# Remarks on Nash equilibria for games with additively coupled payoffs (revision, previous title: An unusual Nash equilibrium result and its application to games with allocations in infinite dimensions)

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**Summary.** If the payoffs of a game are affine, then they are additively coupled. In this situation both the Weierstrass theorem and the Bauer maximum principle can be used to produce existence results for a Nash equilibrium, since each player is faced with an individual, independent optimization problem. We consider two instances in the literature where these simple observations immediately lead to substantial generalizations.

# 1 Nash equilibria for additively coupled payoffs

Let $\Gamma := \{(S_i, \pi_i)\}_{i=1}^m$ denote a game in normal form with $m$ players; $S_i$, a topological space, is the *strategy set* and $\pi_i : S \to \mathbb{R}$ the *payoff* of player $i \in I := \{1, \cdots, m\}$. A strategy vector $\bar{s} := (\bar{s}_1, \cdots, \bar{s}_m)$ in $S := \Pi_{j=1}^m S_j$ is called a *Nash equilibrium* for $\Gamma$ if for each $i \in I$

$$\pi_i(\bar{s}) \geq \pi_i(\bar{s}^{-i}, s_i) \text{ for all } s_i \in S_i. \tag{1}$$

Here $(\bar{s}^{-i}, s_i)$ is the usual notation for the strategy vector whose $j$-th component is $\bar{s}_j$ for $j \neq i$ and $s_i$ for $j = i$; corresponding notation for product sets is $S^{-i} := \Pi_{j \neq i} S_j$, etc.

Although the existence problem for Nash equilibria is in general quite nontrivial, a much simpler problem is encountered when the payoffs are *additively coupled* in the following sense: for each $i \in I$ there are component functions $\pi_{i,j} : S_j \to \mathbb{R}$, $j \in I$, such that $\pi_i$ decomposes as follows:

$$\pi_i(s_1, \cdots, s_m) = \sum_{j=1}^m \pi_{i,j}(s_j). \tag{2}$$

The following proposition is a trivial consequence of (1) and (2):

**Proposition 1.1** *Let the payoffs of $\Gamma$ be additively coupled as in (2). Then $(\bar{s}_1, \cdots, \bar{s}_m) \in S$ is a Nash equilibrium in the sense of (1) if and only if for each $i \in I$*

$$\pi_{i,i}(\bar{s}_i) = \sup_{s_i \in S_i} \pi_{i,i}(s_i).$$

The following two existence results are an immediate consequence of combining Proposition 1.1 with the Weierstrass theorem and the Bauer maximum principle respectively. Observe that these games need only have payoffs that are semicontinuous in one variable. Games with discontinuous payoffs have been a subject of increasing interest in the recent past; e.g., cf. [3, 6].

**Corollary 1.2** *Let the payoffs of $\Gamma$ be additively coupled as in (2) and let each $S_i$, $i \in I$, be equipped with a topology. If for each $i \in I$*

*$S_i$ is compact,*

*$\pi_{i,i}$ is upper semicontinuous,*

*then there exists a Nash equilibrium for $\Gamma$.*

*Proof.* By the Weierstrass theorem the maximum of $\pi_{i,i}$ over $S_i$ is attained for each $i \in I$. Apply Proposition 1.1. Q.E.D.

**Corollary 1.3** *Let the payoffs of $\Gamma$ be additively coupled as in (2) and let each $S_i$, $i \in I$, be a subset of a Hausdorff locally convex topological vector space. If for each $i \in I$*

*$S_i$ is compact and convex,*

*$\pi_{i,i}$ is upper semicontinuous and convex on $S_i$,*

*then there exists a Nash equilibrium $(\bar{s}_1, \cdots, \bar{s}_m) \in S$ for $\Gamma$ such that for each $i \in I$*

*$\bar{s}_i$ is an extreme point of $S_i$.*

*Proof.* By the Bauer maximum principle [5, Theorem 25.9] the maximum of $\pi_{i,i}$ over $S_i$ is attained at some extreme point of $S_i$ for each $i \in I$. Apply Proposition 1.1. Q.E.D.

The next existence result is essentially contained in Corollary 1.3; it forms a solution to a question posed to the author by N. Yannelis. Compared to Corollary 1.3, the only new aspect which is offered is the realization that additive coupledness is an inherent aspect of affinity. While this is trivial in finite dimensions, this aspect seems to have been overlooked in some infinite-dimensional situations in the literature.

**Corollary 1.4** *Let each $S_i$, $i \in I$, be a subset of a Hausdorff locally convex topological vector space. If for each $i \in I$*

$$S_i \text{ is compact and convex,}$$

$$\pi_i \text{ is affine on } S = \Pi_{j=1}^m S_j,$$

$$\pi_i(s^{-i}, \cdot) \text{ is upper semicontinuous on } S_i \text{ for each } s^{-i} \in S^{-i},$$

*then there exists a Nash equilibrium $(\bar{s}_1, \cdots, \bar{s}_m) \in S$ for $\Gamma$ such that for each $i \in I$*

$$\bar{s}_i \text{ is an extreme point of } S_i.$$

*Proof.* It is enough to demonstrate that the situation of Corollary 1.3 obtains. Let $\tilde{s} = (\tilde{s}_1, \cdots, \tilde{s}_m)$ be an arbitrary fixed element of $S$. Fix $i \in I$; we show that $\pi_i$ is additively coupled as in (2) by defining functions $\pi_{i,j} : S_j \to \mathbb{R}$ as follows:

$$\pi_{i,j}(s_j) := \pi_i(\tilde{s}^{-j}, s_j) - \frac{m-1}{m}\pi_i(\tilde{s}).$$

Indeed, affinity of $\pi_i$ gives

$$\sum_{j=1}^m \frac{1}{m}\pi_{i,j}(s_j) = \pi_i(\frac{m-1}{m}\tilde{s} + \frac{1}{m}s) - \frac{m-1}{m}\pi_i(\tilde{s}) = \frac{1}{m}\pi_i(s),$$

which directly implies (2). The remaining conditions of Corollary 1.3 are easily seen to be implied by the above definition of the $\pi_{i,j}$'s. Q.E.D.

# 2 Applications

Two applications of Corollary 1.4 will be discussed briefly. The first of these extends a pure strategy equilibrium existence result of Yannelis and Rustichini [15, Theorem 5.2]. The second application extends a Nash equilibrium existence result in original controls of Parthasarathy and Raghavan [12, Theorem 2] for a differential game with relaxed control functions.

## 2.1 Application to a Bayesian game [15]

The following model is considered by Yannelis and Rustichini [15]. Let $(\Omega, \mathcal{T}, P)$ be a probability space. Consider $m$ sub-$\sigma$-algebras $\mathcal{S}_1, \cdots, \mathcal{S}_m$ of $\mathcal{T}$; here $\mathcal{S}_i$ represents the way in which player $i \in I$ observes the uncertain state of the world as modelled by the probability space. Upon learning about the state of the world, players can take actions in a separable Banach space $(Y, \|\cdot\|)$, the *action space.* The actions of each player $i \in I$ are restricted by means of a given multifunction $X_i : \Omega \to 2^Y$ that is supposed to be $\mathcal{S}_i$-measurable, $i = 1, \cdots, m$. Namely, the *strategy set* $\mathbf{S}_i$ for player $i \in I$ consists of all $\mathcal{S}_i$-measurable functions $\mathbf{x}_i : \Omega \to Y$ such that

$$\mathbf{x}_i(\omega) \in X_i(\omega) \text{ for } P\text{-a.e. } \omega \text{ in } \Omega.$$

We shall write $X(\omega) := \Pi_{i=1}^m X_i(\omega)$. The *expected payoff* for player $i \in I$ is

$$p_i(\mathbf{x}_1, \cdots, \mathbf{x}_m) := \int_\Omega u_i(\omega, \mathbf{x}_1(\omega), \cdots, \mathbf{x}_m(\omega))P(d\omega),$$

assuming that the integral exists. Here $u_i : \Omega \times Y^m \to \mathbb{R}$ is the *utility function* of player $i$, supposed to be $\cap_{i=1}^m \mathcal{S}_i \times \mathcal{B}(Y^m)$-measurable, with $\mathcal{B}(Y^m) = (\mathcal{B}(Y))^m$ denoting the Borel $\sigma$-algebra on $Y^m$.

**Theorem 2.1** *Suppose that for each $i \in I$ and for a.e. $\omega \in \Omega$*

$$X_i(\omega) \text{ is nonempty, } \sigma(Y, Y^*)\text{-compact and convex,}$$

$$u_i(\omega, \cdot) \text{ is affine on } X(\omega),$$

$$u_i(\omega, x^{-i}, \cdot) \text{ is } \sigma(Y, Y^*)\text{-upper semicontinuous on } X_i(\omega) \text{ for every } x^{-i} \in \Pi_{j \neq i} X_j(\omega).$$

*Suppose also that for each $i \in I$ there exist integrable $\phi_i,\ \psi_i : \Omega \to \mathbb{R}_+$ with*

$$\begin{aligned}
\sup_{x_i \in X_i(\omega)} \|x_i\| &\leq \phi_i(\omega) \text{ for a.e. } \omega \in \Omega,\\
\sup_{x \in X(\omega)} |u_i(\omega, x)| &\leq \psi_i(\omega) \text{ for a.e. } \omega \in \Omega.
\end{aligned}$$

*Then there exists a Nash equilibrium $(\tilde{\mathbf{x}}_1, \cdots, \tilde{\mathbf{x}}_m)$ for the game $\{(\mathbf{S}_i, p_i)\}_{i=1}^m$ such that for each $i \in I$*

$$\tilde{\mathbf{x}}_i(\omega) \text{ is an extreme point of } X_i(\omega) \text{ for a.e. } \omega \in \Omega.$$

This substantially generalizes Theorem 5.2 of [15], where additional conditions were imposed: there $(\Omega, \mathcal{T}, P)$ is complete and nonatomic and $Y$ is finite-dimensional.

Below we prove Theorem 2.1 by a straightforward application of Corollary 1.4. In view of the generality afforded by that corollary, the actual model of [15] could now also be expanded. However, we shall leave this aspect unexplored.

Let $L_Y^1$ be the vector space $L_Y^1(\Omega, \mathcal{T}, P)$ of all Bochner integrable (equivalence classes of) functions $x : \Omega \to Y$. For each $i \in I$ let $S_i$ be the corresponding quotient of $\mathbf{S}_i$ in $L_Y^1$. By the integrable boundedness condition for the multifunctions $X_i$ and the inclusion $\mathcal{S}_i \subset \mathcal{T}$, the sets $S_1, \cdots, S_m$ are contained in $L_Y^1$. By [10, IV] we know that the dual space of $L_Y^1$ is (identifiable with) the space $M$ of all (equivalence classes) of scalarly measurable functions $z : \Omega \to Y^*$ which are essentially bounded (here $Y^*$ denotes the dual of $Y$). The duality between $L_Y^1$ and $M$ is given by

$$\prec x, z \succ := \int_\Omega < x(\omega), z(\omega) > P(d\omega).$$

By equipping $L_Y^1$ with the weak topology $\sigma(L_Y^1, M)$, it is made into a Hausdorff locally convex topological vector space.

The following lemmas show that, with the above topology, Corollary 1.4 can be brought to bear in the proof of Theorem 2.1:

**Lemma 2.2** *The hypotheses of Theorem 2.1 imply for each $i \in I$ the following:*

*(i) $S_i$ is nonempty, $\sigma(L_Y^1, M)$-compact and convex,*

*(ii) $p_i$ is affine on $S = \Pi_{j=1}^m S_j$,*

*(iii) $p_i(x^{-i}, \cdot)$ is $(\sigma(L_Y^1, M))$-upper semicontinuous on $S_i$ for each $x^{-i} \in S^{-i}$.*

*Proof.* i. Nonemptiness of $S_i$ follows directly by the von Neuman-Aumann measurable selection theorem [4, III.22], in view of the given measurability and other properties of the multifunctions $X_i$. Weak compactness follows by a well-known result of Diestel [7] (see also [14]). Convexity is trivial.

ii. Trivial by linearity of the integral.

iii. Because $p_i(x^{-i}, \cdot)$ is affine, its upper semicontinuity in the topology $\sigma(L_Y^1, M)$ is equivalent to upper semicontinuity in the $L_Y^1$-norm (Mazur's theorem). Such semicontinuity follows immediately by Fatou's lemma and a well-known corollary of Egorov's theorem [11, II.4.3]. QED

**Lemma 2.3** *For each $i \in I$ the set of extreme points of $S_i$ consists precisely of all equivalence classes of functions $\mathbf{x}_i$ in $\mathbf{S}_i$ such that*

$$\mathbf{x}_i(\omega) \text{ is an extreme point of } X_i(\omega) \text{ for a.e. } \omega \in \Omega. \tag{3}$$

*Proof.* We reiterate well-known measurable selection arguments involving extreme points [4, pp. 109-110]: First, observe that when $x_i \in S_i$ has a representant $\mathbf{x}_i$ with (3), then $x_i$ is trivially extreme in $S_i$. Conversely, suppose that $x_i \in S_i$ is such that it has a a representant $\mathbf{x}_i \in \mathbf{S}_i$ which does not satisfy (3). Then there exists a set $A \in \mathcal{T}$, $P(A) > 0$, such that for every $\omega \in A$ the set $\Phi(\omega)$ is nonempty, where

$$\Phi(\omega) := \{(y, y') \in X_i(\omega)^2 \backslash \Delta : \frac{1}{2}(y + y') = \mathbf{x}_i(\omega)\}.$$

Here $\Delta := \{(y, y') \in Y^2 : y \neq y'\}$. By elementary arguments [4, Lemma IV.10] it follows that the graph of $\Phi$ is $\mathcal{T} \times \mathcal{B}(Y^2)$-measurable. So by the von Neuman-Aumann measurable selection theorem [4, III.22] it follows from the above that there exists a $\mathcal{T}$-measurable mapping $(\mathbf{x}, \mathbf{x}') : A \to Y^2$ such that $(\mathbf{x}(\omega), \mathbf{x}'(\omega)) \in \Phi(\omega)$ for a.e. $\omega$. Extend $\mathbf{x}$ and $\mathbf{x}'$ to all of $\Omega$ by setting them equal to $\mathbf{x}_i$ both on the exceptional null set involved in the previous statement and on $\Omega \backslash A$. Then it is easy to verify that $\mathbf{x}$, $\mathbf{x}'$ both belong to $\mathbf{S}_i$, that $\mathbf{x}_i = \frac{1}{2}(\mathbf{x} + \mathbf{x}')$, and that $\mathbf{x}$ and $\mathbf{x}'$ are essentially different elements of $\mathbf{S}_i$. Therefore, $\mathbf{x}_i$ is not an extreme element of $\mathbf{S}_i$. QED

*Proof of Theorem 2.1.* Lemma 2.2 allows us to apply Corollary 1.4. This gives the existence of $(\bar{x}_1, \cdots, \bar{x}_m) \in S$ which is a Nash equilibrium for the game $\{(S_i, p_i)\}_{i=1}^m$ and satisfies

$$\bar{x}_i \text{ is an extreme point of } S_i,\ i \in I.$$

Let $\bar{\mathbf{x}}_i$ be any representant of the equivalence class $\bar{x}_i$, $i \in I$. Since the values of the payoffs (which we treated with some abuse of notation) are unaffected by this return to the original prequotient setting, $(\bar{\mathbf{x}}_1, \cdots, \bar{\mathbf{x}}_m)$ is a Nash equilibrium for the game $\{(\mathbf{S}_i, p_i)\}_{i=1}^m$ and satisfies for a.e. $\omega \in \Omega$

$$\bar{\mathbf{x}}_i(\omega) \text{ is an extreme point of } X_i(\omega),\ i \in I$$

by Lemma 2.3. Q.E.D.

## 2.2 Application to a differential game [12]

In this section we briefly sketch an application of Corollary 1.4 to a differential game considered by Parthasarathy and Raghavan in [12], which involves two players. The extension to $m$ players would be immediate, and there exist several other possibilities to generalize; however, we shall remain within the model used in [12].

The two players are allowed to use relaxed control functions (i.e., mixing of the control action is allowed – see [13] and [2] for the general background). For such games well-known counterexamples [13, IX.2] are known to essentially restrict considerations to payoffs that are additively coupled in the relaxed controls. This explains why Parthasarathy and Raghavan only study payoffs of the kind

$$P_i(\sigma_1, \sigma_2) := \mu_i(y_{\sigma_1, \sigma_2}) + \int_0^1 [\int_{U_1} F_i(t, u_1)\sigma_1(t)(du_1)]dt + \int_0^1 [\int_{U_2} G_i(t, u_2)\sigma_2(t)(du_2)]dt,\ i = 1, 2,$$

where $[0, 1]$ is the time interval, $F_i$ and $G_i$ are continuous functions, $U_i$ is the space of control points for player $i$, $\mu_i$ is a continuous linear functional on the set $\mathcal{C}[0, 1]$ of all continuous functions on $[0, 1]$ and $y_{\sigma_1, \sigma_2}$ is the solution of a differential equation (equation (4) in [12]) that is semilinear and has an additively coupled right-hand side. Since the $\mu_i$'s are linear, it follows that $P_1$ and $P_2$ are both affine functions. Thus, it is not surprising that Corollary 1.4 turns out to apply to their model. The extreme point properties in Corollary 1.4 imply here that the resulting Nash equilibrium is in *original* (i.e., nonmixed) control functions, which is in agreement with [12, Theorem 2] (e.g., cf. [1]). The mild conditions of Corollary 1.4 also present the possibility to relax some of the conditions immediately; for instance, the continuity conditions for $F_i$ and $G_i$ can be substantially reduced: $F_i(t, u_1)$ needs only to be upper semicontinuous in $u_1$, integrably bounded above and jointly measurable in $(t, u_1)$; a similar observation applies to $G_i$, $i = 1, 2$.

# 3 Additional observations

A slightly more involved variant of Corollary 1.4 is as follows:

**Proposition 3.1** *Let each $S_i$, $i \in I$, be a subset of a Hausdorff locally convex topological vector space. If for each $i \in I$*

$$S_i \text{ is compact and convex,}$$

$$\pi_i \text{ is concave on } S = \Pi_{j=1}^m S_j,$$

$$\pi_i(\cdot, s_i) \text{ is affine on } S^{-i} \text{ for every } s_i \in S_i,$$

$$\pi_i(s^{-i}, \cdot) \text{ is upper semicontinuous on } S_i \text{ for each } s^{-i} \in S^{-i},$$

*then there exists a Nash equilibrium for $\Gamma$.*

Here the additional extreme point condition of Corollary 1.4 disappears for obvious reasons (e.g., consider the situation $m = 1$). The above result forms a natural transpose of a well-konwn version of Glicksberg's Nash equilibrium existence result [9] by switching the roles of $(i)$ upper semicontinuity and (quasi)concavity, $(ii)$ continuity and affinity.

*Proof of Proposition 3.1.* Define the aggregate function $q : S \times S \to \mathbb{R}$ as follows. For $s' := (s'_1, \cdots, s'_m)$, $s := (s_1, \cdots, s_m)$, set

$$q(s', s) := \sum_{i=1}^{m} [\pi_i(s') - \pi_i(s'^{-i}, s_i)].$$

Then we can easily see that for every $s', s \in S$, $q(\cdot, s)$ is concave, $q(s', \cdot)$ is lower semicontinuous and $q(s', s') = 0$. Therefore, by Ky Fan's inequality [8, Theorem 5] there exists $\bar{s} \in S$ such that

$$q(s', \bar{s}) \leq 0 \text{ for all } s' \in S.$$

By substitution of $s' := (\bar{s}^{-j}, s_j)$, $j \in I$, one sees that $\bar{s}$ constitutes a Nash equilibrium. QED

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# References

[1] Balder, E.J.: New existence results for optimal controls in the absence of convexity: the importance of extremality. SIAM J. Control Optim. **32**, 890-916 (1994)

[2] Balder, E.J.: Lectures on Young measures. Cahiers du Centre de Recherche de Mathématiques de la Décision **9517**, CEREMADE, Université de Paris-Dauphine, Paris (1995)

[3] Baye, M.R., Tian, G. and Zhou, J.: Characterizations of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs. Rev. Econ. Studies **60**, 935-948 (1993)

[4] Castaing, C. and Valadier, M.: Convex Analysis and Measurable Multifunctions. Lecture Notes in Mathematics **580**. Berlin: Springer 1977

[5] Choquet, G.: Lectures on Analysis. Reading, Mass.: Benjamin 1969

[6] Dasgupta, P. and Maskin, E.: The existence of equilibrium in discontinuous economic games I: theory. Rev. Econ. Studies **53**, 1-26 (1986)

[7] Diestel, J.: Remarks on weak compactness in $L_1(\mu, X)$. Glasgow Math. J. **18**, 87-91 (1977)

[8] Fan Ky: Some properties of convex sets related to fixed point theorems. *Math. Ann.* **266**, 519-537 (1984)

[9] Glicksberg, I.L.: A further generalization of the Kakutani fixed point theorem, with applications to Nash equilibrium points. Proc. Amer. Math. Soc. **3**, 170-174 (1952)

[10] Ionescu Tulcea, A. and C.: Topics in the theory of lifting. Berlin: Springer 1969

[11] Neveu, J.: Bases mathématiques du calcul des probabilités. Paris: Masson 1964

[12] Parthasarathy, T. and Raghavan, T.E.S.: Existence of saddle points for differential games. SIAM J. Control **13**, 977-980 (1975)

[13] Warga, J.: Optimal control of differential and functional equations. New York: Academic Press 1972

[14] Yannelis, N.C.: Integration of Banach-valued correspondences. In: Khan, M.A., Yannelis, N.C. (eds.) Equilibrium theory in infinite dimensional spaces. Berlin: Springer 1991

[15] Yannelis, N.C. and Rustichini, A. Equilibrium points of non-cooperative random and Bayesian games. In: Aliprantis, C.D., Border, K, Luxemburg, W.A.J. (eds.) Positive Operators, Riesz spaces and Economics Berlin: Springer 1991