathematical characterization of chaos and ergodicity in dynamical systems, in conservative, dissipative and in other settings. This is a tough problem as can already be seen when considering 2-dimensional diffeomorphisms. In particular we refer to the still unproven ergodicity conjecture of [9] and to the conjectures around Hénon like attractors and the principle 'Hénon everywhere', compare with [38, 42]. For a discussion see §8.2. In higher dimension this problem is even harder to handle, e.g., compare with [44, 45] and references therein. In the conservative case a related problem concerns a better understanding of Arnold diffusion.

Somewhat related to this is the analysis of dynamical systems without an explicit perturbation setting. Here numerical and symbolic tools are expected to become useful to develop computer assisted proofs in extended perturbation settings, diagrams of Lyapunov exponents, symbolic dynamcics, etc. Compare with [131]. Also see [44, 45] for applications and further reference. This part of the theory is important for understanding concrete models, that often are not given in 'perturbation format'.

Regarding nearly-integrable Hamiltonian systems, several problems are in order. Continuing the above line of thought, one interest is the development of Hamiltonian bifurcation theory without integrable normal form and, likewise, of KAM theory without action angle coordinates [90]. One big related issue also is to develop KAM theory outside the 'perturbation format'.

The previous section addressed persistence of Diophantine tori involved in a bifurcation. Similar to Cremer's example in §5.3.2 the dynamics in the gaps between persistent tori displays new phenomena. A first step has been made in [89] where internally resonant parabolic tori involved in a quasi-periodic Hamiltonian pitchfork bifurcation are considered. The resulting large dynamical instabilities may be further amplified for tangent (or flat) parabolic resonances, which fail to satisfy the iso-energetic non-degeneracy condition.

The construction of solenoids in [15, 97] uses elliptic periodic orbits as starting points, the simplest example being the result of a period-doubling sequence. This construction should carry over to elliptic tori, where normal-internal resonances lead to 'encircling' tori of the same dimension, while internal resonances lead to elliptic tori of smaller dimension and excitation of normal modes increases the torus dimension. In this way one might be able to construct solenoid-type invariant sets that are limits of tori with varying dimension.

Concerning the global theory of nearly-integrable torus bundles [24], it is of interest to understand the effects of quasi-periodic bifurcations on the geometry and its invariants. Also it is of interest to extend the results of [137] when passing to semi-classical approximations. In that case two small parameters play a role, namely Planck's constant as well as the distance away from integrability.

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