From weak to strong convergence in L_1-spaces via K-convergence

Erik J. Balder
Mathematical Institute, University of Utrecht
Utrecht, the Netherlands

June 11, 1997

A generalized version of Komlós' theorem [6], combined with a useful property of denting points in the style of [17, 22], gives a new, very efficient proof of Visintin's theorem and its generalizations [24, 2, 10, 21, 7], on equivalence of weak and strong convergence in L_1-spaces under denting point conditions.

1 Introduction

It is of some analytical interest to investigate the conditions under which weak convergence of a sequence of L_1-functions implies its strong convergence. Roughly speaking, this means that any oscillatory behavior of the sequence has to be suppressed. One way of achieving this is by requiring the values of the limit function to be pointwise extreme in some sense with respect to the "limit values" of the sequence of functions.

The origins of these ideas lie in work of Tartar (who already demonstrated the usefulness of Young measure theory for this purpose when the functions are uniformly bounded) and DiPerna on existence results for hyperbolic conservation laws. See [13, II] for an exposition of these ideas.

A different but related formulation of the problem was given by Visintin [24], who gave a result of this nature, and applied it to a Stefan type problem, a homogenization problem and a well-posedness problem. Visintin's proof follows a rather classical pattern (see also [22, pp. 5.9-5.10]). Subsequently, Visintin's result was extended to infinite dimensions and given a sharper form – in both finite and infinite dimensions – by the present author [2], using Young measure theory. (Actually, this follows from an observation by Valadier in [23, Theorem 22]; see also [22, section 4] and [8, 7].) The Young measure approach has also been pursued in further extensions by Castaing [10] and the present author [7, 8]. The latter two papers involve generalizations of results by Rzežuchowski, viz. an extension of Visintin's result to extremal faces [20] and an infinite-dimensional extension using denting points instead of extreme points [21]. (In both these works Rzežuchowski gives a proof along the more classical lines followed by Visintin.)

The relevance of Young measures for Visintin's problem can be understood as follows: the relevant Young measures, which are the generalized limits of (subsequences of) the sequence of ordinary functions, provide a clear 'fingerprint' of the oscillatory behavior of the sequence (one could say that at each point they measure of each value the relative frequency with which it is attained by the sequence 'in the limit'). The extremality conditions necessary to prevent oscillations can then be simply seen to force a probability measure (i.e., the pointwise value of the Young measure describing these pointwise relative frequencies) to become degenerate.

Although it remains the firm conviction of the present author that the Young measure approach is superior in terms of transparency and overall generality (see our concluding comments), the present note introduces yet another approach to the subject. This approach applies very directly to *denting point* conditions, and then produces some strong results. However, its effectiveness seems limited to situations where this strenghtened extreme point condition can be found present (fortunately, in a number of interesting cases, the notions of extreme and denting point coincide.) In this connection mention should be made of recent work of Amrani-Castaing-Valadier [1], where denting properties are combined with certain truncation arguments.

The approach to the problem taken in this note is based on changing the mode of weak convergence in a new way: instead of looking at underlying narrow convergence in the sense of Young measures, we apply Komlós' theorem, which transforms weak convergence of the whole sequence into *K-convergence*, i.e., pointwise a.e. convergence of *arithmetic averages* for subsequences. This theorem of Komlós is quite well-known in finite dimensions [16, 14]; it has been instrumental in the formulation of the so-called subsequence principle [11]. For pointwise strong convergence it was extended by Garling [15] to superreflexive Banach spaces, and by the present author to – inter alia – pointwise weak convergence in reflexive Banach spaces [3, 6, 4].

In contrast to Young measure convergence, K-convergence gives less information in the limit (for the limit function remains an ordinary function, which merely reflects an average taken over oscillations), but is rather more informative in terms of the limiting process itself, for it makes *microscopic* information available about a very orderly kind of pointwise a.e. convergence of arithmetic averages, while the weak convergence of the original sequence, as well as the narrow convergence towards the limiting Young measures, is purely *macroscopic* (i.e., stated in terms of integrals). This makes the approach by means of K-convergence more dynamic than the Young measure approach. [1] From this description it will be clear that the properties of *neighborhoods* of extreme points become important; for this reason denting properties of extreme points play such an essential role here.

2 Preliminaries

Let $(E, \|\cdot\|)$ be a Banach space. The topological dual of E is denoted by E^*, the corresponding duality by $<\cdot,\cdot>$, and the weak topology on E by $\sigma(E, E^*)$.

[1] For the record: it is also possible to approach Young measure convergence by K-convergence (pointwise one then encounters classical, narrow convergence of probability measures); cf. [6, 7].

Recall that a point e in a subset C of E is said to be a *denting point* of C if

$$e \notin \text{cl co}(C \backslash B(e;\epsilon)) \text{ for every } \epsilon > 0.$$

Here $B(e;\epsilon) := \{x \in E : \|x - e\| < \epsilon\}$, and cl, co form the usual notation for closures and convex hulls in E. Of course, a denting point of a set is always an extreme point of that set. When the set in question is convex and closed, the converse is also known to hold in finite dimensions (e.g., see [19] or [22, Lemma 1]). Some more general results in this direction are as follows:

Recall that $e \in C$ is a *point of continuity* for $C \subset E$ if the identity mapping $\iota : (C, \sigma(E, E^*)) \to (C, \|\cdot\|)$ is continuous at e [17]. The following result extends the equivalence $(i) \Leftrightarrow (iii)$ in [17, Theorem, p.526] – of which the proof can be found in [18]) – to the unbounded case.

Proposition 2.1 *Suppose that $C \subset E$ is closed and convex. Then e is a denting point of C if and only if e is both an extreme point of C and a point of continuity for C.*

Proof. If C is in addition bounded, the result follows by [17, Theorem, p.526]. Let us see how the present unbounded case can be reduced to this situation.

Notice first that e is a denting point of C if and only if e is a denting point of $C \cap$cl $B(e;\rho)$ for some $\rho > 0$. To see that the nontrivial implication is true, we consider arbitrary $\epsilon, 0 < \epsilon < \rho$. By the Hahn-Banach theorem there exists a closed hyperplane separating e and $D := C \cap \text{cl } B(e;\rho) \backslash B(e;\epsilon)$. So there exists an open halfspace H containing e such that D is contained in $E \backslash H$. But then $E \backslash H$ also contains $C \backslash B(e;\epsilon)$, for if there were any x in $H \cap C \backslash B(e;\epsilon)$, then $y := e + \epsilon(x - e)/\|x - e\|$ would be in $D \cap H$, which is clearly impossible.

Next, notice that when e is a point of continuity for C, it is so for $C \cap \text{cl } B(e;\rho)$ *a fortiori*. Also, when e is an extreme point of C, it must be an extreme point of $C \cap \text{cl } B(e;\rho)$. So, combined with the above, this goes to show that e is a denting point of C when e is point of continuity and extreme in C. The converse implication follows directly, by the Hahn-Banach theorem, from the definition of denting point.

Q.E.D.

Corollary 2.2 *Suppose that $C \subset E$ is closed and convex and suppose that for $e \in C$ there exists $\rho > 0$ such that $C \cap \text{cl } B(e;\rho)$ is norm-compact. Then e is a denting point of C if and only if e is an extreme point of C.*

Proof. The identity $\iota : (C', \sigma(E, E^*)) \to (C', \|\cdot\|)$ is continuous; here $C' := C \cap \text{cl } B(e;\rho)$. Hence, e is a denting point of C' by Proposition 2.1; the first part of the proof of that proposition shows then that e is also a denting point of C.

Q.E.D.

The following result, our first main tool, was extracted from the proof of an extension to the unbounded case, due to Valadier [22, Theorem 15] of the equivalence $(i) \Leftrightarrow (ii)$ in a theorem of Lin-Lin-Troyanski [17, p.526].

Proposition 2.3 *Suppose that (x_i) is a sequence in E such that*

$$\sigma(E,E^*)\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}x_i = e,$$

where $e \in E$ is a denting point of the set $C := \mathrm{cl\ co}\,\{x_i : i \in \mathbb{N}\}$. Then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\|x_i - e\| = 0.$$

Proof. For $\epsilon > 0$ we define

$$\begin{aligned} I_n^\epsilon &:= \{1 \le i \le n : \|x_i - e\| \ge \epsilon\}, \\ y_n &:= |I_n^\epsilon|^{-1}\sum_{i\in I_n^\epsilon} x_i, \\ z_n &:= |n - I_n^\epsilon|^{-1}\sum_{1\le i\le n, i\notin I_n^\epsilon} x_i. \end{aligned}$$

Here $|I_n^\epsilon|$ stands for the cardinality of I_n^ϵ, etc. First, we claim that for every $\epsilon > 0$

$$\lim_{n\to\infty} |I_n^\epsilon|/n = 0. \tag{1}$$

Suppose this were not true, then, taking a suitable subsequence and renumbering, we can suppose that $\eta := \lim_n |I_n^\epsilon|/n$ exists and is strictly positive. But η cannot be 1, since in that case the identity

$$\frac{1}{n}\sum_{i=1}^{n}x_i = \frac{|I_n^\epsilon|}{n}y_n + (1 - \frac{|I_n^\epsilon|}{n})z_n \tag{2}$$

would give in the limit $e \in \mathrm{cl\ co}\,(C\backslash B(e;\epsilon))$, which is impossible. Now either $\|z_n - e\| \to 0$ or not. In the former case, elementary algebraic manipulations in (2) show that in the limit $\eta^{-1}(e - (1-\eta)e) \in \mathrm{cl\ co}\,(C\backslash B(e;\epsilon))$, which again is impossible. In the latter case, we have, by taking a suitable subsequence and renumbering, that (z_n) stays away from e, say at a distance of at least $\alpha > 0$. Now the identity (2) gives in the limit the absurd statement $e \in \mathrm{cl\ co}\,(C\backslash B(e;\min(\epsilon,\alpha)))$. This contradiction proves the claim in (1).

Next, we study certain bounds imposed by the denting property of e (these form the essential improvement introduced by Valadier in his proof mentioned above): By the Hahn-Banach theorem there exist a nonnull $x^* \in E^*$ and $\beta \in \mathbb{R}$ such that $< e, x^* >< \beta$ and $< x, x^* >> \beta$ for all $x \in C\backslash B(e;1)$. Thus, for every $x \in C$ the inequality $<x,x^*>\le \beta$ implies that $\|x - e\| < 1$, and the opposite strict inequality $<x,x^*>> \beta$ implies then $\|x-e\| \le m(x)$, where $m(x) :=< x-e, x^*> /(\beta- <e,x^*>)$. This is seen by noting that then $y := x/m(x) + (m(x)-1)e/m(x)$ belongs to C and satisfies $<y,x^*>= \beta$, and rewriting the ensuing inequality $\|y - e\| \le 1$.

Now the proof can be finished. Letting J_n be the set of all $1 \le i \le n$ for which $<x_i,x^*>\le \beta$, it is easy to deduce from the above that the following holds

$$\frac{1}{n}\sum_{i\le n, i\notin I_n^\epsilon}\|x_i - e\| \le \epsilon,$$

so we can concentrate on the remainder, for which

$$\frac{1}{n}\sum_{i\in I_n^\epsilon}\|x_i - e\| \leq |I_n^\epsilon|/n + \frac{1}{n}\sum_{i\in I_n^\epsilon\setminus J_n} m(x_i).$$

By (1) the first term on the right converges to zero. The second term on the right can be decomposed as follows:

$$\frac{1}{n}\sum_{i\in I_n^\epsilon\setminus J_n} m(x_i) = m(\frac{1}{n}\sum_{i=1}^{n} x_i) - \sum_{i\leq n, i\notin I_n^\epsilon} m(x_i)/n - \sum_{i\in I_n^\epsilon\cap J_n} m(x_i)/n.$$

Because the function m is affine and continuous, with $m(e) = 0$, the first term on the right must converge to zero. Also, the second term on the right is majorized in absolute value by $\|x^*\|^*\epsilon/(\beta- < e, x^* >)$, and the third term by $|I_n^\epsilon|\|x^*\|^*/n(\beta- < e, x^* >)$, an expression which converges to zero by (1). Combining the above, we conclude that $\sum_{i=1}^n \|x_i - e\|/n$ can be made arbitrarily small.

Q.E.D.

Of course, in the above proposition e is a point of continuity for C. Therefore, the weak convergence of the averages to e is automatically equivalent to their strong convergence. The result shows that much more can be said: for instance, by Lemma 2.5 below, (x_i) itself converges strongly to e along some subsequence of density 1.

Let us also observe that arithmetic averages form a special kind of barycenter. A more general version of the result, in terms of barycenters of probability measures, can also be distilled from the proof of [22, Theorem 15] (of course, this comes very close to what was actually done there in terms of expectations of a sequence of random E-valued variables on some probability space).

Under a flanking strong compactness condition, it is possible to derive a parallel result whose denting point condition is weakened. Recall that the *limes superior* (for the norm topology) of a sequence $(x_i) \subset E$ is defined by

$$\mathrm{Ls}_{\|\cdot\|}(x_i) := \cap_{p=1}^{\infty}\mathrm{cl}\ \{x_i : i \geq p\};$$

i.e., this is the set of all limit points of the sequence (x_i).

Proposition 2.4 *Suppose that (x_i) is a sequence in E such that*

$$\sigma(E, E^*)\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} x_i = e,$$

where $e \in E$ is an extreme point of the set cl co (Ls(x_i)). *Suppose also that $K := \|\cdot\|\text{-cl}\{x_i : i \in \mathbb{N}\}$ is compact for the norm topology. Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\|x_i - e\| = 0.$$

Proof. Define probability measures ν_n on $(E, \|\cdot\|)$ by setting $\nu_n(A) := |\{x_i \in A, 1 \leq i \leq n\}|/n$, for A in the Borel σ-algebra $\mathcal{B}(E)$ on E. Since all measures ν_n are carried by the compact set K, it follows from [12, 12.7] that every subsequence of (ν_n) has a further subsequence (ν_m) which converges in the vague topology to some probability measure ν_* on K. Hence, for every $x^* \in E^*$ we get $\int_K <x, x^*> \nu_*(dx) = \lim_m <\sum_{i=1}^m x_i/m, x^*>=<e, x^*>$. It follows that the barycenter of ν_* equals e. On the other hand, for every $p \in \mathbb{N}$ we have $1 = \liminf_m \nu_m(F_p) \geq \nu_*(F_p)$ for the closed subset $F_p := \|\cdot\|\text{-cl}\ \{x_i : i \geq p\}$ of K. Hence, ν_* is carried by $\cap_p F_p = \mathrm{Ls}_{\|\cdot\|}(x_i)$, so e, its barycenter, belongs to the closed convex hull of that set. But of that same set e is an extreme point, so ν_* has to be the Dirac measure at e [12, Excercise 26.2]. We therefore conclude that the entire sequence (ν_n) converges narrowly to the Dirac measure at e. Applying this knowledge to the continuous function $x \mapsto \|x - e\|$ on K gives immediately the desired statement.

Q.E.D.

The following lemma can give more insight in this connection:

Lemma 2.5 *Suppose that $(\alpha_i) \subset \mathbb{R}, \alpha_* \in \mathbb{R}$ are such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |\alpha_i - \alpha_*| = 0.$$

Then there exists a subsequence $(\alpha_j)_{j\in J}$ of density 1 (i.e. $|J \cap \{1, \cdots, n\}|/n \to 1$) such that $\lim_{j\in J} \alpha_j = \alpha_*$.

This is the implication (1) $\Rightarrow$ (2) in Theorem 1.8 of [25]. (In [25] additional boundedness is requested, but it can be checked that boundedness does not play a role in the proof of the implication in question.)

The second main tool for the approach of this note is a recent extension of Komlós' theorem [16]. From now on $(\Omega, \mathcal{F}, \mu)$ will denote a finite measure space. [2] The space of all E-valued Bochner-integrable functions on $(\Omega, \mathcal{F}, \mu)$ will be denoted by $\mathcal{L}^1_E(\mu)$. Outer integration over $(\Omega, \mathcal{F}, \mu)$ will be denoted by $\int^*_\Omega$; cf. e.g. [9]. Also, τ will denote a locally convex topology on E that is not weaker than $\sigma(E, E^*)$ and not stronger than the norm topology. [3]

Theorem 2.6 ([5, Theorem 2.1]) *Suppose that $(f_k) \subset \mathcal{L}^1_E(\mu)$ is such that*

$$\sup_k \int^*_\Omega h(\omega, f_k(\omega))\mu(d\omega) < +\infty$$

for some function $h : \Omega \times E \to (-\infty, +\infty]$ with the following properties:

(i) $h(\omega, \cdot)$ is convex and inf-compact for τ on E for every $\omega \in \Omega$,

[2]The results presented here continue to hold – when properly interpreted – if μ is a σ-finite measure; cf. [2, 7].

[3]The topology could vary per point ω (by the pointwise character of Theorem 2.6), but we shall not bring this out; cf. [21].

(ii) $h(\omega, x) \geq \alpha\|x\| + \phi(\omega)$ *for all* $\omega \in \Omega, x \in E$ *for some* $\alpha > 0$ *and* $\phi \in \mathcal{L}^1_{\mathbb{R}}(\mu)$.

Then there exist a subsequence (f_m) *of* (f_k) *and a function* $f_* \in \mathcal{L}^1_E(\mu)$ *such that for every subsequence* (f_{m_i}) *of* (f_m)

$$\tau\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f_{m_i}(\omega) = f_*(\omega) \text{ for a.e. } \omega \in \Omega.$$

In the terminology of [6], the subsequence (f_m) is said to *K-converge* to f_*. For finite-dimensional E, Theorem 2.6 amounts to a deep and famous result of Komlós [16, 11, 14]. The above infinite-dimensional extension, as well as several applications to scalarly measurable functions and multifunctions, was given in [3, 4, 5, 6].

Some relevant situations to which the above result applies instantly, are the following two. They will be used in the next section.

Example 2.7 *Suppose that the Banach space* E *is reflexive and that*

$$\sup_k \int_\Omega \|f_k(\omega)\|\mu(d\omega) < +\infty.$$

Then for $\tau := \sigma(E, E^*)$ *the conditions of the above theorem are met trivially by* $h(\omega, x) := \|x\|, \alpha := 1, \phi := 0$.

Example 2.8 *Suppose that for a.e.* $\omega \in \Omega$

$$\text{cl co}\,\{f_k(\omega) : k \in \mathbb{N}\} \textit{ has } \tau\textit{-compact intersections with all closed balls,}$$

and that

$$\sup_k \int_\Omega \|f_k(\omega)\|\mu(d\omega) < +\infty.$$

Then the conditions of the above theorem are met by setting $h(\omega, x) := \|x\|$ *if* $x \in \text{cl co}\,\{f_k(\omega) : k \in \mathbb{N}\}$, *and* $h(\omega, x) := +\infty$ *otherwise, and further by setting* $\alpha := 1, \phi := 0$. *Note in particular that* $h(\omega, \cdot)$ *is inf-compact for* τ: *for every* $\beta \geq 0$ *the set* $\{x \in E : h(\omega, x) \leq \beta\}$ *is precisely the intersection of* $\text{cl}\, B(0;\beta)$ *and* $\text{cl co}\,\{f_k(\omega) : k \in \mathbb{N}\}$, *which is* τ*-compact by hypothesis.*

3 Main results

Let $\mathcal{T}$ denote any locally convex topology on $\mathcal{L}^1_E(\mu)$ such that for any sequence $(g_n) \subset \mathcal{L}^1_E(\mu)$

$$\mathcal{T}\text{-}\lim_n g_n = g_0 \text{ and } \sigma(E, E^*)\text{-}\lim_n g_n(\omega) = g^0(\omega) \text{ a.e. imply } g_0(\omega) = g^0(\omega) \text{ a.e.}$$

(For instance, $\mathcal{T}$ could be weak or strong convergence in $\mathcal{L}^1_E(\mu)$ or convergence in measure.)

From Proposition 2.3 and Theorem 2.6 the following abstract result is immediate. It says that when a sequence $\mathcal{T}$-converges to a limit function of which almost all values are denting points of the closed convex hull of the corresponding values of the other functions, then a much stronger form of convergence is actually present:

Theorem 3.1 *Suppose that $(f_k) \subset \mathcal{L}^1_E(\mu)$ converges in $\mathcal{T}$ to $f_0 \in \mathcal{L}^1_E(\mu)$ having the following property:*

$$f_0(\omega) \textit{ is a denting point of } \mathrm{cl\ co}\ \{f_k(\omega) : k \in \mathbb{N}\} \textit{ a.e.} \tag{3}$$

Suppose also that

$$\sup_k \int_\Omega^* h(\omega, f_k(\omega))\mu(d\omega) < +\infty$$

for some function $h : \Omega \times E \to (-\infty, +\infty]$ with the following properties:

(i) $h(\omega, \cdot)$ is convex and inf-compact for $\sigma(E, E^*)$ on E for every $\omega \in \Omega$,

(ii) $h(\omega, x) \geq \alpha\|x\| + \phi(\omega)$ for all $\omega \in \Omega, x \in E$

for some $\alpha > 0$ and $\phi \in \mathcal{L}^1_E(\mu)$.

Then every subsequence of (f_k) has a subsequence (f_m) such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \|f_m(\omega) - f_0(\omega)\| = 0 \textit{ a.e.} \tag{4}$$

Moreover, if (f_k) is uniformly integrable, this implies

$$\lim_{k\to\infty} \int_\Omega \|f_k - f_0\| d\mu = 0.$$

Proof. Apply Theorem 2.6 to obtain the K-convergent subsequence (f_m) with K-limit f_*. By the local convexity of $\mathcal{T}$ we have $\sum_{m=1}^n f_m/n \to f_0$ in $(\mathcal{L}^1_E(\mu), \mathcal{T})$. Hence, by the other property of $\mathcal{T}$ and K-convergence it follows that $f_*(\omega) = f_0(\omega)$ a.e. So Theorem 2.6 now gives for (f_m) (and more generally for every subsequence of (f_m)):

$$\sigma(E, E^*)\text{-} \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f_m(\omega) = f_0(\omega) \text{ a.e.}$$

In view of (3), Proposition 2.3 turns this directly into (4).

Under the additional condition, we define $\gamma := \limsup_k \int \|f_k - f_0\| d\mu$. There is a subsequence (f_p) of (f_k) with $\gamma = \lim_p \int \|f_p - f_0\| d\mu$. Applying the above, we find that there exists a subsequence (f_m) of (f_p) with

$$\lim_{n\to\infty} \int_\Omega \frac{1}{n} \sum_{m=1}^{n} \|f_m(\omega) - f_0(\omega)\|\mu(d\omega) = 0,$$

by a version of the dominated convergence theorem. On the other hand, the choice of (f_p) – of which (f_m) is a subsequence – gives $\gamma = \lim_n \frac{1}{n} \sum_{m=1}^n \int \|f_m - f_0\| d\mu$. Hence $\gamma = 0$, and the result follows.

Q.E.D.

Theorem 3.2 *Suppose that $(f_k) \subset \mathcal{L}^1_E(\mu)$ converges in $\mathcal{T}$ to $f_0 \in \mathcal{L}^1_E(\mu)$ having the following property:*

$$f_0(\omega) \text{ is an extreme point of } \mathrm{cl\ co\ Ls}(f_k(\omega)) \text{ a.e.} \tag{5}$$

Suppose also that

$$\sup_k \int_\Omega^* h(\omega, f_k(\omega))\mu(d\omega) < +\infty$$

for some function $h : \Omega \times E \to (-\infty, +\infty]$ with the following property:

$$h(\omega, x) \geq \alpha\|x\| + \phi(\omega) \text{ for all } \omega \in \Omega, x \in E$$

for some $\alpha > 0$ and $\phi \in \mathcal{L}^1_E(\mu)$, and suppose that

$$\mathrm{cl}\ \{f_k(\omega) : k \in \mathbb{N}\} \text{ is } \|\cdot\|\text{-compact a.e.}$$

Then every subsequence of (f_k) has a subsequence (f_m) such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \|f_m(\omega) - f_0(\omega)\| = 0 \text{ a.e.}$$

Moreover, if (f_k) is uniformly integrable, this implies

$$\lim_{k\to\infty} \int_\Omega \|f_k - f_0\|\|d\mu = 0.$$

Proof. The only difference with the previous proof is that one has to modify h so as to get the desired inf-compactness: define $h'(\omega, x) := h(\omega, x)$ for $x \in \mathrm{cl\ co}\{f_k(\omega) : k \in \mathbb{N}\}$ and $h'(\omega, x) := +\infty$ if not. Note that by Krein's theorem the set $\mathrm{cl\ co}\{f_k(\omega) : k \in \mathbb{N}\}$ is also compact. Therefore, h' satisfies the demands of Theorem 2.6, and by using Proposition 2.4 the proof is now an easy adaptation of the previous one.

Q.E.D.

Remark 3.3 *By Lemma 2.5 the convergence statement (4) in Theorem 3.1 and its counterpart in Theorem 3.2 imply that for the subsequence (f_m) for a.e. ω the following holds:*

$$\|f_j(\omega) - f_0(\omega)\| \to 0 \text{ along some subsequence } J_\omega \text{ of } \mathbb{N} \text{ with density 1.}$$

Remark 3.4 *In Theorems 3.1-3.2 there is, under the additional uniform integrability hypothesis, actually* equivalence *between the K-convergence statement and the final L_1-convergence statement $\int_\Omega \|f_k - f_0\|d\mu \to 0$. For the latter implies that $f_k \to f_0$ in measure. Hence each subsequence of (f_k) contains a further subsequence (f_m) such that $\|f_m(\omega) - f_0(\omega)\| \to 0$ a.e.*

We now give some applications of Theorems 3.1, 3.2.

In the first application, which is Visintin's original result [24, Theorem 1], the denting point property appears in the form of an (equivalent) extreme point condition:

Corollary 3.5 (Visintin [24]) *Suppose that E is finite-dimensional and suppose that $(f_k) \subset \mathcal{L}^1_E(\mu)$ converges weakly to $f_0 \in \mathcal{L}^1_E(\mu)$ having the following property:*

$$f_0(\omega) \text{ is an extreme point of } \mathrm{cl\ co}\ \{f_k(\omega) : k \in \mathbb{N}\} \text{ a.e.}$$

Then $\lim_{k\to\infty} \int_\Omega \|f_k(\omega) - f_0(\omega)\| \mu(d\omega) = 0$.

Proof. By Corollary 2.2 $f_0(\omega)$ is a denting point of cl co $\{f_k(\omega) : k \in \mathbb{N}\}$ as well. So the result follows from Theorem 3.1 and Example 2.7.

Q.E.D.

By almost the same proof one obtains the following infinite-dimensional version [21, Theorem 2], [7, Proposition 3.2]:

Corollary 3.6 *Suppose that E is a reflexive Banach space and suppose that $(f_k) \subset \mathcal{L}^1_E(\mu)$ converges weakly to $f_0 \in \mathcal{L}^1_E(\mu)$ having the following property:*

$$f_0(\omega) \text{ is a denting point of } \mathrm{cl\ co}\ \{f_k(\omega) : k \in \mathbb{N}\} \text{ a.e.}$$

Then $\lim_{k\to\infty} \int_\Omega \|f_k(\omega) - f_0(\omega)\| \mu(d\omega) = 0$.

The following corollary generalizes [1, Lemma 10], where the extreme point condition involves, instead of cl co $\mathrm{Ls}(f_k(\omega))$, the larger set $\cap_p$cl co $\{f_k(\omega) : k \geq p\}$.

Corollary 3.7 *Suppose that $(f_k) \subset \mathcal{L}^1_E(\mu)$ converges weakly to $f_0 \in \mathcal{L}^1_E(\mu)$ having the following property:*

$$f_0(\omega) \text{ is an extreme point of } \mathrm{cl\ co\ Ls}(f_k(\omega)) \text{ a.e.}$$

Suppose furthermore that

$$\mathrm{cl\ co}\ \{f_k(\omega) : k \in \mathbb{N}\} \text{ is } \|\cdot\|\text{-compact.}$$

Then $\lim_{k\to\infty} \int_\Omega \|f_k(\omega) - f_0(\omega)\| \mu(d\omega) = 0$.

Proof. Apply Theorem 3.2 with $h(\omega, x) := \|x\|$.

Q.E.D.

It is interesting to note, following [1, Example, p.15] that in the above result the flanking strong compactness condition cannot be replaced by a weak compactness condition of the same sort. Also, using the present methods, we were not able to derive the related result [1, Theorem 11], which extends Corollary 3.7 by the present methods (even though Theorem 2.6 applies). Yet the Young measure approach immediately applies to that result. The same can be said about the main result of [2]. Therefore, although the results obtained here in Theorems 3.1-3.2 and Remark 3.3 are of an exceptionally strong pointwise character when uniform integrability is replaced by uniform L_1-boundedness, the author is led to the initial conclusion that the method described in this note can only be of limited value, and that the Young measure approach seems to be preferable in terms of allround generality [7].

Evidently, there remains a good deal of work to be done towards unifying the various convergence methods (classical methods [24, 20, 21], Young measure approach [2, 10, 23, 7], truncation methods [1], and the K-convergence method introduced here).

References

[1] Amrani, A., Castaing, C. and Valadier, M. (1991). Méthodes de troncature appliquées à des problèmes de convergence faible ou forte dans L^1, preprint, Department of Mathematics, University of Montpellier II.

[2] Balder, E.J. (1986). On weak convergence implying strong convergence in L_1-spaces, Bull. Austral. Math. Soc. 33, 363-368.

[3] — (1989). Infinite-dimensional extension of a theorem of Komlós, Probab. Th. Rel. Fields 81, 185-188.

[4] — (1989). Unusual applications of a.e. convergence, in : Almost Everywhere Convergence (E.A. Edgar, L. Sucheston, eds.), Academic Press, New York, pp. 31-53.

[5] — (1990). On uniformly bounded sequences in Orlicz spaces, Bull. Austral. Math. Soc. 41, 495-502.

[6] — (1990). New sequential compactness results for spaces of scalarly integrable functions, J. Math. Anal. Appl. 151, 1-16.

[7] — (1989). On equivalence of strong and weak convergence in L_1-spaces under extreme point conditions, Preprint 599, Department of Mathematics, Utrecht, forthcoming.

[8] — (1990). On weak convergence implying strong convergence under extremal conditions, Preprint 610, Department of Mathematics, Utrecht, to appear in J. Math. Anal. Appl.

[9] Bertsekas, D.P. and Shreve, S.E. (1978). Stochastic Optimal Control: The Discrete Time Case, Academic Press, New York.

[10] Castaing, C. (1988). Convergence faible et sections extrémales, Travaux Sém. Anal. Convexe 18, 2.1-2.18.

[11] Chatterji, S.D. (1985). A subsequence principle theory, Jber. d. Dt. Math.-Verein. 87, 91-107.

[12] Choquet, G. (1969). Lectures on Analysis, Benjamin, Reading, Mass.

[13] Dacorogna, B. (1982). Weak Continuity and Weak Lower Semicontinuity of Non-Linear Functionals, Lecture Notes in Mathematics 922, Springer -Verlag, Berlin.

[14] Diestel, J. (1984). Sequences and Series in Banach Spaces, Springer-Verlag, Berlin.

[15] Garling, D.J.H. (1979). Subsequence principles for vector-valued random variables, Math. Proc. Cambridge Philos. Soc. 86, 301-311.

[16] Komlós, J. (1967). A generalisation of a problem of Steinhaus. Acta Math. Acad. Sci. Hungar. 18, 217-229.

[17] Lin, B.-L., Lin, P.-K. and Troyanski, S.L. (1988). Characterizations of denting points, Proc. Amer. Math. Soc. 102, 526-528.

[18] Lin, B.-L., Lin, P.-K. and Troyanski, S.L. (1986). A characterization of denting points of a closed bounded convex set, Longhorn Notes, Functional Analysis Seminar, 1985-1986, University of Texas, Austin, pp. 99-101.

[19] Olech, C. (1976). Existence theory in optimal control, in: Control Theory and Topics in Functional Analysis, IAEA, Vienna, pp. 291-328.

[20] Rzežuchowski, T. (1989). Strong convergence of selections implied by weak, Bull. Austral. Math. Soc. 39, 201-214.

[21] Rzežuchowski, T. (1990). Impact of dentability on weak convergence in L^1, preprint, Warsaw Technical University.

[22] Valadier, M. (1989). Differents cas ou, grâce à une propriété d'extremalité, une suite de fonctions intégrables faiblement convergente converge fortement, Travaux Sém. Anal. Conves faiblement convergente converge fortement, Travaux Sém. Anal. Convexe 19, 5.1-5.20.

[23] Valadier, M. (1990). Young measures, in: Methods of Nonconvex Analysis (A. Cellina, ed.), Lecture Notes in Mathematics 1446, Springer-Verlag, Berlin, pp. 152-188.

[24] Visintin, A. (1984). Strong convergence results related to strict convexity, Comm. Partial Differential Equations 9, 439-466.

[25] Walters, P. (1975). Ergodic Theory, Lecture Notes in Mathematics 458, Springer-Verlag, Berlin.