# A unifying approach to existence of Nash equilibria

Erik J. Balder
Mathematical Institute, University of Utrecht
Utrecht, the Netherlands

June 11, 1997

An approach initiated in [4] is shown to unify results about the existence of (i) Nash equilibria in games with at most countably many players, (ii) Cournot-Nash equilibrium distributions for large, anonymous games, and (iii) Nash equilibria (both mixed and pure) for continuum games. A new, central notion of *mixed externality* is developed for this purpose.

## 1 Introduction

In [4] a new analysis of Cournot-Nash equilibrium distributions was given by characterizing these as solutions of an associated variational inequality in terms of transition probabilities. In that paper the use of some key results from Young measure theory made it possible to formulate a rather powerful existence result for equilibrium distributions. This was shown to generalize equilibrium results in [15, 17] (and also those of [15], as was shown recently [5]). Recall that Young measure theory is basically a theory of narrow convergence for transition probabilities [2, 3, 7, 22, 23] which extends the classical notion of narrow (or weak) convergence for probability measures.

In this paper the ideas of [4] will be expanded considerably, and it will be shown that a whole class of Nash equilibrium results can be obtained in this way. In itself, it is not surprising that Young measure theory should play an important role in equilibrium existence questions for game theory. Rather, it seems surprising that the narrow topology for transition probabilities had not been used before for such purposes. Indeed, if we think of a set of players $T$, then it is standard to let each player $t \in T$ choose a probability measure, say $\delta(t)$, on the set of all actions available to him/her. Therefore, the combined effect of these choices of the players is to yield a transition probability, viz. the mapping $t \mapsto \delta(t)$. Since it is evident that Nash equilibrium questions for such games can be cast into the form of some fixed point problem for the $\delta$'s, one is led naturally to consider the topologization of the space of all transition probabilities, for which the narrow topology turns out to be an ideal candidate. As could be expected, when the set $T$ of players is finite or countably infinite, use of the Young measure theory adds nothing of interest, for then its topology is simply equivalent to the classical narrow topology for (products of) probability measures. It is rather when $T$ is uncountable that the Young measure topology adds new insights to the study of Nash equilibria, and this the present paper will demonstrate.

To make suitable use of the Young measure topology, a key notion of *mixed externality* is formulated here. For some of the equilibrium results considered such a mixed externality has a known form. For other results, phrased in terms of pure Nash equilibria, the mixed externality is both new and artificial. The basic pattern is then as follows: instead proving the existence of a pure Nash equilibrium solution right away, the existence question is first resolved for a mixed version of the problem. Once this has been done, it is easy to derive existence of a pure equilibrium solution from it by means of well-known methods of purification. In this way we obtain a new, powerful approach which simultaneously addresses several existence questions for classical noncooperative games. Until now, a coherent approach to these subjects was not available. Along the way, we shall also obtain some real improvements of existing results in this area.

The setup of this paper is as follows: First, notions and terminology are established concerning the mixed externality notion for games in normal form. Then Theorem 2.1, the main theorem for mixed Nash equilibrium profiles, is stated, as is Proposition 2.1, the main supporting tool for purification. Next, these central results are then used to derive the existence (i) of a mixed Nash equilibrium solution for a classical game (subsection 3.1), (ii) of a Cournot-Nash equilibrium distribution (subsection 3.2), and (iii) of pure Nash equilibria for continuum games in two essentially different situations (subsections 3.3, 3.4). As for (i), our Theorem 3.1.1 is rather classical. With regard to (ii), Theorem 3.2.1 coincides with the main equilibrium distribution existence result of [4]. In turn, the latter result is known to generalize the equilibrium distribution existence results of Mas-Colell [17] and Khan-Rustichini [15] (as explained in [4]) and of Khan-Rustichini [16] (as explained in [5]). Further, concerning (iii), Theorem 3.3.1 contains a generalization of a well-known result of Schmeidler [21, Theorem 1]; our result also partly generalizes the extension of this result given by Khan in [14, Theorem 5.1]. Finally, Theorem 3.4.1, a rather different result, generalizes Schmeidler's [21, Theorem 2] and the recent extension of his result by Rath [20].

## 2 Central notions and results

This section starts with an introduction of the mixed externality notion for games in normal form. After this, the main equilibrium existence result is stated for a *mixed* version of the game (Theorem 2.1). This is followed by Proposition 2.1, the main purification tool.

Let $(T, \mathcal{T}, \mu)$ be a finite measure space of *players*; it is convenient to suppose $\mu(T) = 1$. Let $S$ be a metric space of *actions*. Each player $t$ has to restrict her/his actions to a certain subset of $S$, denoted by $S_t$. Each set $S_t$ is supposed to belong to the Borel $\sigma$-algebra $\mathcal{B}(S)$, i.e., the $\sigma$-algebra generated by all open subsets of $S$. The set of all probability measures on $(S, \mathcal{B}(S))$ is denoted by $M_1^+(S)$. On some occasions we shall also write $\Sigma(t) := S_t$, so as to emphasize the fact that $\Sigma : t \mapsto S_t$ forms a multifunction. We shall consider transition probabilities (alias *Young measures*) $\delta : T \to M_1^+(S)$, such that for $\mu$-a.e. $t$ one has $\delta(t)(S_t) = 1$. Such transition probabilities will be called *mixed (action) profiles*, and the set of all of these is denoted by $\mathcal{R}$. See [18, III.2] for general measure-theoretical details on transition probabilities; here we just recall that a function $\delta : T \to M_1^+(S)$ is called a *transition probability* if $t \mapsto \delta(t)(B)$ is $\mathcal{T}$-measurable for every fixed set $B \in \mathcal{B}(S)$. The profile $\delta$ expresses that each player $t$ has chosen $\delta(t) \in M_1^+(S)$ for her/his mixed action; moreover, apart from some null set of players, each player $t$ has chosen $\delta(t)$ in such a way so as to result in an action in the proper subset $S_t$. A special subset of $\mathcal{R}$ is made up by the *pure* action profiles; namely, a profile $\delta \in \mathcal{R}$ is said to be *pure* if it corresponds to a measurable function from $T$ into $S$ such that for all $t$

$$\delta(t) = \epsilon_f(t) := \text{ Dirac probability at } f(t).$$

It is clear that in this case the definition of $\mathcal{R}$ forces $f$ to belong to the set $\mathcal{S}_\Sigma$ of all measurable a.e. selections of the multifunction $\Sigma : t \mapsto S_t$. Let $P_t : \mathcal{R} \to [-\infty, +\infty)$ be player $t$'s *payoff function*; $P_t(\delta)$ measures $t$'s personal benefit if the mixed profile $\delta \in \mathcal{R}$ is somehow realized.

Naturally, for every player $t$ it is important to distinguish the "internal" part $\delta(t)$ of $\delta \in \mathcal{R}$, over which $t$ has total control, from any other part, called "external" for contrast, over which player $t$ may have at most partial influence. In fact, for the notion of a Nash equilibrium (see below) this distinction is vital. A canonical way to distinguish is obtained by requiring the following *internal-external form* for the payoff function $P_t$: there are supposed to exist (i) a space $Y$, (ii) a function $U_t : S_t \times Y \to [-\infty, +\infty)$, and (iii) a mapping $e_t : \mathcal{R} \to Y$ such that $P_t$ decomposes as follows:

$$P_t(\delta) = \int_{S_t} U_t(x, e_t(\delta))\delta(t)(dx),\ t \in T.$$

Our basic assumptions, to be encountered later, will ensure that the above integral expression is meaningful. For technical reasons (see section 4) we assume that the space $Y$ is common to all players; this sometimes requires reformulating a little. In several applications, $e_t$ does not really depend on the player variable $t$; in such a case we shall simply write $e : \mathcal{R} \to Y$, etc. The space $Y$

will be called the space of *profile statistics* of the game. We shall call $U_t$ the *utility* function and $e_t$ the *mixed externality* of player $t$. As a particular consequence of the internal-external form, we have for a pure profile $\epsilon_f \in \mathcal{R}$ that

$$P_t(\epsilon_f) = U_t(f(t), e_t(\epsilon_f)),\ t \in T.$$

The above internal-external form of the payoffs can be found in some important instances:

**Example 2.1** *Take $T := I := \{1, 2, \dots n\}$ as the index set for a game with $n$ players; take $\mathcal{T} := 2^I$ and $\mu(\{i\}) = 1/n$ for each $i \in I$ (the precise nature of $\mu$ is not very relevant, as long as the empty set is the only null set). Let $S_i$ denote the set of actions available to player $i$. Rather than taking the set-theoretical sum (and, later, the topological sum) of the $S_i$'s, we suppose without loss that all $S_i$ are subsets (measurable by later assumptions) of a common set $S$. A mixed action profile $\delta$ can be considered as an $n$-vector $(\delta_1, \delta_2, \dots, \delta_n)$ of probability measures $\delta_i \in M_1^+(S_i) \subset M_1^+(S)$, simply by setting $\delta_i := \delta(i)$. Let $V_i : S^n \to [-\infty, +\infty]$ be player $i$'s ordinary (i.e., unmixed) payoff function for the normal form game. Then the expected payoff for player $i$ under the mixed action profile $(\delta_1, \delta_2, \dots, \delta_n)$ is*

$$P_i(\delta) = \int_{S_i} [\int_{S^{-i}} V_i(x_i, x^{-i})\delta^{-i}(dx^{-i})]\delta_i(dx_i),$$

*where $S^{-i} := \Pi_{j \neq i} S_j$, $\delta^{-i} := \Pi_{j \neq i} \delta_j$ (product measure), etc. Since $\delta^{-i}$ is a probability measure on $S^{-i}$, it can also be regarded as a probability measure on the larger set $S^{n-1}$. So the the internal-external form obtains if we set $Y := M_1^+(S^{n-1})$ (the set of all probability measures on $S^{n-1}$), $U_i(x_i, y) := \int_{S^{n-1}} V_i(x_i, x^{-i})y(dx^{-i})$ and*

$$e_i(\delta) := \delta^{-i}.$$

*(This is slightly less straightforward than might have been expected, because of our intention to keep the space $Y$ common to all players.) A game with countably infinitely many players can, of course, be treated in essentially the same way.*

**Example 2.2** *Consider $T = [0, 1]$ as the set of players, equipped with the Borel or Lebesgue $\sigma$-algebra $\mathcal{T}$ and the Lebesgue measure $\lambda$. Let $S$ be a separable Banach space $E$; suppose that for every $t$ the set $\Sigma(t) := S_t \subset E$ is closed and convex. Suppose also that there exists an integrable function $\phi : [0, 1] \to \mathbf{R}$ such that $\|x\| \leq \phi(t)$ for all $x \in S_t$ (here $\|x\|$ denotes the norm on $E$). By definition, each $\delta$ in $\mathcal{R}$ is a transition probability from $\delta : [0, 1] \to M_1^+(E)$ such that $\delta(t)(S_t) = 1$ for $\lambda$-a.e. $t$. By the integrable boundedness condition it follows that*

$$\int_T [\int_{S_t} \|x\|\delta(t)(dx)]\lambda(dt) \leq \int_T \phi \, d\lambda < +\infty.$$

*So in the first place we conclude that $\int_{S_t} \|x\|\delta(t)(dx) < +\infty$ for a.e. $t$. By [23, I.4.29] this guarantees for a.e. $t$ the existence of the* barycenter

$$\text{bar } \delta(t) := \int_{S_t} x \, \delta(t)(dx)$$

*of the probability measure $\delta(t)$, and by [23, I.6.3] this point lies in the closed convex subset $S_t$ of $E$.*

*On the exceptional null set involved here, we set* bar $\delta(t) := 0$. *It is easy to see that $t \mapsto$* bar $\delta(t)$, *thus defined, is integrable. Therefore, a mixed externality mapping $e$ from $\mathcal{R}$ into $Y := L^1_\Sigma[0, 1]$ is well-defined by*

$$e(\delta) := \pi(\text{bar } \delta),$$

*where $\pi : \mathcal{L}^1_E[0, 1] \to L^1_E[0, 1]$ is precisely defined by $\pi(f) := \{f' \in \mathcal{L}^1_E[0, 1] : f'(t) = f(t) \text{ for a.e. } t\}$ (observe that $\pi(\text{bar } \delta)$ is independent of the way in which we redefined* bar $\delta$ *on the exceptional null set above). Recall that $\mathcal{L}^1_E[0, 1]$ is the space of all Bochner-integrable functions from $[0, 1]$ into $E$ [23, I.4.29], that $L^1_E[0, 1]$ is exactly defined as the space of all* equivalence classes *with respect to* *the equivalence relation* $f = f'$ *a.e. on the latter space, and that* $L^1_\Sigma[0,1]$ *is defined by* $L^1_\Sigma[0,1] := \pi(\mathcal{S}_\Sigma \cap \mathcal{L}^1_E[0,1])$. *The reader's attention is called to the following notational rule, which is obeyed throughout:* prequotient spaces are denoted by script $\mathcal{L}$'s, and quotient spaces by straight $L$'s. *So as to have internal-external form with respect to the above specification of* $e$, *the payoff* $P_t$ *must be as follows:*

$$P_t(\delta) = \int_{S_t} U_t(x, \pi(\text{bar }\delta))\delta(t)(dx).$$

*In particular, for pure profiles* $\epsilon_f$ *this entails*

$$P(\epsilon_f) = U_t(f(t), \pi(f)),$$

*and this coincides with the form of the payoff postulated in Schmeidler's article [21].*

**Example 2.3** *In Example 2.2 one can, by way of alternative, also consider the following mixed externality mappings* $e$: *For* $r$ *fixed Lebesgue-measurable subsets* $T_1, T_2, \dots, T_r$ *of* $[0,1]$ *define*

$$e(\delta) := (\int_{T_i} \text{bar }\delta(t)\lambda(dt))_{i=1}^r,$$

*and set* $Y := E^r$. *For pure profiles* $\epsilon_f$ *this gives*

$$P_t(\epsilon_f) = U_t(f(t), (\int_{T_i} f\, d\lambda)_{i=1}^r),$$

*which is the form considered in Rath's article [20]. More generally, one could consider* $r$ *measurable functions* $g_1, g_2, \dots, g_r : D \to \mathbf{R}$ *and set*

$$e(\delta) := (\int_T [\int_{S_t} g_i(t,x)\delta(t)(dx)]\mu(dt))_{i=1}^r.$$

Nonstandard examples of the internal-external form can also be given:

**Example 2.4** *Consider in Example 2.2 the situation where half of the players, say for* $t \in [0, \frac{1}{2}]$, *act in complete isolation from all their opponents. In this case one can model* $e_t(\delta)$ *to be a constant (say identically equal to* $\infty$*) for all* $t \in [0, \frac{1}{2}]$, *and keep* $e_t(\delta) := \pi(\text{bar}\delta)$ *for* $t \in (\frac{1}{2}, 1]$. *Of course, the point* $\infty$ *should now be added to the space of profile statistics:* $Y := L^1_\Sigma[0,1] \cup \{+\infty\}$ *(topological sum).*

Our main concern will be with the following classical equilibrium notion, which is due to Nash; for the games with payoffs in the above internal-external form it runs as follows:

**Definition 2.1** *A mixed profile* $\delta_* \in \mathcal{R}$ *is said to be a* mixed Nash equilibrium profile *if*

$$\delta_*(t)(\text{arg max}_{x \in S_t} U_t(x, e_t(\delta_*))) = 1 \text{ for } \mu\text{-a.e. } t \text{ in } T.$$

The fact that a null set of players is allowed to escape the above requirement might be less desirable for certain models; however, we stress that the present analysis is strictly tied to the definition as given above. We shall now prepare our main Nash equilibrium existence result by listing the assumptions that must be satisfied.

**Assumption 2.1** $Y$ *is a Suslin metric space.*

**Assumption 2.2** $S$ *is a Suslin metric space.*

Recall here that a metric space is *Suslin* if it is the continuous image of a Polish (i.e., complete separable and metric) space [9, III].

**Assumption 2.3** $S_t$ *is nonempty and compact in* $S$ *for every* $t \in T$.

**Assumption 2.4** $U_t : S_t \times Y \to [-\infty, +\infty)$ *is upper semicontinuous on* $S_t \times Y$ *for every* $t \in T$.

Together with the previous assumption, this guarantees that $U_t(\cdot, e_t(\delta))$ is bounded from above on $S_t$ by a constant; therefore, the integral in the internal-external form representation of $P_t(\delta)$ is well-defined.

**Assumption 2.5** $D := \{(t,x) \in T \times S : x \in S_t\}$ *is* $\mathcal{T} \times \mathcal{B}(S)$*-measurable.*

Observe that, together with Assumptions 2.2 and 2.3, this guarantees the nonemptiness of $\mathcal{R}$: By the von Neuman-Aumann measurable selection theorem [8] there exists at least one measurable a.e. selection $f$ of $\Sigma$ (i.e., $f \in \mathcal{S}_\Sigma$); correspondingly, $\epsilon_f$ then belongs to $\mathcal{R}$.

**Assumption 2.6** $U_t(x, \cdot)$ *is continuous on* $Y$ *for every* $t \in T$, $x \in S_t$.

**Assumption 2.7** $(t,x) \mapsto U_t(x,y) : D \to [-\infty, +\infty)$ *is* $\mathcal{D}$*-measurable for every* $y \in Y$.

Here $\mathcal{D}$ stands for the $\sigma$-algebra on $D$, formed by all $\mathcal{T} \times \mathcal{B}(S)$-measurable subsets of $D$.

**Assumption 2.8** *For every* $t \in T$ *and* $\delta \in \mathcal{R}$ *the mapping* $t \mapsto e_t(\delta)$ *is* $\mathcal{T}$*-measurable.*

**Assumption 2.9** *For every* $t \in T$ *the mixed externality mapping* $e_t : \mathcal{R} \to Y$ *is continuous for the narrow topology.*

Recall from [2, 3] that, given Assumptions 2.3 and 2.5), the *narrow* topology (alias *weak* or *Young measure* topology) on $\mathcal{R}$ is defined as the coarsest topology for which the integral functionals $\delta \mapsto \int_T [\int_{S_t} g(t,x)\delta(t)(dx)]\mu(dt)$ are lower semicontinuous, for all $\mathcal{D}$-measurable $g : D \to (-\infty, +\infty]$ such that $g$ is integrably bounded from below (i.e., $\inf_{x \in S_t} g(t,x) \geq \phi(t)$, for some $\phi \in \mathcal{L}^1_{\mathbf{R}}(T)$) and $g(t,\cdot)$ is lower semicontinuous on $S_t$ for every $t \in T$.

**Theorem 2.1 (mixed Nash equilibrium existence result)** *If Assumptions 2.1–2.9 hold, then there exists a mixed Nash equilibrium profile.*

Section 4 is devoted to the proof of Theorem 2.1, which follows essentially the approach of [4]. The usefulness of this existence result will become apparent in the next section, sometimes in conjunction with the following sufficient condition for the existence of a pure Nash equilibrium profile.

**Proposition 2.1 (sufficient conditions for purification)** *Suppose that Assumptions 2.1–2.9 hold. Let* $\delta_*$ *be the mixed Nash equilibrium profile (guaranteed to exist by Theorem 2.1). Suppose that a pure profile* $\epsilon_{f_*} \in \mathcal{R}$ *satisfies*

$$e_t(\epsilon_{f_*}) = e_t(\delta_*) \text{ for } \mu\text{-a.e. } t$$

*and that either* [1]

$$\int_T [\int_{S_t} \arctan U_t(x, e_t(\delta_*))\delta_*(t)(dx)]\mu(dt) = \int_T \arctan U_t(f_*(t), e_t(\delta_*))\mu(dt) \tag{2.1}$$

*or, equivalently,*

$$U_t(f_*(t), e_t(\delta_*)) = \int_{S_t} U_t(x, e_t(\delta_*))\delta_*(t)(dx) \text{ for } \mu\text{-a.e. } t. \tag{2.2}$$

*Then* $\epsilon_{f_*}$ *is a pure Nash equilibrium profile, i.e.,*

$$f_*(t) \in \arg\max_{x \in S_t} U_t(x, e_t(\epsilon_{f_*})) \text{ for } \mu\text{-a.e. } t \text{ in } T.$$

[1] The arctangent is used here to ensure boundedness – whence integrability – of the integrands; this avoids making unnecessary additional assumptions.

*Proof.* First, let us establish that the function $u : t \mapsto \max_{x \in S_t} g_*(t,x)$ is measurable with respect to the $\mu$-completion $\mathcal{T}_\mu$ of the $\sigma$-algebra $\mathcal{T}$. Here $g_*(t,x) := U_t(x, e_t(\delta_*))$ is $\mathcal{D}$-measurable on $D$ by the assumptions (the composition of measurable functions is measurable). Now set $\tilde{g} \equiv g_*$ on $D$ and $\tilde{g} \equiv -\infty$ on $(T \times S)\backslash D$. By Assumption 2.5 and the above, $\tilde{g}$ is $\mathcal{T} \times \mathcal{B}(S)$-measurable. Evidently, we have $u(t) = \max_{x \in S} \tilde{g}(t,x)$, so $\mathcal{T}_\mu$-measurability of $u$ follows by [8, III.39], in view of Assumption 2.2. By a well-known property of the completion [9, II.15], the above fact also implies that there exists a $\mathcal{T}$-measurable function $v : T \to \mathbf{R}$ such that $v(t) = u(t)$ for $\mu$-a.e. $t$ in $T$.

Since $\delta_*$ is mixed Nash, it must be that for $\mu$-almost all $t$ the probability measure $\delta_*(t)$ is carried by the set $\arg\max_{x \in S_t} U_t(x, e_t(\delta_*))$. Hence, taking arctangents it is clear that

$$\int_T [\int_{S_t} \arctan U_t(x, e_t(\delta_*))\delta_*(t)(dx)]\mu(dt) = \int_T \arctan v(t)\mu(dt). \tag{2.3}$$

By the hypotheses for $f_*$, this gives

$$\int_T \arctan U_t(f_*(t), e_t(\epsilon_{f_*}))\mu(dt) = \int_T \arctan v(t)\mu(dt),$$

and since $U_t(f_*(t), e_t(\epsilon_{f_*})) \leq v(t)$ a.e., this implies that $U_t(f_*(t), e_t(\epsilon_{f_*})) = v(t)$ a.e. This proves that $\epsilon_{f_*}$ is a Nash equilibrium profile.

Finally, note that the equivalence of (2.1) and (2.2) follows immediately from (2.3) and the definition of the maximum functions $u$ and $v$. Q.E.D.

# 3 Applications

We shall now consider essentially four different applications of Theorem 2.1. The first application addresses the rather classical situation considered in Example 2.1. The second one works with Mas-Colell's notion of a Cournot-Nash equilibrium distribution [17]. The third application places additional convexity and quasiconcavity conditions on the basic ingredients of the game, in a setting for continuum games which is somewhat more general than the one used by Schmeidler [21] (see Example 2.2). The fourth application, formulated for continuum games in the same setup, is based on the requirement that $Y$, the space of profile statistics, is finite-dimensional and the measure $\mu$ is nonatomic.

## 3.1 Classical $n$-person games

In this subsection we consider the situation of Example 2.1.

**Assumption 3.1.1** *$S_i$ is a nonempty compact metric space for every $i \in I$.*

**Assumption 3.1.2** *$V_i$ is upper semicontinuous and bounded above on $\Pi_j S_j$ for every $i \in I$.*

**Assumption 3.1.3** *$V_i(x_i, \cdot)$ is bounded and continuous on $\Pi_{j \neq i} S_i$ for every $x_i \in S_i$ and $i \in I$.*

**Theorem 3.1.1** *Suppose that Assumptions 3.1.1–3.1.3 hold. Then there exists an $n$-vector $\delta_* := (\delta_{*1}, \delta_{*2}, \ldots, \delta_{*n})$, consisting of probability measures $\delta_{*i} \in M_1^+(S_i)$, such that for each $i \in I$*

$$P_i(\delta_*) \geq P_i(\delta_i \times \delta_*^{-i}) \text{ for every } \delta_i \in M_1^+(S_i).$$

*Proof.* As in Example 2.1, rather than taking $S$ to be the topological sum of the $S_i$'s (which is now obviously compact and metrizable by Assumption 3.1.1), we suppose without loss of generality that all $S_i$'s are subsets of a compact metric space $S$. We apply Theorem 2.1 to $Y := M_1^+(S^{n-1})$, equipped with the classical weak topology. As in Example 2.1, we set

$$U_i(x_i, y) := \int_{S^{-i}} V_i(x_i, x^{-i})\, y(dx^{-i}).$$

Assumption 2.1 holds by compactness and metrizability of $M_1^+(S^{n-1})$ for the classical weak topology [9, III.60]. Assumption 2.2 is evidently fulfilled. Also, Assumption 2.3 is contained in Assumption 3.1.1. The measurability assumptions hold trivially. Further, Assumption 2.4 holds by Assumption 3.1.2 and and well-known facts about weak convergence (combine [9, III.48] and [6, Theorem 3.2]). Also, Assumption 2.6 holds by definition of the definition of weak convergence, in view of the continuity hypothesis for the $V_i$. Assumption 2.8 holds trivially. Finally, in this case the narrow topology on $\mathcal{R}$ coincides with the product topology, obtained when $M_1^+(S)$ (whence each subspace $M_1^+(S_i)$, $i \in I$) is equipped with the classical weak topology. Therefore, Assumption 2.9 is evidently valid. The Nash equilibrium profile $\delta_* \in \mathcal{R}$, guaranteed by Theorem 2.1, gives the desired $\delta_{*i} := \delta_*(i)$ for each $i$. Since $e_i(\delta_*) = \Pi_{j \neq i}\delta_{*i}$ for each $i \in I$, the obvious identity

$$\sup_{x_i \in S_i} U_i(x_i, e_i(\delta_*)) = \sup_{\delta_i \in M_1^+(S_i)} \int_{\Pi_j S_j} V_i \, d(\delta_i \times \delta_*^{-i})$$

immediately implies the desired result. Q.E.D.

Observe that in standard textbooks on game theory Assumptions 3.1.2–3.1.3 are replaced by the somewhat more stringent continuity condition for the functions $V_i$, $i \in I$; e.g., cf. [11, Theorem 4.1.1].

## 3.2 Large anonymous games

Again, let $M_1^+(S)$ stand for the space of of probability measures on $S$, equipped with the classical narrow (or weak) topology. Recall that this is the coarsest topology for which the functionals $\nu \mapsto \int_S c \, d\nu$ are continuous for all bounded continuous $c : S \to \mathbf{R}$. For any $\delta \in \mathcal{R}$, $\mu \otimes \delta \mid_S$ denotes the *marginal* on $S$ of the product probability $\mu \otimes \delta$ on $D$ [18, III.2]. That is,

$$\mu \otimes \delta \mid_S (B) := \int_T \delta(t)(B)\mu(dt), \; B \in \mathcal{B}(S).$$

We shall now use the mixed externality $e : \delta \mapsto \mu \otimes \delta \mid_S$. Observe how this has the effect of mixing the individual probability measures $\delta(t)$, $t \in T$, which means that in a certain sense the players influence their opponents only anonymously.

**Theorem 3.2.1 (equilibrium distribution existence result)** *Suppose that Assumptions 2.2–2.7 hold for $Y := M_1^+(S)$. Then there exists a $\delta_* \in \mathcal{R}$ such that*

$$\delta_*(t)(\arg\max_{x \in S_t} U_t(x, \mu \otimes \delta_* \mid_S)) = 1 \textit{ for a.e. } t.$$

*Proof.* By [9, III.60] it follows from Assumption 2.2 that $Y = M_1^+(S)$ is metrizable and Suslin for the classical narrow topology. Evidently, all that has to be done is to check Assumption 2.9. This amounts to verifying that for any continuous bounded $c : S \to \mathbf{R}$ the functional $\delta \mapsto \int_S c d(\mu \otimes \delta \mid_S)$ is continuous from $\mathcal{R}$, equipped with the Young measure (alias narrow) topology, to $M_1^+(S)$, equipped with the classical narrow topology. Since

$$\int_S c \, d(\mu \otimes \delta \mid_S) = \int_T [\int_{S_t} c(x)\delta(t)(dx)]\mu(dt),$$

using $g(t, x) := c(x)$ in the definition of narrow convergence on $\mathcal{R}$ shows that said functional is lower semicontinuous. In the same way, upper semicontinuity follows from substituting $g(t, x) := -c(x)$ in that same definition. Q.E.D.

As a consequence of the above result, the probability measure $p_* \in M_1^+(D)$, given by $p_* := \mu \otimes \delta_*$, satisfies

$$p_*(\{(t, x) \in D : x \in \arg\max_{x \in S_t} U_t(x, p_* \mid_S)\}) = 1 \text{ and } p_* \mid_T = \mu.$$

Therefore, $p_*$ is a *Cournot-Nash equilibrium distribution* in the sense of [17, 15, 4]. Theorem 3.2.1 constitutes the main result of [4]. As shown there and in [5], it generalizes existence results for equilibrium distributions in [17, 15, 16].

### 3.3 Continuum games with convexity

As in Example 2.2, we suppose in this subsection that the space of actions $S$ is a separable Banach space $(E, \|\cdot\|)$. This Banach space is equipped with a locally convex topology $\omega$ which is not stronger than the norm topology and not weaker than the weak topology. Unless the contrary is explicitly mentioned, topological references to $S := E$ are understood to be with respect to $\omega$. Observe already that $(E, \omega)$ is a Suslin space, since $(E, \|\cdot\|)$ is Polish.

**Assumption 3.3.1** *$\Sigma(t) := S_t \subset E$ is convex for a.e. $t$.*

**Assumption 3.3.2** *There exists an integrable function $\phi$ from $T$ into $\mathbf{R}$ such that*

$$\sup_{x \in S_t} \|x\| \leq \phi(t) \text{ for a.e. } t.$$

**Assumption 3.3.3** *$U(t, \cdot, y)$ is quasi-concave for every $y \in L^1_\Sigma[0,1]$ for a.e. $t$.*

Clearly, Assumption 3.3.1 causes the set $\mathcal{S}_\Sigma$ of measurable a.e. selectors of $\Sigma$ to be equal to the set $\mathcal{L}^1_\Sigma[0,1]$ of all integrable a.e. selectors of $\Sigma$.

**Theorem 3.3.1 (continuous game equilibrium existence result)** *Suppose that Assumptions 2.3–2.7 and Assumptions 3.3.1–3.3.3 hold. Then there exists $f_* \in \mathcal{L}^1_\Sigma[0,1]$ such that for a.e. $t$*

$$U_t(f_*(t), \pi(f_*)) \geq U_t(x, \pi(f_*)) \text{ for all } x \in S_t.$$

*Proof.* First, note that the Banach space $L^1_E[0,1]$ is separable for the $L_1$-norm; therefore, it is a Polish space. So for the relative weak topology $\sigma(L^1_E[0,1], L^\infty_{E^*}[E]([0,1]))$ the space $Y := L^1_\Sigma[0,1]$ (which is certainly closed and convex) is Suslin. So Assumption 2.1 is valid; Assumption 2.2 was already seen to hold. Recall here [12, IV] that $L^\infty_{E^*}[E]([0,1])$, the set of all (equivalence classes of) scalarly measurable and essentially bounded functions $b : [0,1] \rightarrow E^*$ is the topological dual of $L^1_E[0,1]$. Here $E^*$ is the topological dual of $(E, \|\cdot\|)$. Also, by the definition of $e$ in Example 2.2 it follows easily that Assumption 2.8 holds. Finally, to ensure validity of Assumption 2.9, it is enough to establish that $\delta \mapsto \pi(\text{bar } \delta)$ is continuous from $\mathcal{R}$, equipped with the narrow topology, into $L^1_\Sigma[0,1]$ (still equipped with the relative weak topology). Let $b \in L^\infty_{E^*}[E]([0,1])$ be arbitrary. Define two normal integrands $g$ and $g'$, both integrably bounded from below, by setting $g(t,x) :=< x, b(t) >$ and $g'(t,x) := - < x, b(t) >$ on $D$. Then the definition of the narrow topology on $\mathcal{R}$ implies that $\delta \mapsto \int_{[0,1]}[\int_S < x, b(t) > \delta(t)(dx)]\lambda(dt)$ is narrowly continuous, which is to say that $\delta \mapsto \int_{[0,1]} <$ bar $\delta, b > d\lambda$ is narrowly continuous. So Assumption 2.9 holds. Theorem 2.1 may be applied, and this gives existence of a mixed Nash equilbrium profile $\delta_*$. We finish by applying Proposition 2.1: Let $f_* :=$ bar $\delta_*$; then the first condition of the proposition holds trivially. It remains to show that its third condition (being equivalent to the second one) holds: By Assumptions 2.4 and 3.3.3 the set arg $\max_{x \in S_t} U_t(x, e(\delta_*))$ is closed and convex. By Theorem 2.1, $\delta_*(t)$ is carried by this set. Therefore, $f_*(t)$, the barycenter of $\delta_*(t)$, also belongs to it for a.e. $t$. Q.E.D.

Theorem 3.3.1 generalizes a well-known theorem of Schmeidler [21, Theorem 1] completely and its extension by Khan [14, Theorem 7.1] partly in the following sense: Khan supposes $U(t, \cdot, \cdot)$ to be continuous on $S_t \times L^1_\Sigma$, which is certainly more than Assumptions 2.4–2.6 ask for. Also, Khan requires all $S_t$ to lie in one fixed weakly compact subset of $E$, which is much heavier than Assumption 3.3.2 (a fair portion of [14, section 7] is spent on attempts to improve on this). On the other hand, although the above result can almost automatically be extended to a setup where an abstract measure space $(T, \mathcal{T}, \mu)$ replaces $([0,1], \mathcal{T}, \lambda)$ (indeed, Theorem 2.1 naturally deals with this situation), the Suslin Assumption 2.1 for $Y := L^1_\Sigma(T)$ forces certain restrictions on the measure space $(T, \mathcal{T}, \mu)$. For instance, if $\mathcal{T}$ were countably generated, then $L^1_E(T)$ is a Polish space for the $L^1$-norm topology, so $L^1_\Sigma(T)$ becomes Suslin for the weak topology. Even though this restriction might seem fairly weak, it should be observed that [14] requires nothing of this kind.

### 3.4 Continuum games with nonatomicity

In this subsection $S$ is once more supposed to be a metrizable Suslin space. However, we now need to assume that the probability space $(T, \mathcal{T}, \mu)$ is *nonatomic*. Except for the fact that this probability space now replaces $[0,1]$, our present model will be as in Example 2.3 in all other respects.

**Assumption 3.4.1** *The probability space $(T, \mathcal{T}, \mu)$ is nonatomic.*

**Assumption 3.4.2** *The functions $g_1, \ldots, g_r : D \to \mathbf{R}$ are $\mathcal{D}$-measurable, integrably bounded and such that $g_i(t, \cdot)$ is continuous on $S_t$ for each $t$, $i = 1, \ldots, r$.*

**Theorem 3.4.1 (nonatomic game equilibrium result)** *Suppose that Assumptions 2.2–2.7 and 3.4.1–3.4.2 hold. Then there exists $f_* \in \mathcal{S}_\Sigma$ such that*

$$f_*(t) \in \arg\max_{x \in S_t} U_t(x, d(f_*)) \text{ for a.e. } t,$$

*where*

$$d(f) := (\int_T g_i(t, f(t)) \mu(dt))_{i=1}^r, \; f \in \mathcal{S}_\Sigma.$$

This result will be proven by means of Lemma III of [2]; remarks on this possibility can be found in [4, p. 353]. Here we recall this lemma in slightly simplified form:

**Lemma 3.4.1 ([2, Lemma III])** *Suppose that Assumptions 2.2–2.3 and 3.4.1 hold. Let $g_1, \ldots, g_n : T \times S \to (-\infty, +\infty]$ be $\mathcal{T} \times \mathcal{B}(S)$-measurable, integrably bounded from below and such $g_i(t, \cdot)$ is lower semicontinuous on $S$ for each $t$, $i = 1, \ldots, n$. Then for every $\delta \in \mathcal{R}$ there exists $f \in \mathcal{S}_\Sigma$ such that*

$$\int_T g_i(t, f(t)) \mu(dt) \le \int_T [\int_{S_t} g_i(t, s) \delta(t)(ds)] \mu(dt), \; i = 1, \ldots, n.$$

Compared to [2, Lemma III], we have already substituted $h(t, s) := 0$ if $(t, s) \in D$ and $h(t, s) := +\infty$ if $(s, t) \in (T \times S) \backslash D$. Then $h$ belongs to the class $\mathcal{H}(T; S)$ of [2] (by Assumptions 2.3–2.5), and for any $\delta \in \mathcal{R}$ the finiteness condition $\int_T [\int_S h(t, s) \delta(t)(ds)] \mu(dt) = 0 < +\infty$ is automatic. Such finiteness then causes [2, Lemma III] to give a function $f$ which belongs to $\mathcal{S}_\Sigma$, in agreement with what was stated in the lemma above.

*Proof of Theorem 3.4.1.* Clearly, Assumptions 2.1–2.7 are fulfilled. Let us define

$$e(\delta) := (\int_T [\int_{S_t} g_i(t, x) \delta(t)(dx)] \mu(dt))_{i=1}^r.$$

Then it follows that Assumptions 2.8–2.9 are also valid [apply the definition of narrow convergence to both $\delta \mapsto \int_T [\int_S g_i(t, x) \delta(t)(dx)] \mu(dt)$ and $\delta \mapsto -\int_T [\int_S g_i(t, x) \delta(t)(dx)] \mu(dt)$]. So by Theorem 2.1 there exists a mixed Nash equilibrium profile $\delta_* \in \mathcal{R}$. Now we can apply Lemma 3.4.1 to the collection $g_1, \ldots, g_r, -g_1, \ldots, -g_r, \ell_1, \ell_2$. Here the integrands $\ell_1, \ell_2 : D \to (-\infty, +\infty]$ ae defined by

$$\ell_j(t, x) := (-1)^{j-1} \arctan U_t(x, \bar{d}(\delta_*)).$$

for $j = 1, 2$ (by Assumptions 2.5–2.7 they may be included). Application of Lemma 3.4.1 gives the existence of a function $f_* \in \mathcal{S}_\Sigma$ such that $e(\delta_*) = e(\epsilon_{f_*}) = d(f_*)$ and

$$\int_T [\int_S \ell_j(t, x) \delta_*(t)(dx)] \mu(dt) \ge \int_T \ell_j(t, f_*(t)) \mu(dt)$$

for $j = 1, 2$. This implies

$$\int_T [\int_{S_t} \arctan U_t(x, e(\delta_*)) \delta_*(t)(dx)] \mu(dt) = \int_T \arctan U_t(f_*(t), e(\delta_*)) \mu(dt).$$

By $e(\delta_*) = e(\epsilon_{f_*})$, this is precisely (2.1). By Proposition 2.1 it therefore follows that $\epsilon_{f_*}$ is a Nash equilibrium profile. From this the stated result follows directly. Q.E.D.

Theorem 3.4.1, as contained in the remarks on [4, p. 353] and worked out above, substantially generalizes the main result of Rath's recent paper [20, Theorem 2, Remarks 6–8]. The latter also requires Assumption 3.3.2 to hold. If we also adopt Assumption 3.3.2, then Rath's result follows by substituting $S = \mathbf{R}^r$ and $g_i(t,x) := x^i$ ($i$-th coordinate). In fact, from combining subsections 3.2-3.3 it is evident that Rath's result remains valid for $S := E$ (our separable Banach space of section 3.3) if we set

$$d(f) := (\int_T < f(t), s_i^* > \mu(dt))_{i=1}^r,$$

where $s_1^*, \ldots, s_r^*$ are $r$ elements from the dual space $E^*$. Observe that this corresponds to having for the mixed externality

$$e(\delta) := (\int_T [\int_S < x, s_i^* > \delta(t)(dx)]\mu(dt))_{i=1}^r.$$

Of course, the Assumptions 2.3–2.7 and also 3.3.2 (but not the earlier Assumptions 3.3.1 and 3.3.3) must still hold. For yet another obvious but relevant way to purify in quite general situations the reader is referred to [4, p. 353].

# 4 Proof of Theorem 2.1

In this section Theorem 2.1 will be proven. The proof is virtually the same as that of [4, Theorem 1]. It is based on fundamental features of Young measure theory [2, 3, 22] and on Ky Fan's inequality. The first result deals with compactness for the narrow topology on $\mathcal{R}$. This can be found in [4, Lemma 3]. It follows thanks to Assumptions 2.2, 2.3 and 2.7.

**Lemma 4.1** *$\mathcal{R}$ is a compact convex subset of a certain vector space.*

The vector space is specified in the proof of Proposition 4 in [4]. In analogy to [4, p. 350], let us define $p : \mathcal{R} \times \mathcal{R} \to \mathbf{R}$ by

$$p(\delta, \eta) := \int_T [\int_{S_t} \arctan U_t(x, e_t(\delta))\eta(t)(dx)]\mu(dt).$$

Here the double integral is well-defined by Assumptions 2.5, 2.6, 2.7 and 2.9 (use [8, III.14] and [18, III.2]). Our next result could also have been proven by the same method as used to prove Lemma 5 in [4]; here we opt for a somewhat more transparent proof.

**Lemma 4.2** *i. $p$ is upper semicontinuous on $\mathcal{R} \times \mathcal{R}$. ii. $p(\cdot, \eta)$ is continuous on $\mathcal{R}$ for every $\eta \in \mathcal{R}$.*

*Proof.* i. Let $d_Y$ and $d_S$ stand for the metrics on $Y$ and $S$; we may suppose $d_Y \leq 1$ (else take $\min(1, d_Y(y, y'))$ as a new metric, equivalent to the old one). Let $((\delta_\alpha, \eta_\alpha))$ be a generalized sequence, converging in $\mathcal{R} \times \mathcal{R}$ to $(\delta_0, \eta_0)$. Define $g : T \times S \times Y \to (-\infty, +\infty]$ by setting $g(t,x,y) := -\arctan U(t,x,y)$ for $(t,x) \in D$, $y \in Y$, and by $g(t,x,y) := +\infty$ for $(t,x) \notin D$, $y \in Y$. By Assumptions 2.3–2.6, $g$ is a normal integrand on $T \times (X \times Y)$, so by the approximation procedure of [3, p. 268] (and thanks to Assumptions 2.1, 2.2) there is a nondecreasing sequence $(g_n)$ of $\mathcal{T} \times \mathcal{B}(S)$-measurable functions $g_n : T \times S \to (-\infty, +\infty]$ such that for a.e. $t$, $|g_n(t,x,y) - g_n(t,x',y')| \leq n d_S(x,x') + n d_Y(y,y')$ for all $x, x' \in S$ and all $y, y' \in Y$ (Lipschitz property). So to prove

$$\liminf_\alpha \int_T [\int_{S_t} g(t,x,e_t(\eta_\alpha))\delta_\alpha(t)(dx)]\mu(dt) \geq \int_T [\int_{S_t} g(t,x,e_t(\eta_0))\delta_0(t)(dx)]\mu(dt)\mu(dt), \tag{4.1}$$

it is enough to prove the same inequality with $g$ replaced by $g_n$ for any $n$ (indeed, an application of the monotone convergence theorem then easily implies the above inequality). So fix $n$; note that

$$g_n(t,x,e_t(\eta_\alpha)) \geq g_n(t,x,e_t(\eta_0)) - n d_Y(e_t(\delta_\alpha), e_t(\delta_0)),$$

by the Lipschitz-property of $g_n$. Integrating successively over $\delta_\alpha(t)$ and $\mu$ gives

$$\int_T [\int_{S_t} g_n(t,x,e_t(\eta_\alpha))\delta_\alpha(t)(dx)]\mu(dt) \geq \int_T [\int_{S_t} g_n(t,x,e_t(\eta_0))\delta_\alpha(t)(dx)]\mu(dt) - \rho_\alpha$$

where $\rho_\alpha := n\int_T d_Y(e_t(\delta_\alpha), e_t(\delta_0))\mu(dt)$. By the dominated convergence theorem and Assumption 2.9 it follows that $\rho_\alpha \to 0$. Applying [3, Theorem 2.2] now easily gives (4.1).

ii. Using Assumption 2.6, this follows immediately from [3, Theorem 2.2] by a simpler argument than the one above. Q.E.D.

**Lemma 4.3** *For every $\delta_* \in \mathcal{R}$ the following are equivalent:*
*a. $\delta_*$ is a mixed Nash equilibrium profile.*
*b. $p(\delta_*, \delta_*) \geq p(\delta_*, \eta)$ for all $\eta \in \mathcal{R}$.*

*Proof.* The proof runs precisely as the one for [4, Corollary 1]. It is based on using [4, Proposition 3] (essentially a measurable selection argument).

*Proof of Theorem 2.1.* By Lemmas 4.1-4.2 we can apply Ky Fan's inequality (for which no Hausdorff conditions are needed [10, p. 501] – observe that the narrow topology is non-Hausdorff) to the functional $q : \mathcal{R} \times \mathcal{R} \to \mathbf{R}$, defined by

$$q(\delta, \eta) := p(\delta, \eta) - p(\delta, \delta),$$

just as was done in proving [1, Theorem 5]. Indeed, $\mathcal{R}$ is compact and convex (Lemma 4.1) and it was already seen to be nonempty. By Lemma 4.2, $q(\cdot, \eta)$ is lower semicontinuous for every $\eta \in \mathcal{R}$. Finally, $q(\delta, \cdot)$ is trivially affine. So by Ky Fan's inequality it follows that there exists $\delta_* \in \mathcal{R}$ satisfying b in Lemma 4.3, whence a of Lemma 4.3. Q.E.D.

# References


[1] Aubin, J.-P. and Ekeland, I.: *Applied Nonlinear Analysis*, Wiley, New York, 1984

[2] Balder, E.J.: A general approach to lower semicontinuity and lower closure in optimal control theory. *SIAM J. Control Optim.* 22 (1984), 570-598

[3] Balder, E.J.: Generalized equilibrium results for games with incomplete information. *Math. Oper. Res.* 13 (1988), 265-276

[4] Balder, E.J.: On Cournot-Nash equilibrium distributions for games with differential information and discontinuous payoffs. *Econ. Theory* 1 (1991), 339-354

[5] Balder, E.J.: Comments on the existence of equilibrium distributions, Preprint, Department of Mathematics, Utrecht, 1993

[6] Billingsley, P.: *Convergence of Probability Measures*, Wiley, New York, 1968

[7] Berliocchi, H. and Lasry, J.-M.: Intégrandes normales et mesures paramétrées en calcul des variations. *Bull. Soc. Math. France* 101 (1973), 129-184

[8] Castaing, C. and Valadier, M.: *Convex Analysis and Measurable Multifunctions.* Springer, Berlin, 1977

[9] Dellacherie, C. and Meyer, P.-A.: *Probabilities and Potential.* North-Holland, Amsterdam, 1975

[10] Ding, X.P. and Tan, K.-K.: Generalized variational inequalities and generalized quasi-variational inequalities. *J. Math. Anal. Appl.* 148 (1990), 497-508

[11] Ichiishi, T.: *Game Theory for Economic Analysis.* Academic Press, New York, 1983

[12] Ionescu Tulcea, A. and C.: *Topics in the Theory of Lifting.* Springer, Berlin, 1969

[13] Khan, M.A.: On Cournot-Nash equilibrium distributions for games with a nonmetrizable action space and upper semicontinuous payoffs. *Trans. Am. Math. Soc.* 315 (1989), 127-146

[14] Khan, M.A.: On extensions of the Cournot-Nash theorem. In: *Advances in Equilibrium Theory* (Aliprantis, C.D., Burkinshaw, O., and Rothman, N.J., eds.). Springer Lect. Notes Econ. Math. Syst. 244, 1985, pp. 79-106

[15] Khan, M.A. and Rustichini, A.: Cournot-Nash equilibrium distributions for games with differential information. In: *Fixed Point Theory and Applications* (Théra, M.A. and Baillon, J.-B., eds.). Pitman Research Notes 252, 1991, pp. 245-259

[16] Khan, M.A. and Rustichini, A.: Cournot-Nash equilibrium distributions of games with uncertainty and imperfect information. *J. Math. Econom.* 22 (1993), 35-59

[17] Mas-Colell, A.: On a theorem of Schmeidler. *J. Math. Econom.* 13 (1984), 201-206

[18] Neveu, J.: *Mathematical Foundations of the Calculus of Probabilities.* Holden-Day, San Fransisco, 1965

[19] Owen, G.: *Game Theory.* Academic Press, New York, 1982

[20] Rath, K.P.: A direct proof of the existence of pure strategy equilibria in games with a continuum of players. *Econ. Theory* 2 (1992), 427-433

[21] Schmeidler, D.: Equilibrium points of nonatomic games. *J. Statist. Phys.* 7 (1973), 295-300

[22] Valadier, M.: Young measures. In: *Methods of Nonconvex Analysis* (A. Cellina, ed.) Lecture Notes in Mathematics 1446, Springer, Berlin, 1990, pp. 152-188.

[23] Warga, J.: *Optimal Control of Differential and Functional Equations.* Academic Press, New York, 1972.

[24] Yannelis, N.C.: Integration of Banach-valued correspondences. In: *Equilibrium Theory in Infinite Dimensional Spaces* (Khan, M.A. and Yannelis, N.C., eds.) Springer, Berlin, 1991, pp. 2-35.