Attraction properties of the Ginzburg-Landau manifold.

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Abstract

We consider solutions of weakly unstable PDE on an unbounded spatial domain. It has been shown earlier by the first author [J. Nonlinear Sci. Vol. 3] that the set of modulated solutions (called "Ginzburg-Landau manifold") is attracting. We seek to understand "how big" is the domain of attraction. Starting with general initial conditions of order ε^{ν} for the Fourier-transformed version of the given PDE we find that on the time-scale $\frac{T}{\varepsilon^{\eta}}$; $\eta \leq 2$ (that is long in the terms of the original "physical" time t, but shorter than the natural time for the Ginzburg-Landau) the corresponding solutions evolve to the scaling of the clustered modes-distribution peaked at the integer multiples of the critical wave number, with the amplitudes sensitively dependent on ν such that for ν arbitrary close to zero after the time $\frac{T}{\varepsilon^{\eta}}$; $\eta \leq 2$ solutions get on the Ginzburg-Landau manifold.

1 Introduction

In many physical situations (such as the Taylor-Couette problem of flow between concentric rotating cylinders, the Bénard experiment on a layer of fluid heated from below and the Poiseuille flow between parallel walls driven by a pressure gradient) [2], [17], [18] one observes that by changing a control parameter R (Reynolds-number, Taylor's-number, Rayleigh's-number) a basic state looses stability ($R > R_{\mathrm{cr}}$) and get some periodic structure. The famous Ginzburg-Landau (or amplitude, envelope) equation describes the evolution of patterns in these kind of situations through instabilities and bifurcations [6]. The equation is obtained as a result of formal approximating procedure. In mathematical sense the equation is a "universal" approximate equation for large classes of non-linear PDE's of evolution type (see for example [5]). The equation looks as follows:

$$\frac{\partial A_1}{\partial \tau} = (\alpha + \beta \mid A_1 \mid^2) A_1 - \gamma \, \frac{\partial^2 A_1}{\partial \xi^2} \tag{1.1}$$

with $A(\xi, \tau) : \; R \times R_+ \to C$, α is real and β, γ are (in general) complex. All coefficients can be computed explicitly in any particular problem under consideration.

It will be of importance for our considerations to note that the space-like variable ξ and the time-like variable τ are *slow* variables (as compared to the "physical" variables of the original problem). In particular

$$\tau = \varepsilon^2 \, t, \qquad \xi = \varepsilon x \tag{1.2}$$

where ε is a small parameter ($R - R_{\rm cr} = \alpha \varepsilon^2$) and t is the original time variable.

We study solutions $\Psi(x, t)$ of the class of nonlinear evolution PDE's given by

$$\frac{\partial \Psi}{\partial t} = L\Psi + N(\Psi), \tag{1.3}$$

with $x \in (-\infty, \infty)$, $t \geq 0$. L is a real linear differential operator in x, with constant coefficients containing some control parameter R. $N(\Psi)$ are quadratic nonlinear terms. They are of the structure

$$N(\Psi) = 2\pi P(\Psi^2) \tag{1.4}$$

where P is again a linear differential operator in x, with constant coefficients. This choice of the nonlinear terms avoid some non-essential complications. A generalization of $N(\Psi)$ is given in our section 7.

Next we introduce the symbols $\mu(k; R)$, $\rho(k; R)$ of the operators L and N, through the formulas

$$L \cdot e^{-ikx} = e^{-ikx} \, \mu(k; R) \qquad P \cdot e^{-ikx} = e^{-ikx} \, \rho(k; R). \tag{1.5}$$

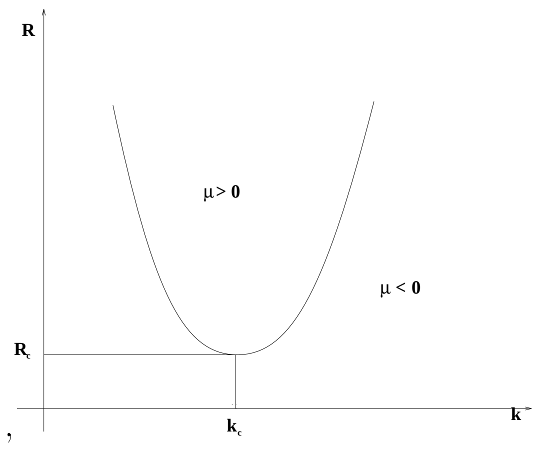
In order to make the analysis transparent we consider the case that μ and ρ are real. However, we emphasize that this is not a restriction for the results. Extension to the complex case is an easy exercise.

L is assumed to be of higher order than P, so that $\rho(k;R)/\mu(k;R)$ tends to zero for $\mid k \mid\rightarrow\infty$. L and P are further arbitrary. A neutral stability curve $\mu(k,R)=0$ is sketched in fig. 1. But only the local behavior near the critical wave length $k=k_c$ is important, where the neutral stability curve is assumed to be parabola-like.

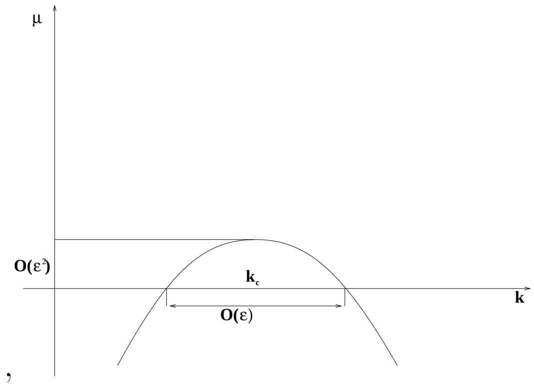
Outside the neutral stability curve liner problem shows stability and instability - inside. We will consider the slightly supercritical situation.

$$R>R_c;\quad R-R_c=\alpha\,\varepsilon^2 \tag{1.6}$$

with ε a small parameter. For simplicity of notation we suppress further the explicit dependence of μ and ρ on R. The basic requirement is that for $R=R_c+\alpha\,\varepsilon^2$ the function $\mu(k)$ has the graph as given in fig. 2.



(a) Neutral stability curve.



(b) For values of R $\mathcal{O}(\varepsilon^2)$ above the critical value R_c ,an $\mathcal{O}(\varepsilon)$-band of wave numbers becomes unstable.

The effect of nonlinearity is analogous to bifurcation, where a single isolated mode becomes unstable. However in our case a continuous band of modes become unstable. In [7], [6], [5], [12] it is shown that this kind of effects can be described by the amplitude equation (1.1). Equation (1.1) is a result of a formal substitution of

$$\Psi = \Psi^b + \varepsilon A_1 e^{ik_c x} + \varepsilon^2 A_0 + \varepsilon^2 A_2 e^{2ik_c x} + c.c. + ... \qquad (1.7)$$

in to the original problem. Where A_j is the function of the slow variables: $A_j(\varepsilon x, \varepsilon^2 t)$ and Ψ^b is a basic solution of the problem (we can take $\Psi^b = 0$) [1].

Recently a lot of work was done to prove the validity of the approximation. For certain specific problems of fluid dynamics a theory was developed by Collet and Eckman [3], Ioose, Mielke and Demay [11], Ioose and Mielke [10] and Schneider [13], [15], [14]. Another approach was introduced by van Harten [9] and followed by Bollerman [1]. Instead of working with the original PDE as given in (1.3) it will be more convenient to study its Fourier transformed version:

$$\frac{\partial \psi}{\partial t} = \mu(k)\ \psi + \rho(k)\ \psi * \psi \qquad (1.8)$$

where $\psi(k,t)$ is the Fourier transform of $\Psi(x,t)$ and "$\star$" denotes the convolution $\psi * \psi := \int_{-\infty}^{\infty}\ \psi(k',t)\ \psi(k-k',t)\ dk'$. The initial value problem for (1.3) is thus transformed into

$$\psi(k,t) = e^{\mu(k)t}\ \left[\psi^0(k) + \rho(k)\ \int_0^t\ e^{-\mu(k)t'}\ \psi * \psi\ dt'\right] \qquad (1.9)$$

where $\psi^0(k)$ is the Fourier-transform of the initial conditions $\Psi(x,0)$. The equation (1.9) will be the main object of our analysis.

Let us introduce a scaling of the Fourier-components

$$\psi = \delta_k(\varepsilon)\ \tilde{\psi}\ ,\ \tilde{\psi} = \mathcal{O}(1) \qquad (1.10)$$

with $\delta_k(\varepsilon)$ sketched in figure 1 and called by [9] a "clustered mode-distribution": the Fourier-components are of the order $\varepsilon^{|n-1|}$ in (L_∞ norm) in intervals $|\ k - n\ k_c\ | = \mathcal{O}(\varepsilon)$ and tail off very rapidly to very small orders of magnitude outside these intervals. This *clustered mode-distribution* was first introduced in [4]. The distribution is invariant under convolution.

[1] Let us note that all A_j for $j \neq \pm 1$ are slaved to the critical modes (in other words, they can be expressed in terms of $A_{\pm 1}$ through algebraic convolution equations).

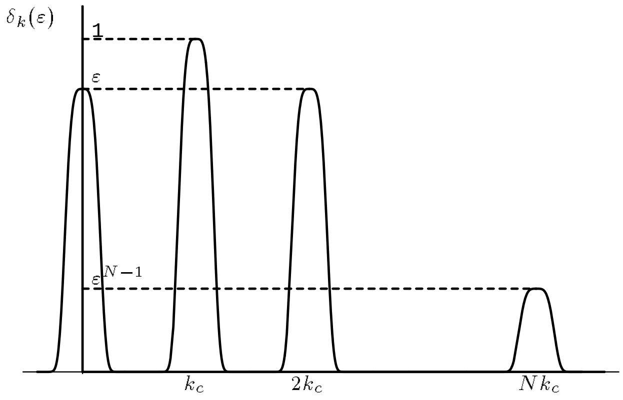


Figure 1: The wave spectrum of a solution has a special structure "clustered mode-distribution" peaked at the integer multiples of the critical wave number with the width of the peaks of order ε

To make Ginzburg-Landau formalism rigorous we have to have two properties satisfied: attractivity and approximation property, which can be essentially described as follows.

Approximation property (based on the results of [9]): *Let the Fourier-transformed version of the GL-equation* (1.1) *has an unique solution for* $\tau \leq \tau_0$. *for some* $\tau_0 > 0$ *And let* ψ_{GL} *be scaled according to the "clustered mode-distribution" (i.e. given by Fourier-transform of (1.7)). Then there exist a solution* ψ of (1.8) *with the same initial conditions* $\psi_{GL}|_{t=0} = \psi_{t=0}$ *and a constant* C *independent of* ε *such that* $\sup_{t \leq \frac{\tau_0}{\varepsilon^2}} ||\psi - \psi_{GL}|| \leq C\varepsilon^2$.

This property was formulated in different forms for particular problems with respect to the norms of the suitable Banach spaces in [3], [13], [15], [14]. To make this property well grounded one needs an

Attractivity property (based on the [8]) *Consider in* (1.9) *initial data* $\Psi^0(k)$ *scaled as follows*

$$\psi^0(k) = \tilde{\delta}_k(\varepsilon)\tilde{\psi}^0(k), \tilde{\psi}^0(k) = \mathcal{O}(1) \tag{1.11}$$

$$\tilde{\delta}_k(\varepsilon) = \max[f(k, k_c), \varepsilon] \tag{1.12}$$

where $f(k, k_c)$ *is of order unity for* $|k - k_c| = \mathcal{O}(\varepsilon)$ *and becomes rapidly small* *outside this interval. Then on time scales given by*

$$0 < t < \frac{\tilde{T}}{\varepsilon^{\eta}}, \eta \in (0,2), \tilde{T} = \mathcal{O}(1) \tag{1.13}$$

the corresponding solutions $\psi(k,t)$ settle to the scaling of the clustered mode-distribution of fig. 3.

For Kuramoto-Shivashinsky equation analogical result was proved in [16]. We note that the time-scales given in (1.13) are long in terms of the original "physical" time t, but are short as compared to the GL time-scale (1.2). Hence, the attractivity property states that from initial conditions scaled by (1.11), (1.12) the solutions of the Fourier - transformed (1.9) collapse to the clustered mode distribution before they start to evolve on the Ginzburg-Landau time scale.

Combining these two results one gets the proper justification of using the Ginzburg-Landau formalism.

The main purpose of this paper is to study the largeness of the domain of attraction. Starting with general initial conditions (we work with functions which Fourier transform is in $L_1 \cap L_\infty$) of order ε^ν for (1.9) we proceed as follows.

1st step: We rescale on ε^ν our equation and show that after the time $\frac{\tilde{T}}{\varepsilon^{\eta_1}}$; $\eta_1 \le \nu$ uncritical modes decay to order $\mathcal{O}(\varepsilon^\nu)$ [2] and in k_c a peak of order one with width $\varepsilon^{\frac{\eta_1}{2}}$ appears.

2nd step: We extend the time-scale till $\frac{\tilde{T}}{\varepsilon^{\eta_2}}$; $\eta_2 \le 2\nu$ and get the formation of the peaks in 0 and $2k_c$ as a result of the modes interaction.

Then we show that for any $\nu \ll 1$ after finite number of steps one can reach the required extension of the time-scale till $\frac{\tilde{T}}{\varepsilon^{\eta}}$; $\eta \le 2$. After this we are ready to get our

Main result : *For any ν, $\nu \ll 1$ "Ginzburg-Landau manifold" [8] (i.e. set of functions of the form (1.7)) is an attractor for the solutions with initial conditions of order ε^ν.*

The method we use is very natural. In essence we follow a line of reasoning of [8], including the "boot-straps strategy" introduced there. But to get the

[2]Symbols $\mathcal{O}(1)$, $o(1)$ will be used in L_∞–norm if another norm was not specified.

result we have to go through some rather technical but essentially interesting complications.

Remarks and comments. Due to the fact that in the proof we don't restrict ourself by the limitation on the sigh of the real part of Landau constant ($Re[\beta]$) it is not possible to handle a wider class of initial conditions (for example spatial periodical initial conditions: $\varepsilon^{\nu} e^{ik_c x} + \varepsilon^{\nu} e^{2ik_c x} +$ c.c.). In this paper the initial conditions don't have the order-one peak near k_c as it was done in [8]. It is possible to show for example that if one starts with initial conditions of order ε^{ν} which have order one peak with width also ε^{ν} then it is not possible to extend the time more than $\frac{\tilde{T}}{\varepsilon^{\eta}}$; $\eta \leq 2\nu$ and corresponding distribution after this time will be peaked with the "right" amplitudes but with width $1/\sqrt{t}$.

2 A priori estimates

Our starting point is the equation (1.9) rewritten for the function $\Phi(k,t) = \varepsilon^{-\nu}\psi(k,t)$, i.e.

$$\Phi(k,t) = e^{\mu(k)t}\left[\Phi^0(k) + \varepsilon^{\nu}\ \rho(k)\ \int_0^t\ e^{-\mu(k)t'}\Phi * \Phi\ dt'\right] \tag{2.1}$$

$$\Phi(k,0) = \Phi^0(k) = \mathcal{O}(1)$$

We assume that $\Phi(k,t)$ is from $L_{\infty}(-\infty,\infty)$ and decays sufficiently fast for $\mid k \mid \rightarrow \infty$ so that

$$\|\Phi\|_{L_1} := \int_{-\infty}^{\infty} \mid \Phi(k,t) \mid dk < \infty \tag{2.2}$$

If this is the case then the following estimate holds:

$$\mid \Phi * \Phi \mid \leq \operatorname*{Sup}_k \mid \Phi(k,t) \mid \cdot \|\Phi\|_{L_1} \tag{2.3}$$

Next we introduce the norms

$$X(\Phi) := \operatorname*{Sup}_{0\leq t\leq T} \|\Phi\|_{L_1}\ , \qquad Y(\Phi) := \operatorname*{Sup}_{0\leq t\leq T} \{\operatorname*{Sup}_k \mid \Phi(k,t) \mid\} \tag{2.4}$$

where T is a parameter to be chosen later on. From (2.1) it follows that, for $t \in [0,T]$,

$$\mid \Phi \mid \leq e^{\mu(k)t} \mid \Phi^0 \mid + \varepsilon^{\nu}\ F(k,t)\ X(\Phi)\ Y(\Phi) \tag{2.5}$$

with

$$F(k,t) = \frac{|\rho(k)|}{\mu(k)}\left[e^{\mu(k)t} - 1\right] \tag{2.6}$$

We intend to derive inequalities for $X(\Phi)$, $Y(\Phi)$ and eventually prove that these norms remain bounded for $\varepsilon \downarrow 0$ on time-interval of order $\tilde{T}/\varepsilon^{\eta_1}$ which are large (for $\varepsilon \downarrow 0$) but are short as compared to the intrinsic time-scale of the Ginzburg-Landau equation given by $\tau = t/\varepsilon^2$. For that purpose one needs estimates of the function $F(k,t)$ in the supremum- and the L_1-norm (with respect to k). Using the results of Appendix 2 one finds from (2.5) the following set of inequalities:

$$Y(\Phi) \leq e^{\mu(k_c)T} \operatorname*{Sup}_k |\Phi^0| + \sigma_1(\varepsilon,T)\, X(\Phi)\, Y(\Phi) \tag{2.7}$$

$$X(\Phi) \leq e^{\mu(k_c)T} \|\Phi^0\|_{L_1} + \sigma_2(\varepsilon,T)\, X(\Phi)\, Y(\Phi) \tag{2.8}$$

with

$$\sigma_1(\varepsilon,T) = \varepsilon^\nu(\rho_0\, T + \mathcal{O}(\varepsilon^2 T^2)) \tag{2.9}$$

$$\sigma_2(\varepsilon,T) = \varepsilon^\nu(\rho_0\, T + \mathcal{O}(1) + \mathcal{O}(\varepsilon^2 T^2)) \tag{2.10}$$

where ρ_0 is a constant.

Remarks: In order to establish the L_1-estimate (2.10) a technical condition on the decay of $|\rho(k)| / \mu(k)$ as $|k| \to \infty$ is used.

We now turn to the analysis of the set of inequalities (2.7), (2.8), with σ_1, σ_2 given by (2.9), (2.10). Clearly, if one considers time-scales defined by

$$T = \frac{\tilde{T}}{\varepsilon^\eta}\,, \qquad \eta \in (0,\nu) \tag{2.11}$$

with $\tilde{T}$ arbitrary numbers, then

$$\sigma_1(\varepsilon,T) = o(1), \qquad \sigma_2(\varepsilon,T) = o(1). \tag{2.12}$$

We introduce the abbreviations

$$A_1 := e^{\mu(k_c)T}\ \mathrm{Sup} \mid \Phi^0 \mid \ , \qquad A_2 := e^{\mu(k_c)T} \|\Phi^0\|_{L_1} \tag{2.13}$$

and rewrite (2.7) as follows

$$[1 - \sigma_1\ X(\Phi)]\ \ Y(\Phi) \leq A_1. \tag{2.14}$$

With the aid of this inequality $Y(\Phi)$ can be eliminated from (2.8) and one finds

$$f(X) := \sigma_1 X^2 - [1 + \sigma_1 A_2 - \sigma_2 A_1]\ \ X + A_2 \geq 0 \tag{2.15}$$

where we have further abbreviated $X(\Phi) := X$.

Let X_1, X_2 denote the two zeros of $f(X)$, given explicitly by

$$X_{1,2} = \frac{1}{2\sigma_1}\left[(1 + \sigma_1 A_2 - \sigma_2 A_1) \mp \sqrt{(1 + \sigma_1 A_2 - \sigma_2 A_1)^2 - 4\sigma_1 A_1}\right] \tag{2.16}$$

On the time-scales (2.11), using (2.12), $X_{1,2}$ simplifies to the following result

$$X_1 = A_2 + o(1)\ , \qquad X_2 = \frac{1}{\sigma_1}(1 + o(1)) \tag{2.17}$$

The graph of $f(x)$ is sketched in fig. 4. Condition (2.15) implies that either $X < X_1$ or $X > X_2$. At initial time $T = 0$ one has $X = A_2 = \mathcal{O}(1)$ and hence $X < X_1$. The norm $X(\Phi)$ depends continuously on T and therefore cannot jump to the branch $X > X_2$. Hence, for all T restricted by (2.11) we have $X < X_1$. Using (2.14) one gets a similar result for the norm $Y(\Phi)$. We remove now the abbreviations and write out in full the results:

Lemma 2.1 *On time intervals $0 \leq t \leq \frac{\tilde{T}}{\varepsilon^\eta}$, $\eta \in (0, \nu)$, the following a priori estimates hold.*

$$X(\Phi) := \mathop{\mathrm{Sup}}_t \|\Phi\|_{L_1} \leq \|\Phi^0\|_{L_1} \exp\ \{\mu(k_c)\frac{\tilde{T}}{\varepsilon^\eta}\} + o(1) \tag{2.18}$$

$$Y(\Phi) := \mathop{\mathrm{Sup}}_t \{\mathop{\mathrm{Sup}}_k (\Phi)\} \leq \mathop{\mathrm{Sup}}_k \mid \Phi^0 \mid \exp\ \{\mu(k_c)\frac{\tilde{T}}{\varepsilon^\eta}\} + o(1) \tag{2.19}$$

We finally note that $\mu(k_c) = \mathcal{O}(\varepsilon^2)$ so that the exponential functions can be replaced by $1 + o(1)$.

Now from (2.5), (2.6) and Lemma 2.1 we have

$$\mid \Phi \mid \leq e^{\mu(k)t} \mid \Phi^0 \mid + \varepsilon^{\nu} \frac{\mid \rho(k) \mid}{\mu(k)} \left[e^{\mu(k)t} - 1 \right] \text{ Sup } \mid \Phi^0 \mid \cdot \|\Phi^0\|_{L_1} + o(1) \tag{2.20}$$

The estimate is valid on time-scales given by (2.11).

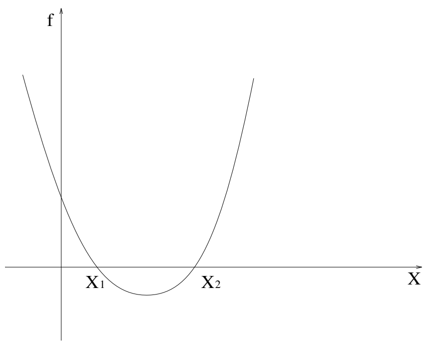


Figure 2: For all T restricted by (2.11) we have $X < X_1$.

3 Analysis and new estimates of the convolution integral.

From now on we assume that we have progressed in time till $t_1 = \tilde{T}/\varepsilon^{\eta_1}$, $\eta_1 = \nu$; and we consider the initial value problem for (2.1) with initial conditions $\Phi(k, 0) = \Phi(k, t_1)$ We introduce the new initial scaling

$$\Phi(k, t) = \delta_k^{(1)}(\varepsilon)\varphi(k, t) \; ; \qquad \delta_{-k}^{(1)} = \delta_k^{(1)} \tag{3.1}$$

$$\delta_k^{(1)}(\varepsilon) = \text{Max}[f_{\eta_1}(k, k_c), \varepsilon^{\nu}], \quad \text{ for } k \geq 0 \tag{3.2}$$

$$f_{\eta}(k, k_0) := \frac{\varepsilon^{\eta}}{(k - k_0)^2 + \varepsilon^{\eta}} \tag{3.3}$$

The function $f_\eta(k,k_0)$ mimics a distribution of orders of magnitude which is of order unity for $|k-k_0| = \mathcal{O}(\varepsilon^{\eta/2})$, and becomes rapidly smaller outside such intervals. In fact

$$f_\eta(k,k_0) = \mathcal{O}(\varepsilon^{\eta-2p})\,, \quad \text{for } |k-k_0| = \mathcal{O}(\varepsilon^p),\ p<\eta/2 \tag{3.4}$$

Which corresponds to the distribution given by (2.22) We are given now that at the initial time $t=0$, $\varphi(k,0) = \mathcal{O}(1)$ for each value of $k \in (-\infty,\infty)$. After the scaling (3.1) we get

$$\Phi * \Phi = \int_{-\infty}^{\infty} \delta_{k'}\delta_{k-k'}\varphi(k')\varphi(k-k')dk' \tag{3.5}$$

where, for the simplicity of notations the dependence of φ on t has temporally been suppressed.

The analysis of $\Phi * \Phi$ is a bit technical, but the ideas are very simple: on small intervals of the k'-axis δ_k, and/or $\delta_{k-k'}$ are of order unity. One separates out these intervals (taking them of order $\varepsilon^{(\eta-\nu)/2}$ so that the decay of $\delta_{k'}\delta_{k-k'}$ to order ε^ν is incorporated). The contribution of each of these small intervals can be bounded by $[\sup_k|\varphi|]^2$ multiplied by an explicitly given integral. On the remainder of the k'-axis $\delta_{k'}\cdot\delta_{k-k'} = \varepsilon^{2\nu}$ and the integral of $|\varphi(k')|\cdot|\varphi(k-k')|$ can be bounded by the product of $\sup_k|\varphi|$ and $\|\varphi\|_{L_1}$. We shall demonstrate in this way the following result:

Lemma 3.1 *For $k \geq 0$,*

$$|\Phi * \Phi| \leq c\varepsilon^{\eta/2}\,\mathrm{Max}[f_\eta(k,0), f_\eta(k,2k_c), \varepsilon^\nu](\sup_k|\varphi|)^2 + \varepsilon^{2\nu}\|\varphi\|_{L_1}\sup_k|\varphi|$$

where c is a constant independent of ε.

For simplicity we drop index 1 from η_1 in the lemma. For the proof of the lemma 3.1 see Appendix A.1.

Introducing the result of Lemma 3.1 in (2.1) and performing the integration with respect to t' produces the basic inequality

$$\begin{aligned}|\Phi| \quad &\leq e^{\mu(k)t}|\Phi^0| + \varepsilon^\nu F(k,t)\left\{\varepsilon^{2\nu}X(\varphi)\right.\\ &\left.+\, c\varepsilon^{\frac{\eta_1}{2}}\,\mathrm{Max}[f_{\eta_1}(k,0), f_{\eta_1}(k,2k_c), \varepsilon^\nu)]Y(\varphi)\right\}Y(\varphi)\end{aligned}$$

with $F(k,t)$ defined in (2.6). We intend to derive inequalities for $X(\varphi)$ and $Y(\varphi)$ and eventually prove that these norms remain bounded on time-intervals $t \leq \tilde{T}/\varepsilon^{\eta_2}$. The analysis will necessarily be somewhat technical, but in essence is again very simple. The results are collected in Lemma 3.2, at the end of this section.

From (3.6) it follows in a straightforward way that

$$Y(\varphi) \leq e^{\mu(k_c)T} Y(\varphi^0) + c\varepsilon^{\nu}\{\varepsilon^{2\nu} Y(F_1) X(\varphi) + \varepsilon^{\frac{\eta_1}{2}} Y(F_2) Y(\varphi)\} Y(\varphi) \tag{3.6}$$

$$X(\varphi) \leq e^{\mu(k_c)T} X(\varphi^0) + c\varepsilon^{\nu}\{\varepsilon^{2\nu} X(F_1) X(\varphi) + \varepsilon^{\frac{\eta_1}{2}} X(F_2) Y(\varphi)\} Y(\varphi) \tag{3.7}$$

with

$$F_1 = F(k,t) \frac{1}{\mathrm{Max}(f_{\eta_1}(k,k_c), \varepsilon^{\nu})} \tag{3.8}$$

$$F_2 = F(k,t) \frac{\mathrm{Max}(f_{\eta_1}(k,0), f_{\eta_1}(k,2k_c), \varepsilon^{\nu})}{\mathrm{Max}(f_{\eta_1}(k,k_c), \varepsilon^{\nu})} \tag{3.9}$$

So the task is to bounded the Sup- and the L_1-norms of the explicitly given functions F_1 and F_2. The analysis is elementary, but somewhat delicate. It is given in the Appendix A.2. The results are as follows:

$$Y(F_1) = \varepsilon^{-\eta_2} \tilde{C}_1(\varepsilon^{\eta_2} T) \ , \quad Y(F_2) = \varepsilon^{-\sigma} \tilde{C}_2(\varepsilon^{\eta_2} T) \ , \quad \sigma = \max\{\eta_2 - \nu, \nu\} \tag{3.10}$$

$$X(F_1) = \varepsilon^{-\eta_2 + \frac{\eta_2 - \nu}{3}} \hat{C}_1(\varepsilon^{\eta_2} T), \qquad X(F_2) = \varepsilon^{-\eta_2 + \nu + \frac{\eta_2 - \nu}{3}} \hat{C}_2(\varepsilon^{\eta_2} T) \tag{3.11}$$

here $\tilde{C}_1, \tilde{C}_2, \hat{C}_1$ and $\hat{C}_2$ are approximately constant when $\varepsilon^{\eta_2} T = o(1)$; these expressions remain bounded when $\varepsilon^{\eta_2} T = \mathcal{O}(1)$ but is numerically small. Overestimating all these constants by some constant C_0 we obtain the following system of inequalities for $\sigma = \eta_2 - \nu$

$$Y \leq A_1 + C_0 \varepsilon^{2\nu - \eta_2} [\varepsilon^{\nu} X + \varepsilon^{\frac{\eta_1}{2}} Y] Y \tag{3.12}$$

$$X \leq A_2 + C_0 \varepsilon^{2\nu - \eta_2 + \frac{\eta_2 - \nu}{3}} [\varepsilon^{\nu} X + \varepsilon^{\frac{\eta_1}{2}} Y] Y \tag{3.13}$$

where we have used abbreviations (2.13) for A_1 and A_2. From (3.13) we deduce

$$X \le \frac{A_2 + \varepsilon^{\frac{1}{6}(10\nu - 4\eta_2 + 3\eta_1)} C_0 Y^2}{1 - \varepsilon^{\frac{2}{3}(4\nu - \eta_2)} C_0 Y} \tag{3.14}$$

For ε small this is permissible if $\eta_2 \le 4\nu$. From (3.12) X can now be eliminated, and (regrouping the terms) we find the inequality:

$$\mathcal{G}(Y) \ge 0 \tag{3.15}$$

$$\begin{aligned} \mathcal{G}(Y) \qquad &:= C_0 Y^2 (\varepsilon^{\frac{2}{3}(4\nu - \eta_2)} + \varepsilon^{2\nu - \eta_2 + \frac{\eta_1}{2}}) \\ &+ (-1 + \varepsilon^{3\nu - \eta_2} C_0 A_2 - \varepsilon^{\frac{2}{3}(4\nu - \eta_2)} C_0 A_1) Y + A_1 \end{aligned}$$

The function $\mathcal{G}(Y)$ has two zeros $Y_{1.2}$, the smaller one is given by

$$Y_{1.2} = A_1 + \mathcal{O}(\varepsilon^p) \ , \quad p > 0, \ \eta_2 \le \frac{5}{2}\nu \tag{3.16}$$

$Y_{1.2}$ are both real and positive. A plot of $\mathcal{G}(Y)$ is the same as on fig. 2.1. So we have got $Y_1 > A_1$. In order to interpret these results we look closer at the definition of A_1 and A_2 in (2.13), and impose the following limitation on T:

$$T = \frac{\tilde{T}}{\varepsilon^{\eta_2}}, \quad \tilde{T} = \mathcal{O}(1), \ \eta_2 = \mathrm{Min}[\frac{5}{2}\nu, 2] \tag{3.17}$$

On these time scales:

$$A_1 = \mathrm{Sup}\ |\varphi^0|[1 + o(1)], \qquad A_2 = \|\varphi^0\|_{L_1} \cdot [1 + o(1)] \tag{3.18}$$

Now we can use the same argument as in the proof of the Lemma 2.1 and to get the following result.

Lemma 3.2 *We consider scaled Fourier-components* $\Phi(k,t) = \delta_k^1(\varepsilon)\varphi(k,t)$ *with* $\delta_k^1(\varepsilon) = \mathrm{Max}[f_{\eta_1}(k,k_c), \varepsilon^\nu]$*,* $\eta_1 = \nu$*, with* T *limited by*

$$T = \frac{\tilde{T}}{\varepsilon^{\eta_2}}, \quad \tilde{T} = \mathcal{O}(1), \ \eta_2 = \mathrm{Min}[\frac{5}{2}\nu, 2]$$

Then the norms $X(\varphi)$, $Y(\varphi)$ *are uniformly bounded, independent of* ε.

4 The appearance of clustered modes-distribution

With the à priori estimate of Lemma 3.2 our basic inequality (3.6) contains a wealth of information on the Fourier-components $\Phi(k,t)$. We repeat this result here for the convenience of further analysis:

$$\begin{aligned} |\Phi| \qquad & \leq e^{\mu(k)t}|\Phi^0| + \varepsilon^{\nu-\eta_2} f_{\eta_2}(k,k_c)\{\varepsilon^{2\nu}X(\varphi) \\ & +c\varepsilon^{\frac{\eta_1}{2}} \operatorname{Max}[f_{\eta_1}(k,0), f_{\eta_1}(k,2k_c), \varepsilon^{\nu}]Y(\varphi)\}Y(\varphi) \end{aligned}$$

where we have used the estimate A.2.5 for the function $F(k,t)$ appearing in (3.6).

We know that $X(\varphi)$ and $Y(\varphi)$ are bounded on the time-scales given by

$$0 \leq t \leq T, \quad T = \frac{\tilde{T}}{\varepsilon^{\eta_2}},\ \eta_2 = \operatorname{Min}(\frac{5}{2}\nu, 2),\ \tilde{T} = \mathcal{O}(1). \tag{4.1}$$

So we immediately deduce from (4.1):

Lemma 4.1 *For all k such that $|k-k_c| \geq d, d = \mathcal{O}(1)$ the influence of initial conditions becomes exponentially small on time-scales (4.2)*

Next, again from (4.1), we find

Lemma 4.2 *On time-scales (4.1) the Fourier-components $\Phi(k,t)$ reach the magnitudes*

$$\Phi = \delta_k^{(2)}(\varepsilon)\varphi^{(2)},\ \varphi^{(2)} = \mathcal{O}(1)$$

with $\delta_k^{(2)}(\varepsilon) = \operatorname{Max}\{f_{\eta_2}(k,k_c),\ \varepsilon^{\frac{\eta_1}{2}+\nu}f_{\eta_1}(k,0),\ \varepsilon^{\frac{\eta_1}{2}+\nu}f_{\eta_1}(k,2k_c),\ \varepsilon^{2\nu+\frac{\eta_1}{2}}\}$.

We see that the clustered mode-distribution begins to appear. It is sketched in fig. 4.1

Remarks. The restriction for η can be easily observed from the contribution of second term in (4.1) near k_c. Which demands from $(2\nu - \frac{\eta_1}{2} - \eta_2)$ to be positive, i.e. $\eta_2 < 2\nu + \frac{\eta_1}{2} \leq \frac{5}{2}\nu$. From the restrictions on the growth the first linear term in (4.1) follows that $\eta \leq 2$.

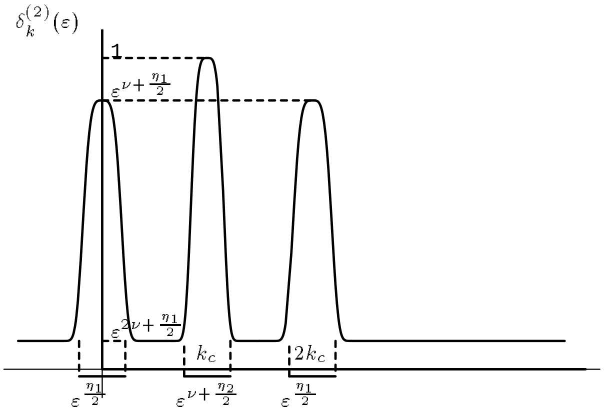


Figure 3: The appearance of the clustered mode distribution.

5 The "Bootstraps strategy"

It is clear now how repeating the same procedure in the third step we start with the scaling

$$\Phi = \delta_k^{(2)}(\varepsilon)\varphi^{(2)}(t,k), \quad \varphi^{(2)}(0,k) = \mathcal{O}(1) \tag{5.1}$$

$$\delta_k^{(2)}(\varepsilon) = \mathrm{Max}\ \{f_{\eta_2}(k,k_c),\ \varepsilon^{\frac{\eta_1}{2}+\nu} f_{\eta_1}(k,0),\ \varepsilon^{\frac{\eta_1}{2}+\nu} f_{\eta_1}(k,2k_c),\ \varepsilon^{2\nu+\frac{\eta_1}{2}}\} \tag{5.2}$$

(i.e. $\Phi|_{t=0}$ for the new problem is $\Phi|_{t=\frac{\tilde{T}}{\varepsilon^{\eta_2}}}$, $\eta_2 = \mathrm{Min}[\frac{5}{2}\nu, 2]$) and get analogous result

Lemma 5.1 *On time-scales*

$$0 \le t \le T, \quad T = \frac{\tilde{T}}{\varepsilon^{\eta_3}},\ \eta_3 = \mathrm{Min}[\frac{\eta_1+\eta_2}{2} + 2\nu, 2],\ \tilde{T} = \mathcal{O}(1)$$

the Fourier-components $\Phi(k,t)$ reach the magnitudes

$$\Phi = \delta_k^{(3)}(\varepsilon)\varphi^{(3)},\ \varphi^{(3)} = \mathcal{O}(1)$$

$$\delta_k^{(3)}(\varepsilon) = \mathrm{Max}\ [f_{\eta_3}^2(k,k_c),\ \varepsilon^{\frac{\eta_2}{2}+\nu} f_{\eta_2}(k,0),\ \varepsilon^{\frac{\eta_2}{2}+\nu} f_{\eta_2}(k,2k_c),\ \varepsilon^{\frac{\eta_1+\eta_2}{2}+2\nu} f_{\eta_1}(k,3k_c),$$
$$\varepsilon^{3\nu+\frac{\eta_1+\eta_2}{2}}]$$

By this procedure in every step one gets extension of the time-scale and improves the results on clustered modes. By induction one can show that starting on the N-th step with initial conditions

$$\Phi|_{t=0} = \Phi|_{t=\frac{\tilde{T}}{\varepsilon^{\eta_N}}} = \delta_k^{(N)}(\varepsilon)\phi^{(N)}, \quad \phi^{(N)} = \mathcal{O}(1)$$

we end up with the following

Lemma 5.2 *On time-scales*

$$0 \le t \le T, \quad T = \frac{\tilde{T}}{\varepsilon^{\eta_{N+1}}}, \; \eta_{N+1} = \mathrm{Min}[\frac{\eta_{N-1}+\eta_N}{2} + 2\nu, 2], \; \tilde{T} = \mathcal{O}(1)$$

the Fourier-components $\Phi(k,t)$ reach the magnitudes

$$\Phi = \delta_k^{(N+1)}(\varepsilon)\varphi^{(N+1)}, \quad \varphi^{(N+1)} = \mathcal{O}(1)$$

$$\delta_k^{(N+1)}(\varepsilon) = \mathrm{Max}\;[f^N_{\eta_{N+1}}(k,k_c),\; \varepsilon^{\frac{\eta_N}{2}+\nu} f_{\eta_N}(k,0),\; \varepsilon^{\frac{\eta_N}{2}+\nu} f_{\eta_N}(k,2k_c),$$
$$\sum_{l=3}^{N} \varepsilon^{\frac{1}{2}\sum_{i=0}^{l-2}\eta_{N-i}+(l-1)\nu} f^N_{\eta_{N-l+2}}(k,lk_c), \varepsilon^{\frac{1}{2}\sum_{i=0}^{N-2}\eta_{N-i}+N\nu}]$$

See fig. 6.

Note that the case that ν is arbitrary close to zero is the most interesting

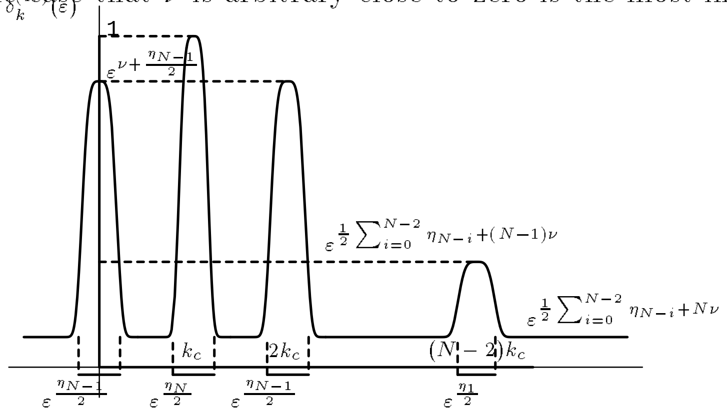


Figure 4: The results of the lemma 5.2.

for our purposes. So for $\eta_{N+1} = \mathrm{Min}[\frac{\eta_N+\eta_{N-1}}{2} + 2\nu, 2]$ we choose the first component. And now we want to show that for any ν arbitrary close to zero there exist N_1 such that η_{N_1} equals 2. In this way we would reach the right time-scale $T \leq \frac{\tilde{T}}{\varepsilon^\eta}$, $\eta \leq 2$.

Let us find the explicit formula for η_N which satisfies to the following recurrent problem

$$\eta_{N+1} = 2\nu + \frac{\eta_N + \eta_{N-1}}{2}$$

$$\eta_1 = \nu$$

$$\eta_2 = 2\nu + \frac{\nu}{2}$$

It is easy exercise to show that

$$\eta_N = \frac{4\nu}{3}N + \frac{2\nu}{9}((-\frac{1}{2})^N - 1)$$

And it is obvious now that for any small $0 < \nu$ we can choose $N_1 \geq \frac{3}{4\nu}$ such that $\mathrm{Min}[\frac{\eta_N + \eta_{N-1}}{2} + 2\nu, 2] = 2$ and further we will be on the time-scale $t < \frac{\tilde{T}}{\varepsilon^\eta}$, $\eta \leq 2$.

6 The main result.

Let us overview what was done. We have begun with the initial conditions of order unity for the scaled equation (2.1) and by Lemma (2.1) we have shown that the uncritical modes decay and the first peak forms. But we have got it on unsatisfactory short time-scale $t \leq \frac{\tilde{T}}{\varepsilon^{\eta_1}}, \eta_1 \leq \nu$. To extend the time and to get the rest of the peaks we started with the rougher scaling exploiting the results of our first step. And recycling the analysis N_1 times as we have shown in the previous section we can get a time-extension till $t \leq \frac{\tilde{T}}{\varepsilon^2}$. Let us now formulate our main statement and then we will show how it can be obtained from the point where we have stopped.

Theorem 6.1 *Let the initial conditions for the Fourier-components $\Phi(k,t)$ in equation (2.1) are of unity i.e.*

$$\Phi(k,0) = \Phi^0(k) = \mathcal{O}(1)$$

Consider the time-instant

$$t = \frac{\tilde{T}}{\varepsilon^2}, \quad \tilde{T} \in \mathbf{R}_+$$

Then:

$$\Phi(k,t) = \tilde{\delta}_k^N(\varepsilon)\tilde{\varphi}(k,t), \quad \tilde{\varphi} = \mathcal{O}(1)$$

$$\tilde{\delta}_k^N(\varepsilon) = \mathrm{Max}\{\sum_{n=0}^{N} \varepsilon^{|1-n|}[f_2(k,nk_c)]^N,\ \varepsilon^N\}$$

where N is an arbitrarily large integer.

Comments. We note that $\psi(k,t) = \varepsilon^\nu \Phi(k,t)$. So we have got the mode-distribution scaled on ε^ν. However for ν close to zero we are arbitrary close to the GL manifold, introduced in [8], [9].

Full proof of the theorem follows by induction starting with initial conditions $\delta_k^{N_1}\Phi(k,t)$. We don't have restrictions on the time any more and automatically can get on every step the boundedness of the corresponding X and Y norms. We leave out the explicit technical details which from now on should be obvious.

7 More general quadratic nonlinearities and weighted norms

We shall consider now more general problems of the structure

$$\frac{\partial \Psi}{\partial t} = L\Psi + (\rho_1\Psi)\ (\rho_2\Psi) \tag{7.1}$$

where L is as before, while ρ_1, ρ_2 are linear differential operators in the space-like variable x, with symbols $\rho_1(k)$, $\rho_2(k)$. Of course, one can also have finite sums of non-linearities of this structure, i.e.

$$\sum_{\ell=1}^{m}(\rho_1^{(\ell)}\Psi)\ (\rho_2^{(\ell)}\Psi). \tag{7.2}$$

For simplicity of presentation we develop the reasoning for $m=1$.

The complication introduced by the more general form 7.1 comes from the fact that taking the Fourier-transform does not lead to the simple and elegant equation (2.1). Instead one gets

$$\Phi(k,t) = e^{\mu(k)t}\left[\Phi^0 + \varepsilon^\nu \int_0^t e^{-\mu(k)t'}\widehat{\Phi * \Phi}\ dt'\right] \tag{7.3}$$

with

$$\widehat{\Phi * \Phi} = \int_{-\infty}^{\infty} G(k', k-k')\ \Phi(k')\ \Phi(k-k')\, dk' \tag{7.4}$$

$$G(k', k-k') = \rho_1(k')\ \rho_2(k-k') + \rho_2(k')\ \rho_1(k-k') \tag{7.5}$$

To avoid this complication Van Harten [9] advocates a reformulation in terms of a vector function of which the components are derivatives of Ψ with respect to x up to a suitably chosen order.

We shall show that, at least for the purpose of our analysis, one can use a simpler approach, in terms of weighted norms (to be defined shortly).

Let us first give some details on the behaviour of $\rho_1(k)$, $\rho_2(k)$ for $|\ k\ | \to \infty$. These functions are polynomials, so we have

$$\begin{array}{l} |\ \rho_1(k)\ | \leq c\ |\ k\ |^{p_1} \\ |\ \rho_2(k)\ | \leq c\ |\ k\ |^{p_2} \end{array} \quad \text{for } k > 0 \ \ p_1, p_2 > 0 \tag{7.6}$$

We introduce a weight-function $g(k)$ by

$$g(k) = 1 + |\ k\ |^p \ \ , \ \ p = p_1 + p_2 \tag{7.7}$$

In the Appendix A.3 we demonstrate that

Lemma 7.1 *The function*

$$\rho_0(k) = \int_{-\infty}^{\infty} \frac{G(k', k-k')}{g(k')\ g(k-k')}\, dk' \tag{7.8}$$

with G and g specified through 7.5, 7.6, 7.7 is uniformly bounded for $k \in (-\infty, \infty)$.

The proof is an (amusing) exercise in elementary analysis but is not altogether trivial, so it could not be left as an exercise for the reader.

We now assume that $\Phi(k,t)$ is a continuous function of k which decays sufficiently fast for $|\ k\ | \to \infty$ so that

$$\operatorname*{Sup}_k \{g(k)\ |\ \Phi(k,t)\ |\} \ \ \text{exists} \tag{7.9}$$

This condition is not unnatural and corresponds *grosso modo* to the assumption that Ψ and its derivatives up to a certain order have Fourier-transforms which decay for $\mid k \mid \to \infty$.

With this preparation we can follow the line of section 2. The first step is

$$\mid \widehat{\Phi * \Phi} \mid \leq \mid \rho_0(k) \mid \left\{ \operatorname{Sup}_k \left[g(k) \cdot \mid \Phi(k,t) \mid \right] \right\}^2 . \tag{7.10}$$

Next introducing

$$Y_g(\Phi) := \operatorname{Sup}_{0 \leq t \leq T} \left\{ \operatorname{Sup}_k \left[g(k) \cdot \mid \Phi(k,t) \mid \right] \right\} \tag{7.11}$$

one gets

$$\mid \Phi \mid \leq e^{\mu(k)t} \mid \Phi^0 \mid + \varepsilon^\nu \; F^0(k,t) \left[Y_g(\Phi) \right]^2 \tag{7.12}$$

$$F^0(k,t) = \frac{\mid \rho_0(k) \mid}{\mu(k)} \; \left[e^{\mu(k)t} - 1 \right] \tag{7.13}$$

In order to derive an inequality for $Y_g(\Phi)$ one must analyse

$$\operatorname{Sup}_{0 \leq t \leq T} \left\{ \operatorname{Sup}_k \left[g(k) \; F^0(k,t) \right] \right\} .$$

However, the function $g(k) \; F^0(k,t)$ is of the same structure as $F(k,t)$ of section 2 so one can just use the results of Appendix 1. Hence:

$$Y_g(\Phi) \leq e^{\mu(k_c)T} \operatorname{Sup} \left[g \mid \Phi^0 \mid \right] + \sigma_1(\varepsilon, T) \; \left[Y_g(\Phi) \right]^2 \tag{7.14}$$

From here an a priori estimate for the weighted supremum norm follows immediately, and the previous results can readily be reproduced.

7.1 Cubic Nonlinearity

Let us show how to deal with some difficulties concerning a cubic nonlinearity: $N(\Psi) = P(\Psi^3)$. In the first step of the proof the same rescaling can be done. Working with the inequality

$$\mid \Phi \mid \leq e^{\mu(k)t} \mid \Phi^0 \mid + \varepsilon^{2\nu} \; F(k,t) \, X^2(\Phi) \; Y(\Phi) \tag{7.15}$$

instead of (2.5) one gets that after the time $t_1 = \tilde{T}/\varepsilon^{\eta_1}$ with $\eta_1 = 2\nu$ one can use the scaling (3.1) with $\delta_k^{(1)}(\varepsilon) = \mathrm{Max}[f_{\eta_1}(k, k_c), \varepsilon^{2\nu}]$. Now in order to get the result analogical to the statement of lemma 3.1 one has to estimate

$$\Phi * \Phi * \Phi = \int_{-\infty}^{\infty} \delta_{k-k'} \varphi(k-k') \int_{-\infty}^{\infty} \delta_{k'-k''} \delta_{k''} \varphi(k'-k'') \varphi(k'')\, dk''\, dk' \tag{7.16}$$

Using analogical reasoning to the quadratic case one gets

$$\begin{aligned} |\Phi| \; & \leq e^{\mu(k)t} |\Phi^0| + \varepsilon^{2\nu} F(k,t) \left\{ \varepsilon^{6\nu} X^2(\varphi) + \varepsilon^{\frac{\eta_1}{2}+4\nu} X(\varphi) Y(\varphi) \right. \\ & \left. + c\varepsilon^{\eta_1} \mathrm{Max}[f_{\eta_1}(k,k_c), f_{\eta_1}(k, 3k_c), \varepsilon^{2\nu})] Y^2(\varphi) \right\} Y(\varphi) \end{aligned}$$

Skipping the details of the proof of the section 3, we end up with the statement of lemma 3.2 valid till $T = \frac{\tilde{T}}{\varepsilon^{\eta_2}}, \quad \tilde{T} = \mathcal{O}(1), \ \eta_2 = \mathrm{Min}[\eta_1 + 2\nu, 2]$ After N steps one can extend this result till $\eta_{N+1} = \mathrm{Min}[\eta_N + 2\nu, 2] = \mathrm{Min}[2\nu(N+1), 2]$ which is much faster than in the quadratic case. Finally one will get the distribution given in figure 5 on the time–scale given by $0 < t < \frac{\tilde{T}}{\varepsilon^{\eta}}, \eta \in (0,2), \tilde{T} = \mathcal{O}(1)$.

A Appendix

A.1 Proof of the lemma 3.1.

Let us introduce the following sub-intervals of the k'-axis:

$$I_{\pm} = \{k' | k' = \pm k_c + \mathcal{O}(\varepsilon^{(\eta-\nu)/2})\} \tag{A.1.1}$$

$$J_{\pm} = \{k' | k - k' = \pm k_c + \mathcal{O}(\varepsilon^{(\eta-\nu)/2})\} \tag{A.1.2}$$

In each of these intervals one of the order functions $\delta_{k'}$, $\delta_{k-k'}$ is of order unity. However, we observe (and one can easily verify this) that:
When $k \neq \mathcal{O}(\varepsilon^{(\eta-\nu)/2})$ and $k \neq \mp 2k_c + \mathcal{O}(\varepsilon^{(\eta-\nu)/2})$ then $I_{\pm}$ and $J_{\pm}$ cannot pairwise coincide and one of the factors in $\delta_{k'}\delta_{k-k'}$ is always $\mathcal{O}(\varepsilon^{\nu})$.

We commence our analysis with this restriction on the values of k. The first step is the estimate

$$|\Phi * \Phi| \leq \varepsilon^{\nu} \int_{I_+ + J_-} \delta_{k'} |\varphi(k')| \cdot |\varphi(k-k')| dk' + \tag{A.1.3}$$

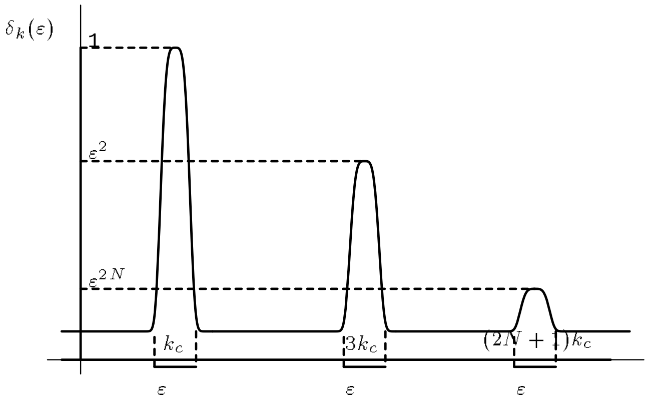


Figure 5: The cluster mode-distribution in the case of cubic nonlinearity.

$$+\varepsilon^{\nu}\int_{J_+ + J_-}\delta_{k-k'}|\varphi(k')|\cdot|\varphi(k-k')|dk' +$$
$$+\varepsilon^{2\nu}\int_{-\infty}^{\infty}|\varphi(k')|\cdot|\varphi(k-k')|dk'$$

Note that in the last integral we have "filled in" the small subintervals removed in the first four integrals, which is consistent with over-estimating $|\Phi * \Phi|$. Next we use, in an obvious way.

$$|\Phi * \Phi| \leq \varepsilon^{\nu}[\int_{I_+ + I_{-1}}\delta_{k'}dk' + \int_{J_+ + J_-}\delta_{k-k'}dk'][\sup_k|\varphi|]^2 + \qquad (A.1.4)$$
$$+\varepsilon^{2\nu}\sup_k|\varphi|\cdot\int_{-\infty}^{\infty}|\varphi|dk'$$

Each of the remaining integrals in A.1.4 is equal to the integral:

$$\int_{k_c - c\varepsilon^{\frac{\eta-\nu}{2}}}^{k_c + c\varepsilon^{\frac{\eta-\nu}{2}}}\frac{\varepsilon^{\eta}}{(k'-k_c)^2+\varepsilon^{\eta}}dk' = \varepsilon^{\eta/2}(\pi + \mathcal{O}(\varepsilon^{\nu/2})) \qquad (A.1.5)$$

This last result is easily obtained by explicit integration.

Next we consider the values of k such that $k = \mathcal{O}(\varepsilon^{(\eta-\nu)/2})$. Then $I_\pm$ and $J_\pm$ coincide pairwise and (skipping a few steps entirely parallel to the preceding analysis) one gets

$$|\Phi * \Phi| \leq 2 \int_{k_c - c\varepsilon^{\frac{\eta-\nu}{2}}}^{k_c + c\varepsilon^{\frac{\eta-\nu}{2}}} \frac{\varepsilon^\eta}{(k'-k_c)^2 + \varepsilon^\eta} \cdot \frac{\varepsilon^\eta}{(k-k'+k_c)^2 + \varepsilon^\eta} dk' [\sup_k |\varphi|]^2 \tag{A.1.6}$$

$$+\varepsilon^{2\nu} \sup_k [\varphi] \cdot \|\varphi\|_{L_1}.$$

The remaining explicit integral we denote

$$I_0 = \int_{k_c - c\varepsilon^{\frac{\eta-\nu}{2}}}^{k_c + c\varepsilon^{\frac{\eta-\nu}{2}}} \frac{\varepsilon^\eta}{(k'-k_c)^2 + \varepsilon^\eta} \frac{\varepsilon^\eta}{(k-k'+k_c)^2 + \varepsilon^\eta} dk' \tag{A.1.7}$$

where c is some (order-one) constant. We introduce the transformation

$$k' = k_c + \frac{1}{2}k + \hat{k} \tag{A.1.8}$$

and obtain

$$I_0 = \int_{-\frac{1}{2}k - c\varepsilon^{\frac{\eta-\nu}{2}}}^{-\frac{1}{2}k + c\varepsilon^{\frac{\eta-\nu}{2}}} \frac{\varepsilon^\eta}{(\hat{k} + \frac{1}{2}k)^2 + \varepsilon^\eta} \cdot \frac{\varepsilon^\eta}{(\hat{k} - \frac{1}{2}k)^2 + \varepsilon^\eta} d\hat{k} \tag{A.1.9}$$

For each $k = \mathcal{O}(\varepsilon^{\frac{\eta-\nu}{2}})$ we can choose c such that the upper integration limit is positive. For reasons which shall become clear shortly, we apply a somewhat more conservative condition

$$-k + c\varepsilon^{\frac{\eta-\nu}{2}} > 0. \tag{A.1.10}$$

Next the integral A.1.9 is reformulated so that the integration variable runs over non-negative values only:

$$I_0 = [\int_0^{-\frac{1}{2}k + c\varepsilon^{\frac{\eta-\nu}{2}}} + \int_0^{\frac{1}{2}k + c\varepsilon^{\frac{\eta-\nu}{2}}}] \frac{\varepsilon^\eta}{(\hat{k} + \frac{1}{2}k)^2 + \varepsilon^\eta} \frac{\varepsilon^\eta}{(\hat{k} - \frac{1}{2}k)^2 + \varepsilon^\eta} d\hat{k} \tag{A.1.11}$$

We can now introduce the obvious estimates

$$I_0 \leq \frac{\varepsilon^\eta}{(\frac{1}{2}k)^2+\varepsilon^\eta}[\int_0^{-\frac{1}{2}k+c\varepsilon^{\frac{\eta-\nu}{2}}} + \int_0^{\frac{1}{2}k+c\varepsilon^{\frac{\eta-\nu}{2}}}]\frac{\varepsilon^\eta}{(\hat{k}-\frac{1}{2}k)^2+\varepsilon^\eta}d\hat{k} \tag{A.1.12}$$

The final step is the transformation of variable

$$\hat{k} = \frac{1}{2}k + \varepsilon\xi \tag{A.1.13}$$

which produces

$$I_0 \leq \frac{\varepsilon^\eta}{(\frac{1}{2}k)^2+\varepsilon^\eta}[\int_{-\frac{1}{2}\frac{k}{\varepsilon}}^{-\frac{k}{\varepsilon}+\frac{c}{\varepsilon^{1-\frac{\eta-\nu}{2}}}} + \int_{-\frac{1}{2}\frac{k}{\varepsilon}}^{\frac{c}{\varepsilon^{1-\frac{\eta-\nu}{2}}}}]\frac{d\xi}{\xi^2+1} \tag{A.1.14}$$

By explicit integration one gets

$$I_0 \leq \frac{\varepsilon^\eta}{(\frac{1}{2k})^2+\varepsilon^\eta}c\varepsilon^{\frac{\eta}{2}}(1+\mathcal{O}(\varepsilon^{\frac{\nu}{2}})) \tag{A.1.15}$$

Hence

$$I_0 \leq \varepsilon^{\frac{\eta}{2}} f(k,0)\cdot c(1+\mathcal{O}(\varepsilon^{\frac{\nu}{2}})) \tag{A.1.16}$$

We consider finally $k = 2k_c + \mathcal{O}(\varepsilon^{\frac{\eta-\nu}{2}})$. Again there are intervals of the k'-axis on which both $\delta_{k'}$ and $\delta_{k-k'}$ are of order unity. Proceeding as above one now gets

$$|\Phi * \Phi| \leq 2\int_{k_c-c\varepsilon^{\frac{\eta-\nu}{2}}}^{k_c+c\varepsilon^{\frac{\eta-\nu}{2}}} \frac{\varepsilon^\eta}{(k'-k_c)^2+\varepsilon^\eta}\cdot\frac{\varepsilon^\eta}{(k-k'-k_c)^2+\varepsilon^\eta}dk'[\sup_k|\varphi|]^2 \tag{A.1.17}$$

$$+\varepsilon^{2\nu}\sup_k[\varphi]\cdot\|\varphi\|_{L_1}.$$

The integral

$$I_1 = \int_{k_c-c\varepsilon^{\frac{\eta-\nu}{2}}}^{k_c+c\varepsilon^{\frac{\eta-\nu}{2}}} \frac{\varepsilon^\eta}{(k'-k_c)^2+\varepsilon^\eta}\frac{\varepsilon^\eta}{(k-k'-k_c)^2+\varepsilon^\eta}dk' \tag{A.1.18}$$

after the transformation

$$k = 2k_c + \tilde{k}, \tilde{k} = \mathcal{O}(\varepsilon^{\frac{\eta-\nu}{2}}) \qquad \text{(A.1.19)}$$

will look as follows

$$I_1 = \int_{k_c - c\varepsilon^{\frac{\eta-\nu}{2}}}^{k_c + c\varepsilon^{\frac{\eta-\nu}{2}}} \frac{\varepsilon^\eta}{(k' - k_c)^2 + \varepsilon^\eta} \frac{\varepsilon^\eta}{(\tilde{k} - k' + k_c)^2 + \varepsilon^\eta} dk' \qquad \text{(A.1.20)}$$

which is identical with I_0 , with k replaced by $\tilde{k}$. Further difference is that $\tilde{k}$ can take negative values, but with condition A.1.10 replaced by

$$-|\tilde{k}| + c\varepsilon^{\frac{\eta-\nu}{2}} > 0 \qquad \text{(A.1.21)}$$

one can just repeat the analysis and gets

$$I_1 \leq \frac{\varepsilon^\eta}{(\frac{1}{2}\tilde{k})^2 + \varepsilon^\eta} \varepsilon^{\frac{\eta}{2}} c(1 + \mathcal{O}(\varepsilon^{\frac{\nu}{2}})) \qquad \text{(A.1.22)}$$

finally

$$I_1 \leq \varepsilon^{\frac{\eta}{2}} f(k, 2k_c) \cdot c(1 + \mathcal{O}(\varepsilon^{\frac{\nu}{2}})) \qquad \text{(A.1.23)}$$

Which complete the proof of the lemma 3.1.

A.2 Coefficients in the inequalities for norms

Our first object is the study of $F(k,t)$ given by

$$F(k,t) := \frac{|\rho(k)|}{\mu(k)}[e^{\mu(k)t} - 1] \qquad \text{(A.2.1)}$$

Note that $\mu(k)$ is a polynomial in k and has a positive maximum of the order ε^2 at $k = k_c$. In the vicinity of $k = k_c$ $\mu(k)$ is monotonic for both $k > k_c$ and $k < k_c$. In fact, for $|k - k_c|$ small we have

$$\mu(k) = \varepsilon^2 \mu_0 - \mu_1 (k - k_c)^2 + 0[(k - k_c)^3]; \quad \mu_0, \mu_1 > 0 \qquad \text{(A.2.2)}$$

By straightforward power series expansion one finds

$$|F(k,t)| = |\rho(k)| t [1 + 0(\mu(k)t)] \qquad \text{(A.2.3)}$$

The error term is of the order ε^η when $|k-k_c| = \mathcal{O}(\varepsilon)$ but becomes larger outside that region. On the other hand, for $(k-k_c)^2 > \frac{\mu_0}{\mu_1}\varepsilon^2$ the function $\mu(k)$ is negative, so that one than has

$$|F(k,t)| \leq \frac{|\rho(k)|}{-\mu(k)} \tag{A.2.4}$$

The order of magnitude of $F(k,t)$, over the whole domain of k, can be described by

$$|F(k,t)| \leq \frac{1}{\varepsilon^\eta} C(\varepsilon^\eta t) f_\eta(k,k_c) \tag{A.2.5}$$

where $C(\varepsilon^\eta t)$ is bounded and of order unity when $\varepsilon^\eta t$ is less or equal order unity.

Next we consider

$$F_1(k,t) := \frac{F(k,t)}{\mathrm{Max}[f_\eta(k,k_c), \varepsilon^\nu]} \tag{A.2.6}$$

Because of the denominator the situation is more complicated. As before we find

$$\text{for } |k-k_c| = \mathcal{O}(\varepsilon^{\frac{\eta-\nu}{2}}), \quad |F_1(k,t)| \leq \frac{c}{\varepsilon^\eta} \tag{A.2.7}$$

$$\text{for } |k-k_c| = \mathcal{O}(\varepsilon^p);\ p \leq \frac{\eta-\nu}{2}, \quad |F_1(k,t)| \leq \frac{c}{\varepsilon^{\nu+2p}} \tag{A.2.8}$$

In the above (and in the sequel) the symbol c denotes constants which (in a sharp estimate) are of course not all the same. Our conclusion is that in the supremum norm

$$Y(F_1) \leq \frac{1}{\varepsilon^\eta} \tilde{C}_1(\varepsilon^\eta T) \tag{A.2.9}$$

$$\tilde{C}_1(\varepsilon^\eta T) = \rho_0 \varepsilon^\eta T + C_0 \tag{A.2.10}$$

In order to deduce useful estimates in the L_1-norm we must be even more careful. We must assume that $|\rho(k)/\mu(k)|$ decays to zero for $|k| \to \infty$ sufficiently fast so that the integral over the whole k-axis (excluding neighbourhood where $\mu(k) = 0$), exists. This condition is automatically satisfied in the

differential operators in the basic equation (1.3) have leading terms of even order (in that case $|\rho/\mu| \sim k^{-2}$ for $|k| \to \infty$) Now to the estimates. The difficulty lies in the fact that the intervals in A.2.7 and A.2.8 contribute to the same order of magnitude, yet we must exploit the fact that the largest contributions come from a $|k-k_c| = \mathcal{O}(\varepsilon^{\frac{\eta-\nu}{2}})$ subinterval.
We divide the integration interval as follows

$$\int_0^\infty |F_1(k,t)|dk = [\int_0^{k_c - c\varepsilon^p} + \int_{k_c - c\varepsilon^p}^{k_c + c\varepsilon^p} + \int_{k_c + c\varepsilon^p}^\infty]|F_1(k,t)|dk \tag{A.2.11}$$

The middle integral is bounded by $Y(F_1)$ times the interval length. So we get

$$\int_0^\infty |F_1(k,t)|dk \leq Y(F_1)2c\varepsilon^p + [\int_0^{k_c - c\varepsilon^p} + \int_{k_c + c\varepsilon^p}^\infty]|F_1(k,t)|dk. \tag{A.2.12}$$

Using A.2.10 and A.2.12 it follows that

$$\int_0^\infty |F_1(k,t)|dk \leq 2c\tilde{C}_1(\varepsilon^\eta T)\varepsilon^{-\eta+p} + \varepsilon^{-\nu-2p}C. \tag{A.2.13}$$

Optimal choice of p is obtained by putting

$$-\eta + p = -\nu - 2p \to p = \frac{\eta - \nu}{3}, \tag{A.2.14}$$

so that the final result is:

$$X(F_1) = \varepsilon^{-\eta + \frac{\eta-\nu}{3}}\hat{C}_1(\varepsilon^\eta T) \tag{A.2.15}$$

We now turn to the analysis of

$$F_2(k,t) := F(k,t)\frac{c\,\mathrm{Max}[f_\eta(k,0), f_\eta(k,2k_c), \varepsilon^\nu]}{\mathrm{Max}[f_\eta(k,k_c), \varepsilon^\nu]} \tag{A.2.16}$$

When $k > c\varepsilon^{\frac{\eta-\nu}{2}}$ and $|k - 2k_c| > c\varepsilon^{\frac{\eta-\nu}{2}}$, then

$$F_2(k,t) = \varepsilon^\nu c F_1(k,t) \tag{A.2.17}$$

On the other hand, for $k = \mathcal{O}(\varepsilon^{\frac{\eta-\nu}{2}})$ or $|k - 2k_c| = \mathcal{O}(\varepsilon^{\frac{\eta-\nu}{2}})$ an easy estimate shows that

$$|F_2(k,t)| \leq \frac{c}{\varepsilon^{\nu}} \tag{A.2.18}$$

Therefore, using the results for F_1, it follows that

$$Y(F_2) = \frac{1}{\varepsilon^{\sigma}} \tilde{C}_2(\varepsilon^{\eta} T), \ \sigma = max(\nu, \eta - \nu) \tag{A.2.19}$$

Finally the L_1-norm of F_2. Near $k = 0$ and $k = 2k_c$ the contribution to the integral is of order $\varepsilon^{\frac{\eta}{2}-\nu}$, because integrals of $f_{\eta}(k,0)$, $f_{\eta}(k,2k_c)$ are of order $\varepsilon^{\frac{\eta}{2}}$. The contribution of the neighbourhood of $k = k_c$ is as established in the analysis of F_1. Therefore:

$$X(F_2) = \varepsilon^{\nu-\eta+\frac{\eta-\nu}{3}} \hat{C}_2(\varepsilon^{\eta} T) \tag{A.2.20}$$

A.3 Proof of Lemma 8.1

In what follows the symbol c denotes constants (which of course are not all the same). Furthermore, without loss of generality, we consider $k > 0$. We study the integral

$$\rho_0(k) = \int_{-\infty}^{\infty} \frac{|\ G(k', k-k')\ |}{g(k')\, g(k-k')}\, dk'.$$

$$G(k', k-k') = \rho_1(k')\ \rho_2(k-k') + \rho_2(k')\ \rho_1(k-k')$$

For sufficiently large values of the argument, say $k > k_0$,

$$\rho_1(k) \leq c\ k^{p_1}\ ,\ \rho_2(k) \leq c\ k^{p_2} \tag{A.3.1}$$

$$p_1 + p_2 = p\ \ ,\ \ p_1, p_2 \geq 0 \tag{A.3.2}$$

The function $g(k)$ is defined by

$$g(k) = 1 + \mid k \mid^{p} \tag{A.3.3}$$

We first consider the case $p = 1$, which can be computed explicitly, and to which the method for $p \geq 2$ does not apply.

With

$$\rho_1 = c_1, \quad \rho_2 = c_2 + c_3 k \tag{A.3.4}$$

we get

$$G(k', k - k') = 2c_1 c_2 + c_3 k \tag{A.3.5}$$

To get rid of the absolute-value signs the integral is decomposed and after some trivial transformations one gets

$$\rho_0(k) = (2c_1 c_2 + c_3 k) \left\{ 2 \int_0^\infty \frac{dk'}{(1+k')(1+k+k')} + \int_0^k \frac{dk'}{(1+k')(1+k-k')} \right\}. \tag{A.3.6}$$

Explicit computation then produces

$$\rho_0(k) = \frac{4(2c_1 c_2 + c_3 k)}{k(2+k)} \, \ell n \left(\frac{1}{1+k} \right) \tag{A.3.7}$$

Hence we even find that

$$\rho_0(k) \to 0 \text{ as } k \to \infty \tag{A.3.8}$$

Next the case $p \geq 2$. Again it's useful to decompose the integral, so we start with

$$\rho_0(k) = 2I_1 + I_2 \tag{A.3.9}$$

$$I_1 = \int_0^\infty \left[\frac{\rho_1(k') \, \rho_2(k+k') + \rho_2(k') \, \rho_1(k+k')}{g(k') \, g(k+k')} \right] dk' \tag{A.3.10}$$

$$I_2 = \int_0^k \left[\frac{\rho_1(k') \, \rho_2(k-k') + \rho_2(k')\rho_1(k-k')}{g(k')g(k-k')} \right] dk' \tag{A.3.11}$$

We first consider I_1. Take $k > k_0$ and decompose further

$$I_1 = \int_0^{k_0} [\dots] \; dk' + \int_{k_0}^{\infty} [\dots] \; dk' \tag{A.3.12}$$

Because of A.2.3 we get

$$\begin{aligned} I_1 \;&\leq\; c \int_0^{k_0} \frac{\rho_1(k')(k+k')^{p_1} + \rho_2(k')(k+k')^{p_2}}{[1+(k')^p][1+(k+k')^p]} \, dk' \\ &+\; c^2 \int_{k_0}^{\infty} \frac{(k')^{p_1}(k+k')^{p_2} + (k')^{p_2}(k+k')^{p_1}}{[1+(k')^p][1+(k+k')^p]} \, dk' \end{aligned} \tag{A.3.13}$$

One easily sees that the first integral can be estimated by a constant (which depends on k_0, but k_0 is fixed). Taking this into account we overestimate the second integral by writing

$$I_1 \leq c + c^2 \int_{k_0}^{\infty} \frac{(k'+k)^{p_1}(k+k')^{p_2} + (k'+k)^{p_2}(k+k')^{p_1} + 2}{[1+(k')^p][1+(k+k')^p]} \, dk' \tag{A.3.14}$$

Hence

$$I_1 \leq c + 2c^2 \int_{k_0}^{\infty} \frac{dk'}{[1+(k')^p]} \tag{A.3.15}$$

The remaining integral exists, because $p \geq 2$.

We now turn to I_2. For $k' \in [0,k]$, $\mid G(k', k-k') \mid$ has a maximum, which cannot be larger than ck^p. Hence

$$I_2 \leq ck^p \int_0^k \frac{dk'}{[1+(k')^p][1+(k-k')^p]} \tag{A.3.16}$$

Again we decompose:

$$\int_0^k [\dots] \; dk' = \int_0^{\frac{1}{2}k} [\dots] \; dk' + \int_{\frac{1}{2}k}^{k} [\dots] \; dk' \tag{A.3.17}$$

This leads to the estimate

$$I_2 \leq ck^p \left\{ \frac{1}{1+(\frac{1}{2}k)^p} \int_0^{\frac{1}{2}k} \frac{dk'}{1+(k')^p} + \frac{1}{1+(\frac{1}{2}k)^p} \int_{\frac{1}{2}k}^{k} \frac{dk'}{1+(k-k')^p} \right\} \tag{A.3.18}$$

Finally

$$I_2 \leq \frac{ck^p}{1+(\frac{1}{2}k)^p} \int_0^\infty \frac{dk'}{1+(k')^p} \tag{A.3.19}$$

which again is bounded. $\square$

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