SCALING FOR A RANDOM POLYMER

R. van der Hofstad
F. den Hollander

Mathematical Institute
University of Utrecht
P.O. Box 80.010
3508 TA Utrecht
The Netherlands

Abstract

Let Q_n^β be the law of the n-step random walk on $\mathbb{Z}^d$ obtained by weighting simple random walk with a factor $e^{-\beta}$ for every self-intersection (Domb-Joyce model of 'soft polymers'). It was proved by Greven and den Hollander (1993) that in $d = 1$ and for every $\beta \in (0, \infty)$ there exist $\theta^*(\beta) \in (0, 1)$ and $\mu_\beta^* \in \{\mu \in l^1(\mathbb{N}) : \|\mu\|_{l^1} = 1, \mu > 0\}$ such that under the law Q_n^β as $n \to \infty$:
(i) $\theta^*(\beta)$ is the limit empirical speed of the random walk;
(ii) μ_β^* is the limit empirical distribution of the local times.
A representation was given for $\theta^*(\beta)$ and μ_β^* in terms of a largest eigenvalue problem for a certain family of $\mathbb{N} \times \mathbb{N}$ matrices. In the present paper we use this representation to prove the following scaling result as $\beta \downarrow 0$:
(i) $\beta^{-\frac{1}{3}}\theta^*(\beta) \to b^*$;
(ii) $\beta^{-\frac{1}{3}}\mu_\beta^*(\lceil \cdot \beta^{-\frac{1}{3}} \rceil) \to^{L^1} \eta^*(\cdot)$.
The limits $b^* \in (0, \infty)$ and $\eta^* \in \{\eta \in L^1(\mathbb{R}^+) : \|\eta\|_{L^1} = 1, \eta > 0\}$ are identified in terms of a Sturm-Liouville problem, which turns out to have several interesting properties.

The techniques that are used in the proof are functional analytic and revolve around the notion of epi-convergence of functionals on $L^2(\mathbb{R}^+)$. Our scaling result shows that the speed of soft polymers in $d = 1$ is not right differentiable at $\beta = 0$, which precludes expansion techniques that have been used successfully in $d \geq 5$ (Hara and Slade (1992a,b)). In simulations the scaling limit is seen for $\beta \leq 10^{-2}$.

Keywords: Random walk with self-repellence, weak interaction scaling limit, epi-convergence, Sturm-Liouville problem.

1991 Mathematical Subject Classification: primary 60J15; secondary 34B24, 47A55, 58E30.

Date: 4-7-1994.

0 Introduction and main results

0.1 Model and motivation

A polymer is a long chain of molecules with two characteristic properties: (i) an irregular shape (due to entanglement); (ii) a certain stiffness (due to sterical hindrance). One way of describing such a polymer is the following model, which is based on a *random walk with self-repellence.*

Let $(S_i)_{i\geq 0}$ be simple random walk on $\mathbb{Z}^d$ ($d \geq 1$), starting at the origin. Let P_n be its law on n-step paths and let E_{P_n} be expectation w.r.t. P_n. Define a new law Q_n^β on n-step paths by setting

$$\frac{dQ_n^\beta}{dP_n}\Big((S_i)_{i=0}^n\Big) = \frac{1}{Z_n^\beta} \exp\Big[-\beta \sum_{\substack{i,j=0\\ i\neq j}}^n 1\{S_i = S_j\}\Big], \tag{0.1}$$

where Z_n^β is the normalizing constant

$$Z_n^\beta = E_{P_n}\Big(\exp\Big[-\beta \sum_{\substack{i,j=0\\ i\neq j}}^n 1\{S_i = S_j\}\Big]\Big) \tag{0.2}$$

and $\beta \in [0,\infty]$ is a parameter. The law Q_n^β is called the *n-polymer measure with strength of repellence* β [1].

Eqs.(0.1-2) define what is called the Domb-Joyce model of 'soft polymers', where the weight factor gives a penalty $e^{-\beta}$ for every self-intersection. The limiting cases $\beta = 0$ and $\beta = \infty$ correspond to simple random walk resp. self-avoiding random walk. For a recent guide to the literature on this model the reader is referred to Madras and Slade (1993) Section 10.1.

It is generally believed that for $\beta \in (0,\infty]$ the mean square displacement behaves like

$$E_{Q_n^\beta}[|S_n|^2] \sim Dn^{2\nu} \quad (n \to \infty), \tag{0.3}$$

where $D = D(\beta, d) > 0$ is some amplitude and $\nu = \nu(d)$ is a *critical exponent.* The latter is believed to be *independent* of β and to assume the values [2]

$$\begin{aligned} \nu &= 1 && d = 1\\ &= \tfrac{3}{4} && d = 2\\ &= 0.588\ldots && d = 3\\ &= \tfrac{1}{2} && d \geq 4. \end{aligned} \tag{0.4}$$

Note that $\nu = \frac{1}{2}$ is the exponent for simple random walk ($\beta = 0$) in any $d \geq 1$ (with $D = 1$). Apparently, the repellence changes the qualitative behavior when $d \leq 3$ but not when $d \geq 4$ [3]. The fact that ν is the same for all $\beta \in (0,\infty]$ says that soft polymers are in the same universality class as self-avoiding walk.

[1] Note that if $\beta > 0$ then $(Q_n^\beta)_{n\geq 0}$ is not a consistent family, i.e., Q_n^β is not the projection on n-step paths of the law of some process evolving in time (like P_n).

[2] The value in $d = 3$ is well below $\max\{\frac{3}{d+2}, \frac{1}{2}\}$, the so-called Flory value (Madras and Slade (1993) Section 2.2).

[3] Actually, $d = 4$ is a critical dimension where it is believed that $E_{Q_n^\beta}\big[|S_n|^2\big] \sim Dn(\log n)^{\frac{1}{4}}$, containing a logarithmic correction to (0.3-4).

Sofar a rigorous proof of (0.3-4) has only been given for $d \geq 5$ (Hara and Slade (1992a,b) [4]) and for $d = 1$ (Greven and den Hollander (1993)). In the latter work there is also a recipe for evaluating the amplitute $D(\beta, 1)$ as a function of β, which we next describe.

0.2 Speed and local times in $d = 1$

Define the random variables

$$
\begin{aligned}
\theta_n &= \frac{1}{n}|S_n| && (0.5)\\
\mu_n &= \frac{1}{|R_n|}\sum_{x \in R_n} \delta_{\ell_n(x)}, && (0.6)
\end{aligned}
$$

where

$$
\begin{aligned}
R_n &= \Big(\min_{0 \leq i \leq n} S_i, \max_{0 \leq i \leq n} S_i\Big) \cap \mathbb{Z}\\
\ell_n(x) &= \#\{0 \leq i < n : S_i = x\}. && (0.7)
\end{aligned}
$$

In words, θ_n is the *empirical speed* and μ_n is the *empirical distribution of local times* after n steps. Theorems 1-3 below are taken from Greven and den Hollander (1993) and are the starting point of the present paper.

Theorem 1 *For every $\beta \in (0, \infty)$ there exists $\theta^*(\beta) \in (0, 1)$ such that*

$$
\lim_{n \to \infty} Q_n^\beta\Big(|\theta_n - \theta^*(\beta)| \leq \epsilon\Big) = 1 \quad \textit{for every} \quad \epsilon > 0, \tag{0.8}
$$

with $\beta \to \theta^*(\beta)$ analytic, $\lim_{\beta \downarrow 0} \theta^*(\beta) = 0$ and $\lim_{\beta \to \infty} \theta^*(\beta) = 1$ [5].

Theorem 2 *For every $\beta \in (0, \infty)$ there exists $\mu_\beta^* \in \{\mu \in l^1(\mathbb{N}) : \|\mu\|_{l^1} = 1, \mu > 0\}$ such that*

$$
\lim_{n \to \infty} Q_n^\beta\Big(\|\mu_n - \mu_\beta^*\|_{l^1} \leq \epsilon\Big) = 1 \quad \textit{for every} \quad \epsilon > 0, \tag{0.9}
$$

with $\beta \to \mu_\beta^*$ analytic, $\lim_{\beta \downarrow 0} \mu_\beta^* = 0$ and $\lim_{\beta \to \infty} \mu_\beta^* = \delta_1$ pointwise.

The limits $\theta^*(\beta)$ and μ_β^* in Theorems 1 and 2 can be found in terms of the following *largest eigenvalue problem.* Let $A_{r,\beta}$ $(r \in \mathbb{R}, \beta > 0)$ be the matrix

$$
A_{r,\beta}(i,j) = e^{r(i+j-1) - \beta(i+j-1)^2} P(i,j) \; (i, j \in \mathbb{N}), \tag{0.10}
$$

where P is the Markov matrix

$$
P(i,j) = \begin{pmatrix} i+j-2 \\ i-1 \end{pmatrix} \left(\frac{1}{2}\right)^{i+j-1}. \tag{0.11}
$$

[4] The proof in Hara and Slade (1992a,b) is for $\beta = \infty$. However, the technique that is used (the so-called 'lace expansion') easily implies the same result for all $\beta \in (0, \infty]$. Brydges and Spencer (1985) earlier used the same technique to prove (0.3-4) for $d \geq 5$ and β sufficiently small.

[5] Note that (0.5) and (0.8) imply (0.3) with $\nu(1) = 1$ and $D(\beta, 1) = [\theta^*(\beta)]^2$.

Let $(\lambda(r,\beta),\tau_{r,\beta})$ be the unique solution of the largest eigenvalue problem [6]

$$
\begin{aligned}
& A_{r,\beta}\tau = \lambda\tau \ (\lambda > 0, \tau \in l^2(\mathbb{N})) \\
& \|\tau\|_{l^2} = 1, \tau > 0.
\end{aligned}
\tag{0.12}
$$

Theorem 3 *Fix $\beta \in (0,\infty)$. Let $r^*(\beta) \in (0,\infty)$ be the unique solution of*

$$\lambda(r,\beta) = 1. \tag{0.13}$$

Then

$$
\begin{aligned}
\frac{1}{\theta^*(\beta)} &= \Big[\frac{\partial}{\partial r}\lambda(r,\beta)\Big]_{r=r^*(\beta)} \\
\mu^*_\beta(k) &= \Big[\sum_{\substack{i,j\in\mathbb{N}\\ i+j-1=k}} \tau_{r,\beta}(i)A_{r,\beta}(i,j)\tau_{r,\beta}(j)\Big]_{r=r^*(\beta)} \ (k\in\mathbb{N}).
\end{aligned}
\tag{0.14}
$$

The representation in Theorem 3 is not easy to manipulate, which is why precise analytical estimates of $\theta^*(\beta)$ and μ^*_β are hard to get. For instance, the intuitively appealing conjecture that $\beta \to \theta^*(\beta)$ is increasing still remains open (see Greven and den Hollander (1993)). However, it is easy to get numerical estimates (see section 0.3). Moreover, we shall see that (0.13-14) provide a good starting point for carrying out a *scaling analysis* as $\beta \downarrow 0$ (see sections 0.4-5), which is the main topic of the present paper.

0.3 Numerical estimates of $r^*(\beta)$ and $\theta^*(\beta)$

Table 1 below lists some numerical estimates of $r^*(\beta)$ and $\theta^*(\beta)$ obtained from (0.13-14), based on a 300×300 truncation of $A_{r,\beta}$ defined in (0.10). We have used a standard iteration method to estimate the largest eigenvalue and corresponding eigenvector for a range of r,β-values.

β	$\beta^{-\frac{2}{3}}r^*(\beta)$	$\beta^{-\frac{1}{3}}\theta^*(\beta)$
2	1.696	0.793
0.5	1.730	1.055
10^{-2}	2.011	1.10938
10^{-3}	2.098	1.10930
10^{-4}	2.144	1.10886
10^{-5}	2.168	1.10910
10^{-6}	2.179	1.10924

Table 1

There is ample evidence for the asymptotic behavior $r^*(\beta) \sim a^*\beta^{\frac{2}{3}}$ and $\theta^*(\beta) \sim b^*\beta^{\frac{1}{3}}$ $(\beta \downarrow 0)$, with estimates $a^* = 2.19 \pm 0.01$ and $b^* = 1.109 \pm 0.001$.

[6] $A_{r,\beta} : l^2(\mathbb{N}) \longmapsto l^2(\mathbb{N})$ is positive, self-adjoint and compact for all $r\in\mathbb{R}, \beta > 0$. Both $(r,\beta)\to\lambda(r,\beta)$ and $(r,\beta)\to\tau_{r,\beta}$ are analytic. Moreover, $r\to\lambda(r,\beta)$ is strictly increasing and log-convex, $\lambda(0,\beta) < 1$ and $\lambda(\infty,\beta)=\infty$ for every $\beta > 0$ (see Greven and den Hollander (1993)).

The value of $\theta^*(\beta)$ has been computed by making use of the identity

$$\begin{aligned}\frac{1}{\theta^*(\beta)} &= \sum_{k\in\mathbb{N}} k\mu^*_\beta(k)\\ &= 2\Big[\sum_{i\in\mathbb{N}} i\tau^2_{r^*(\beta),\beta}(i)\Big]-1\end{aligned}\tag{0.15}$$

(Greven and den Hollander (1993)). Since $\tau_{r,\beta}$ is easier to estimate than $\frac{\partial}{\partial r}\lambda(r,\beta)$, the relation in (0.15) allows for better accuracy than (0.14).

0.4 Main results

The goal of this paper is to turn the numerical observations in section 0.3 into a mathematical statement. Our results are formulated in Theorems 4-7 below.

1. Our main scaling theorem reads:

Theorem 4 *There exist $a^*, b^* \in (0,\infty)$ and $\eta^* \in \{\eta \in L^1(\mathbb{R}^+) : \|\eta\|_{L^1} = 1, \eta > 0\}$ such that as $\beta \downarrow 0$*

$$\begin{array}{lll}\beta^{-\frac{2}{3}} r^*(\beta) & \to & a^*\\ \beta^{-\frac{1}{3}}\theta^*(\beta) & \to & b^*\\ \beta^{-\frac{1}{3}}\mu^*_\beta(\lceil\cdot\beta^{-\frac{1}{3}}\rceil) & \to^{L^1} & \eta^*(\cdot).\end{array}\tag{0.16}$$

2. The limits a^*, b^* and η^* in Theorem 4 can be identified in terms of the following *Sturm-Liouville problem.* For $a \in \mathbb{R}$, let $\mathcal{L}^a$ be the differential operator defined by

$$(\mathcal{L}^a x)(u) = (2au - 4u^2)x(u) + x'(u) + ux''(u) \quad (x \in C^\infty(\mathbb{R}^+)).\tag{0.17}$$

In section 5 we shall show that the largest eigenvalue problem

$$\begin{array}{ll}\mathcal{L}^a x & = \rho x \;\; (\rho \in \mathbb{R}, x \in L^2(\mathbb{R}^+)\cap C^\infty(\mathbb{R}^+))\\ (i) & \|x\|_{L^2} = 1, x > 0\\ (ii) & \int_0^\infty \{u^2[x(u)]^2 + u[x'(u)]^2\}du < \infty\end{array}\tag{0.18}$$

has a unique solution $(x^a, \rho(a))$ with the following properties:

$$\begin{array}{ll}(i) & a \to \rho(a) \text{ is analytic, strictly increasing and strictly convex on } \mathbb{R}\\ (ii) & \rho(0) < 0,\ \lim_{a\uparrow\infty}\rho(a) = \infty \text{ and } \lim_{a\downarrow-\infty}\rho(a) = -\infty\\ (iii) & a \to x^a \text{ is analytic as a map from } \mathbb{R} \text{ to } L^2(\mathbb{R}^+).\end{array}\tag{0.19}$$

The main part of our analysis to prove Theorem 4 will revolve around the following theorem, which is proved in sections 2-5:

Theorem 5 *Fix $a \in \mathbb{R}$. As $\beta \downarrow 0$*

$$\begin{array}{ll}\beta^{-\frac{1}{3}}\big[\lambda(a\beta^{\frac{2}{3}},\beta) - 1\big] & \to \rho(a)\\ \beta^{-\frac{1}{6}}\tau_{a\beta^{\frac{2}{3}},\beta}(\lceil\cdot\beta^{-\frac{1}{3}}\rceil) & \to^{L^2} x^a(\cdot).\end{array}\tag{0.20}$$

We shall show in section 6 that (0.20) identifies the limits in Theorem 4 as follows:

Theorem 6 *a^*, b^* and η^* are given by*

$$\begin{array}{ll} a^* \text{ is the unique solution of } \rho(a) = 0 \\ \frac{1}{b^*} = \rho'(a^*) \\ \eta^*(\cdot) = \frac{1}{2}[x^{a^*}(\frac{1}{2}\cdot)]^2. \end{array} \tag{0.21}$$

3. The analysis in section 5 of the Sturm-Liouville problem will lead to the following additional properties:

Theorem 7 *(i) $u \to x^{a^*}(u)$ is analytic and strictly decreasing on $\mathbb{R}_0^+ = [0, \infty)$.*
(ii) $u \to u\frac{d}{du}x^{a^*}(u)$ is unimodal with a minimum at $u = \frac{1}{2}a^*$.
(iii)

$$\lim_{u\to\infty} u^{-\frac{3}{2}} \log x^{a^*}(u) = -\frac{4}{3}. \tag{0.22}$$

(iv)

$$\frac{1}{b^*} = 2\int_0^\infty u[x^{a^*}(u)]^2 du. \tag{0.23}$$

Theorems 4-7 are proved in sections 2-6. Section 1 contains preparations.

Our result $\theta^*(\beta) \sim b^*\beta^{\frac{1}{3}}$ implies that the speed is not right-differentiable at $\beta = 0$. Thus the limit of weak repellence cannot be treated by perturbation type arguments (i.e., by doing an expansion of (0.1-2) for small β).

0.5 Numerical estimates of a^*, b^* and η^*

Let $y^{a,\rho}$ be the unique *power series solution* of $\mathcal{L}^a y = \rho y$ with $y^{a,\rho}(0) = 1$. We shall see in section 5 that this power series has infinite radius of convergence and has coefficients which satisfy a simple recurrence relation (see (5.23) below). Moreover, we shall see that:

(i) $\rho(a)$ is simple
(ii) $\mathcal{S}_a = \{\rho \in \mathbb{R} : y^{a,\rho} \in L^2(\mathbb{R}^+)\}$ is a countable set which has $\rho(a)$ as a maximum
(iii) $\rho \notin \mathcal{S}_a : \lim_{u\to\infty} y^{a,\rho}(u) = \pm\infty$
(iv) $\rho \in \mathcal{S}_a, \rho \neq \rho(a) : y^{a,\rho}(u) < 0$ for some $u > 0$
(v) $y^{a,\rho(a)} = x^a$, the monotone solution of (0.18).

Properties $(i) - (v)$ give us a way to estimate a^* and x^{a^*}. Namely, put $\rho = 0$ and consider $y^{a,0}$, the unique power series solution of $\mathcal{L}^a y = 0$ ($a \in \mathbb{R}$). Since a^* is the unique value of a for which $y^{a,0} \in L^2(\mathbb{R}^+)$ and $y^{a,0} \geq 0$, we can vary a and tune into a^* by looking at the tail behavior and the sign of $y^{a,0}$. It turns out that this method is very sensitive indeed and that a^* can be estimated by $a^* = 2.189 \pm 0.001$. For a outside this interval it was found that either $y^{a,0}(u) < 0$ for some $u \in [0, 3]$, or $u \to y^{a,0}(u)$ not monotone on $u \in [0, 3]$.

[Figure 1]

Figure 1 compares x^{a^*} with the numerical estimates in section 0.3. The solid line is $u \to y^{a,0}(u)/\|y^{a,0}\|_{L^2}$ for $a = 2.189$. The dots are the values of $\beta^{-\frac{1}{6}}\tau_{r^*(\beta),\beta}(\lceil u\beta^{-\frac{1}{3}}\rceil)$ for $\beta = 10^{-4}$ and $\lceil u\beta^{-\frac{1}{3}}\rceil = 1, \ldots, 64$. The agreement is excellent. (For $\beta = 10^{-5}$ and $\beta = 10^{-6}$ all dots were found to lie on the solid line within printing precision.)

Pick $a = 2.189$. Since $y^{a,0}$ is an approximation of x^{a^*}, we can estimate $\frac{1}{b^*}$ by the integral $2\int_0^\infty u[y^{a,0}(u)]^2 du$ (recall (0.23)). However, we have only computed $y^{a,0}(u)$ for $u \in [0,3]$ and it turns out that this is not enough to get a good estimate of b^* up to the third decimal. A better way is to use (0.15) and estimate

$$\frac{1}{b^*} \approx 2\beta^{\frac{2}{3}} \sum_{i\in\mathbb{N}} i[y^{a,0}(i\beta^{\frac{1}{3}})]^2 - \beta^{\frac{1}{3}}. \tag{0.24}$$

This gives $b^* = 1.109 \pm 0.001$.

0.6 The Edwards model

Westwater (1984) studies Brownian motion on $\mathbb{R}$ with self-repellence, i.e., the Edwards model where (0.1) is replaced by

$$\frac{d\nu_T^g}{d\mu_T}\Big((W_t)_{0\le t\le T}\Big) = \frac{1}{Z_T^g}\exp\Big[-g\int_0^T ds\int_0^T dt\ \delta(W_s - W_t)\Big]. \tag{0.25}$$

Here μ_T is the Wiener measure on Brownian motion paths $(W_t)_{0\le t\le T}$, δ the Dirac-function, $g \in [0,\infty)$ the repellence parameter and Z_T^g the normalizing constant. [7] We give two properties showing that the Edwards model arises as the weak interaction limit of the Domb-Joyce model.

Property 1 *For every $g \in [0,\infty)$*

$$Q_n^{gn^{-\frac{3}{2}}}\Big((n^{-\frac{1}{2}}S_{\lceil tn\rceil})_{0\le t\le 1} \in \cdot\Big) \Rightarrow \nu_1^g\Big((W_t)_{0\le t\le 1}\in\cdot\Big) \text{ as } n\to\infty. \tag{0.26}$$

Proof. See Brydges and Slade (1994) Theorem 1.3. The double sum in (0.1) equals $-(n+1)+\sum_x \ell_n^2(x)$ (recall (0.7)), of which the first term may be absorbed into the normalizing constant Z_n^β in (0.2). The key point is that $n^{-\frac{3}{2}}\sum_x \ell_n^2(x)$ under the law P_n converges to $\int_{\mathbb{R}} \hat\ell_1^2(x)dx$ under the law μ_1 (recall footnote 7). This immediately implies (0.26). The analogous for $T \neq 1$ is obvious. □

Westwater (1984) proves the following result which is analogous to Theorems 1 and 3:

For every $g \in [0,\infty)$

$$\lim_{T\to\infty} \nu_T^g\Big(|\frac{1}{T}|W_T| - \hat\theta^*(g)| \le \epsilon\Big) = 1 \text{ for every } \epsilon > 0, \tag{0.27}$$

where

$$\hat\theta^*(g) = \Big[\frac{\partial}{\partial\lambda}E(g,\lambda)\Big]_{\lambda=0} \tag{0.28}$$

with $E(g,\lambda)$ the smallest eigenvalue in $L^2(\mathbb{R}^+)$ of the operator $\hat{\mathcal{L}}^{g,\lambda}$ given by

$$\Big(\hat{\mathcal{L}}^{g,\lambda}y\Big)(v) = \Big[gv^2 + \lambda v^{-2} - \frac{1}{2}v^{-1}\Big(\frac{d^2}{dv^2} + \frac{1}{4}v^{-2}\Big)v^{-1}\Big]y(v). \tag{0.29}$$

(The term between round brackets equals $v^{\frac{1}{2}}\Delta_{rad}^{(2)}v^{-\frac{1}{2}}$ with $\Delta_{rad}^{(2)}$ the 2-dimensional Laplace operator.)

[7] The double integral in (0.25) should be read as $\int_{\mathbb{R}} \hat\ell_T^2(x)dx$, where $\hat\ell_T(x) = \int_0^T dt\ \delta(W_t - x)$ is the density of the occupation time measure w.r.t. Lebesgue measure.

Property 2 *For every $g \in [0,\infty)$*

$$\begin{aligned} E(g,0) &= a^* g^{\frac{2}{3}} \\ \left[\tfrac{\partial}{\partial\lambda}E(g,\lambda)\right]_{\lambda=0} &= b^* g^{\frac{1}{3}}, \end{aligned} \tag{0.30}$$

with a^*, b^* the same constants as in Theorems 4 and 6.

Proof. Take the eigenvalue problem

$$\left(\hat{\mathcal{L}}^{g,\lambda} y\right)(v) = E(g,\lambda) y(v). \tag{0.31}$$

Substitute into (0.31) the following change of variables:

$$\begin{aligned} y(v) &= v^{\frac{1}{2}} x\left(\tfrac{1}{2} g^{\frac{1}{3}} v^2\right) \\ u &= \tfrac{1}{2} g^{\frac{1}{3}} v^2. \end{aligned} \tag{0.32}$$

Then, after a small computation, we obtain the Sturm-Liouville problem in (0.17-18)

$$\left(\mathcal{L}^a x\right)(u) = \rho x(u), \tag{0.33}$$

with

$$\begin{aligned} a &= g^{-\frac{2}{3}} E(g,\lambda) \\ \rho &= g^{-\frac{1}{3}} \lambda. \end{aligned} \tag{0.34}$$

Think of (0.34) as a parametrization of the curve $a \to \rho(a)$ in terms of λ. Recalling the definition of a^*, b^* in (0.21), we now get from (0.33-0.34) that

$$\rho(a^*) = 0 \Leftrightarrow a^* = g^{-\frac{2}{3}} E(g,0) \tag{0.35}$$

and

$$\begin{aligned} \left[\tfrac{\partial}{\partial\lambda}E(g,\lambda)\right]_{\lambda=0} &= g^{-\frac{1}{3}} \left[\tfrac{\partial}{\partial\rho} E(g,\rho g^{\frac{1}{3}})\right]_{\rho=0} \\ &= g^{-\frac{1}{3}} \left[\tfrac{\partial}{\partial\rho}\left(a(\rho) g^{\frac{2}{3}}\right)\right]_{\rho=0} \\ &= g^{\frac{1}{3}} a'(0) \\ &= g^{\frac{1}{3}} \frac{1}{\rho'(a^*)} \\ &= g^{\frac{1}{3}} b^*, \end{aligned} \tag{0.36}$$

where $\rho \to a(\rho)$ is the inverse function of $a \to \rho(a)$. $\square$

Properties 1 and 2 show that Theorems 4 and 6 connect up nicely with the Edwards model.

We close this section with a *heuristic* explanation of the power $\frac{1}{3}$ in our result $\theta^*(\beta) \sim b^*\beta^{\frac{1}{3}}$ $(\beta \downarrow 0)$. First, by Brownian scaling

$$E_{\nu_1^{gT^{\frac{3}{2}}}}(W_1^2) = \frac{1}{T} E_{\nu_T^g}(W_T^2). \tag{0.37}$$

Since, according to (0.27),

$$[\hat{\theta}^*(g)]^2 = \lim_{T\to\infty} \frac{1}{T^2} E_{\nu_T^g}(W_T^2), \tag{0.38}$$

it follows that

$$\hat{\theta}^*(g) = g^{\frac{1}{3}} \hat{\theta}^*(1). \tag{0.39}$$

Next, according to Theorem 1,

$$[\theta^*(g)]^2 = \lim_{n\to\infty} \frac{1}{n^2} E_{Q_n^g}(S_n^2). \tag{0.40}$$

Moreover, by Property 1 we know that for g, T fixed

$$\frac{1}{n} E_{Q_n^{g(\frac{T}{n})^{\frac{3}{2}}}}(S_n^2) \sim E_{\nu_1^{gT^{\frac{3}{2}}}}(W_1^2) \quad (n \to \infty). \tag{0.41}$$

Now, if we assume that (0.41) continues to hold for g fixed and $T = n$, then by using (0.40-41) resp. (0.37-38) we arrive at

$$\begin{aligned} [\theta^*(g)]^2 &\sim \tfrac{1}{n^2} E_{Q_n^g}(S_n^2) \\ &\sim E_{\nu_1^{gT^{\frac{3}{2}}}}(W_1^2) \\ &= \tfrac{1}{T^2} E_{\nu_T^g}(W_T^2) \\ &\sim [\hat{\theta}^*(g)]^2 \quad (T = n \to \infty). \end{aligned} \tag{0.42}$$

The above argument has uniformity problems because (0.39) and (0.42) would imply $\theta^*(g) = g^{\frac{1}{3}}\theta^*(1)$ for all g. However, this cannot be true because $\theta^*(g) \le 1$ for all g. Nevertheless, it explains the power $\frac{1}{3}$ without using the explicit solution.

1 Preparations

In this section we formulate the functional analytic framework in which we are going to approach our scaling theorem. Section 1.1 shows that our key result, Theorem 5 in section 0.4, is equivalent to convergence of a variational problem involving a certain functional F_β^a to a variational problem involving some limit functional F^a (Lemma 1 and Proposition 1 below). Section 1.2 shows that this convergence holds when F_β^a epi-converges to F^a and certain compactness properties are satisfied (Proposition 2 below). In this section we also formulate the main steps that have to be checked in order to prove these facts (Proposition 3 below). In section 1.3 we collect some properties of the matrix P, defined in (0.11), that will be needed in the proofs.

1.1 A variational representation

Rayleigh's formula for the pair $(\lambda(r,\beta),\tau_{r,\beta})$ defined in (0.12) reads

$$\begin{aligned} &(i) \quad \lambda(r,\beta)=\max_{\substack{y\in l^2(\mathbb{N}),y\geq 0,\\ \|y\|_{l^2}\leq 1}}\langle y,A_{r,\beta}y\rangle_{l^2} \\ &(ii) \quad \tau_{r,\beta} \text{ is the unique maximizer.}\end{aligned} \tag{1.1}$$

In anticipation of the scaling suggested by Table 1, we pick $r=a\beta^{\frac{2}{3}}$ $(a\in\mathbb{R})$ and rewrite (1.1) in the following form. Define the functional $F^a_\beta : L^2(\mathbb{R}^+)\to\mathbb{R}$ as

$$F^a_\beta(x)=\beta^{-\frac{2}{3}}\int_0^\infty du\int_0^\infty dv\; x(u)x(v)A_{a\beta^{\frac{2}{3}},\beta}\big(\lceil u\beta^{-\frac{1}{3}}\rceil,\lceil v\beta^{-\frac{1}{3}}\rceil\big)-\beta^{-\frac{1}{3}}\|x\|^2_{L^2}. \tag{1.2}$$

Lemma 1 *For all $\beta>0$*

$$\begin{aligned} &(i)\ \ \beta^{-\frac{1}{3}}\big[\lambda(a\beta^{\frac{2}{3}},\beta)-1\big]=\max_{\substack{x\in L^2(\mathbb{R}^+),x\geq 0,\\ \|x\|_{L^2}=1}}F^a_\beta(x) \\ &(ii)\ \ \beta^{-\frac{1}{6}}\tau_{a\beta^{\frac{2}{3}},\beta}(\lceil\cdot\beta^{-\frac{1}{3}}\rceil) \textit{ is the unique maximizer.}\end{aligned} \tag{1.3}$$

Proof. (i) Fix $\beta>0$. For $x\in L^2(\mathbb{R}^+)$ define

$$\hat{x}(i)=\beta^{-\frac{1}{6}}\int_{(i-1)\beta^{\frac{1}{3}}}^{i\beta^{\frac{1}{3}}}x(u)du\ (i\in\mathbb{N}). \tag{1.4}$$

Then the first term in (1.2) equals $\beta^{-\frac{1}{3}}\langle\hat{x},A_{a\beta^{\frac{2}{3}},\beta}\hat{x}\rangle_{l^2}$. Hence using (1.1)(i) we may write

$$\beta^{-\frac{1}{3}}\big[\lambda(a\beta^{\frac{2}{3}},\beta)-1\big]=\max_{\substack{y\in l^2(\mathbb{N}),y\geq 0,\\ \|y\|_{l^2}\leq 1}}\ \max_{\substack{x\in L^2(\mathbb{R}^+),x\geq 0,\\ \|x\|_{L^2}=1,\hat{x}=y}}F^a_\beta(x). \tag{1.5}$$

Note that, by Cauchy-Schwarz, we have $\|\hat{x}\|_{l^2}\leq\|x\|_{L^2}$ and so the restrictions $\|y\|_{l^2}\leq 1,\|x\|_{L^2}=1,\hat{x}=y$ in (1.5) are compatible. Interchange the two maxima in (1.5) to get the claim.
(ii) Use that $\|\hat{x}\|_{l^2}=\|x\|_{L^2}$ iff $x(u)=\beta^{-\frac{1}{6}}\hat{x}(i)$ for $u\in((i-1)\beta^{\frac{1}{3}},i\beta^{\frac{1}{3}}]$. □

In sections 2-5 we shall prove:

Proposition 1 *As $\beta\downarrow 0$*

$$\begin{aligned} &(i)\quad \max_{\substack{x\in L^2(\mathbb{R}^+),x\geq 0,\\ \|x\|_{L^2}=1}}F^a_\beta(x)\to\max_{\substack{x\in L^2(\mathbb{R}^+),x\geq 0,\\ \|x\|_{L^2}=1}}F^a(x) \\ &(ii)\quad \textit{unique maximizer l.h.s.}\ \to^{L^2}\ \textit{unique maximizer r.h.s.},\end{aligned} \tag{1.6}$$

where the limit functional $F^a : L^2(\mathbb{R}^+)\to\overline{\mathbb{R}}$ is given by

$$F^a(x)=\int_0^\infty\Big\{(2au-4u^2)[x(u)]^2-u[x'(u)]^2\Big\}du, \tag{1.7}$$

with the understanding that $F^a(x)=-\infty$ if the integral is not defined.

Note that $F^a(x)=\langle x,\mathcal{L}^a x\rangle_{L^2}$ for all x where both sides are finite, with $\mathcal{L}^a$ as defined in (0.17).

Lemma 1 and Proposition 1 imply Theorem 5. To prove Proposition 1, we shall need the notion of epi-convergence, which we next explain.

1.2 Epi-convergence

Let (X,τ) be a metrizable topological space and let $Y \subset X$ be dense in X. Let

$$\begin{array}{l} G_\beta : X \to \mathbb{R} \;\; (\beta > 0) \\ G : X \to \overline{\mathbb{R}}. \end{array} \tag{1.8}$$

Definition 1 *The family $(G_\beta)_{\beta>0}$ is said to be epi-convergent to G on Y, written*

$$e - \lim_{\beta \downarrow 0} G_\beta = G \text{ on } Y, \tag{1.9}$$

if the following properties hold:

$$\begin{array}{ll} (i) & \forall x_\beta \to^\tau x \text{ in } Y : \limsup_{\beta\downarrow 0} G_\beta(x_\beta) \le G(x) \\ (ii) & \exists x_\beta \to^\tau x \text{ in } Y : \liminf_{\beta\downarrow 0} G_\beta(x_\beta) \ge G(x). \end{array} \tag{1.10}$$

The importance of the notion of epi-convergence is contained in the following proposition:

Proposition 2 *Suppose that*

(1) $e - \lim_{\beta\downarrow 0} G_\beta = G$ *on* Y

(2) $\forall \beta > 0 : G_\beta$ *is continuous on* X *and has a unique maximizer* $\bar{x}_\beta \in X$

(3) $\exists K \subset Y$ *such that*

  (i) K *is* τ*-relatively compact in* X

  (ii) G *has a unique maximizer* $\bar{x} \in \overline{K}$

  (iii) $\exists (x_\beta)_{\beta>0} \subset \overline{K}$ *such that* $x_\beta - \bar{x}_\beta \to^\tau 0$ *and* $G_\beta(x_\beta) - G_\beta(\bar{x}_\beta) \to 0$ *as* $\beta \downarrow 0$.

Then as $\beta \downarrow 0$

$$\sup_{x\in X} G_\beta(x) \to \sup_{x\in X} G(x) \tag{1.11}$$

$$\bar{x}_\beta \to^\tau \bar{x}. \tag{1.12}$$

Proof. See Attouch (1984) Theorem 1.10 and Proposition 1.14. □

Remark: Epi-convergence differs from pointwise convergence: $\lim_{\beta\downarrow 0} G_\beta(x) = G(x)$ for all $x \in Y$. Namely, (1.10)(i),(ii) are weaker in the sense that they require only inequalities, but stronger in the sense that they involve limits in neighborhoods rather than single points. Epi-convergence is a unilateral notion. We have chosen the direction that is suitable for suprema rather than infima.

Fix $a \in \mathbb{R}$. We are going to apply Proposition 2 with the following choices:

$$\begin{array}{rcl} X & = & \{x \in L^2(\mathbb{R}^+) : x \ge 0, \|x\|_{L^2} = 1\} \\ Y & = & X \cap C^1(\mathbb{R}_0^+) \\ \tau & = & \text{topology induced by } \|\cdot\|_{L^2} \\ K & = & K_C^a = \{x \in Y : F^a(x) \ge -C\} \\ G_\beta & = & F_\beta^a \\ G & = & F^a \end{array} \tag{1.13}$$

with F_β^a and F^a defined in (1.2) and (1.7) and with C large enough so that $K_C^a \neq \emptyset$. Our main result is:

Proposition 3 *Assumptions (1)-(3) in Proposition 2 hold for the choice in (1.13).*

We prove Assumption (1) in section 2, (3)(i),(ii) in section 5 and (3)(iii) in section 3. We already know (2) to be true because of Lemma 1(ii).

Proposition 3 proves Proposition 1 in section 1.1.

1.3 Properties of P

We list a few identities and estimates for the matrix P, defined in (0.11), that will be needed later on.

Lemma 2 *For every $i \geq 1, k \geq 0$*

$$\sum_{j\geq 1} \frac{(i+j+k-2)!}{(i+j-2)!} P(i,j) = 2^k \frac{(i+k-1)!}{(i-1)!}. \tag{1.14}$$

Proof. Elementary. Use that the summands in the l.h.s. can be rewritten as $P(i+k,j)$ times the r.h.s. Then use that $\sum_{j\geq 1} P(i+k,j) = 1$. □

Lemma 3 *(i) For $i,j \to \infty$ such that $i - j = o((i+j)^{\frac{2}{3}})$*

$$P(i,j) = \Big\{ \frac{1}{\sqrt{2\pi(i+j)}} \exp\Big[-\frac{(i-j)^2}{2(i+j)} \Big] \Big\} \Big[1 + \mathcal{O}((i+j)^{-\frac{1}{3}}) \Big]. \tag{1.15}$$

(ii) There exist $0 < c_1 < c_2 < \infty$ such that

$$\exp\Big[-c_2 \frac{(i-j)^2}{(i+j)} \Big] \leq P(i,j) \leq \exp\Big[-c_1 \frac{(i-j)^2}{(i+j)} \Big] \text{ for all } i,j \geq 1. \tag{1.16}$$

Proof. Via Stirling's formula. See also Révész (1990) Theorem 2.8. □

Lemma 2 allows us to compute the following moments, which we shall need in section 2:

$$\begin{array}{rcll} \sum_{j\geq 1}(i+j-1)^n P(i,j) & = & 2i & (n=1) \\ & & 4i^2 + 2i & (n=2) \\ & & 8i^3 + 12i^2 + 6i & (n=3) \\ & & 16i^4 + 48i^3 + 72i^2 + 32i & (n=4). \end{array} \tag{1.17}$$

Lemma 3(i) is a Gaussian approximation of P, while Lemma 3(ii) shows that $P(i,j)$ is small away from the diagonal.

Lemma 4 *For all $i,j \geq 0$ with $(i,j) \neq (0,0)$*

$$P(i+1,j) + P(i,j+1) - 2P(i+1,j+1) = 0 \tag{1.18}$$

with the convention $P(i,0) = P(0,j) = 0$.

Proof. Elementary. □

Lemma 4 will be needed in sections 2 and 3 to obtain monotonicity properties and estimates of $\tau_{a\beta^{\frac{2}{3}},\beta}$, the eigenvector of $A_{a\beta^{\frac{2}{3}},\beta}$.

2 $(F^a_\beta)_{\beta>0}$ is epi-convergent to F^a

In this section we prove Assumption (1) in Proposition 2 for the choice in (1.13).

This section is technically somewhat involved, as it consists of a chain of estimates and inequalities that are needed to handle the epi-convergence. The proof is contained in Lemmas 5-8 below. Throughout sections 2 and 3 we fix $a \in \mathbb{R}$ and we write the abbreviations $F_\beta = F^a_\beta$, $F = F^a$, $A_\beta = A_{a\beta^{\frac{2}{3}},\beta}$, $\lambda(\beta) = \lambda(a\beta^{\frac{2}{3}},\beta)$, $\tau_\beta = \tau_{a\beta^{\frac{2}{3}},\beta}$.

We begin by splitting F_β, F into two parts, namely (recall (1.2) and (1.7))

$$\begin{aligned} F_\beta &= F^1_\beta + F^2_\beta \\ F &= F^1 + F^2 \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} F^1_\beta(x) &= \beta^{-\frac{2}{3}} \int_0^\infty du \int_0^\infty dv\ x^2(u)[A_\beta - P]\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) \\ F^2_\beta(x) &= -\tfrac{1}{2}\beta^{-\frac{2}{3}} \int_0^\infty du \int_0^\infty dv\ [x(u)-x(v)]^2 A_\beta\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} F^1(x) &= \int_0^\infty du\ (2au - 4u^2)\ x^2(u) \\ F^2(x) &= -\int_0^\infty du\ u[x'(u)]^2. \end{aligned} \tag{2.3}$$

Lemma 5 $\forall x_\beta \to^{L^2} x$ *in* X : $\limsup_{\beta\downarrow 0} F^1_\beta(x_\beta) \le F^1(x)$.

Proof. Abbreviate

$$e_\beta(i,j) = a\beta^{\frac{2}{3}}(i+j-1) - \beta(i+j-1)^2, \tag{2.4}$$

which is the exponent appearing in $A_\beta(i,j)$, i.e., $A_\beta = e^{e_\beta}P$ (see (0.10)). We note that e_β has the following properties:

$$\begin{aligned} &(i)\ e_\beta(i,j) \le 0 \text{ for } i \ge a\beta^{-\frac{1}{3}}, j \ge 1 \\ &(ii)\ e_\beta(i,j) \le \tfrac{1}{4}a^2\beta^{\frac{1}{3}} \text{ for } i,j \ge 1. \end{aligned} \tag{2.5}$$

Hence, for small enough β and large enough N

$$\begin{aligned} F^1_\beta(x_\beta) \ &\le \beta^{-\frac{2}{3}} \int_0^N du \int_0^\infty dv\ x^2_\beta(u) \\ &\times\Big\{e_\beta\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) + e^2_\beta\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big)\Big\} P\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) \end{aligned} \tag{2.6}$$

(use that $e^t \le 1 + t + t^2$ for $t \ll 1$ and $t \le 0$). The integral over v can be transformed into the following sum:

$$\beta^{\frac{1}{3}} \sum_{j\ge 1} \Big\{e_\beta(i,j) + e^2_\beta(i,j)\Big\} P(i,j) \text{ with } i = \lceil u\beta^{-\frac{1}{3}}\rceil. \tag{2.7}$$

Using (1.17), we can carry out the summation. Namely,

$$\begin{aligned} \textstyle\sum_{j\ge 1} e_\beta(i,j)P(i,j) \ &= a\beta^{\frac{2}{3}}(2i) - \beta(4i^2 + 2i) \\ \textstyle\sum_{j\ge 1} e^2_\beta(i,j)P(i,j) \ &= a^2\beta^{\frac{4}{3}}(4i^2 + 2i) - 2a\beta^{\frac{5}{3}}(8i^3 + 12i^2 + 6i) \\ &\quad + \beta^2(16i^4 + 48i^3 + 72i^2 + 32i). \end{aligned} \tag{2.8}$$

Since $i = \lceil u\beta^{-\frac{1}{3}} \rceil \le (N+1)\beta^{-\frac{1}{3}}$, the contribution to (2.6) of the second sum can be estimated above by

$$\beta^{\frac{1}{3}}(6a^2(N+1)^2 + 168(N+1)^4) \int_0^N du\ x_\beta^2(u) = \mathcal{O}(\beta^{\frac{1}{3}}), \tag{2.9}$$

where we use that $\|x_\beta\|_{L^2} = 1$. The error term is uniform in x_β for fixed N. Hence we get

$$\begin{aligned} F_\beta^1(x_\beta) &\le \beta^{-\frac{1}{3}} \textstyle\int_0^N du\ x_\beta^2(u) \\ &\quad \times \left\{ a\beta^{\frac{2}{3}}(2\lceil u\beta^{-\frac{1}{3}} \rceil) - \beta(4\lceil u\beta^{-\frac{1}{3}}\rceil^2 + 2\lceil u\beta^{-\frac{1}{3}}\rceil)\right\} + \mathcal{O}(\beta^{\frac{1}{3}}) \\ &= \textstyle\int_0^N du\ x_\beta^2(u)(2au - 4u^2) + \mathcal{O}(\beta^{\frac{1}{3}}). \end{aligned} \tag{2.10}$$

Now let $\beta \downarrow 0$. Then we obtain, recalling that $x_\beta \to^{L^2} x$,

$$\begin{aligned} \limsup_{\beta\downarrow 0} F_\beta^1(x_\beta) &\le \limsup_{\beta\downarrow 0} \textstyle\int_0^N du\ x_\beta^2(u)(2au - 4u^2) \\ &= \textstyle\int_0^N du\ x^2(u)(2au - 4u^2). \end{aligned} \tag{2.11}$$

Finally, let $N \to \infty$ and note that the r.h.s. of (2.11) converges to $F^1(x)$. □

Lemma 6 *$\forall x \in X:\quad \liminf_{\beta\downarrow 0} F_\beta^1(x) \ge F^1(x)$.*

Proof. Estimate

$$F_\beta^1(x) \ge \beta^{-\frac{2}{3}} \int_0^\infty du \int_0^\infty dv\ x^2(u) e_\beta\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) P\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) \tag{2.12}$$

(use that $e^t \ge 1 + t$ for all t). The integral over v is $\beta^{\frac{1}{3}}$ times the first sum computed in (2.8) with $i = \lceil u\beta^{-\frac{1}{3}}\rceil$. Hence

$$\begin{aligned} F_\beta^1(x) &\ge \beta^{-\frac{1}{3}} \textstyle\int_0^\infty du\ x^2(u) \\ &\quad \times \left\{ a\beta^{\frac{2}{3}}(2u\beta^{-\frac{1}{3}}) - \beta\big(4(u\beta^{-\frac{1}{3}} + 1)^2 + 2(u\beta^{-\frac{1}{3}} + 1)\big)\right\} \\ &= \textstyle\int_0^\infty du\ x^2(u)(2au - 4u^2) + \mathcal{O}(\beta^{\frac{1}{3}}). \end{aligned} \tag{2.13}$$

Now let $\beta \downarrow 0$. Then the claim follows. □

Lemma 7 *$\forall x_\beta \to^{L^2} x$ in X with $x \in Y : \limsup_{\beta\downarrow 0} F_\beta^2(x_\beta) \le F^2(x)$.*

Proof. The proof is in Steps 1-3 below.

STEP 1 *For every $\epsilon > 0$ and N, M finite*

$$F_\beta^2(x_\beta) \le -\frac{1}{2}(1 + \mathcal{O}(\beta^{\frac{1}{9}})) \int_\epsilon^N du \int_{-M}^M dw\ \Big[\frac{1}{\beta^{\frac{1}{6}}}\{x_\beta(u) - x_\beta(u + w\beta^{\frac{1}{6}})\}\Big]^2 N_{2u}(w), \tag{2.14}$$

where N_{2u} is the Gaussian with mean zero and variance $2u$.

Proof. Pick $\epsilon > 0$ and N, M finite. Then

$$F_\beta^2(x_\beta) \le -\frac{1}{2}\beta^{-\frac{2}{3}} e^{-9N^2\beta^{\frac{1}{3}}} \int_\epsilon^N du \int_{u-M\beta^{\frac{1}{6}}}^{u+M\beta^{\frac{1}{6}}} dv\ [x_\beta(u) - x_\beta(v)]^2 P\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big), \quad (2.15)$$

where we use that $A_\beta = e^{e_\beta}P$ with $e_\beta\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) \ge -9N^2\beta^{\frac{1}{3}}$ on the integration area (see (2.4)). Put $w = \beta^{-\frac{1}{6}}(v-u)$. Then by Lemma 3(i)

$$\begin{aligned} F_\beta^2(x_\beta) \quad &\le -\tfrac{1}{2}\beta^{-\frac{2}{3}} e^{-9N^2\beta^{\frac{1}{3}}} \textstyle\int_\epsilon^N du \int_{-M}^M dw\ \beta^{\frac{1}{6}}\ [x_\beta(u) - x_\beta(u + w\beta^{\frac{1}{6}})]^2 \\ &\qquad \times \Big\{\frac{1}{\sqrt{2\pi 2u\beta^{-\frac{1}{3}}}} \exp\big[-\tfrac{w^2}{4u}\big]\Big\}(1 + o(\beta^{\frac{1}{9}})), \end{aligned} \quad (2.16)$$

where the error term is uniform on the integration area. Collecting all the powers of β, we get the claim. □

To investigate the limit of the integral in (2.14) as $\beta \downarrow 0$, we proceed with a technical fact contained in Steps 2 and 3 below. Let T_h be the translation operator defined by $T_h x_\beta(\cdot) = x_\beta(\cdot + h)$.

STEP 2 *For every* $0 < a < b < \infty$

$$\liminf_{h\to 0, \beta\downarrow 0} \int_a^b \ \Big\{\frac{1}{h}[T_h x_\beta - x_\beta](u)\Big\}^2 \ge \int_a^b \ [x'(u)]^2\ du. \quad (2.17)$$

Proof. Since (2.17) is trivial when the liminf is infinite, we may assume that the liminf is finite, say L. Pick any subsequence h_n, β_n along which the liminf is reached, and put $y_n = \frac{1}{h_n}[T_{h_n}x_{\beta_n} - x_{\beta_n}]$. Then, because $\|y_n\|_{L^2[a,b]} \le L + 1 < \infty$ for n large enough, it follows from the Banach-Alaoglu theorem (Rudin (1991) Theorem 3.15) that there exists a subsequence (y_{n_k}) and a $y \in L^2[a,b]$ such that

$$y_{n_k} \to y \text{ weakly in } L^2[a,b] \ \ (k \to \infty). \quad (2.18)$$

Thus, for any $\phi \in C_c^1(a,b) = \{\phi \in C^1(a,b) : supp(\phi) \subset (a,b)\}$

$$\int_a^b y_{n_k}(u)\phi(u)du \to \int_a^b y(u)\phi(u)du \ \ (k \to \infty). \quad (2.19)$$

Next, the l.h.s. of (2.19) can be rewritten as

$$\begin{aligned} \textstyle\int_a^b y_{n_k}(u)\phi(u)du \quad &= \textstyle\int_a^b \ \frac{1}{h_n}[T_{h_n}x_{\beta_n} - x_{\beta_n}](u)\phi(u)du \\ &= \textstyle\int_{a+h_n 1_{\{h_n<0\}}}^{b+h_n 1_{\{h_n>0\}}} x_{\beta_n}(u)\frac{1}{h_n}[T_{-h_n}\phi - \phi](u)du \\ &= \textstyle\int_a^b \ x_{\beta_n}(u)\frac{1}{h_n}[T_{-h_n}\phi - \phi](u)du + o(1) \ \ (n \to \infty). \end{aligned} \quad (2.20)$$

The last equality holds because $\|x_{\beta_n}\|_{L^2(\mathbb{R}^+)} = 1$ and $|\frac{1}{h_n}[T_{-h_n}\phi - \phi]| \le \max_{u\in\mathbb{R}^+} |\phi'(u)| < \infty$. Let $n \to \infty$ and note that by the latter property

$$\frac{1}{h_n}[T_{-h_n}\phi - \phi] \to -\phi' \text{ pointwise and weakly in } L^2[a,b]. \quad (2.21)$$

Together with $x_{\beta_n} \to^{L^2} x$, (2.21) implies that the last integral in (2.20) tends to $\int_a^b x(u)[-\phi'(u)]du = \int_a^b x'(u)\phi(u)du$ (recall from (1.13) that $x \in Y \subset C^1(\mathbb{R}_0^+)$). Since $C_c^1(a,b)$ is dense in $L^2[a,b]$ in the weak topology, we thus have from (2.19)

$$y = x' \ a.e. \ on \ [a,b]. \tag{2.22}$$

The claim now follows by combining (2.18) and (2.22), and noting that $\|\cdot\|_{L^2[a,b]}$ is lower semicontinuous in the weak topology: $L = \lim_{k\to\infty} \|y_{n_k}\|_{L^2[a,b]} \geq \|y\|_{L^2[a,b]} = \|x'\|_{L^2[a,b]}$. □

STEP 3 *For every $\epsilon > 0$ and N finite, every $f : \mathbb{R}^+ \to \mathbb{R}^+$ bounded and continuous, and every $w \in \mathbb{R}$*

$$\liminf_{\beta\downarrow 0} \int_\epsilon^N du\ f(u)\Big[\frac{1}{\beta^{\frac{1}{6}}}\{x_\beta(u) - x_\beta(u + w\beta^{\frac{1}{6}})\}\Big]^2 \geq \int_\epsilon^N du\ f(u)[wx'(u)]^2. \tag{2.23}$$

Proof. Pick any sequence (f_n) of functions on $\mathbb{R}^+$ such that

$$\begin{array}{l} (i) \quad f_n(u) = f_{n,k} \text{ for } c_{n,k-1} < u \leq c_{n,k} \ (k = 1,\ldots,n; c_{n,0} = \epsilon, c_{n,n} = N) \\ (ii) \quad f_n \leq f \\ (iii) \ f_n \uparrow f \text{ in sup-norm on } [\epsilon, N] \text{ as } n \to \infty. \end{array} \tag{2.24}$$

Then, by (i) and (ii),

$$\begin{aligned} l.h.s.\ (2.23) \quad &\geq \liminf_{\beta\downarrow 0} \textstyle\int_\epsilon^N du\ f_n(u)\Big[\frac{1}{\beta^{\frac{1}{6}}}\{x_\beta(u) - x_\beta(u + w\beta^{\frac{1}{6}})\}\Big]^2 \\ &\geq \textstyle\sum_{k=1}^n f_{n,k} \liminf_{\beta\downarrow 0} \int_{c_{n,k-1}}^{c_{n,k}} du\ \Big[\frac{1}{\beta^{\frac{1}{6}}}\{x_\beta(u) - x_\beta(u + w\beta^{\frac{1}{6}})\}\Big]^2 \\ &\geq \textstyle\sum_{k=1}^n f_{n,k} \int_{c_{n,k-1}}^{c_{n,k}} du\ [wx'(u)]^2 \\ &= \textstyle\int_\epsilon^N du\ f_n(u)[wx'(u)]^2, \end{aligned} \tag{2.25}$$

where in the third inequality we use (2.17) with $h = w\beta^{\frac{1}{6}}$ and $a = c_{n,k-1}, b = c_{n,k}$ $(k = 1,\ldots,n)$. Now let $n \to \infty$ and use (iii) together with Fatou to get the claim in (2.23). □

Using (2.23) we can now finish the proof of Lemma 7. Indeed, continuing with (2.14), we get

$$\begin{aligned} \limsup_{\beta\downarrow 0} F_\beta^2(x_\beta) \quad &\leq -\tfrac{1}{2}\textstyle\int_{-M}^M dw\ \int_\epsilon^N du\ N_{2u}(w)[wx'(u)]^2 \\ &= -\tfrac{1}{2}\textstyle\int_\epsilon^N du\ [x'(u)]^2 \int_{-M}^M dw\ w^2 N_{2u}(w). \end{aligned} \tag{2.26}$$

Finally, let $M \to \infty$ and note that $\int_{-\infty}^\infty dw\ w^2 N_{2u}(w) = 2u$. Then let $N \to \infty$ and $\epsilon \downarrow 0$ to get the claim in Lemma 7. □

Lemma 8 *$\forall x \in Y$ such that $\int_0^\infty u^2 x^2(u)du < \infty : \liminf_{\beta\downarrow 0} F_\beta^2(x) \geq F^2(x)$.*

Proof. The double integral defining $F_\beta^2(x)$ is split into three parts, which we estimate separately in Steps 1-3 below.

STEP 1

$$\lim_{\beta\downarrow 0} -\frac{1}{2}\beta^{-\frac{2}{3}}\int_0^\infty du \int_0^\infty dv\ 1_{\{u>\beta^{-\frac{1}{6}}\ or\ v>\beta^{-\frac{1}{6}}\}}[x(u)-x(v)]^2 A_\beta\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) = 0. \quad (2.27)$$

Proof. First consider the part where $u > \beta^{-\frac{1}{6}}, v \geq 0$. By (2.5)(ii) and Lemma 3(ii)

$$\begin{aligned}
&-\tfrac{1}{2}\beta^{-\frac{2}{3}} \textstyle\int_{\beta^{-\frac{1}{6}}}^\infty du \int_0^\infty dv\ [x(u)-x(v)]^2 A_\beta\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) \\
&\quad \geq -\tfrac{1}{2}\beta^{-\frac{2}{3}} e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}}\Big\{ \textstyle\int_{\beta^{-\frac{1}{6}}}^\infty du \int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty dv\ [x(u)-x(v)]^2 P\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) \\
&\qquad + \textstyle\int_{\beta^{-\frac{1}{6}}}^\infty du \int_0^{\frac{1}{2}\beta^{-\frac{1}{6}}} dv\ [x(u)-x(v)]^2 e^{-c_1\beta^{-\frac{1}{3}}\frac{(u-v)^2}{(u+v)}}\Big\} \qquad (2.28) \\
&\quad = -\tfrac{1}{2}\beta^{-\frac{2}{3}} e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}}\Big\{ \textstyle\int_{\beta^{-\frac{1}{6}}}^\infty du \int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty dv\ [x(u)-x(v)]^2 P\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) \\
&\qquad + \mathcal{O}(e^{-\frac{1}{12}c_1\beta^{-\frac{1}{2}}})\Big\},
\end{aligned}$$

where c_1 is the constant in Lemma 3(ii). To get the error term we have used that $(u-v)^2/(u+v) \geq \frac{1}{3}(u-v)$ on the integration area. The double integral in the r.h.s. of (2.28) can be bounded above by

$$\begin{aligned}
&\textstyle\int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty du \int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty dv\ [x(u)-x(v)]^2 P\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) \\
&\quad \leq 2\textstyle\int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty du\ x^2(u) \int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty dv\ P\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) \qquad (2.29) \\
&\quad \leq 2\beta^{\frac{1}{3}} \textstyle\int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty du\ x^2(u).
\end{aligned}$$

Hence

$$\begin{aligned}
r.h.s.\ (2.28)\ &\geq -\big[1+\mathcal{O}(\beta^{\frac{1}{3}})\big]\beta^{-\frac{1}{3}} \textstyle\int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty du\ x^2(u) \\
&\geq -4\big[1+\mathcal{O}(\beta^{\frac{1}{3}})\big] \textstyle\int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty du\ u^2 x^2(u) \qquad (2.30) \\
&= o(1).
\end{aligned}$$

By symmetry, the same estimate holds for the part with $u \geq 0, v > \beta^{-\frac{1}{6}}$. □

STEP 2

$$\lim_{\beta\downarrow 0} -\frac{1}{2}\beta^{-\frac{2}{3}}\int_0^\infty du \int_0^\infty dv\ 1_{\{u,v\leq\beta^{-\frac{1}{6}}, |u-v|>\beta^{\frac{1}{24}}\}}[x(u)-x(v)]^2 A_\beta\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) = 0. \quad (2.31)$$

Proof. By (2.5)(ii) and Lemma 3(ii), the integral in the l.h.s. of (2.31) can be bounded below by

$$\begin{aligned}
&-\tfrac{1}{2}\beta^{-\frac{2}{3}} e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}} \textstyle\int_0^{\beta^{-\frac{1}{6}}} du \int_0^{\beta^{-\frac{1}{6}}} dv\ 1_{\{|u-v|>\beta^{\frac{1}{24}}\}}[x(u)-x(v)]^2 e^{-c_1\beta^{-\frac{1}{3}}\frac{(u-v)^2}{(u+v)}} \\
&\quad \geq -\tfrac{1}{2}\beta^{-\frac{2}{3}} e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}} e^{-\frac{1}{2}c_1\beta^{-\frac{1}{12}}} \textstyle\int_0^{\beta^{-\frac{1}{6}}} du \int_0^{\beta^{-\frac{1}{6}}} dv\ [x^2(u)+x^2(v)] \qquad (2.32) \\
&\quad = \mathcal{O}(e^{-\frac{1}{4}c_1\beta^{-\frac{1}{12}}}),
\end{aligned}$$

where c_1 is the constant in Lemma 3(ii). $\square$

STEP 3

$$\liminf_{\beta\downarrow 0} -\tfrac{1}{2}\beta^{-\frac{2}{3}}\int_0^{\beta^{-\frac{1}{6}}} du \int_0^{\beta^{-\frac{1}{6}}} dv\ 1_{\{|u-v|\le\beta^{\frac{1}{24}}\}}[x(u)-x(v)]^2 A_\beta\big(\lceil u\beta^{-\frac{1}{3}}\rceil,\lceil v\beta^{-\frac{1}{3}}\rceil\big)$$

$$= \liminf_{\beta\downarrow 0} F_\beta^2(x)$$

$$\ge F^2(x). \tag{2.33}$$

Proof. By (2.5)(ii) and the mean value theorem we have

$$\begin{aligned}
&-\tfrac{1}{2}\beta^{-\frac{2}{3}}\int_0^{\beta^{-\frac{1}{6}}} du \int_0^{\beta^{-\frac{1}{6}}} dv\ 1_{\{|u-v|\le\beta^{\frac{1}{24}}\}}[x(u)-x(v)]^2 A_\beta\big(\lceil u\beta^{-\frac{1}{3}}\rceil,\lceil v\beta^{-\frac{1}{3}}\rceil\big)\\
&\quad\ge -\tfrac{1}{2}\beta^{-\frac{2}{3}}e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}}\int_0^{\infty} du \int_0^{\infty} dv\ 1_{\{|u-v|\le\beta^{\frac{1}{24}}\}}[(u-v)x'(\xi_{uv})]^2 P\big(\lceil u\beta^{-\frac{1}{3}}\rceil,\lceil v\beta^{-\frac{1}{3}}\rceil\big)
\end{aligned} \tag{2.34}$$

for some ξ_{uv} between u and v. Let, as in (2.34),

$$I_\beta(u) = \beta^{-\frac{2}{3}}\int_0^\infty dv\ 1_{\{|u-v|\le\beta^{\frac{1}{24}}\}}(u-v)^2 P\big(\lceil u\beta^{-\frac{1}{3}}\rceil,\lceil v\beta^{-\frac{1}{3}}\rceil\big). \tag{2.35}$$

Then, because $x\in C^1(\mathbb{R}_0^+)$, it follows that

$$r.h.s.\ (2.34) = -\frac{1}{2}e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}}\int_0^\infty du\ I_\beta(u)[x'(\xi_u)]^2 \tag{2.36}$$

for some $\xi_u \in [u-\beta^{\frac{1}{24}}, u+\beta^{\frac{1}{24}}]\cap\mathbb{R}_0^+$.

Next, using (1.17) we can estimate

$$\begin{aligned}
I_\beta(u) &\le \beta^{-\frac{2}{3}}\int_0^\infty dv\ (u-v)^2 P\big(\lceil u\beta^{-\frac{1}{3}}\rceil,\lceil v\beta^{-\frac{1}{3}}\rceil\big)\\
&= \beta^{\frac{1}{3}}\sum_{j\ge 1} P\big(\lceil u\beta^{-\frac{1}{3}}\rceil, j\big)\Big\{(u\beta^{-\frac{1}{3}})^2-(u\beta^{-\frac{1}{3}})(2j-1)+(j^2-j+\tfrac{1}{3})\Big\}\\
&= \beta^{\frac{1}{3}}\Big\{(u\beta^{-\frac{1}{3}})^2-(u\beta^{-\frac{1}{3}})(2\lceil u\beta^{-\frac{1}{3}}\rceil+1)+(\lceil u\beta^{-\frac{1}{3}}\rceil^2+3\lceil u\beta^{-\frac{1}{3}}\rceil+\tfrac{1}{3})\Big\}\\
&\le 2u+\tfrac{13}{3}\beta^{\frac{1}{3}}.
\end{aligned} \tag{2.37}$$

Combining (2.34) and (2.36-37) with the estimates obtained in Steps 1 and 2, we now have

$$F_\beta^2(x) = l.h.s.\ (2.34)+o(1) \ge -(1+\mathcal{O}(\beta^{\frac{1}{3}}))\int_0^\infty du\ (u+\frac{13}{6}\beta^{\frac{1}{3}})[x'(\xi_u)]^2+o(1). \tag{2.38}$$

Next, pick $\delta>0$ and define

$$z_k^\delta = \sup_{(k-1)\delta<u\le k\delta} u[x'(u)]^2 \quad (k\ge 1). \tag{2.39}$$

Since $\xi_u \in [u - \beta^{\frac{1}{24}}, u + \beta^{\frac{1}{24}}] \cap \mathbb{R}_0^+$, it follows that for β small enough

$$\begin{aligned}
&\textstyle\int_0^\infty du\ (u + \frac{13}{6}\beta^{\frac{1}{3}})[x'(\xi_u)]^2 \\
&\qquad \le (1+\delta) \textstyle\int_\delta^\infty du\ \xi_u [x'(\xi_u)]^2 + 2\delta \int_0^\delta du\ [x'(\xi_u)]^2 \\
&\qquad \le (1+\delta) \textstyle\sum_{k\ge 1} (\delta z_k^\delta + 2\beta^{\frac{1}{24}} \max\{z_k^\delta, z_{k+1}^\delta\}) + 2\delta \int_0^\delta du\ \sup_{v \le 2\delta} [x'(v)]^2 \\
&\qquad \le (1+\delta)(1 + 4\delta^{-1}\beta^{\frac{1}{24}}) \textstyle\sum_{k\ge 1} \delta z_k^\delta + 2\delta^2 \sup_{v\le 2\delta} [x'(v)]^2.
\end{aligned} \tag{2.40}$$

Now let $\beta \downarrow 0$ followed by $\delta \downarrow 0$. Because $x \in C^1(\mathbb{R}_0^+)$, we have

$$\lim_{\delta\downarrow 0} \sum_{k\ge 1} \delta z_k^\delta = \int_0^\infty u[x'(u)]^2\, du = -F^2(x) \tag{2.41}$$

$$\lim_{\delta\downarrow 0} \sup_{v\le 2\delta} [x'(v)]^2 = [x'(0+)]^2 < \infty \tag{2.42}$$

and so

$$\liminf_{\beta\downarrow 0}\ r.h.s.\ (2.38) \ge F^2(x). \tag{2.43}$$

□

Lemmas 5-8 show that F_β epi-converges to F on Y. To see why, recall (2.1) and note that if $\int_0^\infty u^2 x^2(u) du = \infty$, then $F(x) \le F^1(x) = -\infty$. This proves Assumption (1) in Proposition 2 as was claimed in Proposition 3.

3 An approximate maximizer of F_β^a

Again we fix $a \in \mathbb{R}$ and suppress it from the notation. Like section 2, this section is technically somewhat involved, as it consists of a chain of estimates and inequalities that are needed to handle the approximation.

Define the scaled form of the eigenvector τ_β of A_β as

$$\bar\tau_\beta(u) = \beta^{-\frac{1}{6}} \tau_\beta(i) \text{ for } (i-1)\beta^{\frac{1}{3}} < u \le i\beta^{\frac{1}{3}} \quad (i \ge 1). \tag{3.1}$$

By Lemma 1, $\bar\tau_\beta$ is the unique maximizer of F_β. However, $\bar\tau_\beta$ is a step function and therefore $F(\bar\tau_\beta)$ is not defined, i.e., $\bar\tau_\beta \notin \overline{K} = \{x \in X : F(x) \ge -C\}$ (recall (1.13)). Thus, to apply Proposition 2, we must find an approximation of $\bar\tau_\beta$ that lies in $\overline{K}$ and approximates $F_\beta(\bar\tau_\beta)$ (i.e., we must prove Assumption 3(iii) in Proposition 2).

Proposition 4 *$\exists(\tilde\tau_\beta) \subset \overline{K}$ such that as $\beta \downarrow 0$*

$$\begin{aligned}
&(i)\ \ \|\bar\tau_\beta - \tilde\tau_\beta\|_{L^2} \to 0 \\
&(ii)\ 0 \le F_\beta(\bar\tau_\beta) - F_\beta(\tilde\tau_\beta) \to 0.
\end{aligned} \tag{3.2}$$

The proof of Proposition 4 is contained in Lemmas 9-13 below. We shall see that it suffices to pick for $\tilde\tau_\beta$ the following *linear* and *renormed* interpolation of $\bar\tau_\beta$:

$$\begin{aligned}
&\tilde\tau_\beta = \hat\tau_\beta \|\hat\tau_\beta\|_{L^2}^{-1} \\
&\hat\tau_\beta(u) = \beta^{-\frac{1}{6}} \Big\{ \tau_\beta(i) + (u\beta^{-\frac{1}{3}} - i)(\tau_\beta(i) - \tau_\beta(i-1)) \Big\} \text{ for } (i-1)\beta^{\frac{1}{3}} < u \le i\beta^{\frac{1}{3}} \quad (i \ge 1).
\end{aligned} \tag{3.3}$$

(put $\tau_\beta(0)=\tau_\beta(1)$).

We begin with two lemmas showing what is needed about τ_β in order to prove Proposition 4. Abbreviate $\Delta\tau_\beta(i)=\tau_\beta(i)-\tau_\beta(i-1)$ $(i\geq 1)$.

Lemma 9 *(i)* $\|\bar\tau_\beta-\tilde\tau_\beta\|_{L^2}\leq\|\Delta\tau_\beta\|_{l^2}+\tau_\beta^2(0)$.
(ii) $0\leq F_\beta(\bar\tau_\beta)-F_\beta(\tilde\tau_\beta)\leq\lambda(\beta)\beta^{-\frac{1}{3}}\|\Delta\tau_\beta\|_{l^2}^2[1-\frac{1}{6}\|\Delta\tau_\beta\|_{l^2}^2+\frac{1}{2}\tau_\beta^2(0)]^{-1}$.

Proof. (i) From (3.1) and (3.3) we compute

$$\|\bar\tau_\beta-\hat\tau_\beta\|_{L^2}^2 = \frac{1}{3}\|\Delta\tau_\beta\|_{l^2}^2 \tag{3.4}$$

$$\|\hat\tau_\beta\|_{L^2}^2 = \|\tau_\beta\|_{l^2}^2-\langle\tau_\beta,\Delta\tau_\beta\rangle_{l^2}+\frac{1}{3}\|\Delta\tau_\beta\|_{l^2}^2. \tag{3.5}$$

Using the relation $\langle\tau_\beta,\Delta\tau_\beta\rangle_{l^2}=\frac{1}{2}\|\Delta\tau_\beta\|_{l^2}^2-\frac{1}{2}\tau_\beta^2(0)$, together with (3.4-5) and $\|\tau_\beta\|_{l^2}=1$, we get

$$\begin{aligned}\|\bar\tau_\beta-\tilde\tau_\beta\|_{L^2} &\leq \|\bar\tau_\beta-\hat\tau_\beta\|_{L^2}+\|\hat\tau_\beta-\tilde\tau_\beta\|_{L^2}\\ &= \|\bar\tau_\beta-\hat\tau_\beta\|_{L^2}+|\|\hat\tau_\beta\|_{L^2}-1|\\ &= (\tfrac{1}{3}\|\Delta\tau_\beta\|_{l^2}^2)^{\frac{1}{2}}+|[1-\tfrac{1}{6}\|\Delta\tau_\beta\|_{l^2}^2+\tfrac{1}{2}\tau_\beta^2(0)]^{\frac{1}{2}}-1|\\ &\leq (\tfrac{1}{3})^{\frac{1}{2}}\|\Delta\tau_\beta\|_{l^2}+\tfrac{1}{6}\|\Delta\tau_\beta\|_{l^2}^2+\tfrac{1}{2}\tau_\beta^2(0)\\ &\leq ((\tfrac{1}{3})^{\frac{1}{2}}+\tfrac{1}{3})\|\Delta\tau_\beta\|_{l^2}+\tfrac{1}{2}\tau_\beta^2(0)\end{aligned} \tag{3.6}$$

where we use that $\|\Delta\tau_\beta\|_{l^2}\leq 2, \tau_\beta^2(0)\leq 1$.
(ii) From the definition of F_β in (1.2) we get, after substitution of (3.1) and (3.3),

$$\begin{aligned}F_\beta(\bar\tau_\beta) &= \beta^{-\frac{1}{3}}\langle\tau_\beta,A_\beta\tau_\beta\rangle_{l^2}-\beta^{-\frac{1}{3}}\|\tau_\beta\|_{l^2}^2\\ F_\beta(\hat\tau_\beta) &= \beta^{-\frac{1}{3}}\langle(\tau_\beta-\tfrac{1}{2}\Delta\tau_\beta),A_\beta(\tau_\beta-\tfrac{1}{2}\Delta\tau_\beta)\rangle_{l^2}-\beta^{-\frac{1}{3}}\|\hat\tau_\beta\|_{L^2}^2.\end{aligned} \tag{3.7}$$

It follows from (3.7) that

$$\begin{aligned}F_\beta(\bar\tau_\beta)-F_\beta(\tilde\tau_\beta) &= F_\beta(\bar\tau_\beta)-\tfrac{1}{\|\hat\tau_\beta\|_{L^2}^2}F_\beta(\hat\tau_\beta)\\ &= \beta^{-\frac{1}{3}}\tfrac{1}{\|\hat\tau_\beta\|_{L^2}^2}\Big\{\tfrac{1}{3}\lambda(\beta)\|\Delta\tau_\beta\|_{l^2}^2-\tfrac{1}{4}\langle\Delta\tau_\beta,A_\beta\Delta\tau_\beta\rangle_{l^2}\Big\},\end{aligned} \tag{3.8}$$

where in the second equality we use the symmetry of A_β and the relations $A_\beta\tau_\beta=\lambda(\beta)\tau_\beta$ and (3.5). Finally, observe that $|\langle\Delta\tau_\beta,A_\beta\Delta\tau_\beta\rangle_{l^2}|\leq\langle|\Delta\tau_\beta|,A_\beta|\Delta\tau_\beta|\rangle_{l^2}\leq\lambda(\beta)\|\Delta\tau_\beta\|_{l^2}^2$ to get the claim. □

Lemma 10

$$\begin{aligned}F(\tilde\tau_\beta) &\geq -2\sqrt{5}|a|\big(\beta^{\frac{2}{3}}\textstyle\sum_{i\geq 1}i^2\tau_\beta^2(i)\big)^{\frac{1}{2}}[1-\tfrac{1}{6}\|\Delta\tau_\beta\|_{l^2}^2+\tfrac{1}{2}\tau_\beta^2(0)]^{-\frac{1}{2}}\\ &\quad+\Big\{20\beta^{\frac{2}{3}}\textstyle\sum_{i\geq 1}i^2\tau_\beta^2(i)+\beta^{-\frac{1}{3}}\sum_{i\geq 1}i\Delta\tau_\beta^2(i)\Big\}[1-\tfrac{1}{6}\|\Delta\tau_\beta\|_{l^2}^2+\tfrac{1}{2}\tau_\beta^2(0)]^{-1}.\end{aligned} \tag{3.9}$$

Proof. According to (2.1) and (2.3)

$$F(\hat{\tau}_\beta) = \int_0^\infty du \, \Big\{ (2au - 4u^2) \, \hat{\tau}_\beta^2(u) - u[\hat{\tau}'_\beta(u)]^2 \Big\}. \tag{3.10}$$

Use (3.3) to obtain the estimates

$$\begin{aligned} &\textstyle\int_0^\infty du \; u^2 \hat{\tau}_\beta^2(u) \le \beta^{\frac{2}{3}} \sum_{i \ge 1} i^2 \max\{\tau_\beta^2(i), \tau_\beta^2(i-1)\} \\ &\textstyle\int_0^\infty du \; u[\hat{\tau}'_\beta(u)]^2 \le \beta^{-\frac{1}{3}} \sum_{i \ge 1} i \Delta \tau_\beta^2(i). \end{aligned} \tag{3.11}$$

Since $\int_0^\infty du \; u \hat{\tau}_\beta^2(u) \le (\int_0^\infty du \; u^2 \hat{\tau}_\beta^2(u))^{\frac{1}{2}} \|\hat{\tau}_\beta\|_{L^2}$, we get the claim because $F(\tilde{\tau}_\beta) = \frac{1}{\|\hat{\tau}_\beta\|_{L^2}^2} F_\beta(\hat{\tau}_\beta)$. □

Lemmas 9 and 10 set the stage for the proof of Proposition 4. Namely, we now see that it suffices to prove the following estimates:

Lemma 11 *There exist C_1, C_2, C_3, C_4 such that for β small enough*

$$\begin{array}{ll} (i) & \sum_{i \ge 1} i^2 \tau_\beta^2(i) \le C_1 \beta^{-\frac{2}{3}} \\ (ii) & \sum_{i \ge 1} i \Delta \tau_\beta^2(i) \le C_2 \beta^{\frac{1}{3}} \\ (iii) & \tau_\beta^2(0) \le C_3 \beta^{\frac{1}{3}} \log \frac{1}{\beta} \\ (iv) & \|\Delta \tau_\beta\|_{l^2}^2 \le C_4 \beta^{\frac{2}{3}} \log \frac{1}{\beta}. \end{array} \tag{3.12}$$

Indeed, Lemmas 11(iii-iv) and 9(i-ii) imply (3.2), while Lemmas 11(i-ii) and Lemma 10 imply that $F(\tilde{\tau}_\beta) \ge -C$ for β small enough and C sufficiently large, which guarantees that $\tilde{\tau}_\beta \in \overline{K} = \overline{K_C^a}$.

In the proof of Lemma 11 we shall make use of the following two additional lemmas, the proof of which is deferred to section 4:

Lemma 12 $\forall \beta > 0 : \Delta \tau_\beta(i) = \tau_\beta(i) - \tau_\beta(i-1) \le 0$ *for all* $i \ge \frac{a}{2} \beta^{-\frac{1}{3}}$.

Lemma 13

$$\begin{array}{lcll} \limsup_{\beta \downarrow 0} \beta^{-\frac{1}{3}} [\lambda(\beta) - 1] & \le & \frac{1}{4} a^2 & \\ \liminf_{\beta \downarrow 0} \beta^{-\frac{1}{3}} [\lambda(\beta) - 1] & \ge & \frac{1}{2\pi} a^2 - \frac{2}{a} & (a > 1) \\ & & \frac{1}{\pi} a - \frac{1}{2\pi} - 2 & (a \le 1). \end{array} \tag{3.13}$$

Proof of Lemma 11(i).

STEP 1 *For every $\epsilon > 0$ small enough there exists C_5 such that*

$$\sum_{i \le \epsilon \beta^{-\frac{1}{2}}} i^2 \tau_\beta^2(i) \le C_5 \beta^{-\frac{2}{3}} \text{ for } \beta \text{ small enough.} \tag{3.14}$$

Proof. We start with the trivial inequality

$$\frac{1}{2} \sum_{i,j} [\tau_\beta(i) - \tau_\beta(j)]^2 A_\beta(i,j) \ge 0. \tag{3.15}$$

The l.h.s. of (3.15) can be written out and estimated from above as follows:

$$
\begin{aligned}
&[1-\lambda(\beta)] + \textstyle\sum_{i,j} \tau_\beta^2(i)\big[A_\beta(i,j) - P(i,j)\big] \\
&\qquad \le [1-\lambda(\beta)] + \textstyle\sum_{i\le \epsilon\beta^{-\frac12}} \tau_\beta^2(i) \sum_j \left[e^{e_\beta(i,j)} - 1\right] P(i,j) \\
&\qquad \le [1-\lambda(\beta)] + \textstyle\sum_{i\le \epsilon\beta^{-\frac12}} \tau_\beta^2(i) \sum_j \left[e_\beta(i,j) + e_\beta^2(i,j)\right] P(i,j).
\end{aligned}
\tag{3.16}
$$

For the two inequalities we refer to (2.4-5) (use that $e^t \le 1+t+t^2$ for $t \ll 1$ and $t \le 0$). The sum over j has been evaluated in (2.8). Using that $i \le \epsilon\beta^{-\frac12}$, we get

$$
\begin{aligned}
&r.h.s.\ (3.16) \\
&\quad \le [1-\lambda(\beta)] + \beta^{\frac13} \textstyle\sum_{i\le \epsilon\beta^{-\frac12}} \tau_\beta^2(i) \big\{2a(i\beta^{\frac13}) - (4 - 168\epsilon^2 - 6a\beta^{\frac13})(i\beta^{\frac13})^2\big\}.
\end{aligned}
\tag{3.17}
$$

Combining (3.15-17) we arrive at the following inequality:

$$
\begin{aligned}
&(4 - 168\epsilon^2 - 6a\beta^{\frac13})\beta^{\frac23} \textstyle\sum_{i\le \epsilon\beta^{-\frac12}} i^2\tau_\beta^2(i) \\
&\qquad \le \beta^{-\frac13}[1-\lambda(\beta)] + 2a\beta^{\frac13} \textstyle\sum_{i\le \epsilon\beta^{-\frac12}} i\tau_\beta^2(i).
\end{aligned}
\tag{3.18}
$$

Now, $\beta^{-\frac13}[1-\lambda(\beta)] \le C_6$ by Lemma 13. Moreover, by Cauchy-Schwarz

$$
\sum_{i\le \epsilon\beta^{-\frac12}} i\tau_\beta^2(i) \le \Big(\sum_{i\le \epsilon\beta^{-\frac12}} i^2\tau_\beta^2(i)\Big)^{\frac12}. \tag{3.19}
$$

Hence the claim in (3.14) follows for ϵ such that $168\epsilon^2 < 4$. $\square$

STEP 2 *For every $\epsilon > 0$ small enough there exists C_7 such that*

$$
\sum_{i > \epsilon\beta^{-\frac12}} \tau_\beta^2(i) \le C_7\beta^{\frac13} \ \textit{for } \beta \textit{ small enough.} \tag{3.20}
$$

Proof. Rewrite (3.15) as (recall also (3.16))

$$
\sum_{i>\epsilon\beta^{-\frac12}} \tau_\beta^2(i) \sum_j \left[1 - e^{e_\beta(i,j)}\right] P(i,j) \le [1-\lambda(\beta)] + \sum_{i\le \epsilon\beta^{-\frac12}} \tau_\beta^2(i) \sum_j \left[e^{e_\beta(i,j)} - 1\right] P(i,j). \tag{3.21}
$$

Since $e_\beta(i,j) \le \frac14 a^2\beta^{\frac13}$ for $i,j \ge 1$ and $e_\beta(i,j) \le -\frac12\epsilon^2$ for $i > \epsilon\beta^{-\frac12}, j \ge 1$ (see (2.4-5)), we get

$$
(1 - e^{-\frac12\epsilon^2}) \sum_{i>\epsilon\beta^{-\frac12}} \tau_\beta^2(i) \le C_6\beta^{\frac13} + (e^{\frac14 a^2\beta^{\frac13}} - 1). \tag{3.22}
$$

This implies the claim in (3.20). $\square$

STEP 3 *For every $\epsilon > 0$ there exists C_8 such that*

$$
\sum_{i>\epsilon\beta^{-\frac12}} i^2\tau_\beta^2(i) \le C_8\beta^{-\frac23} \ \textit{for } \beta \textit{ small enough.} \tag{3.23}
$$

Proof. Pick $i > \epsilon\beta^{-\frac{1}{2}}$ and $\delta > 0$ arbitrary. Then, using (2.4) and Lemma 3(ii), we see that there exists $C(\delta) > 0$ such that

$$\begin{aligned}\tau_\beta(i) &= \tfrac{1}{\lambda(\beta)}\textstyle\sum_j A_\beta(i,j)\tau_\beta(j)\\ &= \tfrac{1}{\lambda(\beta)}\textstyle\sum_j e^{e_\beta(i,j)}P(i,j)\tau_\beta(j)\\ &\le \tfrac{1}{\lambda(\beta)}e^{-\frac{1}{2}\epsilon^2}\textstyle\sum_j P(i,j)\tau_\beta(j)\\ &\le (1+\delta)e^{-\frac{1}{2}\epsilon^2}\textstyle\sum_{j>(1-\delta)i} P(i,j)\tau_\beta(j) + \mathcal{O}(e^{-iC(\delta)}).\end{aligned} \tag{3.24}$$

Using (3.24) we get

$$\begin{aligned}\textstyle\sum_{i>\epsilon\beta^{-\frac{1}{2}}} i^2\tau_\beta^2(i) &\le (1+\delta)e^{-\frac{1}{2}\epsilon^2}\textstyle\sum_{i>\epsilon\beta^{-\frac{1}{2}}} i^2 \sum_{j>(1-\delta)i}\tau_\beta(i)P(i,j)\tau_\beta(j) + o(1)\\ &\le \tfrac{(1+\delta)}{(1-\delta)^2}e^{-\frac{1}{2}\epsilon^2}\textstyle\sum_{i>\epsilon\beta^{-\frac{1}{2}}}\lceil(1-\delta)i\rceil^2\tau_\beta^2(\lceil(1-\delta)i\rceil) + o(1)\\ &\le \tfrac{(1+\delta)}{(1-\delta)^3}e^{-\frac{1}{2}\epsilon^2}\textstyle\sum_{i>(1-\delta)\epsilon\beta^{-\frac{1}{2}}} i^2\tau_\beta^2(i) + o(1)\\ &\le \tfrac{(1+\delta)}{(1-\delta)^3}e^{-\frac{1}{2}\epsilon^2}\Big\{\textstyle\sum_{i>\epsilon\beta^{-\frac{1}{2}}} i^2\tau_\beta^2(i) + \epsilon^2\beta^{-1}C_7\beta^{\frac{1}{3}}\Big\} + o(1).\end{aligned} \tag{3.25}$$

The second and the third inequality use Lemma 12, the fourth uses (3.20) with ϵ replaced by $(1-\delta)\epsilon$. Now pick δ so small that $\frac{(1+\delta)}{(1-\delta)^3} < e^{\frac{1}{4}\epsilon^2}$. Then we obtain

$$(1-e^{-\frac{1}{4}\epsilon^2})\sum_{i>\epsilon\beta^{-\frac{1}{2}}} i^2\tau_\beta^2(i) \le e^{-\frac{1}{4}\epsilon^2}\epsilon^2 C_7\beta^{-\frac{2}{3}} + o(1). \tag{3.26}$$

This proves the claim in (3.23). □

Steps 1-3 complete the proof of Lemma 11(i). □

Proof of Lemma 11(ii).

STEP 4 *For all* β

$$\sum_{i\ge1} i\Delta\tau_\beta^2(i+1) = \frac{1}{\lambda(\beta)}\sum_{i,j\ge1}(i+j-1)[1-e^{e_\beta(i+1,j)-e_\beta(i,j)}]\tau_\beta(i)A_\beta(i,j)\tau_\beta(j). \tag{3.27}$$

Proof. Write out

$$\begin{aligned}\textstyle\sum_i i\Delta\tau_\beta^2(i+1) &= \textstyle\sum_i i[\tau_\beta(i+1)-\tau_\beta(i)]^2\\ &= \textstyle\sum_i i[\tau_\beta^2(i+1)+\tau_\beta^2(i)] - \tfrac{1}{\lambda(\beta)}\sum_{i,j} 2i\tau_\beta(i)A_\beta(i+1,j)\tau_\beta(j).\end{aligned} \tag{3.28}$$

Now substitute the relation (see (0.10-11))

$$\begin{aligned}A_\beta(i+1,j) &= e^{e_\beta(i+1,j)}P(i+1,j)\\ &= e^{e_\beta(i+1,j)}\tfrac{i+j-1}{2i}P(i,j)\\ &= e^{e_\beta(i+1,j)-e_\beta(i,j)}\tfrac{i+j-1}{2i}A_\beta(i,j).\end{aligned} \tag{3.29}$$

This gives

$$\begin{aligned}\sum_i i\Delta\tau_\beta^2(i+1) &= r.h.s.\ (3.27)+\sum_i i[\tau_\beta^2(i+1)+\tau_\beta^2(i)] \\ &\quad -\tfrac{1}{\lambda(\beta)}\sum_{i,j}(i+j-1)\tau_\beta(i)A_\beta(i,j)\tau_\beta(j).\end{aligned} \tag{3.30}$$

Both sums in the r.h.s. are equal to $\sum_i(2i-1)\tau_\beta^2(i)$ and therefore cancel out. □

STEP 5 *For β small enough*

$$\frac{1}{\lambda(\beta)}\sum_{i,j\geq 1}(i+j-1)[1-e^{e_\beta(i+1,j)-e_\beta(i,j)}]\tau_\beta(i)A_\beta(i,j)\tau_\beta(j)\leq C_9\beta^{\frac{1}{3}}. \tag{3.31}$$

Proof. By (2.4) we have $e_\beta(i+1,j)-e_\beta(i,j)=a\beta^{\frac{2}{3}}-\beta(2i+2j-1)$. Hence

$$\begin{aligned} l.h.s.\ (3.31) &\leq \tfrac{1}{\lambda(\beta)}\sum_{i,j\geq 1}(i+j-1)[e_\beta(i,j)-e_\beta(i+1,j)]\tau_\beta(i)A_\beta(i,j)\tau_\beta(j) \\ &\leq \tfrac{1}{\lambda(\beta)}2\beta\sum_{i,j}(i+j)^2\tau_\beta(i)A_\beta(i,j)\tau_\beta(j) \\ &\leq 8\beta\sum_i i^2\tau_\beta^2(i)\end{aligned} \tag{3.32}$$

(use that $e^t\geq 1+t$ for all t). In the third inequality we use the symmetry of A_β and the fact that $\|A_\beta\|_{l^2}=\lambda(\beta)$. The claim now follows from Lemma 11(i). □

Steps 4-5 complete the proof of Lemma 11(ii). □

Proof of Lemma 11(iii).

STEP 6

$$\tau_\beta^2(0)\leq C_{10}\beta^{\frac{1}{3}}\log\frac{1}{\beta}\ \textit{for } \beta \textit{ small enough.} \tag{3.33}$$

Proof. By Cauchy-Schwarz, we have for every N

$$\begin{aligned}\tau_\beta(0) &= \tau_\beta(N)-\sum_{i=1}^N\Delta\tau_\beta(i) \\ &\leq \tau_\beta(N)+(\sum_{i=1}^N\tfrac{1}{i})^{\frac{1}{2}}(\sum_{i=1}^N i\Delta\tau_\beta^2(i))^{\frac{1}{2}}.\end{aligned} \tag{3.34}$$

Pick $N=\lceil\beta^{-\frac{1}{2}}\rceil$. Lemma 11(i) gives $\tau_\beta(\lceil\beta^{-\frac{1}{2}}\rceil)\leq C_1\beta^{\frac{1}{3}}$. Together with Lemma 11(ii) and the estimate $\sum_{i=1}^{\lceil\beta^{-\frac{1}{2}}\rceil}\frac{1}{i}\leq\log\frac{1}{\beta}$, the claim follows. □

Step 6 completes the proof of Lemma 11(iii). □

Proof of Lemma 11(iv).

STEP 7 *For all β*

$$\begin{aligned}\sum_{i\geq 1}\Delta\tau_\beta^2(i+1) &= \tfrac{2}{\lambda(\beta)}\sum_{\substack{i,j\\(i,j)\neq(1,1)}}[1-e^{e_\beta(i-1,j)-e_\beta(i,j)}]\tau_\beta(i)A_\beta(i,j)\tau_\beta(j) \\ &\quad -\tau_\beta^2(1)[1-\tfrac{2}{\lambda(\beta)}A_\beta(1,1)].\end{aligned} \tag{3.35}$$

Proof. By Lemma 4 we have the following relation:

$$A_\beta(i,j)-A_\beta(i-1,j)=A_\beta(i,j-1)-A_\beta(i,j)+2A_\beta(i,j)[1-e^{e_\beta(i-1,j)-e_\beta(i,j)}] \tag{3.36}$$

(note that $e_\beta(i-1,j)=e_\beta(i,j-1)$). Hence

$$\begin{aligned}\sum_i \Delta\tau_\beta^2(i+1) &= \sum_i[\tau_\beta(i+1)-\tau_\beta(i)]^2\\ &= \tau_\beta^2(1)+2\sum_{i\geq 2}\tau_\beta(i)[\tau_\beta(i)-\tau_\beta(i-1)]\\ &= \tau_\beta^2(1)+\tfrac{2}{\lambda(\beta)}\sum_{i\geq 2}\sum_j\tau_\beta(i)[A_\beta(i,j)-A_\beta(i-1,j)]\tau_\beta(j)\\ &= \tau_\beta^2(1)+\tfrac{2}{\lambda(\beta)}\sum_{i\geq 2}\sum_j\tau_\beta(i)[A_\beta(i,j-1)-A_\beta(i,j)]\tau_\beta(j)\\ &\quad+\tfrac{4}{\lambda(\beta)}\sum_{i\geq 2}\sum_j[1-e^{e_\beta(i-1,j)-e_\beta(i,j)}]\tau_\beta(i)A_\beta(i,j)\tau_\beta(j).\end{aligned} \tag{3.37}$$

The third term in the last expression is twice the sum in the r.h.s. of (3.35) except for the part with $i=1, j\geq 2$. The second term can be rewritten by carrying out the sum over i, namely (use that $A(i,0)=0$)

$$\begin{aligned}j=1:\quad & -\tfrac{2}{\lambda(\beta)}\sum_{i\geq 2}\tau_\beta(i)A_\beta(i,1)\tau_\beta(1)\\ &= -2\tau_\beta^2(1)+\tfrac{2}{\lambda(\beta)}\tau_\beta^2(1)A_\beta(1,1)\\ j\geq 2:\quad & \tfrac{2}{\lambda(\beta)}\sum_{i\geq 2}\tau_\beta(i)[A_\beta(i,j-1)-A_\beta(i,j)]\tau_\beta(j)\\ &= 2\tau_\beta(j)[\tau_\beta(j-1)-\tau_\beta(j)]-\tfrac{2}{\lambda(\beta)}\tau_\beta(1)[A_\beta(1,j-1)-A_\beta(1,j)]\tau_\beta(j).\end{aligned} \tag{3.38}$$

Thus, after carrying out the sum over j, we see that (3.37) becomes

$$\begin{aligned}\sum_i \Delta\tau_\beta^2(i+1) &= -\sum_j\Delta\tau_\beta^2(j+1)\\ &\quad+2\Big\{r.h.s.\ (3.35)-\tfrac{2}{\lambda(\beta)}\sum_{j\geq 2}[1-e^{e_\beta(0,j)-e_\beta(1,j)}]\tau_\beta(1)A_\beta(1,j)\tau_\beta(j)\\ &\qquad+\tau_\beta^2(1)[1-\tfrac{2}{\lambda(\beta)}A_\beta(1,1)]\Big\}\\ &\quad+\tfrac{2}{\lambda(\beta)}\tau_\beta^2(1)A_\beta(1,1)-\tfrac{2}{\lambda(\beta)}\sum_{j\geq 2}\tau_\beta(1)[A_\beta(1,j-1)-A_\beta(1,j)]\tau_\beta(j).\end{aligned} \tag{3.39}$$

Now, by (3.36) for $i=1$,

$$2[1-e^{e_\beta(0,j)-e_\beta(1,j)}]A_\beta(1,j)=-[A_\beta(1,j-1)-A_\beta(1,j)]+A_\beta(1,j). \tag{3.40}$$

Hence (3.39) simplifies to

$$\begin{aligned}\sum_i \Delta\tau_\beta^2(i+1) &= -\sum_j\Delta\tau_\beta^2(j+1)+\ 2\ r.h.s.\ (3.35)\\ &\quad+\Big\{2\tau_\beta^2(1)-\tfrac{2}{\lambda(\beta)}\tau_\beta^2(1)A_\beta(1,1)-\tfrac{2}{\lambda(\beta)}\sum_{j\geq 2}\tau_\beta(1)A_\beta(1,j)\tau_\beta(j)\Big\}.\end{aligned} \tag{3.41}$$

But the term between braces is zero. □

STEP 8 *For β small enough*

$$r.h.s.\ (3.35) \leq C_{11}\beta^{\frac{2}{3}} \log \frac{1}{\beta}. \tag{3.42}$$

Proof. The first term in (3.35) is easy to bound. Indeed, we have $e_\beta(i-1,j) - e_\beta(i,j) = -a\beta^{\frac{2}{3}} + \beta(2i+2j-3)$ and hence we get

$$\begin{aligned} 1^{st} \text{ term in } (3.35) \ &\leq \tfrac{2}{\lambda(\beta)} \textstyle\sum_{\substack{i,j \\ (i,j)\neq(1,1)}} a\beta^{\frac{2}{3}} \tau_\beta(i) A_\beta(i,j) \tau_\beta(j) \\ &\leq 2a\beta^{\frac{2}{3}} \end{aligned} \tag{3.43}$$

(in the first inequality use that $e^t \geq 1+t$ for all t). For the second term in (3.35), use that $P(1,1) = \frac{1}{2}$ and $e_\beta(1,1) = a\beta^{\frac{2}{3}} - \beta$. Together with $\lambda(\beta) \geq 1 - C_6\beta^{\frac{1}{3}}$ (see below (3.18)) we get

$$2^{nd} \text{ term in } (3.35) \leq 2\tau_\beta^2(1) C_6 \beta^{\frac{1}{3}} \text{ for } \beta \text{ small enough.} \tag{3.44}$$

Finally, use Step 6 to get the claim (recall that $\tau_\beta(0) = \tau_\beta(1)$ in (3.3)). □

Steps 7-8 complete the proof of Lemma 11(iv). □

Lemma 11 completes the proof of Proposition 4. Lemmas 12 and 13 will be proved in section 4.

Proposition 4 shows that Assumption 3(iii) in Proposition 2 holds. We shall prove Assumptions 3(i),(ii) in section 5.

4 Proof of Lemmas 12 and 13

4.1 Proof of Lemma 12

Let $(e_i)_{i\geq 1}$ be the canonical base of $l^2(\mathbb{N})$. Let $s = (s(i))_{i\geq 1}$ be any sequence of numbers in $(0,\infty)$ and let $t = (t(i))_{i\geq 1}$ be given by $t(1) = 1, t(i) = \prod_{k=1}^{i-1} s(k)$ $(i \geq 2)$. Define

$$\begin{aligned} B_s \ &= \{x \in l^2(\mathbb{N}) : x \geq 0, x(i+1) \leq s(i)x(i)\} \\ B_s^0 \ &= \{x = \textstyle\sum_j c_j f_j : c_j \geq 0, c_j \neq 0 \text{ finitely often}\} \end{aligned} \tag{4.1}$$

where $f_j \in l^2(\mathbb{N})$ is defined by

$$f_j = \sum_{i=1}^{j} t(i) e_i. \tag{4.2}$$

Lemma 14 *(i) B_s is a closed convex cone.*
(ii) B_s is the closure of B_s^0.

Proof. Elementary. □

Recall footnote 6. Since, for every $\beta > 0$, A_β is a continuous operator on $l^2(\mathbb{N})$, we have from Lemma 14(ii) that

$$A_\beta B_s \subset B_s \Leftrightarrow A_\beta f_j \in B_s \text{ for all } j \geq 1. \tag{4.3}$$

Since, for every $\beta > 0$, A_β is symmetric and has a spectral gap, we also know that $\lambda(\beta)^{-n} A_\beta^n x \to^{l^2} \langle x, \tau_\beta \rangle_{l^2} \tau_\beta$ $(n \to \infty)$ for any $x \in l^2(\mathbb{N})$. Pick any $x \in B_s$ with $x \neq 0$ to get that

$$A_\beta B_s \subset B_s \Rightarrow \tau_\beta \in B_s. \tag{4.4}$$

Below we shall prove the following:

Lemma 15 *If s satisfies*

$$\frac{is(i) + j\dfrac{1}{s(j)}}{i+j} \geq e^{a\beta^{\frac{2}{3}} - \beta(2i+2j+1)} \text{ for all } i \geq 1, j \geq 0, \tag{4.5}$$

then $A_\beta f_j \in B_s$ for all $j \geq 1$.

Lemma 15 combined with (4.3-4) shows that

$$\tau_\beta \in \bigcap_{\{s:\ s\ satisfies\ (4.5)\}} B_s. \tag{4.6}$$

The r.h.s. of (4.5) is ≤ 1 when $i + j \geq \frac{a}{2}\beta^{-\frac{1}{3}}$. One therefore easily sees that the following choice of s satisfies (4.5):

$$\begin{aligned} s(i) &= 1 && \text{for } i > \tfrac{a}{2}\beta^{-\frac{1}{3}} \\ &= N\beta^{-\frac{1}{3}} && \text{for } i \leq \tfrac{a}{2}\beta^{-\frac{1}{3}}, N \text{ large enough.} \end{aligned} \tag{4.7}$$

This proves Lemma 12.

Proof of Lemma 15. We must show that for all $i, j \geq 1$

$$\begin{aligned} 0 &\leq s(i)(A_\beta f_j)(i) - (A_\beta f_j)(i+1) \\ &= \textstyle\sum_{k=1}^{j} \Big[s(i)A_\beta(i,k)t(k) - A_\beta(i+1,k-1)t(k-1) \Big] - A_\beta(i+1,j)t(j) \end{aligned} \tag{4.8}$$

(recall from Lemma 4 that $A_\beta(i+1, 0) = 0$ by convention). In order to do so, define

$$\psi_i(j) = \sum_{k=1}^{j} \Big[s(i)A_\beta(i,k)t(k) - A_\beta(i+1,k-1)t(k-1) \Big] - 2A_\beta(i+1,j)t(j). \tag{4.9}$$

The following lemma gives a sufficient criterion for $\psi_i(j) \geq 0$, which implies (4.8):

Lemma 16 *If*

$$s(i)A_\beta(i,j+1) + \frac{1}{s(j)} A_\beta(i+1,j) - 2A_\beta(i+1,j+1) \geq 0 \text{ for all } i \geq 1, j \geq 0, \tag{4.10}$$

then

$$\begin{aligned} &(i)\ \ j \to \psi_i(j) \text{ is nondecreasing for all } i \geq 1 \\ &(ii)\ \psi_i(1) \geq 0 \text{ for all } i \geq 1. \end{aligned} \tag{4.11}$$

Proof. (i) Write out

$$\begin{aligned}\psi_i(j+1)-\psi_i(j) &= s(i)A_\beta(i,j+1)t(j+1)+A_\beta(i+1,j)t(j)-2A_\beta(i+1,j+1)t(j+1)\\ &= t(j+1)\Big[s(i)A_\beta(i,j+1)+\tfrac{1}{s(j)}A_\beta(i+1,j)-2A_\beta(i+1,j+1)\Big]\geq 0.\end{aligned} \tag{4.12}$$

(ii) Similarly, by (4.10) with $j=0$ (since $t(1)=1$ and $A_\beta(i+1,0)=0$)

$$\psi_i(1)=s(i)A_\beta(i,1)-2A_\beta(i+1,1)\geq 0. \tag{4.13}$$

□

To complete the proof of Lemma 15, it remains to rewrite (4.10) in the form of (4.5). Abbreviate $f(i)=\exp[a\beta^{\frac{2}{3}}i-\beta i^2]$. Then we have $A_\beta(i,j)=f(i+j-1)P(i,j)$. Use Lemma 4 to write

$$\begin{aligned}l.h.s.\ (4.10) &= f(i+j)[s(i)P(i,j+1)+\tfrac{1}{s(j)}P(i+1,j)]-2f(i+j-1)P(i+1,j+1)\\ &= P(i,j+1)[s(i)f(i+j)-f(i+j+1)]\\ &\quad +P(i+1,j)[\tfrac{1}{s(j)}f(i+j)-f(i+j+1)].\end{aligned} \tag{4.14}$$

Next use that $P(i,j+1)/P(i+1,j)=i/j$. Then (4.10) is seen to be equivalent to

$$\frac{is(i)+j\dfrac{1}{s(j)}}{i+j}\geq\frac{f(i+j+1)}{f(i+j)}. \tag{4.15}$$

□

4.2 Proof of Lemma 13

To prove the upper bound in (3.13), use (2.5)(ii) to get

$$\begin{aligned}\lambda(\beta) &= \textstyle\sum_{i,j}\tau_\beta(i)A_\beta(i,j)\tau_\beta(j)\\ &= \textstyle\sum_{i,j}\tau_\beta(i)e^{e_\beta(i,j)}P(i,j)\tau_\beta(j)\\ &\leq e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}}\textstyle\sum_{i,j}\tau_\beta(i)P(i,j)\tau_\beta(j)\\ &\leq e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}},\end{aligned} \tag{4.16}$$

where the last inequality follows from $\|P\|_{l^2}\leq 1$. This immediately gives the claim.

To prove the lower bound in (3.13), use (1.3)(i) to get that for any $x\in L^2(\mathbb{R}^+)$ with $\|x\|_{L^2}=1$

$$\beta^{-\frac{1}{3}}[\lambda(\beta)-1]\geq F^a_\beta(x). \tag{4.17}$$

Pick for x

$$x_\sigma(u)=\Big(\frac{2}{\pi\sigma^2}\Big)^{\frac{1}{4}}e^{-\frac{u^2}{4\sigma^2}}\ (\sigma>0). \tag{4.18}$$

Now, we know from Lemmas 5-8 that

$$\lim_{\beta \downarrow 0} F^a_\beta(x_\sigma) = F^a(x_\sigma). \tag{4.19}$$

Hence $\liminf_{\beta \downarrow 0} \beta^{-\frac{1}{3}}[\lambda(\beta) - 1] \geq F^a(x_\sigma)$. Compute

$$\begin{aligned} F^a(x_\sigma) &= \int_0^\infty \Big\{(2au - 4u^2)[x_\sigma(u)]^2 - u[x'_\sigma(u)]^2\Big\} du \\ &= \left(\tfrac{2}{\pi\sigma^2}\right)^{\frac{1}{2}} \int_0^\infty (2au - 4u^2 - \tfrac{u^3}{4\sigma^4}) e^{-\frac{u^2}{2\sigma^2}} du \\ &= \left(\tfrac{8}{\pi}\right)^{\frac{1}{2}} a\sigma - 4\sigma^2 - \tfrac{1}{(2\pi)^{\frac{1}{2}}\sigma}. \end{aligned} \tag{4.20}$$

Pick $\sigma = \sigma(a) = \frac{a \vee 1}{(8\pi)^{\frac{1}{2}}}$ to get the claim. $\square$

5 Analysis of the limit variational problem

Recall the notation in (1.13)

$$\begin{aligned} X &= \{x \in L^2(\mathbb{R}^+) : x \geq 0, \|x\|_{L^2} = 1\} \\ Y &= X \cap C^1(\mathbb{R}_0^+) \\ K &= K_C^a = \{x \in Y : F^a(x) \geq -C\}. \end{aligned} \tag{5.1}$$

In this section we analyze the limit variational problem appearing in (1.6), i.e.,

$$\sup_{x \in X} F^a(x). \tag{5.2}$$

In section 5.1 we show that $x \to F^a(x)$ is upper semicontinuous and K_C^a is relatively compact in X (in the L^2-topology). This implies that F^a achieves a maximum in $\overline{K_C^a} = \{x \in X : F^a(x) \geq -C\}$ ($\neq \emptyset$ for C large enough). In section 5.2 we show that all maxima of F^a in X are solutions of the Sturm-Liouville problem

$$\mathcal{L}^a x = \rho x \ (\rho \in \mathbb{R}, x \in X \cap C^\infty(\mathbb{R}^+)), \tag{5.3}$$

where $\mathcal{L}^a$ is defined in (0.17). In section 5.3 we analyze (5.3) and show that it has a unique solution x^a satisfying $F^a(x^a) > -\infty$ and $x^a > 0$, with corresponding eigenvalue $\rho(a)$. This identifies x^a as the unique maximizer of (5.2) and $\rho(a)$ as the maximum. We also study $a \to x^a$ and $a \to \rho(a)$ to prove the claims that were made in (0.19).

5.1 Existence of a maximizer of F^a in $\overline{K_C^a}$

It will be expedient to transform $F^a, \mathcal{L}^a, K_C^a$ as follows. Define (recall (1.7))

$$\begin{aligned} \hat{F}^a(x) &= -F^a(x) + (\tfrac{a^2}{4} + 1)\|x\|_{L^2}^2 \\ &= \int_0^\infty \{q(u)[x(u)]^2 + p(u)[x'(u)]^2\} du \end{aligned} \tag{5.4}$$

with

$$
\begin{aligned}
p(u) &= u \\
q(u) &= (2u - \tfrac{1}{2}a)^2 + 1.
\end{aligned} \tag{5.5}
$$

$\hat{F}^a$ is the "energy" functional corresponding to the Sturm-Liouville differential operator $\hat{\mathcal{L}}^a$ defined by (recall (0.17))

$$
\begin{aligned}
(\hat{\mathcal{L}}^a x)(u) &= -(\mathcal{L}^a x)(u) + (\tfrac{a^2}{4} + 1)x(u) \\
&= q(u)x(u) - [p(u)x'(u)]'.
\end{aligned} \tag{5.6}
$$

Define (recall (1.13))

$$
\begin{aligned}
\hat{K}^a_C &= K^a_{C-(\frac{a^2}{4}+1)} \\
&= \{x \in Y : \hat{F}^a(x) \leq C\}.
\end{aligned} \tag{5.7}
$$

Lemma 17 *For every $a \in \mathbb{R}$*
(i) $\hat{K}^a_C \neq \emptyset$ for C large enough
(ii) $\hat{K}^a_C$ is relatively compact in $L^2(\mathbb{R}^+)$ for all $C \in \mathbb{R}$
(iii) $x \to \hat{F}^a(x)$ is lower semicontinuous on X.

Proof. Standard.
(i) Trivial.
(ii) We check the conditions in Dunford and Schwartz (1964) Theorem IV.8.20.
(a) $\hat{K}^a_C$ is bounded in $L^2(\mathbb{R}^+)$.
(b) By Cauchy-Schwarz

$$
\begin{aligned}
\int_0^\infty (x(u+v) - x(u))^2 du &= \int_0^\infty (\int_u^{u+v} x'(t)dt)^2 \\
&\leq \int_0^\infty du\ [\log(u+v) - \log u] \int_u^{u+v} dt\ t[x'(t)]^2 \\
&= \int_0^\infty dt\ t[x'(t)]^2 I(t,v) 1_{\{t \geq v\}},
\end{aligned} \tag{5.8}
$$

where

$$
I(t,v) = (t+v)\log(1+\frac{v}{t}) + (t-v)\log(1-\frac{v}{t}). \tag{5.9}
$$

Since $t \to I(t,v)$ is decreasing and $I(v,v) = 2v\log 2$, it follows that

$$
\lim_{v \downarrow 0} \int_0^\infty (x(u+v) - x(u))^2 du = 0 \text{ uniformly for } x \in \hat{K}^a_C. \tag{5.10}
$$

(c) From $p(u) \geq 0$ and $\lim_{u\to\infty} q(u) = \infty$ follows

$$
\lim_{N\to\infty} \int_N^\infty x^2(u) du = 0 \text{ uniformly for } x \in \hat{K}^a_C. \tag{5.11}
$$

Conditions (a)-(c) imply that $\hat{K}^a_C$ is relatively compact.
(iii) Define

$$
V^a = \{x \in L^2(\mathbb{R}^+) : \hat{F}^a(x) < \infty\}. \tag{5.12}
$$

On V^a define the inner product

$$\langle x, y\rangle_{V^a} = \int_0^\infty \{q(u)x(u)y(u) + p(u)x'(u)y'(u)\}du. \tag{5.13}$$

Then $(V^a, \langle\cdot,\cdot\rangle_{V^a})$ is a Hilbert space, $\|x\|_{V^a} \geq \|x\|_{L^2}$ and

$$\hat{F}^a(x) = \langle x, x\rangle_{V^a} = \|x\|^2_{V^a}. \tag{5.14}$$

Thus we must prove that $\liminf_{n\to\infty} \|x_n\|_{V^a} \geq \|x\|_{V^a}$ for any $x_n \to^{L^2} x$.

Let $L = \liminf_{n\to\infty} \|x_n\|_{V^a}$. The case $L = \infty$ being trivial, assume $L < \infty$. Then, by the Banach-Alaoglu theorem (Rudin (1991) Theorem 3.15), there exists a subsequence (x_{n_k}) and a $y \in V^a$ such that $L = \lim_{k\to\infty} \|x_{n_k}\|_{V^a}$ and $x_{n_k} \to y$ weakly in V^a $(k \to \infty)$. Hence $L \geq \|y\|_{V^a}$ by Fatou. But, by (ii), weak convergence in V^a implies strong convergence in $L^2(\mathbb{R}^+)$. Hence $x_{n_k} \to^{L^2} y$. Together with $x_n \to^{L^2} x$ this implies $y = x$ and hence the claim follows.

Incidentally, note from (5.4-5) that V^a does not depend on a because it is nothing other than the collection of $x \in L^2(\mathbb{R}^+)$ for which $\int_0^\infty \{u^2[x(u)]^2 + u[x'(u)]^2\}du < \infty$ (recall (0.18)). □

Lemma 17 implies that $\hat{F}^a$ achieves a minimum in $\overline{\hat{K}^a_C}$ (for C large enough).

5.2 Characterization of the minimizer(s) of $\hat{F}^a$

Lemma 18 *Any minimizer $\bar{x}$ of $\hat{F}^a$ in X is a solution of $\hat{\mathcal{L}}^a x = \rho x$ for $\rho = \hat{\rho}(a) \in \mathbb{R}$, the minimal eigenvalue of $\hat{\mathcal{L}}^a$ in V^a.*

Proof. Standard.

Define $\hat{\rho}(a)$ by

$$\hat{\rho}(a) = \min_{x\in X} \hat{F}^a(x). \tag{5.15}$$

Let $\bar{x} \in V^a$ be any minimizer. Then for any $h \in L^2(\mathbb{R}^+)$ and $\epsilon > 0$

$$\hat{F}^a(\bar{x} + \epsilon h) \geq \hat{\rho}(a)\|\bar{x} + \epsilon h\|^2_{L^2}. \tag{5.16}$$

Writing out both sides of (5.16) and using that $\hat{F}^a(\bar{x}) = \hat{\rho}(a)$, we obtain (see (5.13-14))

$$2\epsilon\langle \bar{x}, h\rangle_{V^a} + \epsilon^2\|h\|^2_{V^a} \geq \hat{\rho}(a)\{2\epsilon\langle\bar{x}, h\rangle_{L^2} + \epsilon^2\|h\|^2_{L^2}\}. \tag{5.17}$$

Let $\epsilon \downarrow 0$ to obtain

$$\langle\bar{x}, h\rangle_{V^a} \geq \hat{\rho}(a)\langle\bar{x}, h\rangle_{L^2} \text{ for all } h \in V^a. \tag{5.18}$$

Replace h by $-h$ to get the reverse inequality. Thus

$$\langle\bar{x}, h\rangle_{V^a} = \hat{\rho}(a)\langle\bar{x}, h\rangle_{L^2} \text{ for all } h \in V^a. \tag{5.19}$$

Now note that we have from (5.6) and (5.13) after partial integration

$$\langle\bar{x}, h\rangle_{V^a} = \langle\bar{x}, \hat{\mathcal{L}}^a h\rangle_{L^2} \text{ for all } h \in C^2_c(\mathbb{R}^+) \tag{5.20}$$

It follows from (5.19-20) and the symmetry of $\hat{\mathcal{L}}^a$ that $\bar{x}$ is a weak solution of $\hat{\mathcal{L}}^a x = \hat{\rho}(a)x$. This in turn implies that $\bar{x}$ is a strong solution.

To see that $\hat{\rho}(a)$ is the minimal eigenvalue of $\hat{\mathcal{L}}^a$ in V^a, note that if $\hat{\mathcal{L}}^a x = \rho x$, then by (5.6), (5.13-14) and integration by parts

$$\hat{F}^a(x) = \langle x, x\rangle_{V^a} = \langle x, \hat{\mathcal{L}}^a x\rangle_{L^2} = \rho\|x\|_{L^2} = \rho. \tag{5.21}$$

□

5.3 Analysis of the Sturm-Liouville problem

Lemmas 17-18 show that F^a has a maximizer in $\overline{K_C^a}$ and that each maximizer is a solution of $\mathcal{L}^a x = \rho x$ for $\rho = \rho(a)$, the maximal eigenvalue of $\mathcal{L}^a$ in V^a (recall (5.4-7)).

Lemma 19 *(i) All solutions of $\mathcal{L}^a x = \rho x$ are of the form*

$$x^{a,\rho}(u) = f^{a,\rho}(u) + g^{a,\rho}(u) \log u, \tag{5.22}$$

where $f^{a,\rho}$ and $g^{a,\rho}$ are power series with infinite radius of convergence.
(ii) $F^a(x^{a,\rho}) = -\infty$ if $g^{a,\rho} \not\equiv 0$.

Proof. (i) Formally substitute $f^{a,\rho}(u) = \sum_{n\geq 0} f_n u^n$ and $g^{a,\rho}(u) = \sum_{n \geq 0} g_n u^n$. Then the coefficients are found to satisfy the recurrence relations

$$\begin{aligned} g_n &= \tfrac{1}{n^2}(\rho g_{n-1} - 2a g_{n-2} + 4 g_{n-3}) \ (n \geq 1) \\ f_n &= \tfrac{1}{n^2}(\rho f_{n-1} - 2a f_{n-2} + 4 f_{n-3} - 2n g_n) \ (n \geq 1) \end{aligned} \tag{5.23}$$

(with $f_{-1} = f_{-2} = g_{-1} = g_{-2} = 0$). Note that $g^{a,\rho}$ is a solution of (5.3) and that $f^{a,\rho}$ depends on $g^{a,\rho}$. By induction on n, (5.23) is easily shown to give the following bounds:

$$\begin{aligned} |f_n| &\leq K_1^n (n!)^{-\frac{2}{3}} \ (n \geq 1) \\ |g_n| &\leq K_2^n (n!)^{-\frac{2}{3}} \ (n \geq 1) \end{aligned} \tag{5.24}$$

with K_1, K_2 large enough (depending on ρ, a and f_0, g_0). This implies that the formal solution exists everywhere.
(ii) Trivial, since $\frac{d}{du} x^{a,\rho}(u) \sim g_0 u^{-1}$ $(u \downarrow 0)$ with $g_0 \neq 0$ implies $F^a(x^{a,\rho}) = -\infty$, while $g_0 = 0$ implies that $g_n \equiv 0$. □

At this stage we know from Lemma 19 that all maximizers of F^a are of the form $x^{a,\rho}(u) = f^{a,\rho}(u)$ and, in particular, are analytic on $\mathbb{R}_0^+$.

Our next step is to find the asymptotic behavior of the solutions of (5.3) as $u \to \infty$. This will be needed to get uniqueness of the maximizer.

Lemma 20 *$\mathcal{L}^a x = \rho x$ has two independent solutions $x_-^{a,\rho}$ and $x_+^{a,\rho}$ satisfying*

$$\lim_{u \to \infty} u^{-\frac{3}{2}} \log x_\pm^{a,\rho}(u) = \pm \frac{4}{3}. \tag{5.25}$$

Proof. We use Coddington and Levinson (1955) Theorem 2.1 page 143-144. Define

$$\begin{aligned} w_1(u) &= x(u^2) \\ w_2(u) &= u^{-2} w_1'(u). \end{aligned} \tag{5.26}$$

Then (5.3) can be written as

$$w'(u) = u^{-r} B(u) w(u), \tag{5.27}$$

where $r = 2$ and

$$\begin{aligned} w(u) &= \begin{pmatrix} w_1(u) \\ w_2(u) \end{pmatrix} \\ B(u) &= \begin{pmatrix} 0 & 1 \\ 16 - \frac{8a}{u^2} + \frac{4\rho}{u^4} & -\frac{3}{u^3} \end{pmatrix}. \end{aligned} \tag{5.28}$$

Note that $B(u) = \sum_{n \geq 0} u^{-n} B_n$ ($B_0 \neq 0$) is a convergent power series in u^{-1}, with B_0 having eigenvalues $\lambda_{1,2} = \pm 4$. Therefore (5.27) has a formal solution of the form

$$w(u) = P(u) u^R e^{Q(u)}, \tag{5.29}$$

where $P(u) = \sum_{n=0}^{\infty} u^{-n} P_n$ ($\det P_0 \neq 0$) is a formal power series in u^{-1}, R is a complex diagonal matrix and $Q = \frac{u^{r+1}}{r+1} Q_0 + \ldots + u Q_r$ is a matrix polynomial with Q_i diagonal and $Q_0 = diag\{\lambda_1, \lambda_2\}$. From the proof of the theorem it follows that P, Q, R can be chosen to be real because $B, \lambda_{1,2}$ are real. On p.151 of Coddington and Levinson (1955) there is the further remark that for every formal solution there exists an actual solution with the same asymptotics. □

We see from Lemma 20 that $x_+^{a,\rho} \notin L^2(\mathbb{R}^+)$ and so (5.3) has a unique solution in $L^2(\mathbb{R}^+)$ up to multiplicative constants.

Lemma 21 *Define*

$$\mathcal{S}_a = \{\rho \in \mathbb{R} : f^{a,\rho} \in L^2(\mathbb{R}^+), f^{a,\rho}(0) = 1\}. \tag{5.30}$$

Then
(i) $\mathcal{S}_a$ *is countable, bounded from above and has a maximum*
(ii) $\rho(a) = \max\ \mathcal{S}_a$ *is geometrically simple*
(iii) $f^{a,\rho(a)} > 0$
(iv) $\forall \rho \in \mathcal{S}_a, \rho < \max\ \mathcal{S}_a : f^{a,\rho}$ *changes sign in* $\mathbb{R}^+$.

Proof. Standard Sturm-Liouville theory.
(i),(ii) By Lemma 17(ii), V^a is compactly imbedded in $L^2(\mathbb{R}^+)$ (compare (5.7) and (5.12)). Therefore the eigenfunctions of $\mathcal{L}^a$ in V^a form an orthogonal basis of V^a. Since V^a is separable, this in turns implies that $\mathcal{S}_a$ is countable. We know from Lemmas 19-20 that $\mathcal{L}^a$ has a unique eigenvector in V^a with eigenvalue $\rho(a)$, i.e., $\rho(a)$ is geometrically simple. Since $\rho(a) = \max_{x \in V^a} F^a(x) = \max \mathcal{S}^a$ by Lemma 18, we also know that $\mathcal{S}_a$ is bounded from above and has a maximum.
(iii) From (1.7) one sees that $F^a(|f^{a,\rho(a)}|) = F^a(f^{a,\rho(a)})$. Therefore it follows from the uniqueness of the maximizer that $f^{a,\rho} = |f^{a,\rho}| \geq 0$. Let $u_0 = \inf\{u > 0 : f^{a,\rho(a)}(u) = 0\} > 0$. If $u_0 < \infty$, then we must have $\frac{d}{du} f^{a,\rho(a)}(u_0) = 0$ and $\frac{d^2}{du^2} f^{a,\rho(a)}(u_0) > 0$. However, this contradicts $(\mathcal{L}^a f^{a,\rho(a)})(u) = \rho(a) f^{a,\rho(a)}(u)$ at the point $u = u_0$ (see (0.17)).
(iv) This follows from (iii) and the fact that the eigenfunctions of $\mathcal{L}^a$ in V^a form an orthogonal basis. □

Lemmas 17-18 and 21 show that Assumptions 3(i), (ii) in proposition 2 hold.

5.4 Dependence on a

The maximal eigenvalue and eigenvector of (0.17-18) are

$$\begin{aligned} \rho(a) &= \max\ \mathcal{S}_a \\ x^a &= \frac{f^{a,\rho(a)}}{\|f^{a,\rho(a)}\|_{L^2}}. \end{aligned} \tag{5.31}$$

We can now prove the following properties:

Lemma 22 *(i) $a \to \rho(a)$ and $a \to x^a$ are analytic*
(ii) $a \to \rho(a)$ is strictly increasing and strictly convex on $\mathbb{R}$
(iii) $\rho(0) < 0$, $\lim_{a\uparrow\infty} \rho(a) = \infty$ and $\lim_{a\downarrow-\infty} \rho(a) = -\infty$.

Proof. (i) We give the proof by applying Crandall and Rabinowitz (1973) Lemma 1.3 in the following setting. Pick $a \in \mathbb{R}$ and consider the Hilbert space $(V, \langle\cdot,\cdot\rangle_V)$ with $V = V^0$. Then, from (5.5-6) and (5.13),

$$\begin{aligned} \langle x^a, y\rangle_{V^a} &= \langle \mathcal{L}^a x^a, y\rangle_{L^2} = \rho(a)\langle x^a, y\rangle_{L^2} \\ \langle x^a, y\rangle_{V^a} &= \langle x^a, y\rangle_V - 2ab(x^a, y) + \tfrac{a^2}{4}\langle x^a, y\rangle_{L^2}, \end{aligned} \tag{5.32}$$

where $b : V \times V \to \mathbb{R}$ is the bilinear form defined by

$$b(x, y) = \int_0^\infty u x(u) y(u) du. \tag{5.33}$$

For every $x \in V$ the functional $y \to b(x, y)$ is continuous and linear. Hence it follows from the Riesz representation theorem (Rudin (1987) Theorem 6.19) that there exists a unique linear operator $B : V \to V$ such that

$$b(x, y) = \langle Bx, y\rangle_V \text{ for all } x, y \in V. \tag{5.34}$$

B is symmetric because b is. B is bounded because

$$\begin{aligned} \|Bx\|_V^2 &= b(x, Bx) \\ &\leq \left(\textstyle\int_0^\infty u^2 x^2(u) du\right)^{\frac{1}{2}} \|Bx\|_{L^2} \\ &\leq \tfrac{1}{2}\|x\|_V \|Bx\|_{L^2} \\ &\leq \tfrac{1}{2}\|x\|_V \|Bx\|_V \end{aligned} \tag{5.35}$$

(see (5.5) and (5.13)), so that $\|Bx\|_V \leq \frac{1}{2}\|x\|_V$. To see that B is compact, let (x_n) be a bounded sequence in V. Then, by Lemma 17(ii), there exists a subsequence (x_{n_k}) and an $x \in V$ such that $x_{n_k} \to^{L^2} x$ $(k \to \infty)$. Hence, as in (5.35),

$$\begin{aligned} \|Bx_{n_k} - Bx\|_V^2 &= b(x_{n_k} - x, B(x_{n_k} - x)) \\ &\leq \|x_{n_k} - x\|_{L^2} \tfrac{1}{2}\|B(x_{n_k} - x)\|_V \\ &\leq \|x_{n_k} - x\|_{L^2} \tfrac{1}{4}\|x_{n_k} - x\|_V \\ &\to 0 \ (k \to \infty). \end{aligned} \tag{5.36}$$

In the same manner we can prove that there exists a unique linear, symmetric and compact operator $C : V \to V$ such that

$$\langle x, y\rangle_{L^2} = \langle Cx, y\rangle_V \text{ for all } x, y \in V. \tag{5.37}$$

Now rewrite (5.32) as follows

$$\langle \big[Id - 2aB - \big(\rho(a) - \frac{a^2}{4}\big)C\big]x^a, y\rangle_V = 0 \text{ for all } y \in V. \tag{5.38}$$

Hence, $(V, \langle\cdot,\cdot\rangle_V)$ being a Hilbert space, we have

$$\begin{array}{l} x^a \text{ is a } C\text{-eigenfunction of } Id - 2aB \\ \text{with (largest) eigenvalue } \rho(a) - \frac{a^2}{4}. \end{array} \tag{5.39}$$

Next note that $a \to Id - 2aB$ is analytic in the operator norm. Therefore, to get the claim from Crandall and Rabinowitz (1973) Lemma 1.3, it suffices to check that $\rho(a) - \frac{a^2}{4}$ is a C-simple eigenvalue of $Id - 2aB$, i.e.,
(a) $dim(N(A^a)) = codim(R(A^a)) = 1$
(b) $Cx^a \notin R(A^a)$,
where $A^a = Id - 2aB - (\rho(a) - \frac{a^2}{4})C$ and $N(A^a), R(A^a)$ denote the null space resp. range of A^a.

We have $dim(N(A^a)) = 1$ because of Lemma 21(ii). Moreover, because $2aB + (\rho(a) - \frac{a^2}{4})C$ is compact we have $dim(N(A^a)) = codim(R(A^a))$ (Rudin (1991) Theorem 4.25). This proves (a). To prove (b), first use that A^a is symmetric and bounded to get that $N(A^a) = R(A^a)^\perp$ (the orthogonal complement of $R(A^a)$) and $R(A^a) = \overline{R(A^a)}$ (Rudin (1991) Theorems 4.12 and 4.23). Since $\overline{R(A^a)} = R(A^a)^{\perp\perp}$, it follows that $N(A^a)^\perp = R(A^a)$. Hence (b) is equivalent to $\langle Cx^a, x^a\rangle_V \neq 0$. But $\langle Cx^a, x^a\rangle_V = \langle x^a, x^a\rangle_{L^2} = 1$ by (5.37).
(ii) Because

$$\rho(a) = \sup_{x \in X} F^a(x) \tag{5.40}$$

with unique maximizer $x = f^{a,\rho(a)}$, we immediately see from (1.7) that

$$\frac{\rho(a+\epsilon) - \rho(a)}{\epsilon} \geq \int_0^\infty 2u[f^{a,\rho(a)}(u)]^2 du > 0 \tag{5.41}$$

(pick $\|f^{a,\rho(a)}\|_{L^2} = 1$). This demonstrates that $\rho'(a)$ is everywhere strictly positive. Moreover, since $a \to F^a(x)$ is linear for every x we have from (5.40) that $a \to \rho(a)$ is convex. Because of analyticity, it follows that either $a \to \rho(a)$ is strictly convex or $\rho(a) = C_1 a + C_2$. However, the latter is impossible because of Lemma 13.
(iii) Trivial. Let $\epsilon \to \pm\infty$ in (5.41) or else see (1.7). □

6 Proof of Theorems 4-7

We can now collect the results from sections 2-5 and give the proofs of our theorems in section 0.4.

Proof of Theorem 5. Combine Propositions 1-3 with (1.13). The proof of Proposition 3 was given in Lemma 1 and in sections 2, 3 and 5. □

Proof of Theorems 4 and 6.

1. $r^*(\beta) \sim a^*\beta^{\frac{2}{3}}$.

According to (0.13), $r^*(\beta)$ is defined as the unique solution of

$$\lambda(r,\beta) = 1. \tag{6.1}$$

From (0.20) we know that for every $a \in \mathbb{R}$

$$\beta^{-\frac{1}{3}}\Big[\lambda(a\beta^{\frac{2}{3}},\beta) - 1\Big] \to \rho(a). \tag{6.2}$$

Let $a^* > 0$ be the solution of $\rho(a) = 0$ (see Lemma 22). Now, because $r \to \lambda(r,\beta)$ is increasing (as is obvious from (0.10)), we have for every $\epsilon > 0$

$$\begin{array}{lcl} \lambda(r,\beta) & \geq & 1 + \beta^{\frac{1}{3}}\rho(a^*+\epsilon) + o(\beta^{\frac{1}{3}}) \text{ for } r \geq (a^*+\epsilon)\beta^{\frac{2}{3}} \\ \lambda(r,\beta) & \leq & 1 + \beta^{\frac{1}{3}}\rho(a^*-\epsilon) + o(\beta^{\frac{1}{3}}) \text{ for } r \leq (a^*-\epsilon)\beta^{\frac{2}{3}}. \end{array} \tag{6.3}$$

Since $\rho(a^*-\epsilon) < 0 < \rho(a^*+\epsilon)$ for every $\epsilon > 0$ (see Lemma 22(ii)), (6.1) combined with (6.3) implies

$$(a^*-\epsilon)\beta^{\frac{2}{3}} \leq r^*(\beta) \leq (a^*+\epsilon)\beta^{\frac{2}{3}} \text{ for } \beta \text{ small enough.} \tag{6.4}$$

Let $\epsilon \downarrow 0$ to get the claim.

2. $\theta^*(\beta) \sim b^*\beta^{\frac{1}{3}}$.

According to (0.14), $\theta^*(\beta)$ is defined as

$$\frac{1}{\theta^*(\beta)} = \Big[\frac{\partial}{\partial r}\lambda(r,\beta)\Big]_{r=r^*(\beta)}. \tag{6.5}$$

Define

$$\xi(r,\beta) = \frac{\frac{\partial}{\partial r}\lambda(r,\beta)}{\lambda(r,\beta)} = \frac{\partial}{\partial r}\log\lambda(r,\beta). \tag{6.6}$$

Because $r \to \lambda(r,\beta)$ is increasing and log-convex (see footnote 6), we have that for all $h,\beta > 0$ and $a \in \mathbb{R}$

$$\begin{array}{lcl} \xi(a\beta^{\frac{2}{3}},\beta) & \leq & \frac{1}{h\beta^{\frac{2}{3}}}\Big[\log\lambda((a+h)\beta^{\frac{2}{3}},\beta) - \log\lambda(a\beta^{\frac{2}{3}},\beta)\Big] \\ \xi(a\beta^{\frac{2}{3}},\beta) & \geq & \frac{1}{h\beta^{\frac{2}{3}}}\Big[\log\lambda(a\beta^{\frac{2}{3}},\beta) - \log\lambda((a-h)\beta^{\frac{2}{3}},\beta)\Big]. \end{array} \tag{6.7}$$

Together with (6.2) this gives

$$\begin{array}{lcl} \limsup_{\beta\downarrow 0}\beta^{\frac{1}{3}}\xi(a\beta^{\frac{2}{3}},\beta) & \leq & \dfrac{\rho(a+h)-\rho(a)}{h} \\ \liminf_{\beta\downarrow 0}\beta^{\frac{1}{3}}\xi(a\beta^{\frac{2}{3}},\beta) & \geq & \dfrac{\rho(a)-\rho(a-h)}{h}. \end{array} \tag{6.8}$$

Let $h \downarrow 0$ to get (use Lemma 22)

$$\lim_{\beta\downarrow 0}\beta^{\frac{1}{3}}\xi(a\beta^{\frac{2}{3}},\beta) = \rho'(a). \tag{6.9}$$

Next, because $r \to \xi(r,\beta)$ is increasing we have, via (6.4), for β small enough

$$\begin{array}{lcl} \xi(r^*(\beta),\beta) & \leq & \xi((a^*+\epsilon)\beta^{\frac{2}{3}},\beta) = \beta^{-\frac{1}{3}}\rho'(a^*+\epsilon) + o(\beta^{-\frac{1}{3}}) \\ \xi(r^*(\beta),\beta) & \geq & \xi((a^*-\epsilon)\beta^{\frac{2}{3}},\beta) = \beta^{-\frac{1}{3}}\rho'(a^*-\epsilon) + o(\beta^{-\frac{1}{3}}). \end{array} \tag{6.10}$$

Since (recall that $\lambda(r^*(\beta),\beta)=1$)

$$\frac{1}{\theta^*(\beta)} = \xi(r^*(\beta),\beta), \tag{6.11}$$

it follows that

$$\rho'(a^*-\epsilon) \le \frac{1}{\beta^{-\frac{1}{3}}\theta^*(\beta)} \le \rho'(a^*+\epsilon) \text{ for } \beta \text{ small enough.} \tag{6.12}$$

Let $\epsilon \downarrow 0$ to get the claim with $\frac{1}{b^*} = \rho'(a)$.

3. $\beta^{-\frac{1}{6}}\tau_{r^*(\beta),\beta}(\lceil \cdot \beta^{-\frac{1}{3}} \rceil) \to^{L^2} x^{a^*}(\cdot)$.

Put $a^*(\beta) = \beta^{-\frac{2}{3}} r^*(\beta)$. Then, similarly as in Lemma 1,

$$\beta^{-\frac{1}{6}}\tau_{r^*(\beta),\beta}(\lceil \cdot \beta^{-\frac{1}{3}} \rceil) \text{ is the unique maximizer of } F_\beta^{a^*(\beta)}, \tag{6.13}$$

where the parameter a is replaced by $a^*(\beta)$.

Lemma 23 *Assumptions (1)-(3) in Proposition 2 hold for the following choice replacing (1.13):*

$$\begin{aligned} K &= K_C^{a^*} \ (C \textit{ sufficiently large}) \\ G_\beta &= F_\beta^{a^*(\beta)} \\ G &= F^{a^*}. \end{aligned} \tag{6.14}$$

Proof. The point is that $\lim_{\beta\downarrow 0} a^*(\beta) = a^*$. It is trivial to check that all estimates in sections 2 and 3 remain valid when the fixed parameter a is replaced by $a + o(1)$ $(\beta \downarrow 0)$. See, in particular, the proofs of Lemmas 5, 6, 11-13. □

The claim in **3** now follows from Proposition 2.

4. $\beta^{-\frac{1}{3}}\mu^*_\beta(\lceil \cdot \beta^{-\frac{1}{3}} \rceil) \to^{L^1} \frac{1}{2}[x^{a^*}(\frac{1}{2}\cdot)]^2$.

The proof is in Steps 1-2 below.

Abbreviate $A_\beta = A_{r^*(\beta),\beta}$ and $\tau_\beta = \tau_{r^*(\beta),\beta}$. According to (0.14)

$$\mu^*_\beta(k) = \sum_{\substack{i,j \\ i+j-1=k}} \tau_\beta(i) A_\beta(i,j)\tau_\beta(j). \tag{6.15}$$

STEP 1 *There exists c such that*

$$\int_N^\infty |\beta^{-\frac{1}{3}}\mu^*_\beta(\lceil u\beta^{-\frac{1}{3}} \rceil) - \frac{1}{2}[x^{a^*}(\frac{1}{2}u)]^2| du \le cN^{-2} \textit{ for } \beta \textit{ small enough.} \tag{6.16}$$

Proof. Estimate (recall that $\lambda(r^*(\beta),\beta)=1$)

$$\begin{aligned}\textstyle\int_N^\infty \beta^{-\frac13}\mu_\beta^*(\lceil u\beta^{-\frac13}\rceil)du &= \textstyle\sum_{k\geq N\beta^{-\frac13}} \mu_\beta^*(k)\\ &= \textstyle\sum_{i,j:\ i+j-1\geq N\beta^{-\frac13}} \tau_\beta(i)A_\beta(i,j)\tau_\beta(j)\\ &\leq 2\textstyle\sum_{i\geq\frac12 N\beta^{-\frac13}} \tau_\beta^2(i) \qquad (6.17)\\ &\leq 8N^{-2}\beta^{\frac23}\textstyle\sum_{i\geq 1} i^2\tau_\beta^2(i)\\ &\leq 8C_1N^{-2}.\end{aligned}$$

The last inequality is Lemma 11(i). Furthermore,

$$\int_N^\infty \frac12[x^{a^*}(\frac12 u)]^2du \leq 4N^{-2}\int_N^\infty u^2[x^{a^*}(\frac12 u)]^2du. \qquad (6.18)$$

Since $x^{a^*}\in K_C^{a^*}$, the integral in the r.h.s. is finite and so the claim follows. □

STEP 2 $\lim_{\beta\downarrow 0}\int_0^N|\beta^{-\frac13}\mu_\beta^*(\lceil u\beta^{-\frac13}\rceil)-\frac12[x^{a^*}(\frac12 u)]^2|du=0$ *for every fixed* N.

Proof. Use the triangle inequality to split the integal into three parts:

$$\int_0^N|\beta^{-\frac13}\mu_\beta^*(\lceil u\beta^{-\frac13}\rceil)-\frac12[x^{a^*}(\frac12 u)]^2|du\leq I_\beta^{1,N}+I_\beta^{2,N}+I_\beta^{3,N} \qquad (6.19)$$

with

$$\begin{aligned}I_\beta^{1,N} &= \beta^{\frac13}\textstyle\sum_{i,j:\ i+j-1\leq N\beta^{-\frac13}}|\bar\tau_\beta(i\beta^{\frac13})-x^{a^*}(i\beta^{\frac13})|A_\beta(i,j)x^{a^*}(j\beta^{\frac13})\\ I_\beta^{2,N} &= \beta^{\frac13}\textstyle\sum_{i,j:\ i+j-1\leq N\beta^{-\frac13}}\bar\tau_\beta(i\beta^{\frac13})A_\beta(i,j)|\bar\tau_\beta(j\beta^{\frac13})-x^{a^*}(j\beta^{\frac13})| \qquad (6.20)\\ I_\beta^{3,N} &= \textstyle\int_0^N|\sum_{i,j:\ i+j-1=\lceil u\beta^{-\frac13}\rceil}x^{a^*}(i\beta^{\frac13})A_\beta(i,j)x^{a^*}(j\beta^{\frac13})-\frac12[x^{a^*}(\frac12 u)]^2|du.\end{aligned}$$

Here $\bar\tau_\beta$ is the scaled form of τ_β given by the same relation as (3.1).

For $I_\beta^{1,N}$ use Cauchy-Schwarz and (2.5)(ii) to estimate

$$\begin{aligned}I_\beta^{1,N} &\leq \beta^{\frac13}\Big(\textstyle\sum_{i,j:\ i+j-1\leq N\beta^{-\frac13}}\big[\bar\tau_\beta(i\beta^{\frac13})-x^{a^*}(i\beta^{\frac13})\big]^2A_\beta(i,j)\Big)^{\frac12}\\ &\quad\times\Big(\textstyle\sum_{i,j:\ i+j-1\leq N\beta^{-\frac13}}[x^{a^*}(j\beta^{\frac13})]^2A_\beta(i,j)\Big)^{\frac12}\\ &\leq e^{\frac14 a^2\beta^{\frac13}}\Big(\beta^{\frac13}\textstyle\sum_{i\leq N\beta^{-\frac13}}\big[\bar\tau_\beta(i\beta^{\frac13})-x^{a^*}(i\beta^{\frac13})\big]^2\Big)^{\frac12} \qquad (6.21)\\ &\quad\times\Big(\beta^{\frac13}\textstyle\sum_{j\leq N\beta^{-\frac13}}[x^{a^*}(j\beta^{\frac13})]^2\Big)^{\frac12}.\end{aligned}$$

Define $\bar x_\beta^{a^*}$ by $\bar x_\beta^{a^*}(u)=x^{a^*}(i\beta^{\frac13})$ for $(i-1)\beta^{\frac13}\leq u\leq i\beta^{\frac13}(i\geq 1)$, in analogy with (3.1). Then (6.21) becomes

$$I_\beta^{1,N}\leq e^{\frac14 a^2\beta^{\frac13}}\|\bar\tau_\beta-\bar x_\beta^{a^*}\|_{L^2[0,N]}\ \|\bar{x_\beta}^{a^*}\|_{L^2[0,N]}. \qquad (6.22)$$

Now let $\beta \downarrow 0$ and use that $\bar{\tau}_\beta \to^{L^2} x^{a^*}$ and $\bar{x_\beta}^{a^*} \to^{L^2} x^{a^*}$ to get $\limsup_{\beta\downarrow 0} I_\beta^{1,N} = 0$. The same argument gives that $\limsup_{\beta\downarrow 0} I_\beta^{2,N} = 0$.

To estimate $I_\beta^{3,N}$, we use the mean value theorem to expand $x^{a^*}(i\beta^{\frac{1}{3}})$ and $x^{a^*}(j\beta^{\frac{1}{3}})$ around $\frac{1}{2}u$. Namely

$$\begin{aligned} I_\beta^{3,N} &= \int_0^N \Big| \sum_{i,j:\ i+j-1=\lceil u\beta^{-\frac{1}{3}}\rceil} \Big\{x^{a^*}(\tfrac{1}{2}u) + (i\beta^{\frac{1}{3}} - \tfrac{1}{2}u)\tfrac{d}{du}x^{a^*}(\xi)\Big\} \\ &\quad \times A_\beta(i,j)\Big\{x^{a^*}(\tfrac{1}{2}u) + (j\beta^{\frac{1}{3}} - \tfrac{1}{2}u)\tfrac{d}{du}x^{a^*}(\eta)\Big\} - \tfrac{1}{2}[x^{a^*}(\tfrac{1}{2}u)]^2\Big| du \end{aligned} \tag{6.23}$$

with ξ, η between $\frac{1}{2}u$ and $i\beta^{\frac{1}{3}}$ resp. $j\beta^{\frac{1}{3}}$. Next note that $x^{a^*}(u), |\frac{d}{du}x^{a^*}(u)| \le M < \infty$ for all $u \in \mathbb{R}^+$. Hence

$$\begin{aligned} I_\beta^{3,N} &\le M^2 \int_0^N \Big| \sum_{i,j:\ i+j-1=\lceil u\beta^{-\frac{1}{3}}\rceil} A_\beta(i,j) - \tfrac{1}{2}\Big| du \\ &\quad + 2M^2 \int_0^N \sum_{i,j:\ i+j-1=\lceil u\beta^{-\frac{1}{3}}\rceil} \Big\{|i\beta^{\frac{1}{3}} - \tfrac{1}{2}u| + |i\beta^{\frac{1}{3}} - \tfrac{1}{2}u|^2\Big\} A_\beta(i,j). \end{aligned} \tag{6.24}$$

Next we insert $A_\beta = e^{e_\beta}P$ and use that (recall (2.4) and (0.11))

$$|e_\beta(i,j)| \le (|a| + N)N\beta^{\frac{1}{3}} \text{ for } i,j \le N\beta^{-\frac{1}{3}} \tag{6.25}$$

$$\sum_{\substack{i,j \\ i+j-1=k}} P(i,j) = \frac{1}{2}\ (k \ge 1) \tag{6.26}$$

$$\sum_{\substack{i,j \\ i+j-1=k}} (i - \frac{1}{2}k)^2 P(i,j) = \frac{1}{2}(\frac{1}{4}k + 1)\ (k \ge 1) \tag{6.27}$$

Then (6.24) yields

$$I_\beta^{3,N} \le M^2 N \tfrac{1}{2}(e^{(|a|+N)N\beta^{\frac{1}{3}}} - 1) + 2M^2 e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}} \int_0^N du\ \Big\{(z_\beta(u))^{\frac{1}{2}} + z_\beta(u)\Big\} \tag{6.28}$$

where

$$z_\beta(u) = \beta^{\frac{2}{3}} \sum_{i,j:\ i+j-1=\lceil u\beta^{-\frac{1}{3}}\rceil} (i - \frac{1}{2}u\beta^{-\frac{1}{3}})^2 P(i,j) = \mathcal{O}(\beta^{\frac{1}{3}}). \tag{6.29}$$

Let $\beta \downarrow 0$ to get $\limsup_{\beta\downarrow 0} I_\beta^{3,N} = 0$. □

Steps 1-2 prove the claim in **4**.

Results **1**-**4** complete the proof of Theorems 4 and 6. □

Proof of Theorem 7. The asymptotic behavior of x^{a^*} in (iii) was proved in Lemma 20 (pick $a = a^*$ and $\rho = 0$). To prove (i) and (ii), we recall that x^{a^*} solves (see (0.17))

$$0 = (\mathcal{L}^{a^*}x)(u) = (2a^*u - 4u^2)x(u) + [ux']'(u) \tag{6.30}$$

and has a power series representation (see (5.23))

$$\begin{aligned} x^{a^*}(u) &= \sum_{n\ge 0} x_n u^n \\ x_n &= \tfrac{1}{n^2}(-2a^*x_{n-2} + 4x_{n-3})\ (n \ge 1) \\ x_{-1} &= x_{-2} = 0. \end{aligned} \tag{6.31}$$

We observe that $u \to 2a^*u - 4u^2$ changes sign from positive to negative at $u = \frac{1}{2}a^*$. Since $x^{a^*}(u) > 0$ for all $u \geq 0$, it follows from (6.30) that $u \to u\frac{d}{du}x^{a^*}(u)$ is unimodal with a minimum at $u = \frac{1}{2}a^*$. It is clear that $u\frac{d}{du}x^{a^*}(u) \to 0$ as $u \downarrow 0$. By the unimodality we must have that $u\frac{d}{du}x^{a^*}(u) \to c$ as $u \to \infty$. However, c must be 0 otherwise $\int_0^\infty u[\frac{d}{du}x^{a^*}(u)]^2 du = \infty$, which is impossible since $F^{a^*}(x^{a^*}) = \rho(a^*) = 0 > -\infty$ (see (1.7)). Thus we conclude that $u\frac{d}{du}x^{a^*}(u) < 0$ for all $u > 0$, which implies that $u \to x^{a^*}(u)$ is strictly decreasing.

To prove (iv), use (0.15) to write

$$\begin{aligned} \frac{\beta^{\frac{1}{3}}}{\theta^*(\beta)} &= \beta^{\frac{1}{3}} \textstyle\sum_i (2i-1)\tau^2_{r^*(\beta),\beta}(i) \\ &= \textstyle\int_0^\infty 2u\{\beta^{-\frac{1}{6}}\tau_{r^*(\beta),\beta}(\lceil u\beta^{-\frac{1}{3}}\rceil)\}^2 du. \end{aligned} \tag{6.32}$$

As $\beta \downarrow 0$ the l.h.s. tends to $\frac{1}{b^*}$. Thus we must show that the r.h.s. tends to $\int_0^\infty 2u[x^{a^*}(u)]^2 du$. To prove this claim, first note that

$$\begin{aligned} \textstyle\int_N^\infty 2u\{\beta^{-\frac{1}{6}}\tau_{r^*(\beta),\beta}(\lceil u\beta^{-\frac{1}{3}}\rceil)\}^2 du &= \beta^{\frac{1}{3}} \textstyle\sum_{i \geq N\beta^{-\frac{1}{3}}} (2i-1)\tau^2_{r^*(\beta),\beta}(i) \\ &\leq \tfrac{2}{N}\beta^{\frac{2}{3}} \textstyle\sum_{i \geq N\beta^{-\frac{1}{3}}} i^2\tau^2_{r^*(\beta),\beta}(i) \\ &\leq \tfrac{2}{N}C_1 \text{ for } \beta \text{ sufficiently small} \end{aligned} \tag{6.33}$$

where we use Lemma 11(i). Similarly $\int_N^\infty 2u[x^{a^*}(u)]^2 du \leq \frac{2}{N}\int_N^\infty u^2[x^{a^*}(u)]^2 du = o(N^{-1})$ as $N \to \infty$. Next, recall **3** in the proof of Theorems 4 and 6 to see that

$$\lim_{\beta \downarrow 0} \int_0^N 2u\{\beta^{-\frac{1}{6}}\tau_{r^*(\beta),\beta}(\lceil u\beta^{-\frac{1}{3}}\rceil)\}^2 du = \int_0^N 2u[x^{a^*}(u)]^2 du \; for \; all \; N. \tag{6.34}$$

Let $N \to \infty$ to get the claim. □

Acknowledgments

We thank Ph. Clément for help with the functional analysis in this paper and J.-B. Baillon, E. Balder, J. Duistermaat and R. Gill for useful remarks during various stages of this work. We thank D. Brydges, G Slade and J. Westwater for their contribution to section 0.6 clarifying the connection between the Domb-Joyce model and the Edwards model.

References

[1] H. Attouch, *Variational Convergence for Functions and Operators*, Pitman, Boston 1984.

[2] D.C. Brydges and T. Spencer, Self-avoiding walk in 5 or more dimensions, Commun. Math. Phys. 97, 125-148, 1985.

[3] D.C. Brydges and G. Slade, The diffusive phase of a model of self interacting walks. Preprint 1994.

[4] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York 1955.

[5] M.G. Crandall and P.H. Rabinowitz, *Bifurcations, perturbations of simple eigenvalues, and linearized stability*, Arch. Rat. Mech. Anal. 52, 162-180, 1973.

[6] N. Dunford and J.T. Schwartz, *Linear Operators. Part I: General Theory*, 2^{nd} printing, Interscience Publishers, New York 1964.

[7] A. Greven and F. den Hollander, A variational characterization of the speed of a one-dimensional self-repellent random walk, Ann. Appl. Probab. 3, 1067-1099, 1993.

[8] T. Hara and G. Slade, Self-avoiding walk in five or more dimensions. I. The critical behaviour, Commun. Math. Phys. 147, 101-136, 1992.

[9] T. Hara and G. Slade, The lace expansion for self-avoiding walk in five or more dimensions, Reviews in Math. Phys. 4, 235-327, 1992.

[10] N. Madras and G. Slade, *The Self-Avoiding Walk*, Birkhäuser, Boston 1993.

[11] P. Révész, *Random Walks in Random and Non-Random Environments*, World Scientific, Singapore 1990.

[12] W. Rudin, *Functional Analysis*, 2^{nd} edition, McGraw-Hill, New York 1991.

[13] W. Rudin, *Real and Complex Analysis*, 3^{rd} edition, McGraw-Hill, New York 1987.

[14] J. Westwater, *On Edwards model for polymer chains*, in *Trends and Developments in the Eighties* (S. Albeverio and P. Blanchard, eds.), Bielefeld Encounters in Math. Phys. 4/5. World Scientific, Singapore 1984.