FOURIER TRANSFORMS ON A SEMISIMPLE SYMMETRIC SPACE

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November 29, 1994

Introduction. Let G/H be a semisimple symmetric space, that is, G is a connected semisimple real Lie group with an involution σ, and H is an open subgroup of the group of fixed points for σ in G. The main purpose of this paper is to study an explicit Fourier transform on G/H. In terms of general representation theory the ('abstract') Fourier transform of a compactly supported smooth function $f \in C_c^\infty(G/H)$ is given by (see [5])

$$\hat{f}(\pi)(\eta) = \pi(f)\eta = \int_{G/H} f(x)\pi(x)\eta\, dx, \tag{1}$$

for $(\pi, \mathcal{H}_\pi)$ a unitary irreducible representation of G and $\eta \in (\mathcal{H}_\pi^{-\infty})^H$ an H-invariant distribution vector for π. Here dx is the invariant measure on G/H. Thus $\hat{f}(\pi)(\eta)$ is a smooth vector for $\mathcal{H}_\pi$, depending linearly on η. Our goal is to obtain an explicit version of the restriction of this Fourier transform to representations $(\pi, \mathcal{H}_\pi)$ in the (minimal) unitary principal series $(\pi_{\xi,\lambda}, \mathcal{H}_{\xi,\lambda})$ for G/H, under the assumption that the center of G is finite. In a sequel [9] to this paper it will be proved that a function $f \in C_c^\infty(G/H)$ is uniquely determined by the restriction of $\hat{f}$ to this series (a priori it is known that f is determined by $\hat{f}$).

Let θ be a Cartan involution of G commuting with σ, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$ be the ± 1 eigenspace decompositions of the Lie algebra $\mathfrak{g}$ of G, corresponding to θ and σ, respectively. Let $K = G^\theta$, then K is a maximal compact subgroup of G. The unitary principal series for G/H is a series of parabolically induced representations $\pi_{\xi,\lambda} = \mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$, with $P = MAN$ a minimal $\sigma\theta$-stable parabolic subgroup with the indicated Langlands decomposition, ξ a finite dimensional irreducible unitary representation of M and $\lambda \in i\mathfrak{a}_{\mathrm{q}}^*$. Here $\mathfrak{a}_{\mathrm{q}}$ is the intersection of the Lie algebra $\mathfrak{a}$ of A with $\mathfrak{q}$ – this is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. The unitary principal series for G/H and its non-unitary extension (allowing $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$) was studied in [2]. For the Fourier transform on G/H it is important to determine the space $C^{-\infty}(\xi{:}\lambda)^H$ of H-invariant distribution vectors for $\pi_{\xi,\lambda}$; this is done for λ in generic position in [2]. More precisely an explicit bijective linear map $j(\xi{:}\lambda)$ from a finite dimensional vector space $V(\xi)$ (independent of λ) to $C^{-\infty}(\xi{:}\lambda)^H$ is determined. Moreover, the dependence of $j(\xi{:}\lambda)$ on $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$ is meromorphic. An explicit Fourier transform can then be obtained by composing the map $(\xi, \lambda) \mapsto j(\xi{:}-\lambda)$ with the map $\eta \mapsto \hat{f}(\pi_{\xi,-\lambda})(\eta)$ in (1) (see also [18]). If G/H is a Riemannian symmetric space (or equivalently, if $H = K$) we obtain in this way Helgason's Fourier transform for G/K (see [15]). In this case the Fourier transform is holomorphic as a function of λ, but in general it is only meromorphic, and it need not make sense for all the representations in the unitary principal series.

The main result of this paper is the determination of a normalization $j^\circ(\xi{:}\lambda)$ of $j(\xi{:}\lambda)$ (with a meromorphic λ-dependent normalizing factor), which is regular on $i\mathfrak{a}_{\mathrm{q}}^*$. The *Fourier transform* $\hat{f}$ of $f \in C_c^\infty(G/H)$ can then be defined as above, but with $j(\xi{:}\lambda)$ replaced by $j^\circ(\xi{:}\lambda)$, that is we define

$$\hat{f}(\xi{:}\lambda) = \pi_{\xi,\lambda}(f) \circ j^\circ(\xi{:}-\lambda) \in \mathrm{Hom}_{\mathbb{C}}(V(\xi), \mathcal{H}_{\xi,-\lambda}^\infty) \tag{2}$$

for ξ as above and $\lambda \in i\mathfrak{a}_{\mathrm{q}}^*$; this Fourier transform is then real analytic as a function of λ. When G/H is Riemannian the normalization amounts to a division of the usual Fourier transform by Harish-Chandra's c-function (which is known to be non-zero for imaginary λ, see [11, Lemma 29]).

Let us explain in some detail the construction of $j^\circ(\xi{:}\lambda)$ and the proof of its regularity. Let $A(\bar{P}{:}P{:}\xi{:}\lambda)$ be the standard intertwining operator from $\pi_{P,\xi,\lambda} = \pi_{\xi,\lambda}$ to the principal series $\pi_{\bar{P},\xi,\lambda}$ induced from the parabolic subgroup $\bar{P}$ opposite to P. It is well known that $A(\bar{P}{:}P{:}\xi{:}\lambda)$ depends meromorphically on λ and is bijective for generic λ. We define $j^\circ(\xi{:}\lambda)$ by $j^\circ(\xi{:}\lambda) = A(\bar{P}{:}P{:}\xi{:}\lambda)^{-1} j(\bar{P}{:}\xi{:}\lambda)$, where $j(\bar{P}{:}\xi{:}\lambda)$ is constructed as $j(\xi{:}\lambda)$, but with P replaced by $\bar{P}$. It follows that $j^\circ(\xi{:}\lambda){:}\ V(\xi) \to C^{-\infty}(\xi{:}\lambda)^H$ is again a bijection, for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbf{c}}^*$. Let $\Sigma \subset \mathfrak{a}_{\mathrm{q}}^*$ denote the set of roots of $\mathfrak{a}_{\mathrm{q}}$ in $\mathfrak{g}$ and put

$$\mathfrak{a}_{\mathrm{q}}^*(\epsilon) = \bigl\{\lambda \in \mathfrak{a}_{\mathrm{q}\mathbf{c}}^* \bigm| |\langle \mathrm{Re}\,\lambda, \alpha\rangle| < \epsilon,\ (\forall \alpha \in \Sigma)\bigr\}. \tag{3}$$

for $\epsilon > 0$. The main result mentioned above is

Theorem 1. *There exists, for each finite dimensional unitary representation ξ of M, a constant $\epsilon > 0$ such that the function $\lambda \mapsto j^\circ(\xi{:}\lambda)$ is holomorphic on $\mathfrak{a}_{\mathrm{q}}^*(\epsilon)$. In particular, it is regular on $i\mathfrak{a}_{\mathrm{q}}^*$.*

In order to prove the regularity of $j^\circ(\xi{:}\lambda)$ it suffices to consider all the matrix coefficients formed by it and the K-finite vectors in $\mathcal{H}_{\xi,\lambda}$. The set of matrix coefficients formed by the unnormalized $j(\xi{:}\lambda)$ with the K-finite vectors in $\mathcal{H}_{\xi,\lambda}$ is spanned by (components of) the *Eisenstein integrals*, which were defined in [3]. By construction the Eisenstein integrals $E(\psi{:}\lambda)$ are K-spherical functions on G/H (that is, functions f taking their values in a finite dimensional representation space V_τ for K and satisfying $f(kx) = \tau(k)f(x)$ for all $k \in K, x \in G/H$). The Eisenstein integral depends linearly on the parameter ψ in a certain finite dimensional vector space ${}^\circ\mathcal{C}(\tau)$, and meromorphically on $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbf{c}}^*$. The first step in the proof of Theorem 1 consists of the identification of a normalization $E^\circ(\psi{:}\lambda)$ of $E(\psi{:}\lambda)$, such that the set of matrix coefficients formed by $j^\circ(\xi{:}\lambda)$ with the K-finite vectors in $\mathcal{H}_{\xi,\lambda}$ is spanned by components of the $E^\circ(\psi{:}\lambda)$. It turns out that $E^\circ(\psi{:}\lambda)$ is essentially the same as the *normalized Eisenstein integral* which was introduced in [3]. The normalizing factor, which is $\mathrm{End}({}^\circ\mathcal{C}(\tau))$-valued and meromorphic in λ, is determined from the asymptotic behavior of $E(\psi{:}\lambda)$ towards infinity along minimal $\sigma\theta$-stable parabolic subgroups (see Proposition 2).

The principal step in the proof of Theorem 1 is given in Theorem 2, which states that $\lambda \mapsto E^\circ(\psi{:}\lambda)$ is regular on $i\mathfrak{a}_{\mathrm{q}}^*$. This result is obtained by induction on the split rank of G/H (the dimension of $\mathfrak{a}_{\mathrm{q}}$). The induction results from the existence of asymptotic expansions of the normalized Eisenstein integral along maximal parabolic subgroups, in which the principal part is given by a linear combination of normalized Eisenstein integrals corresponding to symmetric spaces of lower split rank (see Theorem 4). The results of [3], in particular the unitarity of the c-functions, recalled here in Proposition 5, are crucial. These preliminary tools and some further basic properties of the Eisenstein integrals are collected in Sections 1-7. In particular, the proof of Theorem 1 is reduced to Theorem 2 in Section 5. In Section 8 we establish the framework for the parabolic induction, in Sections 9-11 we discuss some general results about asymptotic expansions of eigenfunctions depending holomorphically on λ, and in Section 12 we define the notion of the principal part. In Section 13 we state our main result (the above mentioned Theorem 4) about the asymptotic expansion of the Eisenstein integrals. The proof of this result is carried out in the subsequent three sections. Finally in Section 17 we give the proof of Theorem 2.

The group G is itself a symmetric space for the left times right action of $G \times G$. In this case ('the group case') the spaces $V(\xi)$ are all zero or one dimensional, and the distribution vectors $j^\circ(\xi{:}\lambda)\eta$ can be identified essentially as multiples of the distribution kernel of the inverse of a standard intertwining operator (for details, see [8, Lemma 2]). Thus in this case the regularity in Theorem 1 comes down to the injectivity of this operator for λ purely imaginary, or equivalently, to the regularity of Harish-Chandra's Plancherel factor $\mu(\lambda)$ (for minimal parabolic subgroups). This regularity is stated in [14, p. 142, Lemma 2]. The normalized Eisenstein integrals and their regularity can be found in [12, Thm. 6].

1. The Fourier transform. Let G/H be a reductive symmetric space of Harish-Chandra's class, that is, G is a real reductive Lie group of Harish-Chandra's class (cf. [13]), σ an involution of G, and H an open subgroup of the group G^σ of its fixed points. This assumption on G/H is somewhat more general than that of the introduction. Apart from this we use notation as defined above. As usual, the Killing form on $[\mathfrak{g},\mathfrak{g}]$ is extended to an invariant bilinear form B on $\mathfrak{g}$, for which the inner product $\langle X, Y\rangle = -B(X,\theta Y)$ is positive definite. We also require the extension to be compatible with σ, that is, $B(\sigma X, Y) = B(X, \sigma Y)$ for all $X, Y \in \mathfrak{g}$. The inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ is extended linearly to the complexification $\mathfrak{g}_{\mathbf{c}}$ of $\mathfrak{g}$.

As above $\mathfrak{a}_{\mathrm{q}}$ denotes a fixed maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$, and $P = MAN$ is a parabolic subgroup whose Levi part $M_1 = MA$ is the centralizer in G of $\mathfrak{a}_{\mathrm{q}}$. We denote the set of such parabolic subgroups by $\mathcal{P}_\sigma^{\min}$. Notice that for $P \in \mathcal{P}_\sigma^{\min}$ we have that M and M_1 are invariant under both involutions θ and σ. Let

$$K_M = M \cap K = M_1 \cap K, \quad H_M = M \cap H, \quad \text{and} \quad H_{M_1} = M_1 \cap H,$$

then the quotients M/K_M, M_1/K_M, M/H_M and M_1/H_{M_1} are symmetric spaces.

Furthermore we denote by $\pi_{\xi,\lambda} = \pi_{P,\xi,\lambda}$ the associated representations of the principal series. Here ξ belongs to $\widehat{M_{\mathrm{fu}}}$, the set of (equivalence classes of) irreducible finite dimensional unitary representations of M, and λ belongs to $\mathfrak{a}_{\mathrm{qc}}^*$ (which is viewed as a subspace of $\mathfrak{a}_{\mathbf{c}}^*$ by means of $\langle\cdot\,,\cdot\rangle$). We use the following model ('induction on the left') for $\pi_{\xi,\lambda}$: Let $\mathcal{H}_\xi$ denote a finite dimensional Hilbert space on which $\xi(M)$ acts (unitarily), and let $C^\infty(\xi{:}\lambda) = C^\infty(P{:}\xi{:}\lambda)$ denote the space of smooth $\mathcal{H}_\xi$-valued functions f on G satisfying

$$f(mang) = a^{\lambda+\rho_P}\xi(m)f(g), \qquad \text{for } m \in M, a \in A, n \in N, g \in G;$$

then $\pi_{\xi,\lambda}$ is the right regular representation of G on this space. Here $\rho_P \in \mathfrak{a}^*$ denotes half the trace of the adjoint action on $\mathfrak{n}$; it is easily seen that actually $\rho_P \in \mathfrak{a}_{\mathrm{q}}^*$. Similarly we denote by $C^{-\infty}(\xi{:}\lambda)$ the space of $\mathcal{H}_\xi$-valued generalized functions on G satisfying the above rule of transformation, and by $C^\infty(K{:}\xi)$ and $C^{-\infty}(K{:}\xi)$ the spaces of smooth, respectively generalized, $\mathcal{H}_\xi$-valued functions on K transforming according to $f(mk) = \xi(m)f(k)$ for all $m \in K_M$, $k \in K$. Then restriction to K gives rise to bijective linear maps from $C^\infty(\xi{:}\lambda)$ and $C^{-\infty}(\xi{:}\lambda)$ onto the corresponding function spaces $C^\infty(K{:}\xi)$ and $C^{-\infty}(K{:}\xi)$ on K, for all $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$. The Hilbert space $L^2(K{:}\xi)$ is defined similarly; the inner product is given by

$$\langle f|g\rangle = \int_K \langle f(k)|g(k)\rangle_\xi \, dk \tag{4}$$

with respect to invariant measure on K. (Here and in the following sesqui-linear Hilbert space inner products will be denoted $\langle\cdot|\cdot\rangle$, and the anti-linearity is in the second variable.) When viewed as the representation space for $\pi_{\xi,\lambda}$ we also denote $L^2(K{:}\xi)$ by $\mathcal{H}_{\xi,\lambda}$.

We denote by Σ the root system of $\mathfrak{a}_{\mathrm{q}}$ in $\mathfrak{g}$, and by W the group $N_K(\mathfrak{a}_{\mathrm{q}})/Z_K(\mathfrak{a}_{\mathrm{q}})$ which is naturally identified with the reflection group of Σ. For the time being we fix a set $\mathcal{W} \subset N_K(\mathfrak{a}_{\mathrm{q}})$ of representatives for the double quotient $Z_K(\mathfrak{a}_{\mathrm{q}})\backslash N_K(\mathfrak{a}_{\mathrm{q}})/N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$; the image of $\mathcal{W}$ in W is then a set of representatives for $W/W_{K\cap H}$, where $W_{K\cap H}$ is the subgroup $N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})/Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$ of W. The map $\mathcal{W} \ni w \mapsto PwH$ sets up a bijective correspondence of $\mathcal{W}$ with the set of open $P \times H$ cosets in G. Given an irreducible unitary representation ξ of M we denote by $V(\xi)$ the Hilbert space given by the formal orthogonal sum

$$V(\xi) = \bigoplus_{w\in\mathcal{W}} \mathcal{H}_\xi^{wH_Mw^{-1}} \tag{5}$$

of the spaces of wH_Mw^{-1}-fixed vectors for ξ. Here we notice that each $w \in N_K(\mathfrak{a}_{\mathrm{q}})$ normalizes M and K (but in general not H). Moreover it follows from the lemma below that if $V(\xi)$ is non-zero then the restriction of ξ to K_M is irreducible (so in particular $\xi \in M_{\mathrm{fu}}^\wedge$), and that we have

$$\mathcal{H}_\xi^{wH_Mw^{-1}} = \mathcal{H}_\xi^{w(M\cap H\cap K)w^{-1}}, \qquad (w \in \mathcal{W}).$$

Lemma 1. *There exists a connected normal closed subgroup M_n of M such that*

$$M = K_M M_n, \quad M_n \subset H_M \quad \text{and} \quad wM_nw^{-1} = M_n$$

for all $w \in N_K(\mathfrak{a}_{\mathrm{q}})$. In particular

$$M/wH_Mw^{-1} \simeq K_M/w(K_M \cap H_M)w^{-1}.$$

Proof. Let $\mathfrak{m}_n$ be the Lie subalgebra of $\mathfrak{m}$ generated by $\mathfrak{m}\cap\mathfrak{p}$. Obviously this is an ideal in $\mathfrak{m}$. Moreover it is $\operatorname{Ad} w$-invariant for $w \in N_K(\mathfrak{a}_{\mathrm{q}})$, and since $\mathfrak{a}_{\mathrm{q}}$ is maximal abelian in $\mathfrak{p}\cap\mathfrak{q}$ we have $\mathfrak{m}\cap\mathfrak{p}\subset\mathfrak{h}$ and hence $\mathfrak{m}_n\subset\mathfrak{h}$. Let M_n be the corresponding analytic subgroup of M, then $wM_nw^{-1} = M_n$ and $M_n \subset H$ are obvious, and $M = K_M M_n$ follows from $M = K_M \exp(\mathfrak{m}\cap\mathfrak{p})$. $\square$

Remark. Under quite general assumptions on the pair (G, H) one has that each of the summands of $V(\xi)$ in (5) has dimension at most one. For details about this result, which we shall not be using here, see [8] and the references given there.

Clearly the elements of $C^{-\infty}(P{:}\xi{:}\lambda)^H$ restrict to smooth functions on the open $P\times H$ cosets in G, and hence they can be evaluated at each $w \in \mathcal{W}$. Moreover, the value at w belongs to $\mathcal{H}_\xi^{wH_Mw^{-1}}$, and according to [2, Cor. 5.3] the map $\operatorname{ev}\colon C^{-\infty}(P{:}\xi{:}\lambda)^H \to V(\xi)$, given by the product of all these evaluations, is bijective for generic $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbf{c}}$. Furthermore, by [2, Thm. 5.10] this map allows an inverse map

$$j(\xi{:}\lambda) = j(P{:}\xi{:}\lambda)\colon V(\xi) \to C^{-\infty}(P{:}\xi{:}\lambda)^H$$

which depends meromorphically on λ (as a $C^{-\infty}(K{:}\xi)$ valued map). Thus by definition we have for $\eta \in V(\xi)$ that the $\mathcal{H}_\xi$-valued generalized function $j(\xi{:}\lambda)\eta$ on G restricts to the smooth function

$$j(\xi{:}\lambda)(\eta)(manwh) = a^{\lambda+\rho_P}\xi(m)\eta_w \tag{6}$$

on $PwH = MANwH$. (Here η_w denotes the w-component of η, viewed as an element of $\mathcal{H}_\xi$.) For any real number R we denote

$$\mathfrak{a}^*_{\mathrm{q}}(P;R) = \{\lambda\in\mathfrak{a}^*_{\mathrm{q}\mathbf{c}} \mid \operatorname{Re}\langle\lambda,\alpha\rangle < R,\ (\forall\alpha\in\Sigma(P))\} \tag{7}$$

where $\Sigma(P)$ is the positive system for Σ corresponding to P. It follows from [2, Prop. 5.6] (and the remark succeeding its proof) that if $\lambda+\rho_P \in \mathfrak{a}^*_{\mathrm{q}}(P;0)$ then $j(\xi{:}\lambda)\eta$ is the continuous function on G given by (6) on PwH, $w\in\mathcal{W}$, and by 0 elsewhere. Notice that if G/H is Riemannian then $V(\xi)$ is non-zero if and only if ξ is the trivial M-representation, in which case $V(\xi)\simeq\mathbb{C}$. Moreover in this case $j(\xi{:}\lambda)$ is essentially the map obtained from the Iwasawa decomposition $G = ANK$ by $ank\mapsto a^{\lambda+\rho}$. In particular it is smooth for all $\lambda\in\mathfrak{a}^*_{\mathrm{q}\mathbf{c}}$.

Given two minimal $\sigma\theta$-stable parabolic subgroups $P, P'\in\mathcal{P}_\sigma^{\min}$ (so that their Langlands decompositions $P = MAN$ and $P' = MAN'$ share the M and the A), there is a *standard intertwining operator* $A(P'{:}P{:}\xi{:}\lambda)$ from $C^{-\infty}(P{:}\xi{:}\lambda)$ to $C^{-\infty}(P'{:}\xi{:}\lambda)$ (formally given by integration over $N'\cap\bar N$ on the left). It depends meromorphically on λ and is bijective for generic λ, and by the intertwining property it maps H-invariant vectors to H-invariant vectors. Consequently we obtain an $\operatorname{End}(V(\xi))$-valued meromorphic map $\lambda\mapsto B(P'{:}P{:}\xi{:}\lambda)$ by requiring commutativity of the diagram

$$\begin{tikzcd} C^{-\infty}(P{:}\xi{:}\lambda)^H \arrow[r, "A(P':P:\xi:\lambda)"] & C^{-\infty}(P'{:}\xi{:}\lambda)^H \\ V(\xi) \arrow[u, "j(P:\xi:\lambda)"] \arrow[r, "B(P':P:\xi:\lambda)"'] & V(\xi). \arrow[u, "j(P':\xi:\lambda)"'] \end{tikzcd} \tag{8}$$

By construction the operator $B(P'{:}P{:}\xi{:}\lambda)$ is invertible for generic λ. It is a simple consequence of the definition of $A(P'{:}P{:}\xi{:}\lambda)$ that

$$A(P'{:}P{:}\xi{:}\lambda)^* = A(P{:}P'{:}\xi{:}-\bar{\lambda}). \tag{9}$$

The main result of [2] (see also [4]) asserts the much deeper analog

$$B(P'{:}P{:}\xi{:}\lambda)^* = B(P{:}P'{:}\xi{:}-\bar{\lambda}), \tag{10}$$

with respect to the Hilbert space structure of $V(\xi)$ defined by the orthogonal sum (5).

The normalized map

$$j^\circ(\xi{:}\lambda) = j^\circ(P{:}\xi{:}\lambda)\colon\; V(\xi) \to C^{-\infty}(P{:}\xi{:}\lambda)^H$$

is now defined as follows. In the diagram (8) let P' be the parabolic subgroup $\bar{P}$ opposite to P, and let $j^\circ(P{:}\xi{:}\lambda)$ be the map that goes diagonally from the lower right to the upper left corner, that is

$$j^\circ(P{:}\xi{:}\lambda) = A(\bar{P}{:}P{:}\xi{:}\lambda)^{-1} \circ j(\bar{P}{:}\xi{:}\lambda) = j(P{:}\xi{:}\lambda) \circ B(\bar{P}{:}P{:}\xi{:}\lambda)^{-1}. \tag{11}$$

The two equalities in (11) can be generalized as follows

$$j^\circ(P{:}\xi{:}\lambda) = A(P'{:}P{:}\xi{:}\lambda)^{-1} \circ j(P'{:}\xi{:}\lambda) \circ B(\bar{P}{:}P'{:}\xi{:}\lambda)^{-1}, \tag{12}$$

where $P' \in \mathcal{P}_\sigma^{\min}$ is arbitrary (use that

$$A(\bar{P}{:}P{:}\xi{:}\lambda) = A(\bar{P}{:}P'{:}\xi{:}\lambda) \circ A(P'{:}P{:}\xi{:}\lambda), \tag{13}$$

and similarly for B, cf. [2, Props. 4.6 and 6.2]).

From (12) and the analogue of (13) for B we obtain the following commutative diagram, which is analogous to (8)

$$\begin{array}{ccc} C^{-\infty}(P{:}\xi{:}\lambda)^H & \xrightarrow{A(P'{:}P{:}\xi{:}\lambda)} & C^{-\infty}(P'{:}\xi{:}\lambda)^H \\ {\scriptstyle j^\circ(P:\xi:\lambda)}\big\uparrow & & \big\uparrow{\scriptstyle j^\circ(P':\xi:\lambda)} \\ V(\xi) & \xrightarrow[B(\bar{P}'{:}\bar{P}{:}\xi{:}\lambda)]{} & V(\xi). \end{array} \tag{14}$$

We shall now discuss the singular sets of the meromorphic maps $\lambda \mapsto j(\xi{:}\lambda)$ and $\lambda \mapsto j^\circ(\xi{:}\lambda)$. Let $\Pi_\Sigma(\mathfrak{a}_\mathrm{q})$ denote the set of elements $p \in S(\mathfrak{a}_\mathrm{q}) = P(\mathfrak{a}_\mathrm{q}^*)$ which are products of polynomials of the form $\lambda \mapsto \langle\lambda, \alpha\rangle - c$, where $\alpha \in \Sigma$ and $c \in \mathbb{C}$. The product may be empty; this means just that $1 \in \Pi_\Sigma(\mathfrak{a}_\mathrm{q})$.

Lemma 2. *Let $P \in \mathcal{P}_\sigma^{\min}$, $\xi \in M_{\mathrm{fu}}^\wedge$ and $R > 0$ be given. Then there exists a polynomial $p \in \Pi_\Sigma(\mathfrak{a}_\mathrm{q})$ such that $\lambda \mapsto p(\lambda) j(P{:}\xi{:}\lambda)$ is holomorphic on the set $\mathfrak{a}_\mathrm{q}^*(P;R)$. Furthermore there exists $p \in \Pi_\Sigma(\mathfrak{a}_\mathrm{q})$ such that $\lambda \mapsto p(\lambda) j^\circ(P{:}\xi{:}\lambda)$ is holomorphic on $\mathfrak{a}_\mathrm{q}^*(R) = \mathfrak{a}_\mathrm{q}^*(P;R) \cap \mathfrak{a}_\mathrm{q}^*(\bar{P};R)$.*

Proof. For $j(\xi{:}\lambda)$ this is is a consequence of [3, Thm. 9.1]. For $j^\circ(\xi{:}\lambda)$ we must also isolate the singularities of the inverse intertwining operator $A(\bar{P}{:}P{:}\xi{:}\lambda)^{-1}$ used in the normalization (11). Recall (cf. [17, Prop. 7.3], [2, Prop. 4.7]) that for $P, P' \in \mathcal{P}_\sigma^{\min}$ one defines the meromorphic function $\lambda \mapsto \eta(P'{:}P{:}\xi{:}\lambda) \in \mathbb{C}$ on $\mathfrak{a}_{\mathrm{q}\mathbf{c}}^*$ by

$$A(P{:}P'{:}\xi{:}\lambda) \circ A(P'{:}P{:}\xi{:}\lambda) = \eta(P'{:}P{:}\xi{:}\lambda) I.$$

It follows from [17, Thm. 6.6] that there exists $p_1 \in \Pi_\Sigma(\mathfrak{a}_\mathrm{q})$ such that $\lambda \mapsto p_1(\lambda) A(P{:}\bar{P}{:}\xi{:}\lambda)$ is regular on $\mathfrak{a}_\mathrm{q}^*(R)$. Furthermore, by restricting to any K-type occurring in the principal series for ξ we infer from [3, Lemma 16.6] that there exists $p_2 \in \Pi_\Sigma(\mathfrak{a}_\mathrm{q})$ such that $\lambda \mapsto p_2(\lambda)\eta(\bar{P}{:}P{:}\xi{:}\lambda)^{-1}$ is regular on $\mathfrak{a}_\mathrm{q}^*(R)$. Hence with $p = p_1 p_2$ we find that $\lambda \mapsto p(\lambda) A(\bar{P}{:}P{:}\xi{:}\lambda)^{-1} = p(\lambda)\eta(\bar{P}{:}P{:}\xi{:}\lambda)^{-1} A(P{:}\bar{P}{:}\xi{:}\lambda)$ is regular on $\mathfrak{a}_\mathrm{q}^*(R)$. □

Let dx be a fixed invariant measure on G/H. The *Fourier transform* $f \mapsto \hat{f}(\xi{:}\lambda)$ is defined by (2) for $f \in C_c^\infty(G/H)$, $\xi \in M_{\mathrm{fu}}^\wedge$ and $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbf{c}}^*$, that is by

$$\hat{f}(\xi{:}\lambda)\eta = \int_{G/H} f(x) j^\circ(\xi{:}-\lambda)(\eta)(\cdot\, x)\, dx \in C^\infty(K{:}\xi), \qquad (\eta \in V(\xi)). \tag{15}$$

Then $\hat{f}(\xi{:}\lambda)$ is linear as a function of η and meromorphic as a function of λ, and it is regular on $i\mathfrak{a}_\mathrm{q}^*$ by Theorem 1 (to be proved later). Clearly $f \mapsto \hat{f}(\xi{:}\lambda)\eta$ is a G-equivariance, when $C^\infty(K{:}\xi)$ is identified with the representation space $\mathcal{H}_{\xi,-\lambda}^\infty$ for $\pi_{\xi,-\lambda}$.

Notice that strictly speaking the Fourier transform as defined above depends on the choice of the set $\mathcal{W}$ of representatives for $Z_K(\mathfrak{a}_\mathrm{q})\backslash N_K(\mathfrak{a}_\mathrm{q})/N_{K\cap H}(\mathfrak{a}_\mathrm{q})$. More precisely, if $\mathcal{W}' \subset N_{K\cap H}$ is another set of representatives, and we define the space $V'(\xi)$ as $V(\xi)$ above but with $\mathcal{W}'$ in place of $\mathcal{W}$, then there is a natural isometry $R_\xi \colon V(\xi) \to V'(\xi)$ (see [2, Lemma 5.8]). The map $j'(\xi{:}\lambda) \colon V'(\xi) \to C^{-\infty}(P{:}\xi{:}\lambda)^H$, defined as $j(\xi{:}\lambda)$ above, but with $\mathcal{W}'$ in place of $\mathcal{W}$, is then related to $j(\xi{:}\lambda)$ as follows:

$$j'(\xi{:}\lambda) \circ R_\xi = j(\xi{:}\lambda). \tag{16}$$

Moreover, the same formula holds with j replaced by j° on both sides. The corresponding Fourier transform $\hat{f}'(\xi{:}\lambda)$ is then related to $f(\xi{:}\lambda)$ by $\hat{f}'(\xi{:}\lambda) \circ R_\xi = \hat{f}(\xi{:}\lambda)$.

2. Eisenstein integrals. Eisenstein integrals for G/H were defined and analyzed in [3]. In this section we shall give a slightly more general definition and relate it to the previous one. This more general definition is necessary for the induction procedure in the proof of Theorem 1.

Instead of working with scalar-valued K-finite functions on G/H it is more convenient to consider τ-spherical functions f on G/H, that is V_τ-valued functions satisfying

$$f(kx) = \tau(k) f(x), \qquad (k \in K, x \in G/H).$$

Here (τ, V_τ) is a finite dimensional unitary representation of K. We denote by $C(G/H{:}\tau)$ the space of τ-spherical continuous functions from G/H into V_τ. The function spaces $C^\infty(G/H{:}\tau)$, $C_c^\infty(G/H{:}\tau)$ and $L^2(G/H{:}\tau)$ are defined similarly, and they are topologized in the obvious fashion with the induced topologies from $C(G/H)\otimes V_\tau$, $C^\infty(G/H)\otimes V_\tau$, etc. The Eisenstein integral, to be defined below, is a τ-spherical function on G/H.

Let M be as in the previous section. We shall now define a space ${}^\circ\mathcal{C}(\tau)$ which is analogous to the space ${}^\circ\mathcal{C}(M,\tau_M)$ of cusp forms in [13, Sect 19]. However, since we are only dealing with minimal $\sigma\theta$-stable parabolic subgroups, the actual notion of cusp forms is inessential. Let τ_M denote the restriction of τ on K_M, then $C^\infty(M/wH_Mw^{-1}{:}\tau_M)$ is the space of τ_M-spherical functions on M/wH_Mw^{-1}, for each $w \in N_K(\mathfrak{a}_{\mathrm{q}})$. We define ${}^\circ\mathcal{C}(\tau)$ to be the formal direct sum over $w \in \mathcal{W}$ of these spaces:

$$(17) \qquad {}^\circ\mathcal{C}(\tau) = \bigoplus_{w\in\mathcal{W}} C^\infty(M/wH_Mw^{-1}{:}\tau_M).$$

Given $w \in \mathcal{W}$ we accordingly write ${}^\circ\mathcal{C}_w(\tau)$ for the w-component of the space ${}^\circ\mathcal{C}(\tau)$, and if $\psi \in {}^\circ\mathcal{C}(\tau)$, we write ψ_w for its w-component. Notice that it follows from Lemma 1 that evaluation at e yields a linear bijection of ${}^\circ\mathcal{C}_w(\tau)$ onto $V_\tau^{w(M\cap K\cap H)w^{-1}}$, hence we have

$$(18) \qquad {}^\circ\mathcal{C}(\tau) \simeq \bigoplus_{w\in\mathcal{W}} V_\tau^{w(M\cap K\cap H)w^{-1}}$$

(again the sum is formal; it is not taken inside V_τ). In particular this shows that ${}^\circ\mathcal{C}(\tau)$ is finite dimensional. We equip $C^\infty(M/wH_Mw^{-1}{:}\tau_M)$ as a Hilbert space by means of the L^2-inner product with respect to the normalized invariant measure on the compact symmetric space M/wH_Mw^{-1}. Regarding (17) as an orthogonal sum we obtain a Hilbert space structure on ${}^\circ\mathcal{C}(\tau)$. The map (18) is an isometry if we similarly regard the sum on its right-hand side as an orthogonal sum.

To a pair (ψ,λ) of elements $\psi \in {}^\circ\mathcal{C}(\tau)$ and $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbf{c}}^*$ we associate a V_τ-valued function $\tilde\psi(\lambda)$ on G/H by

$$(19) \qquad \tilde\psi(\lambda{:}x) = \begin{cases} a^{\lambda+\rho_P}\psi_w(m) & \text{for } x = manwH \\ & (m\in M, a\in A, n\in N, w\in\mathcal{W}); \\ 0 & \text{for } x \notin \cup_{w\in\mathcal{W}} PwH. \end{cases}$$

It follows from [2, Prop. 5.6] that if $\lambda + \rho_P \in \mathfrak{a}_{\mathrm{q}}^*(P,0)$ then $\tilde\psi(\lambda)$ is continuous on G. The τ-Eisenstein integral is then defined by

$$(20) \qquad E(\psi{:}\lambda)(x) = E_\tau(P{:}\psi{:}\lambda)(x) = \int_K \tau(k)\tilde\psi(\lambda{:}k^{-1}x)\,dk,$$

for $x \in G/H$. Then $\psi \mapsto E(\psi{:}\lambda)$ is a linear map from ${}^\circ\mathcal{C}(\tau)$ to $C(G/H{:}\tau)$.

Notice that the construction of the Eisenstein integrals is 'functorial' in the following sense: Let (τ_1, V_1) and (τ_2, V_2) be unitary finite dimensional representations of K, and let

$\Phi \in \mathrm{Hom}_K(V_1, V_2)$. Then Φ induces a natural map, also denoted by Φ, from ${}^\circ\mathcal{C}(\tau_1)$ to ${}^\circ\mathcal{C}(\tau_2)$, for which we have

$$E_{\tau_2}(\Phi(\psi_1){:}\lambda)(x) = \Phi(E_{\tau_1}(\psi_1{:}\lambda)(x)), \qquad (\psi_1 \in {}^\circ\mathcal{C}(\tau_1)). \tag{21}$$

In particular this means that if (τ, V_τ) is reducible then the Eisenstein integral E_τ decomposes as the sum of the Eisenstein integrals corresponding to the reduction components of τ.

We shall now determine the relation of the Eisenstein integrals $E(\psi{:}\lambda)$ to the distributions $j(\xi{:}\lambda)$ of the previous section. First we relate the spaces $V(\xi)$ and ${}^\circ\mathcal{C}(\tau)$ to each other.

Let the finite dimensional unitary representations (τ, V_τ) and $(\xi, \mathcal{H}_\xi)$ be given as above. In what follows the Frobenius reciprocity theorem plays a prominent role. We shall be using it in the following formulation. Recall that $C(K{:}\xi)$ is the space of continuous functions $f\colon K \to \mathcal{H}_\xi$ transforming according to $f(mk) = \xi(m)f(k)$ for $k \in K$ and $m \in K_M$, and that K acts from the right on this space, thus providing a model for the induced representation $\mathrm{ind}_{K_M}^K \xi|_{K_M}$. Similarly, let $C(K{:}\xi{:}\tau)$ denote the space of continuous functions $f\colon K \to \mathcal{H}_\xi \otimes V_\tau$ transforming according to the rule:

$$f(mkk') = [\xi(m) \otimes \tau(k')^{-1}]\, f(k), \qquad (k, k' \in K,\ m \in K_M).$$

Then $C(K{:}\xi{:}\tau) \simeq [C(K{:}\xi) \otimes V_\tau]^K$, and Frobenius reciprocity asserts that evaluation at the identity element of K yields an isomorphism of $C(K{:}\xi{:}\tau)$ onto the finite dimensional space $[\mathcal{H}_\xi \otimes V_\tau]^{K_M}$. We denote this map by $\mathbf{e}$. Regarding $C(K{:}\xi{:}\tau)$ as a Hilbert space by means of the L^2-inner product on K we have that $\mathbf{e}$ is an isometry.

We now define, for each $w \in \mathcal{W}$, a sesqui-linear map

$$C(K{:}\xi{:}\tau) \times \mathcal{H}_\xi^{wH_Mw^{-1}} \to {}^\circ\mathcal{C}_w(\tau) = C^\infty(M/wH_Mw^{-1}{:}\tau_M) \tag{22}$$

by mapping the pair (f, η) to the V_τ-valued function $m \mapsto \langle f(e)|\xi(m)\eta\rangle$ on M. Here $\langle\,\cdot\,|\,\cdot\,\rangle\colon [\mathcal{H}_\xi \otimes V_\tau] \times \mathcal{H}_\xi \to V_\tau$ is the sesqui-linear map obtained from contraction by means of $\langle\,\cdot\,|\,\cdot\,\rangle_\xi$. Let $\bar{\mathcal{H}}_\xi$ be the linear space conjugate to $\mathcal{H}_\xi$, then we shall view the above map (22) as a linear map

$$C(K{:}\xi{:}\tau) \otimes \bar{\mathcal{H}}_\xi^{wH_Mw^{-1}} \to {}^\circ\mathcal{C}_w(\tau).$$

Of course one has that $\bar{\mathcal{H}}_\xi \simeq \mathcal{H}_{\xi^\vee}$ (as an M-module). This allows one to avoid the bar in the notations, if one likes. By direct summation over w we get a linear map $T \mapsto \psi_T$ from $C(K{:}\xi{:}\tau) \otimes \bar{V}(\xi)$ to ${}^\circ\mathcal{C}(\tau)$, where $\bar{V}(\xi)$ is the linear space conjugate to $V(\xi)$. We can now state the relation between ${}^\circ\mathcal{C}(\tau)$ and the $V(\xi)$:

Lemma 3. *Let (τ, V_τ) be a finite dimensional unitary representation of K. Define for each $\xi \in M_{\mathrm{fu}}^\wedge$ a linear map $T \mapsto \psi_T$ from $C(K{:}\xi{:}\tau) \otimes \bar{V}(\xi)$ to ${}^\circ\mathcal{C}(\tau)$ as above by*

$$(\psi_{f\otimes\eta})_w(m) = \langle f(e)|\xi(m)\eta_w\rangle \tag{23}$$

for $f \in C(K{:}\xi{:}\tau)$, $\eta \in V(\xi)$, $m \in M$ and $w \in \mathcal{W}$. Then the sum over $\xi \in \widehat{M_{\rm fu}}$ of the maps $(\dim \xi)^{1/2}\psi$ yields a surjective isometry

$$\bigoplus_\xi C(K{:}\xi{:}\tau) \otimes \bar{V}(\xi) \to {}^\circ\mathcal{C}(\tau).$$

Proof. Consider the matrix coefficient map

$$m_w : \mathcal{H}_\xi \otimes \bar{\mathcal{H}_\xi}^{wH_Mw^{-1}} \to C^\infty(M/wH_Mw^{-1}).$$

This map is equivariant for the obvious M-actions and its image is $C^\infty_\xi(M/wH_Mw^{-1})$, the space of functions of left type ξ in $C^\infty(M/wH_Mw^{-1})$. Moreover, $(\dim \xi)^{1/2} m_w$ is an isometry, by Lemma 1 and the Schur orthogonality relations for K_M. Let

$$C^\infty_\xi(M/wH_Mw^{-1}{:}\tau_M) \simeq [C^\infty_\xi(M/wH_Mw^{-1}) \otimes V_\tau]^{K_M}$$

be the space of functions in $C^\infty(M/wH_Mw^{-1}{:}\tau_M)$ of left type ξ. (Notice however that $C^\infty(M/wH_Mw^{-1}{:}\tau_M)$ is actually not invariant under the left action of M.) Then, again by Lemma 1, we have that $C^\infty(M/wH_Mw^{-1}{:}\tau_M)$ is the sum over $\xi \in \widehat{M_{\rm fu}}$ of the spaces $C^\infty_\xi(M/wH_Mw^{-1}{:}\tau_M)$. Now m_w provides us with an onto isomorphism

$$\tilde{m}_w = m_w \otimes I_{V_\tau} : [\mathcal{H}_\xi \otimes V_\tau]^{K_M} \otimes \bar{\mathcal{H}}_\xi^{wH_Mw^{-1}} \to C^\infty_\xi(M/wH_Mw^{-1}{:}\tau_M),$$

and the map $T \mapsto (\psi_T)_w$ from $C(K{:}\xi{:}\tau) \otimes \bar{\mathcal{H}}_\xi^{wH_Mw^{-1}}$ to $C^\infty(M/wH_Mw^{-1}{:}\tau_M)$ is easily identified as $\tilde{m}_w \circ (\mathbf{e} \otimes I)$, where $\mathbf{e}$ is the Frobenius reciprocity map and I the identity map on $\bar{\mathcal{H}}_\xi^{wH_Mw^{-1}}$. The lemma follows immediately. $\square$

We denote by ${}^\circ\mathcal{C}_\xi(\tau)$ the image of $C(K{:}\xi{:}\tau) \otimes \bar{V}(\xi)$ in ${}^\circ\mathcal{C}(\tau)$. Then ${}^\circ\mathcal{C}(\tau) = \oplus_\xi {}^\circ\mathcal{C}_\xi(\tau)$, and according to the proof above we have

$${}^\circ\mathcal{C}_\xi(\tau) = \bigoplus_{w \in \mathcal{W}} C^\infty_\xi(M/wH_Mw^{-1}{:}\tau_M).$$

Moreover ${}^\circ\mathcal{C}_\xi(\tau) \neq 0$ if and only if $V(\xi) \neq 0$ and $\xi^\vee|_{K_M}$ occurs in the decomposition of τ_M into irreducible K_M-types.

Notice that the map $T \mapsto \psi_T$ also depends on τ in a functorial way. In fact let ψ^j be associated to (τ_j, V_j), for $j = 1, 2$, and let $\Phi\colon V_1 \to V_2$ be a K-equivariant map. Then Φ naturally induces maps $\Phi\colon C(K{:}\xi{:}\tau_1) \to C(K{:}\xi{:}\tau_2)$ and $\Phi\colon {}^\circ\mathcal{C}(\tau_1) \to {}^\circ\mathcal{C}(\tau_2)$ and for $f_1 \in C(K{:}\xi{:}\tau_1)$ and $\eta \in \bar{V}(\xi)$ we have $\Phi(\psi^1_{f_1 \otimes \eta}) = \psi^2_{\Phi f_1 \otimes \eta}$.

We are now ready to give the promised relation between E and j. Let ξ be given, and let $T = f \otimes \eta \in C(K{:}\xi{:}\tau) \otimes \bar{V}(\xi)$. Then we obtain from (19) and (23) that

$$\widetilde{\psi_T}(\lambda{:}manwH) = a^{\lambda+\rho_P} \langle f(e) | \xi(m)\eta_w \rangle,$$

and comparing with (6) we see that $\widetilde{\psi_T}(\lambda{:}gH) = \langle f(e) \,|\, [j(\xi{:}\bar{\lambda})\eta](g)\rangle$ for all $g \in G$. Hence by (20) we have

$$E(\psi_T{:}\lambda)(gH) = \int_K \tau(k^{-1})\langle f(e) \,|\, [j(\xi{:}\bar{\lambda})\eta](kg)\rangle dk = \int_K \langle f(k) \,|\, [j(\xi{:}\bar{\lambda})\eta](kg)\rangle dk.$$

Here $\langle \cdot \,|\, \cdot \rangle\colon [\mathcal{H}_\xi \otimes V_\tau] \times \mathcal{H}_\xi \to V_\tau$ is the before-mentioned contraction. Using the same contraction we define a sesqui-linear map $C(K{:}\xi{:}\tau) \times C(K{:}\xi) \to V_\tau$ by

$$\langle f|g\rangle = \int_K \langle f(k)|g(k)\rangle \, dk \tag{24}$$

for $f \in C(K{:}\xi{:}\tau)$, $g \in C(K{:}\xi)$, and we finally have

$$E(\psi_T{:}\lambda)(gH) = \langle f|\pi_{\xi,\bar{\lambda}}(g)j(\xi{:}\bar{\lambda})\eta\rangle, \qquad (g \in G), \tag{25}$$

for $T = f \otimes \eta \in C(K{:}\xi{:}\tau) \otimes \bar{V}(\xi)$. In particular it follows that the Eisenstein integral is a smooth function on G/H. A priori (25) holds when $\lambda + \rho_P \in \mathfrak{a}_{\mathrm{q}}^*(P, 0)$, the range in which we have defined $E(\psi{:}\lambda)$ and in which $j(\xi{:}\bar{\lambda})\eta$ is continuous. However since all elements $f \in C(K{:}\xi{:}\tau)$ are smooth functions on K, the sesqui-linear map in (24) makes sense for $\varphi \in C^{-\infty}(K{:}\xi)$, and hence we get from the results of [2] cited in the previous section, that $\lambda \mapsto E(\psi{:}\lambda)$ extends to a meromorphic $C^\infty(G/H{:}\tau)$-valued function on $\mathfrak{a}_{\mathrm{qc}}^*$ for which (25) holds (in the generalized sense). Notice that this expression shows that the components of the vector valued function $E(\psi{:}\lambda)$ are finite sums of (generalized) matrix coefficients of $j(\xi{:}\bar{\lambda})\eta$ with K-finite vectors, for all $\psi \in {}^\circ\mathcal{C}(\tau)$.

The expression (25) can also be used to relate the Eisenstein integrals $E(P{:}\psi{:}\lambda)$ constructed from different parabolic subgroups P to each other. Using (8) and (9) it is easily seen that

$$E(P{:}\psi_{f\otimes\eta}{:}\lambda) = E(P'{:}\psi_{A(P:P':\xi:-\lambda)^{-1}f\otimes B(P':P:\xi:\bar{\lambda})\eta}{:}\lambda), \qquad (P, P' \in \mathcal{P}_\sigma^{\min}). \tag{26}$$

A priori the intertwining operator $A(P{:}P'{:}\xi{:}-\lambda)$ acts on $C(K{:}\xi)$; the action on $C(K{:}\xi{:}\tau) \simeq [C(K{:}\xi) \otimes V_\tau]^K$ is obtained from tensoring with the trivial action on V_τ.

Notice that if $\mathcal{W}'$ is a second choice of representatives for $Z_K(\mathfrak{a}_{\mathrm{q}})\backslash N_K(\mathfrak{a}_{\mathrm{q}})/N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$, then in analogy with (16) it is easily seen that there is a natural isometry $R_\tau\colon {}^\circ\mathcal{C}(\tau) \to {}^\circ\mathcal{C}(\tau)'$ such that

$$E'(R_\tau\psi{:}\lambda) = E(\psi{:}\lambda), \qquad (\psi \in {}^\circ\mathcal{C}(\tau)), \tag{27}$$

where the quantities with a prime are defined with the new set $\mathcal{W}'$ in place of $\mathcal{W}$.

We shall now relate these Eisenstein integrals to those of [3]. Let ϑ be a finite set of equivalence classes of finite dimensional irreducible representations (μ, V_μ) of K, and let $\mathbf{V}_\vartheta = C(K)_\vartheta$ be the space of K-finite functions on K, whose isotopy types for the left regular representation are contained in ϑ. Thus by Peter-Weyl theory we have the linear isomorphism

$$\mathbf{V}_\vartheta \simeq \sum_{\mu\in\vartheta} V_\mu \otimes \bar{V}_\mu, \tag{28}$$

where $\bar{V}_\mu$ is the conjugate linear space to V_μ. We provide $\mathbf{V}_\vartheta$ with the inner product as a subspace of $L^2(K)$, and define τ_ϑ to be the unitary representation of K on $\mathbf{V}_\vartheta$ obtained from the right action. In the above expression for $\mathbf{V}_\vartheta$ we thus have $\tau_\vartheta \simeq \sum_{\mu\in\vartheta} 1_\mu \otimes \mu^\vee$, where 1_μ is the trivial representation on V_μ, and where $\mu^\vee$ is the representation contragradient to μ, realized on $\bar{V}_\mu$. The Eisenstein integrals of [3] are obtained by specializing the above construction of the V_τ-valued function $E_\tau(\psi{:}\lambda)$ to the case where $(\tau, V_\tau) = (\tau_\vartheta, \mathbf{V}_\vartheta)$.

The map ψ of Lemma 3 can be somewhat simplified in the case when $(\tau, V_\tau) = (\tau_\vartheta, \mathbf{V}_\vartheta)$. Let $C(K{:}\xi)_\vartheta$ denote the subspace of $C(K{:}\xi)$ consisting of the K-finite functions whose (right) K-types belong to ϑ, and let $I \otimes \delta_e \colon C(K{:}\xi{:}\tau_\vartheta) = [C(K{:}\xi) \otimes \mathbf{V}_\vartheta]^K \to C(K{:}\xi)$ be the linear map obtained from evaluation of the elements of $\mathbf{V}_\vartheta = C(K)_\vartheta$ at e, then it is easily seen that $I \otimes \delta_e$ maps $C(K{:}\xi{:}\tau_\vartheta)$ bijectively onto $C(K{:}\xi)_\vartheta$ (use (28)). For $f \in C(K{:}\xi{:}\tau_\vartheta)$ and $\eta \in V(\xi)$ it follows easily from (23) that

$$(\psi_{f\otimes\eta})_w(m)(k) = \langle [(I \otimes \delta_e)f](k^{-1}) | \xi(m)\eta_w \rangle_\xi,$$

for $m \in M, k \in K$. We shall henceforth identify $C(K{:}\xi{:}\tau_\vartheta)$ with $C(K{:}\xi)_\vartheta$ by means of $I \otimes \delta_e$, and we write accordingly (cf. also [3, p. 346])

$$(29) \qquad (\psi_{f\otimes\eta})_w(m)(k) = \langle f(k^{-1}) | \xi(m)\eta_w \rangle_\xi, \qquad (m \in M, k \in K).$$

Specializing (25) to τ_ϑ and applying δ_e we obtain (as in [3, Lemma 4.2])

$$(30) \qquad E_{\tau_\vartheta}(\psi_T{:}\lambda)(gH)(e) = \langle f | \pi_{\xi,\bar\lambda}(g) j(\xi{:}\bar\lambda)\eta \rangle, \qquad (g \in G),$$

for $T = f \otimes \eta \in C(K{:}\xi)_\vartheta \otimes \bar{V}(\xi)$, where $\langle\cdot|\cdot\rangle$ now is the sesqui-linear product (4) on $C(K{:}\xi)$.

On the other hand, for general K-representations (τ, V_τ), the Eisenstein integral $E_\tau(\psi{:}\lambda)$ can be expressed by means of the $E_{\tau_\vartheta}(\psi{:}\lambda)$ as follows. Let ϑ be the set of K-types occurring in $\tau^\vee$, and let $(\tau_\vartheta, \mathbf{V}_\vartheta)$ be constructed as above. Consider the space $\mathbf{V}_\vartheta \otimes V_\tau$ with the K-representations $\tau_\vartheta \otimes 1_\tau$ and $\ell_\vartheta \otimes \tau$, where 1_τ denotes the trivial K-representation on V_τ, and ℓ_ϑ denotes the representation obtained from the left action on $\mathbf{V}_\vartheta$. Clearly these representations commute with each other, and hence the space

$$(31) \qquad (\mathbf{V}_\vartheta \otimes V_\tau)^{(\ell_\vartheta\otimes\tau)(K)}$$

of fixed vectors for the latter action is an invariant subspace with respect to the former action. It is now easily seen that evaluation in the identity in the first factor of the tensor product yields a K-equivariant isomorphism of the space (31) onto V_τ. Let $\Phi \colon V_\tau \hookrightarrow \mathbf{V}_\vartheta \otimes V_\tau$ be the embedding obtained from the inverse of this isomorphism. It follows from the functorial property (21) that we have

$$(32) \qquad \Phi E_\tau(\psi{:}\lambda) = E_{\tau_\vartheta\otimes 1_\tau}(\Phi(\psi){:}\lambda)$$

for $\psi \in {}^\circ\mathcal{C}(\tau)$. Again by functoriality one sees that ${}^\circ\mathcal{C}(\tau_\vartheta \otimes 1_\tau) \simeq {}^\circ\mathcal{C}(\tau_\vartheta) \otimes V_\tau$ and

$$(33) \qquad E_{\tau_\vartheta\otimes 1_\tau}(\psi\otimes u : \lambda) = E_{\tau_\vartheta}(\psi{:}\lambda) \otimes u$$

for $\psi \in {}^\circ\mathcal{C}(\tau_\vartheta)$, $u \in V_\tau$. Using these relations we shall sometimes derive properties of the Eisenstein integrals in the present generality from the corresponding properties in [3].

3. Invariant differential operators. Let $\mathbb{D}(G/H)$ denote the algebra of invariant differential operators on G/H. Let $U(\mathfrak{g})$ be the universal enveloping algebra of the complexification of $\mathfrak{g}$ and recall that the right action of G on $C^\infty(G)$ induces a homomorphism r from $U(\mathfrak{g})^H$ onto $\mathbb{D}(G/H)$, whose kernel is $U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}$. In the following we shall frequently abuse notation by identifying an element $D \in \mathbb{D}(G/H)$ with any $X \in U(\mathfrak{g})^H$ for which $D = r(X)$.

Let $\mathfrak{b}$ be a Cartan subspace for G/H (that is a maximal abelian subspace of $\mathfrak{q}$ consisting of semisimple elements), containing $\mathfrak{a}_{\mathrm{q}}$. Then $\mathfrak{b} \cap \mathfrak{p} = \mathfrak{a}_{\mathrm{q}}$ and $\mathfrak{b} = \mathfrak{b}_{\mathrm{k}} \oplus \mathfrak{a}_{\mathrm{q}}$, where $\mathfrak{b}_{\mathrm{k}} = \mathfrak{b} \cap \mathfrak{k}$. Let $W(\mathfrak{b})$ denote the reflection group of the root system of $\mathfrak{b}_{\mathbf{c}}$ in $\mathfrak{g}_{\mathbf{c}}$, then the Harish-Chandra isomorphism $\gamma = \gamma_{G/H}$ for G/H maps $\mathbb{D}(G/H)$ isomorphically onto $S(\mathfrak{b})^{W(\mathfrak{b})}$, the algebra of invariants for $W(\mathfrak{b})$ in the symmetric algebra $S(\mathfrak{b})$.

We shall now define a similar homomorphism (cf. [3, Sect. 2])

$$\mu\colon \mathbb{D}(G/H) \to \mathbb{D}(M_1/H_{M_1}) \simeq \mathbb{D}(M/H_M) \otimes S(\mathfrak{a}_{\mathrm{q}}).$$

Here $\mathbb{D}(M_1/H_{M_1})$ and $\mathbb{D}(M/H_M)$ denote the algebras of invariant differential operators on the symmetric spaces M_1/H_{M_1} and M/H_M respectively, and the isomorphism between $\mathbb{D}(M_1/H_{M_1})$ and $\mathbb{D}(M/H_M) \otimes S(\mathfrak{a}_{\mathrm{q}})$ is obtained from the decomposition $\mathfrak{m}_1 = \mathfrak{m} \oplus \mathfrak{a}_{\mathrm{q}} \oplus \mathfrak{a}_{\mathrm{h}}$, where $\mathfrak{a}_{\mathrm{h}} = \mathfrak{a} \cap \mathfrak{h}$. Let $P \in \mathcal{P}_\sigma^{\min}$ be given. We first define a homomorphism

$$`\mu_P\colon \mathbb{D}(G/H) \to \mathbb{D}(M_1/H_{M_1})$$

by the requirement

$$D - `\mu_P(D) \in \mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}$$

(it is used that $\mathfrak{g} = \mathfrak{n} + \mathfrak{m}_1 + \mathfrak{h}$, and elements of $\mathbb{D}(M_1/H_{M_1})$ and $U(\mathfrak{m}_1)^{H_{M_1}}$ are identified, as mentioned above for G/H). Let $d_P(m) = |\det(\mathrm{Ad}(m)|_{\mathfrak{n}})|^{1/2}$ for $m \in M_1$, then $d_P(ma) = a^{\rho_P}$ for $m \in MA_{\mathrm{h}}$, $a \in A_{\mathrm{q}}$, where $A_{\mathrm{h}} = \exp \mathfrak{a}_{\mathrm{h}}$ and $A_{\mathrm{q}} = \exp \mathfrak{a}_{\mathrm{q}}$. In particular we may view d_P as a function on M_1/H_{M_1} and define an operator $T_P\colon \mathbb{D}(M_1/H_{M_1}) \to \mathbb{D}(M_1/H_{M_1})$ by $T_P(D) = d_P^{-1} \circ D \circ d_P$. Equivalently T_P is given by $T_P = I_{\mathbb{D}(M/H_M)} \otimes T_{\rho_P}$, where T_{ρ_P} is the automorphism of $S(\mathfrak{a}_{\mathrm{q}})$ given by $T_{\rho_P} u(\lambda) = u(\lambda + \rho_P)$ for $u \in S(\mathfrak{a}_{\mathrm{q}})$, $\lambda \in \mathfrak{a}_{\mathrm{q}}$. Now μ is defined by

$$\mu = T_P \circ `\mu_P. \tag{34}$$

Notice that $\mathfrak{b}$ is also a Cartan subspace for the symmetric space M_1/H_{M_1}. Let $W_{M_1}(\mathfrak{b})$ be the reflection group of its root system and let $\gamma_{M_1/H_{M_1}}$ be the Harish-Chandra isomorphism from $\mathbb{D}(M_1/H_{M_1})$ onto $S(\mathfrak{b})^{W_{M_1}(\mathfrak{b})}$. Then it is easily verified that

$$\gamma_{M_1/H_{M_1}} \circ \mu = \gamma_{G/H}. \tag{35}$$

In particular it follows that μ is injective, and that it is independent of the choice of the parabolic subgroup P (as already indicated by the absence of P as subscript).

The map $`\mu_{\bar P}\colon \mathbb{D}(G/H) \to \mathbb{D}(M_1/H_{M_1})$ is also denoted μ'_P; by (34), the independence of μ on P, and the relation $d_{\bar P} = d_P^{-1}$ we have

$$\mu'_P = T_P \circ \mu. \tag{36}$$

Let $w \in \mathcal{W}$. Then $\mathrm{Ad}(w)$ maps M_1/H_{M_1} onto $M_1/wH_{M_1}w^{-1}$. Moreover, by conjugation with w inside $U(\mathfrak{g})$ we get a map from $U(\mathfrak{m}_1)^{H_{M_1}}$ to $U(\mathfrak{m}_1)^{wH_{M_1}w^{-1}}$, which induces a map from $\mathbb{D}(M_1/H_{M_1})$ to $\mathbb{D}(M_1/wH_{M_1}w^{-1})$. We denote this map by $D \mapsto \mathrm{Ad}(w)D$. Let

$$\mu_w = \mathrm{Ad}(w) \circ \mu \colon \mathbb{D}(G/H) \to \mathbb{D}(M_1/wH_{M_1}w^{-1}) \simeq \mathbb{D}(M/wH_Mw^{-1}) \otimes S(\mathfrak{a}_{\mathrm{q}}).$$

Let $\xi \in M_{\mathrm{fu}}\widehat{\ }$. The algebra $\mathbb{D}(M/wH_Mw^{-1})$ acts naturally on $\mathcal{H}_\xi^{wH_Mw^{-1}}$ by ξ, and thus we have a homomorphism of algebras

$$\xi_w \colon \mathbb{D}(M/wH_Mw^{-1}) \to \mathrm{End}(\mathcal{H}_\xi^{wH_Mw^{-1}}).$$

Let the algebra homomorphism

$$\mu(\xi) \colon \mathbb{D}(G/H) \to \mathrm{End}(V(\xi)) \otimes S(\mathfrak{a}_{\mathrm{q}})$$

be defined as the direct sum over $w \in \mathcal{W}$ of the maps $(\xi_w \otimes I) \circ \mu_w$. For $D \in \mathbb{D}(G/H)$ and $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$ we denote by $\mu(D{:}\xi{:}\lambda)$ the endomorphism of $V(\xi)$ obtained from $\mu(\xi)(D)$ by evaluation of its $S(\mathfrak{a}_{\mathrm{q}})$ components in λ.

For each $w \in \mathcal{W}$ and $D \in \mathbb{D}(G/H)$ we have:

$$D - {}'\mu_{w^{-1}Pw}(D) \in \mathrm{Ad}\, w^{-1}(\mathfrak{n})U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}. \tag{37}$$

By the independence of μ on P we have $T_{w^{-1}Pw} \circ {}'\mu_{w^{-1}Pw} = \mu$, or equivalently

$$(I_{\mathbb{D}(M/wH_Mw^{-1})} \otimes T_\rho) \circ \mathrm{Ad}(w) \circ {}'\mu_{w^{-1}Pw} = \mu_w.$$

From (6) and (37) it then follows that

$$D(j(\xi{:}\lambda)\eta) = j(\xi{:}\lambda)(\mu(D{:}\xi{:}\lambda)\eta), \qquad (\eta \in V(\xi)), \tag{38}$$

for all $D \in \mathbb{D}(G/H)$, as a meromorphic identity in $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$ (cf. also [3, Lemma 4.4]).

Being G-equivariant the operator $A(P'{:}P{:}\xi{:}\lambda)$ is in particular intertwining for the actions of $\mathbb{D}(G/H)$ on $C^{-\infty}(P{:}\xi{:}\lambda)^H$ and $C^{-\infty}(P'{:}\xi{:}\lambda)^H$. By the injectivity of $j(\xi{:}\lambda)$ (for generic λ), and the fact that μ is independent of the choice of parabolic subgroup, we conclude from (38) and the diagram (8) that

$$\mu(D{:}\xi{:}\lambda) \circ B(P'{:}P{:}\xi{:}\lambda) = B(P'{:}P{:}\xi{:}\lambda) \circ \mu(D{:}\xi{:}\lambda) \tag{39}$$

as a meromorphic identity in λ. In particular we have that the relation (38) holds for $j^\circ(\xi{:}\lambda)$ as well:

$$D(j^\circ(\xi{:}\lambda)\eta) = j^\circ(\xi{:}\lambda)(\mu(D{:}\xi{:}\lambda)\eta), \qquad (\eta \in V(\xi)). \tag{40}$$

For $D \in \mathbb{D}(G/H)$ let $D^* \in \mathbb{D}(G/H)$ be its formal (Hermitian) adjoint with respect to the invariant measure dx on G/H. Then by [3, p. 435-436] we have

$$\mu(D^*{:}\xi{:}\lambda) = \mu(D{:}\xi{:} -\bar{\lambda})^* \in \mathrm{End}(V(\xi)), \tag{41}$$

where the asterisk on the right-hand side denotes the adjoint with respect to the Hilbert space structure of $V(\xi)$. It follows immediately from the definition (15) of the Fourier transform and (40) that we have

$$(Df)\hat{}(\xi{:}\lambda) = \hat{f}(\xi{:}\lambda)\circ\mu(\bar{D}{:}\xi{:}\bar{\lambda})^*, \qquad (f\in C_c^\infty(G/H)), \tag{42}$$

where $\bar{D}\in\mathbb{D}(G/H)$ is the conjugate of D, defined by $\bar{D}\phi = \overline{D\bar{\phi}}$ for $\phi\in\mathbb{C}^\infty(G/H)$.

Let (τ, V_τ) be as in Section 2. There is a natural action of the algebra $\mathbb{D}(M/wH_Mw^{-1})$ on the finite dimensional space $C^\infty(M/wH_Mw^{-1}{:}\tau_M)$, for each $w\in\mathcal{W}$. We thus have a homomorphism of algebras $r_w\colon\mathbb{D}(M/wH_Mw^{-1})\to\mathrm{End}({}^\circ\mathcal{C}_w(\tau))$. Let the homomorphism

$$\mu(\tau)\colon\mathbb{D}(G/H)\to\mathrm{End}({}^\circ\mathcal{C}(\tau))\otimes S(\mathfrak{a}_{\mathrm{q}})$$

be the direct sum over $w\in\mathcal{W}$ of the maps $(r_w\otimes I)\circ\mu_w$. Notice that it follows from (23) that $r_w(\psi_{f\otimes\eta})_w = (\psi_{f\otimes\xi_w\eta_w})_w$ and hence

$$\mu(D{:}\tau{:}\lambda)\psi_{f\otimes\eta} = \psi_{f\otimes\mu(\bar{D}{:}\xi{:}\bar{\lambda})\eta}, \tag{43}$$

for $f\in C(K{:}\xi{:}\tau)$, $\eta\in V(\xi)$, and $D\in\mathbb{D}(G/H)$ (recall that $\psi_{f\otimes\eta}$ is anti-linear as a function of $\eta\in V(\xi)$). Using (25) we infer from (38) and (43) that

$$DE_\tau(\psi{:}\lambda) = E_\tau(\mu(D{:}\tau{:}\lambda)\psi{:}\lambda) \tag{44}$$

for all $\psi\in{}^\circ\mathcal{C}(\tau)$, $D\in\mathbb{D}(G/H)$, as a meromorphic identity in $\lambda\in\mathfrak{a}_{\mathrm{qc}}^*$ (cf. [3, Lemma 4.5] for $\tau=\tau_\vartheta$).

The endomorphisms $\mu(D{:}\xi{:}\lambda)$ and $\mu(D{:}\tau{:}\lambda)$ of $V(\xi)$ and ${}^\circ\mathcal{C}(\tau)$, respectively, are diagonalizable. More precisely the following result holds. We view $\mathfrak{b}_{\mathrm{kc}}^*$ and $\mathfrak{a}_{\mathrm{qc}}^*$ as subspaces of $\mathfrak{b}_{\mathrm{c}}^*$, according to the decomposition $\mathfrak{b}=\mathfrak{b}_{\mathrm{k}}\oplus\mathfrak{a}_{\mathrm{q}}$.

Lemma 4. *There exists, for each $\xi\in\widehat{M_{\mathrm{fu}}}$, a finite set L_ξ of elements $\Lambda\in\mathfrak{b}_{\mathrm{kc}}^*$ such that the endomorphisms $\mu(D{:}\xi{:}\lambda)$, for $\lambda\in\mathfrak{a}_{\mathrm{qc}}^*$ and $D\in\mathbb{D}(G/H)$, of $V(\xi)$ are simultaneously diagonalizable with eigenvalues of the form $\gamma(D{:}\Lambda+\lambda)$ with $\Lambda\in L_\xi$.*

Similarly there exists, for each $\tau\in\widehat{M_{\mathrm{fu}}}$, a finite set $L_\tau\subset\mathfrak{b}_{\mathrm{kc}}^*$ such that the endomorphisms $\mu(D{:}\tau{:}\lambda)\in\mathrm{End}({}^\circ\mathcal{C}(\tau))$ are simultaneously diagonalizable with eigenvalues of the form $\gamma(D{:}\Lambda+\lambda)$ with $\Lambda\in L_\tau$.

Proof. It follows from [3, proof of Lemma 4.8 (see the lines following the display (37))] that the endomorphisms $\xi_e(D)$, $D\in\mathbb{D}(M/H_M)$, of $\mathcal{H}_\xi^{H_M}$ are simultaneously diagonalizable, with eigenvalues of the form $\gamma_{M/H_M}(D{:}\Lambda)$ where $\Lambda\in\mathfrak{b}_{\mathrm{kc}}^*$. Conjugating by w we infer that a similar statement holds for $\xi_w(D)$, $D\in\mathbb{D}(M/wH_Mw^{-1})$. The statement about $\mu(D{:}\xi{:}\lambda)$ now follows immediately from (35) and the definition of $\mu(\xi)$ as the direct sum of the maps $(\xi_w\otimes I)\circ\mu_w$, and the statement for τ is then a consequence of Lemma 3 and (43). $\square$

4. Asymptotic expansions. Let (τ, V_τ) be as above. It follows from the differential equation (44) that the components of the vector valued function $E_\tau(\psi:\lambda)$ are $\mathbb{D}(G/H)$-finite functions on G/H. As such they allow converging asymptotic expansions along the $\sigma\theta$-stable parabolic subgroups of G (see [1]). In this section we recall from [3] some properties of these expansions, for the minimal $\sigma\theta$-stable parabolic subgroups.

Recall the following 'KAH'-decomposition of G:

$$G = \mathrm{cl} \bigcup_{w\in\mathcal{W}} KA_{\mathrm{q}}^+ wH, \qquad \text{(disjoint union)},$$

where cl denotes 'closure'. Here $A_{\mathrm{q}}^+ = \exp \mathfrak{a}_{\mathrm{q}}^+$, where $\mathfrak{a}_{\mathrm{q}}^+$ is the positive chamber in $\mathfrak{a}_{\mathrm{q}}$ corresponding to some (fixed) choice of positive system for Σ. Using the decomposition above we see that the asymptotics of a τ-spherical function f on G/H are determined from the behavior of $f(aw)$ for $a \to \infty$ in A_{q}^+ and $w \in \mathcal{W}$ (modulo the behavior 'along the walls' of $A_{\mathrm{q}}^+ w$). In the following we fix two parabolic subgroups $P, Q \in \mathcal{P}_\sigma^{\min}$. The asymptotic expansion to be explored is that of the Eisenstein integral $E(P:\psi:\lambda)$ along $A_{\mathrm{q}}^+(Q)w$, for all $w \in \mathcal{W}$, where $A_{\mathrm{q}}^+(Q)$ corresponds to $\Sigma(Q)$. Notice that, for $a \in A_{\mathrm{q}}$, the function $m \mapsto E(P:\psi:\lambda)(maw)$ belongs to $C^\infty(M/wH_M w^{-1} : \tau_M)$, by sphericality of the Eisenstein integral. By (17) we may view it as an element of ${}^\circ\mathcal{C}(\tau)$.

Let $\mathbb{N}\Sigma(Q)$ denote the set of linear combinations of the elements from $\Sigma(Q)$ with coefficients in $\mathbb{N}$. In view of (32), (33) it follows from [3, Lemma 14.1 and Thm. 14.2] that there exist, for each $\nu \in \mathbb{N}\Sigma(Q)$ and $s \in W$ a unique meromorphic $\mathrm{End}({}^\circ\mathcal{C}(\tau))$ valued function $\lambda \mapsto p_{Q|P,\nu}(s:\lambda)$ on $\mathfrak{a}_{\mathrm{qc}}^*$ such that (generically in λ)

$$E(P:\psi:\lambda)(maw) = a^{-\rho_Q} \sum_{\nu\in\mathbb{N}\Sigma(Q)} \sum_{s\in W} a^{s\lambda-\nu} [p_{Q|P,\nu}(s:\lambda)\psi]_w(m) \tag{45}$$

for $w \in \mathcal{W}$, $m \in M$ and $a \in A_{\mathrm{q}}^+(Q)$. The convergence is absolute and uniform on any subset of $MA_{\mathrm{q}}^+(Q)$ of the form $\{ma \mid \alpha(\log a) > \epsilon, \alpha \in \Sigma(Q)\}$, $\epsilon > 0$. We define the *c-function*

$$C_{Q|P}(s:\lambda) = p_{Q|P,0}(s:\lambda) \in \mathrm{End}({}^\circ\mathcal{C}(\tau)).$$

Moreover we define the τ_M-spherical function $E_{Q,w}(P:\psi:\lambda)$ on $M_1/wH_{M_1}w^{-1}$ by

$$E_{Q,w}(P:\psi:\lambda)(ma) = \sum_{s\in W} a^{s\lambda} [C_{Q|P}(s:\lambda)\psi]_w(m), \qquad (m \in M,\ a \in A_{\mathrm{q}}), \tag{46}$$

and call it the (Q,w)-*principal part* of $E(P:\psi:\lambda)$. It is easily seen from uniqueness of the asymptotic coefficients that the c-functions, as well as the principal parts, of $E(P:\psi:\lambda)$, depend on τ in a functorial way, just as we have earlier seen for $E(P:\psi:\lambda)$ itself (cf. (21)).

The c-functions $C_{Q|P}(s:\lambda)$ allow the following identification in terms of intertwining operators when $s = 1$. Recall that the intertwining operators $A(P:Q:\xi:\lambda)$ act on $C(K:\xi:\tau) \simeq [C(K:\xi) \otimes V_\tau]^K$ by tensoring their usual action on $C(K:\xi)$ with the trivial action on V_τ.

Proposition 1. *Let $\xi \in \widehat{M_{\mathrm{fu}}}$ and let $\psi_{f\otimes\eta} \in {}^\circ\mathcal{C}(\tau)$ be given by (23) with $f \in C(K{:}\xi{:}\tau)$, $\eta \in V(\xi)$. Then*

$$C_{Q|P}(1{:}\lambda)\psi_{f\otimes\eta} = \psi_{A(Q:P:\xi:-\lambda)f\otimes B(\bar{Q}:P:\xi:\bar{\lambda})\eta}, \tag{47}$$

as a meromorphic identity in $\lambda \in \mathfrak{a}^*_{\mathrm{qc}}$.

Proof. Equation (47) with $P = \bar{Q}$ follows from [3, Prop. 15.7] (use functoriality to generalize from τ_ϑ to arbitrary τ). From (26) and uniqueness of the asymptotic coefficients we obtain

$$C_{Q|P}(1{:}\lambda)\psi_{f\otimes\eta} = C_{Q|P'}(1{:}\lambda)\psi_{A(P:P':\xi:-\lambda)^{-1}f\otimes B(P':P:\xi:\bar{\lambda})\eta}.$$

Take $P' = \bar{Q}$, then the result easily follows by application of (47) with $P = \bar{Q}$ and (13). □

In particular we derive from (47), (43) and (39) that

$$\mu(D{:}\tau{:}\lambda) \circ C_{Q|P}(1{:}\lambda) = C_{Q|P}(1{:}\lambda) \circ \mu(D{:}\tau{:}\lambda) \tag{48}$$

for all $D \in \mathbb{D}(G/H)$.

5. The normalized Eisenstein integrals. Let (τ, V_τ) be any unitary finite dimensional K-representation. We define the *normalized Eisenstein integrals* $E^\circ(\psi{:}\lambda) = E^\circ_\tau(P{:}\psi{:}\lambda) \in C^\infty(G/H{:}\tau)$, for $\psi \in {}^\circ\mathcal{C}(\tau)$, $\lambda \in \mathfrak{a}^*_{\mathrm{qc}}$ by (cf. [12, p. 135] in the group case)

$$E^\circ(P{:}\psi{:}\lambda) = E(P{:}C_{P|P}(1{:}\lambda)^{-1}\psi{:}\lambda). \tag{49}$$

Obviously $E^\circ(\psi{:}\lambda)$ is meromorphic as a function of $\lambda \in \mathfrak{a}^*_{\mathrm{qc}}$.

Proposition 2. *Let $\psi \in {}^\circ\mathcal{C}(\tau)$. Then*

$$E^\circ(P{:}\psi{:}\lambda) = E(P'{:}C_{P|P'}(1{:}\lambda)^{-1}\psi{:}\lambda) \tag{50}$$

as a meromorphic identity in $\lambda \in \mathfrak{a}^*_{\mathrm{qc}}$, where $P' \in \mathcal{P}^{\min}_\sigma$ is arbitrary. Moreover in analogy with (25) we have

$$E^\circ(P{:}\psi_T{:}\lambda)(gH) = \langle f|\pi_{P,\xi,\lambda}(g)j^\circ(P{:}\xi{:}\bar{\lambda})\eta\rangle, \qquad (g \in G), \tag{51}$$

for $T = f \otimes \eta \in C(K{:}\xi{:}\tau) \otimes \bar{V}(\xi)$.

Proof. It suffices to prove (50) with $\psi = \psi_T$. It follows easily from (12), (9), (25), in combination with (47) that the right-hand side of (51) equals the right-hand side of (50), for any P'. Taking $P' = P$ we obtain both equations. □

In particular, if we take $P' = \bar{P}$ in (50) we obtain $E^\circ(P{:}\psi{:}\lambda) = E(\bar{P}{:}C_{P|\bar{P}}(1{:}\lambda)^{-1}\psi{:}\lambda)$, which shows that

$$E^\circ(P{:}\psi{:}\lambda) = E^1(\bar{P}{:}\psi{:}\lambda), \tag{52}$$

where E^1 is the Eisenstein integral normalized analogously to [3, Sect. 16].

We can now state our main result about the Eisenstein integrals. Recall that for $\epsilon > 0$ we have defined the set $\mathfrak{a}^*_{\mathrm{q}}(\epsilon)$ by (3).

Theorem 2. *Let (τ, V_τ) be given. There exists $\epsilon > 0$ such that the normalized Eisenstein integral $E^\circ(\psi:\lambda)$ is holomorphic as a function of λ in $\mathfrak{a}_{\mathrm{q}}^*(\epsilon)$, for all $\psi \in {}^\circ\mathcal{C}(\tau)$.*

The theorem will be proved in Section 17. For the time being let us use it to prove the regularity of $j^\circ(\xi:\lambda)$ on $\mathfrak{a}_{\mathrm{q}}^*(\epsilon)$, for some $\epsilon > 0$:

Proof of Theorem 1. Let $\xi \in M_{\mathrm{fu}}^\frown$ be fixed. It follows immediately from Theorem 2 together with the normalized version of (30), which reads

$$E^\circ_{\tau_\vartheta}(\psi_T:\lambda)(gH)(e) = \langle f | \pi_{\xi,\bar\lambda}(g) j^\circ(\xi:\bar\lambda)\eta\rangle, \qquad (T = f \otimes \eta \in C(K:\xi)_\vartheta \otimes \bar V(\xi)), \tag{53}$$

that $\lambda \mapsto \langle f | j^\circ(\xi:\bar\lambda)\eta\rangle \in \mathbb{C}$ is regular on a neighborhood of $i\mathfrak{a}_{\mathrm{q}}^*$, for all K-finite functions $f \in C^\infty(K:\xi)$ and all $\eta \in V(\xi)$. (Notice however that a priori the neighborhood may depend on f.)

By Lemma 2 there exists an element $p \in \Pi_\Sigma(\mathfrak{a}_{\mathrm{q}})$ such that $\lambda \mapsto p(\lambda) j^\circ(\xi:\lambda)\eta$ is regular on $\mathfrak{a}_{\mathrm{q}}^*(1)$, for each $\eta \in V(\xi)$. We claim that p may be chosen such that all its linear factors $\lambda \mapsto \langle\lambda, \alpha\rangle - c$ satisfy $\mathrm{Re}\, c \neq 0$. This will obviously imply the asserted regularity.

In order to prove the above claim, we assume that $p = lp'$ where $p' \in \Pi_\Sigma(\mathfrak{a}_{\mathrm{q}})$ and $l(\lambda) = \langle\lambda, \alpha\rangle - c$ with $\mathrm{Re}\, c = 0$. Then it follows from the above consequence of Theorem 2 that $\lambda \mapsto \langle f | p(\bar\lambda) j^\circ(\xi:\bar\lambda)\eta\rangle$ vanishes for $\bar\lambda \in l^{-1}(0) \cap i\mathfrak{a}_{\mathrm{q}}^*$, for all K-finite functions $f \in C^\infty(K:\xi)$. By the density of the K-finite vectors in $C^\infty(K:\xi)$ we conclude that $\lambda \mapsto p(\lambda) j^\circ(\xi:\lambda)$ vanishes for $\lambda \in l^{-1}(0) \cap i\mathfrak{a}_{\mathrm{q}}^*$, hence also for $\lambda \in l^{-1}(0) \cap \mathfrak{a}_{\mathrm{q}}^*(1)$ by analytic continuation. Hence l is a factor of the holomorphic function $\lambda \mapsto p(\lambda) j^\circ(\xi:\lambda) = l(\lambda) p'(\lambda) j^\circ(\xi:\lambda)$ on $\mathfrak{a}_{\mathrm{q}}^*(1)$, which means that $\lambda \mapsto p'(\lambda) j^\circ(\xi:\lambda)$ is also holomorphic on this set.

Using this argument repeatedly we arrive in a finite number of steps at a polynomial p with the claimed property. $\square$

Notice that it follows from (49), (44), and (48) that

$$DE^\circ(\psi:\lambda) = E^\circ(\mu(D:\tau:\lambda)\psi:\lambda). \tag{54}$$

It follows from this equation and (50) that the normalized Eisenstein integral $E^\circ(P:\psi:\lambda)$ allows asymptotic expansions similar to (45) for all $Q \in \mathcal{P}_\sigma^{\min}$, with coefficients

$$[p_{Q|P',\nu}(s:\lambda) \circ C_{P|P'}(1:\lambda)^{-1}\psi]_w(m).$$

Notice that these coefficients are unique and hence independent of the parabolic subgroup $P' \in \mathcal{P}_\sigma^{\min}$. In particular the operator defined by

$$C^\circ_{Q|P}(s:\lambda) = C_{Q|P'}(s:\lambda) C_{P|P'}(1:\lambda)^{-1} \tag{55}$$

is independent of P' (and hence, in notation analogous to that of [3, eq. (134)], equal to $C^1_{Q|\bar P}(s:\lambda)$). The (Q, w)-principal part of $E^\circ(P:\psi:\lambda)$ is the τ_M-spherical function on $M_1/wH_{M_1}w^{-1}$ given by

$$E^\circ_{Q,w}(P:\psi:\lambda)(ma) = \sum_{s\in W} a^{s\lambda} [C^\circ_{Q|P}(s:\lambda)\psi]_w(m), \qquad (m \in M,\ a \in A_{\mathrm{q}}). \tag{56}$$

Notice that with $P' = \bar{Q}$ and $s = 1$ in (55) it follows from (47) and (13) that

$$C^\circ_{Q|P}(1{:}\lambda)\psi_{f\otimes\eta} = \psi_{A(Q:P:\xi:-\lambda)f\otimes B(\bar{P}:\bar{Q}:\xi:\bar{\lambda})^{-1}\eta}. \tag{57}$$

The operator $C^\circ_{Q|P}(1{:}\lambda)$ can be used to establish a relation between the normalized Eisenstein integrals $E^\circ(P{:}\psi{:}\lambda)$ for different parabolic subgroups P. In analogy with (26) we get from (51) and (14) that

$$E^\circ(P{:}\psi_{f\otimes\eta}{:}\lambda) = E^\circ(P'{:}\psi_{A(P:P':\xi:-\lambda)^{-1}f\otimes B(\bar{P}':\bar{P}:\xi:\bar{\lambda})\eta}{:}\lambda).$$

Combining with (57) we find

$$E^\circ(P{:}\psi{:}\lambda) = E^\circ(P'{:}C^\circ_{P|P'}(1{:}\lambda)^{-1}\psi{:}\lambda), \qquad (P, P' \in \mathcal{P}_\sigma^{\min}). \tag{58}$$

6. The spherical Fourier transform. If f and g are τ-spherical functions on G/H then we define a sesqui-linear pairing by

$$\langle f|g\rangle = \int_{G/H} \langle f(x)|g(x)\rangle_\tau \, dx,$$

whenever the integral makes sense. Furthermore, if $f \in C_c^\infty(G/H{:}\tau)$ then we define the *τ-spherical Fourier transform* $\mathcal{F}f = \mathcal{F}_P f$ of f to be the meromorphic ${}^\circ\mathcal{C}(\tau)$-valued function on $\mathfrak{a}^*_{\mathfrak{q}\mathbf{c}}$ determined by

$$\langle \mathcal{F}_P f(\lambda)|\psi\rangle = \langle f|E^\circ(P{:}\psi{:}-\bar{\lambda})\rangle, \quad (\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbf{c}}, \psi \in {}^\circ\mathcal{C}(\tau)). \tag{59}$$

It follows from Theorem 2 (to be proved later) that $\mathcal{F}f(\lambda)$ is regular on a neighborhood of $i\mathfrak{a}^*_\mathfrak{q}$. Notice that in analogy with (42) we obtain from (41) and (54) that for $D \in \mathbb{D}(G/H)$

$$\mathcal{F}(Df) = \mu(D{:}\tau{:}\lambda) \circ \mathcal{F}f, \qquad (f \in C_c^\infty(G/H{:}\tau)). \tag{60}$$

We shall now describe the relation of this Fourier transform with the Fourier transform $f \mapsto \hat{f}$ on $C_c^\infty(G/H)$. We first transform scalar-valued K-finite functions on G/H into τ-spherical ones, for a suitable K-representation τ (see [13, §26], [16, p. 397] for related constructions). For any scalar-valued K-finite function f on G/H we define a $C(K)$-valued function $\varsigma(f)$ on G/H by $\varsigma(f)(x)(k) = f(kx)$ for $x \in G/H$, $k \in K$. Let $\vartheta \subset \widehat{K}$ be a finite set, and let $\mathbf{V}_\vartheta = C(K)_\vartheta$ be as in Section 2. It is easily seen that if f is K-finite of isotypes from ϑ, then $\varsigma(f)(x) \in \mathbf{V}_\vartheta$ for $x \in G/H$, and $\varsigma(f)$ is τ_ϑ-spherical. We denote by $C_c^\infty(G/H)_\vartheta$ the (closed) subspace of $C_c^\infty(G/H)$ consisting of the K-finite vectors of isotypes from ϑ, and equip it with the induced topology.

Lemma 5. *The map ς is a continuous bijection of $C_c^\infty(G/H)_\vartheta$ onto $C_c^\infty(G/H{:}\tau_\vartheta)$. Its inverse is given by $F \mapsto \delta_e \circ F$, where $\delta_e : \mathbf{V}_\vartheta \to \mathbb{C}$ is the map obtained from evaluation at the identity element $e \in K$.*

Proof. Easy. $\square$

Proposition 3. *Let $f \in C_c^\infty(G/H)_\vartheta$ and let $F = \varsigma(f) \in C_c^\infty(G/H{:}\tau_\vartheta)$. Then for all $\xi \in \widehat{M_{\rm fu}}$, $T \in C(K{:}\xi)_\vartheta \otimes \bar{V}(\xi)$ and $\lambda \in \mathfrak{a}_{\rm qc}^*$ we have*

$$\langle \mathcal{F}F(\lambda)|\psi_T\rangle = \langle \hat{f}(\xi{:}\lambda)|T\rangle,$$

where $\psi_T \in {}^\circ\mathcal{C}(\tau)$ is determined by (29) and linearity.

Proof. The Hilbert space structure on $\mathbf{V}_\vartheta$ is obtained from $L^2(K)$. From (59), the definition of ς, sphericality, and invariance of dx we find that for any $\psi \in {}^\circ\mathcal{C}(\tau)$

$$\begin{aligned}\langle \mathcal{F}F(\lambda)|\psi\rangle &= \int_{G/H}\int_K F(x)(k)\overline{E^\circ_{\tau_\vartheta}(\psi{:}-\bar{\lambda})(x)(k)}\,dk\,dx \\ &= \int_{G/H}\int_K f(kx)\overline{E^\circ_{\tau_\vartheta}(\psi{:}-\bar{\lambda})(kx)(e)}\,dk\,dx \\ &= \int_{G/H} f(x)\overline{E^\circ_{\tau_\vartheta}(\psi{:}-\bar{\lambda})(x)(e)}\,dx.\end{aligned}$$

Let $\varphi \in C(K{:}\xi)_\vartheta$ and $\eta \in V(\xi)$, and suppose that $\psi = \psi_T$ with $T = \varphi \otimes \eta$. Applying (53) we now have

$$\langle \mathcal{F}F(\lambda)|\psi\rangle = \int_{G/H} f(gH)\langle \pi_{\xi,-\lambda}(g)j^\circ(\xi{:}-\lambda)\eta|\varphi\rangle d(gH) = \langle \hat{f}(\xi{:}\lambda)\eta|\varphi\rangle = \langle \hat{f}(\xi{:}\lambda)|\psi\rangle.$$

For general T the result follows by linearity. $\square$

7. The action of the Weyl group. Let $w \in N_K(\mathfrak{a}_{\rm q})$. Since w normalizes M it acts on (equivalence classes of) representations of M. If $f \in C(P{:}\xi{:}\lambda)$ then the left translate given by $[L(w)f](x) = f(w^{-1}x)$ belongs to $C(wPw^{-1}{:}w\xi{:}w\lambda)$, and the map $f \mapsto L(w)f$ is a bijective intertwining operator for the right actions. Moreover, this map extends to generalized functions, and hence gives rise to a linear bijection

$$L(w)\colon C^{-\infty}(P{:}\xi{:}\lambda)^H \to C^{-\infty}(wPw^{-1}{:}w\xi{:}w\lambda)^H.$$

According to [2, Lemma 6.10] there exists a unitary linear bijection $L(\xi,w)\colon V(\xi) \to V(w\xi)$, independent of P and λ, such that the diagram

$$\begin{array}{ccc} C^{-\infty}(P{:}\xi{:}\lambda)^H & \xrightarrow{L(w)} & C^{-\infty}(wPw^{-1}{:}w\xi{:}w\lambda)^H \\ {\scriptstyle j(P:\xi:\lambda)}\uparrow & & \uparrow{\scriptstyle j(wPw^{-1}:w\xi:w\lambda)} \\ V(\xi) & \xrightarrow[L(\xi,w)]{} & V(w\xi) \end{array} \tag{61}$$

is commutative (in a meromorphic sense in λ). Explicitly the map $L(\xi,w)$ is constructed as follows. By transference of the left multiplication under the canonical bijection $\mathcal{W} \to Z_K(\mathfrak{a}_{\rm q})\backslash N_K(\mathfrak{a}_{\rm q})/N_{K\cap H}(\mathfrak{a}_{\rm q})$ we equip $\mathcal{W}$ with a $N_K(\mathfrak{a}_{\rm q})$-action (recall that $Z_K(\mathfrak{a}_{\rm q})$ is a normal subgroup of $N_K(\mathfrak{a}_{\mathrm{q}})$). This action is denoted by $(w, v) \mapsto w \cdot v$. If $w \in N_K(\mathfrak{a}_{\mathrm{q}})$, $v \in \mathcal{W}$, we choose an element $u(w, v) \in Z_K(\mathfrak{a}_{\mathrm{q}}) = K_M$ such that

$$w \cdot v = u(w, v)wv \mod N_{K \cap H}(\mathfrak{a}_{\mathrm{q}}).$$

The map $L(\xi, w)\colon V(\xi) \to V(w\xi)$ is given by

$$[L(\xi, w)\eta]_{w \cdot v} = (w\xi)(u(w, v))\eta_v \in \mathcal{H}_{w\xi}^{(w \cdot v)H_M(w \cdot v)^{-1}}, \qquad (v \in \mathcal{W}, \eta \in V(\xi)).$$

The intertwining operator $L(w)$ commutes with the standard intertwining operators in the sense that we have

$$L(w) \circ A(P'{:}P{:}\xi{:}\lambda) = A(wP'w^{-1}{:}wPw^{-1}{:}w\xi{:}w\lambda) \circ L(w). \tag{62}$$

Combining this with the diagrams (8) and (61) we obtain (62) with A replaced by B and $L(w)$ by $L(\xi, w)$. Moreover, we obtain that the commutativity of the diagram (61) holds with j replaced by j°. Consequently we also have that

$$L(w) \circ \hat{f}(P{:}\xi{:}\lambda) = \hat{f}(wPw^{-1}{:}w\xi{:}w\lambda) \circ L(\xi, w), \tag{63}$$

for all $f \in C_c^\infty(G/H)$. Another consequence of the diagram (61) is the following relation (use (38)):

$$L(\xi, w) \circ \mu(D{:}\xi{:}\lambda) = \mu(D{:}w\xi{:}w\lambda) \circ L(\xi, w), \qquad (D \in \mathbb{D}(G/H)). \tag{64}$$

Using Lemma 3 we can combine the maps $L(\xi, w)$ to a linear endomorphism $\mathcal{L}(w)$ of ${}^\circ\mathcal{C}(\tau)$, for any unitary finite dimensional representation τ of K. This is defined by

$$\mathcal{L}(w)\psi_T = \psi_{[L(w) \otimes L(\xi, w)]T}, \qquad (T \in C(K{:}\xi{:}\tau) \otimes V(\xi)\,) \tag{65}$$

for all $\xi \in \widehat{M_{\mathrm{fu}}}$. Here $L(w)\colon C(K{:}\xi{:}\tau) \to C(K{:}w\xi{:}\tau)$ is simply given by the left regular action on functions on K.

By a straightforward calculation, using the relevant definitions, one sees that for every $\psi \in {}^\circ\mathcal{C}(\tau)$ one has

$$[\mathcal{L}(w)\psi]_{w \cdot v}(m) = \tau(w)\psi_v(w^{-1}m\, u(w, v)w), \qquad (w \in N_K(\mathfrak{a}_{\mathrm{q}}),\, v \in \mathcal{W},\, m \in M). \tag{66}$$

Lemma 6. *The map $\mathcal{L}$ is a homomorphism of $N_K(\mathfrak{a}_{\mathrm{q}})$ into the unitary group $\mathrm{U}({}^\circ\mathcal{C}(\tau))$. Its kernel contains $Z_K(\mathfrak{a}_{\mathrm{q}})$.*

Proof. It is a straightforward consequence of the definitions that $\mathcal{L}$ is a homomorphism whose image consists of unitary operators. The assertion about the kernel follows from (66): If $w \in Z_K(\mathfrak{a}_{\mathrm{q}})$ then $u(w, v) = w^{-1}$ and $\tau(w)\psi_v(w^{-1}m) = \psi_v(m)$ by K_M-sphericality. □

We denote the induced unitary representation of W in ${}^\circ\mathcal{C}(\tau)$ by $\mathcal{L}$ as well.

Lemma 7. *Let $P, Q \in \mathcal{P}_\sigma^{\min}$, $s, w \in W$ and $\psi \in {}^\circ\mathcal{C}(\tau)$. Then*

$$C_{Q|P}(s:\lambda) = C_{Q|wPw^{-1}}(sw^{-1}:w\lambda) \circ \mathcal{L}(w) = \mathcal{L}(w) \circ C_{w^{-1}Qw|P}(w^{-1}s:\lambda), \tag{67}$$

as a meromorphic identity in $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$. In particular

$$C_{Q|P}(s:\lambda) = C_{Q|sPs^{-1}}(1:s\lambda) \circ \mathcal{L}(s) = \mathcal{L}(s) \circ C_{s^{-1}Qs|P}(1:\lambda). \tag{68}$$

All these relations hold as well with C replaced by C°.

Proof. The identities in (68) follow from those in (67) by taking $w = s$.

It follows easily from (25) and the diagram (61) that we have

$$E(wPw^{-1}:\mathcal{L}(w)\psi:w\lambda) = E(P:\psi:\lambda) \tag{69}$$

for all $\psi \in {}^\circ\mathcal{C}(\tau)$ (cf. also [3, Lemma 15.4]). Since as remarked the diagram holds as well for the normalized operator j°, we get from (51) that the relation (69) holds for the normalized Eisenstein integrals as well. By uniqueness of asymptotics we obtain the first identity in (67), with C as well as C°.

Let $w \in N_K(\mathfrak{a}_\mathrm{q})$, then by (45) we have for each $v \in \mathcal{W}$

$$E(P:\psi:\lambda)(av) = a^{-\rho_{w^{-1}Qw}} \sum_{\mu\in\mathbb{N}\Sigma(w^{-1}Qw)} \sum_{t\in W} a^{t\lambda-\mu}[p_{w^{-1}Qw|P,\mu}(t:\lambda)\psi]_v(e)$$

for $a \in A_\mathrm{q}^+(w^{-1}Qw)$. Applying $\tau(u(w,v)w)$ to this expression and using sphericality we obtain

$$\begin{aligned}
&E(P:\psi:\lambda)(u(w,v)wav)\\
&\quad = a^{-w^{-1}\rho_Q} \sum_{\mu\in\mathbb{N}\Sigma(w^{-1}Qw)} \sum_{t\in W} a^{t\lambda-\mu}\, \tau(w)[p_{w^{-1}Qw|P,\mu}(t:\lambda)\psi]_v(w^{-1}u(w,v)w),\\
&\quad = a^{-w^{-1}\rho_Q} \sum_{\mu\in\mathbb{N}\Sigma(w^{-1}Qw)} \sum_{t\in W} a^{t\lambda-\mu}[\mathcal{L}(w)p_{w^{-1}Qw|P,\mu}(t:\lambda)\psi]_{w\cdot v}(e),
\end{aligned}$$

by (66). On the other hand since $waw^{-1} \in A_\mathrm{q}^+(Q)$ we also have

$$\begin{aligned}
&E(P:\psi:\lambda)(u(w,v)wav) = E(P:\psi:\lambda)((waw^{-1})w\cdot v)\\
&\quad = (waw^{-1})^{-\rho_Q} \sum_{\nu\in\mathbb{N}\Sigma(Q)} \sum_{s\in W} (waw^{-1})^{s\lambda-\nu}[p_{Q|P,\nu}(s:\lambda)\psi]_{w\cdot v}(e)\\
&\quad = a^{-w^{-1}\rho_Q} \sum_{\nu\in\mathbb{N}\Sigma(Q)} \sum_{s\in W} a^{w^{-1}s\lambda-w^{-1}\nu}[p_{Q|P,\nu}(s:\lambda)\psi]_{w\cdot v}(e),
\end{aligned}$$

and hence by uniqueness of asymptotics we conclude

$$p_{Q|P,\nu}(s:\lambda) = \mathcal{L}(w) \circ p_{w^{-1}Qw|P,w^{-1}\nu}(w^{-1}s:\lambda)$$

for all $\nu \in \mathbb{N}\Sigma(Q)$, $s \in W$. Taking $\nu = 0$ we obtain the second expression for $C_{Q|P}(s:\lambda)$ in (67). Finally, for C° this expression is now immediate from (55). $\square$

The importance of (68) lies in the fact that it allows us to recover the c-functions $C_{Q|P}(s:\lambda)$ and $C^\circ_{Q|P}(s:\lambda)$ from the simpler case $s = 1$, where they are explicitly known from (47) and (57), respectively. We shall now derive some consequences of (68). First of all, by combining the normalized versions of (69) and (68) with (58) we arrive at the following functional equation for the normalized Eisenstein integrals (cf. [3, Prop. 16.4]):

Proposition 4. *We have*

$$E^\circ(Q{:}C^\circ_{Q|P}(s{:}\lambda)\psi{:}s\lambda) = E^\circ(P{:}\psi{:}\lambda) \tag{70}$$

for all $\psi \in {}^\circ\mathcal{C}(\tau)$, $s \in W$, $Q, P \in \mathcal{P}_\sigma^{\min}$, as a meromorphic identity in $\lambda \in \mathfrak{a}^*_{\mathrm{qc}}$.

By uniqueness of asymptotics it follows from the above that

$$C^\circ_{Q'|Q}(t{:}s\lambda) \circ C^\circ_{Q|P}(s{:}\lambda) = C^\circ_{Q'|P}(ts{:}\lambda) \tag{71}$$

for any $Q' \in \mathcal{P}_\sigma^{\min}$. In particular, substituting $Q' = P$ and using that $C^\circ_{P|P}(1{:}\lambda)$ is the identity operator on ${}^\circ\mathcal{C}(\tau)$, we obtain

$$C^\circ_{P|Q}(s^{-1}{:}s\lambda) \circ C^\circ_{Q|P}(s{:}\lambda) = I. \tag{72}$$

The following relation is also a consequence of (68):

$$C_{Q|P}(s{:}\lambda) \circ \mu(D{:}\tau{:}\lambda) = \mu(D{:}\tau{:}s\lambda) \circ C_{Q|P}(s{:}\lambda), \qquad (D \in \mathbb{D}(G/H)). \tag{73}$$

For $s = 1$ this is (48), and in general it is obtained using that

$$\mathcal{L}(w) \circ \mu(D{:}\tau{:}\lambda) = \mu(D{:}\tau{:}w\lambda) \circ \mathcal{L}(w).$$

The latter equality follows from (65), (64) and (43). Furthermore, from definition (55) we see that equation (73) holds with C replaced by C°.

Finally another consequence is the equation

$$C_{Q|P}(s{:}\lambda)^* = C_{P|Q}(s^{-1}{:} -s\bar{\lambda}), \tag{74}$$

as well as the normalized version with C replaced by C°. The proof of (74) is reduced to the case $s = 1$ by means of (68) and the unitarity of $\mathcal{L}(w)$. For $s = 1$ it follows from (47), respectively (57), together with (9), and (10).

The following result (essentially from [3, Thm. 16.3]), which follows immediately from (72) and the normalized version of (74), is crucial for the proof of the regularity of E° on $\lambda \in i\mathfrak{a}^*_{\mathrm{q}}$ (Theorem 2). It shows that the (Q, w)-principal part $E^\circ_{Q,w}(P{:}\psi{:}\lambda)$ is regular for any $Q \in \mathcal{P}_\sigma^{\min}$.

Proposition 5. *Let $s \in W$ and $Q, P \in \mathcal{P}_\sigma^{\min}$ be given. We have*

$$C^\circ_{Q|P}(s{:} -\bar{\lambda})^* \circ C^\circ_{Q|P}(s{:}\lambda) = I \tag{75}$$

as a meromorphic identity in $\lambda \in \mathfrak{a}^*_{\mathrm{qc}}$. In particular the operator $C^\circ_{Q|P}(s{:}\lambda)$ on ${}^\circ\mathcal{C}(\tau)$ is unitary for $\lambda \in i\mathfrak{a}^*_{\mathrm{q}}$, and it is regular as a function of λ in this set.

The asserted regularity is a consequence of the unitarity, in view of the Riemann boundedness theorem. Combining (70) and (75) with the definition (59) of the τ-spherical Fourier transform we have

$$\mathcal{F}_Q f(s\lambda) = C^\circ_{Q|P}(s{:}\lambda)\mathcal{F}_P f(\lambda) \tag{76}$$

for all $f \in C_c^\infty(G/H{:}\tau)$, $Q, P \in \mathcal{P}_\sigma^{\min}$, $s \in W$.

8. Non-minimal parabolic subgroups. Up to now we have only considered parabolic subgroups from the set $\mathcal{P}_\sigma^{\min}$ of minimal $\sigma\theta$-stable parabolic subgroups containing A_{q}. In order to prepare for using induction on the split rank of G/H we shall now consider arbitrary $\sigma\theta$-stable parabolic subgroups containing A_{q}. Let $\mathcal{P}_\sigma$ denote the set of these, and let $Q \in \mathcal{P}_\sigma$ be given. Let $Q = M_{1Q}N_Q = M_Q A_Q N_Q$ denote its Langlands decomposition, then σ leaves the reductive group M_Q invariant, and $M_Q/H_{M_Q} = M_Q/M_Q \cap H$ and $M_{1Q}/H_{M_{1Q}} = M_{1Q}/M_{1Q} \cap H$ are reductive symmetric spaces of Harish-Chandra's class. The space $\mathfrak{m}_Q \cap \mathfrak{a}_{\mathrm{q}}$ is a maximal abelian subspace of $\mathfrak{m}_Q \cap \mathfrak{p} \cap \mathfrak{q}$, and its dimension is the split rank of M_Q/H_{M_Q}. If Q is a proper parabolic subgroup of G then this number is strictly smaller than the split rank of G. The above mentioned induction on the split rank will be based on these observations. However, for reasons of convenience we mostly work with M_{1Q} rather than M_Q.

Let $\mathcal{P}_{\sigma,Q}^{\min}$ denote the set of parabolic subgroups $P \in \mathcal{P}_\sigma^{\min}$ contained in Q, then it is easily seen that the map $P \mapsto {}^*P = M_{1Q} \cap P$ is a bijection of $\mathcal{P}_{\sigma,Q}^{\min}$ onto the set $\mathcal{P}_\sigma^{\min}(M_{1Q})$ of minimal $\sigma\theta$-stable parabolic subgroups of M_{1Q} containing A_{q}. Moreover, if P has the Langlands decomposition $P = MAN$ then the Langlands decomposition ${}^*P = {}^*M{}^*A{}^*N$ of ${}^*P$ is given by ${}^*M = M$, ${}^*A = A$, ${}^*N = M_Q \cap N$ (compare [13, p. 113]).

We shall now relate some of the elements constructed above for the pair (G, σ) to the similar elements for $(M_{1Q}, \sigma|_{M_{1Q}})$. We begin with the H-invariant distribution vectors $j(P{:}\xi{:}\lambda)$ on G and their analogues $j({}^*P{:}\xi{:}\lambda)$ on M_{1Q}. Clearly restriction from G to M_{1Q} (or from K to $K_Q = M_Q \cap K$ in the compact picture) gives a map r_Q of the space $C(P{:}\xi{:}\lambda)$ into the analogous space $C({}^*P{:}\xi{:}\lambda)$ for M_{1Q}.

Let W_Q denote the centralizer of A_Q in W. Then $W_Q \simeq N_{K_Q}(\mathfrak{a}_{\mathrm{q}})/Z_{K_Q}(\mathfrak{a}_{\mathrm{q}})$, and we see that W_Q is naturally isomorphic with the Weyl group of the root system $\Sigma(\mathfrak{m}_{1Q}, \mathfrak{a}_{\mathrm{q}})$. In analogy with the set $\mathcal{W}$ we fix a set $\mathcal{W}_Q \subset N_{K_Q}(\mathfrak{a}_{\mathrm{q}})$ of representatives for the double quotient $Z_{K_Q}(\mathfrak{a}_{\mathrm{q}})\backslash N_{K_Q}(\mathfrak{a}_{\mathrm{q}})/N_{K_Q\cap H}(\mathfrak{a}_{\mathrm{q}})$. Obviously the natural map $N_{K_Q}(\mathfrak{a}_{\mathrm{q}}) \to N_K(\mathfrak{a}_{\mathrm{q}})$ induces an injection of $Z_{K_Q}(\mathfrak{a}_{\mathrm{q}})\backslash N_{K_Q}(\mathfrak{a}_{\mathrm{q}})/N_{K_Q\cap H}(\mathfrak{a}_{\mathrm{q}})$ into $Z_K(\mathfrak{a}_{\mathrm{q}})\backslash N_K(\mathfrak{a}_{\mathrm{q}})/N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$, and hence this map induces an injection of $\mathcal{W}_Q$ into $\mathcal{W}$. For simplicity we assume that the choices of $\mathcal{W}_Q$ and $\mathcal{W}$ have been made such that in fact we have $\mathcal{W}_Q \subset \mathcal{W}$. Since as previously mentioned the basic constructions of $j(\xi{:}\lambda)$, $E(\psi{:}\lambda)$ etc. are essentially independent of the choice of $\mathcal{W}$ (cf. (16), (27)), this assumption causes no problems (though of course it cannot be realized for all Q at the same time).

Let $V_Q(\xi) = V(M_{1Q}{:}\xi)$ denote the subspace of $V(\xi) = \oplus_{w\in\mathcal{W}}\mathcal{H}_\xi^{wH_Mw^{-1}}$ corresponding to the direct summands labeled by $w \in \mathcal{W}_Q$, and let $\mathrm{pr}_Q\colon V(\xi) \to V_Q(\xi)$ be the (orthogonal) projection along the remaining components. Then $j({}^*P{:}\xi{:}\lambda)$ maps $V_Q(\xi)$ onto $C^{-\infty}({}^*P{:}\xi{:}\lambda)^{H_{M_{1Q}}}$ for generic $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$, and into $C({}^*P{:}\xi{:}\lambda)^{H_{M_{1Q}}}$ for all $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$ which satisfy $\lambda + \rho_{{}^*P} \in \mathfrak{a}_{\mathrm{q}}^*({}^*P; 0)$. Here $\rho_{{}^*P} = \frac{1}{2}\,\mathrm{tr}(\mathrm{ad}_{{}^*\mathfrak{n}}) \in \mathfrak{a}_{\mathrm{q}}^*$ is the 'rho' of the parabolic subgroup ${}^*P$ of M_{1Q}, and $\mathfrak{a}_{\mathrm{q}}^*({}^*P; 0)$ is defined in analogy with (7). It is easily seen that $\rho_{{}^*P} = \rho_P - \rho_Q$, where $\rho_Q = \frac{1}{2}\,\mathrm{tr}(\mathrm{ad}_{\mathfrak{n}_Q}) \in \mathfrak{a}_{\mathrm{q}}^*$. Moreover we have $\langle \rho_Q, \alpha\rangle = 0$ for all roots $\alpha \in \Sigma(\mathfrak{m}_{1Q}, \mathfrak{a}_{\mathrm{q}})$. It follows that if $\lambda + \rho_P \in \mathfrak{a}_{\mathrm{q}}^*(P; 0)$ then $\lambda + \rho_{{}^*P} \in \mathfrak{a}_{\mathrm{q}}^*({}^*P; 0)$. Hence under this condition on λ we immediately have that

$$\mathrm{r}_Q \circ j(P{:}\xi{:}\lambda) = j({}^*P{:}\xi{:}\lambda) \circ \mathrm{pr}_Q\colon V(\xi) \to C({}^*P{:}\xi{:}\lambda)^{H_{M_{1Q}}}. \tag{77}$$

The condition on λ is important, since in general it does not make sense to restrict a distribution on G to M_{1Q}.

We shall now consider the relation between the standard intertwining operators for G, that is, $A(P_2{:}P_1{:}\xi{:}\lambda)\colon C^\infty(P_1{:}\xi{:}\lambda) \to C^\infty(P_2{:}\xi{:}\lambda)$, and the similar operators for M_{1Q}.

Lemma 8. *Let $Q \in \mathcal{P}_\sigma$ and let $P_1, P_2 \in \mathcal{P}_{\sigma,Q}^{\min}$. Then the following diagram is commutative for all generic $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$:*

$$\begin{array}{ccc} C^\infty(P_1{:}\xi{:}\lambda) & \xrightarrow{A(P_2:P_1:\xi:\lambda)} & C^\infty(P_2{:}\xi{:}\lambda) \\ {\scriptstyle \mathrm{r}_Q}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{r}_Q} \\ C^\infty({}^*\!P_1{:}\xi{:}\lambda) & \xrightarrow{A({}^*P_2:{}^*P_1:\xi:\lambda)} & C^\infty({}^*\!P_2{:}\xi{:}\lambda). \end{array} \tag{78}$$

Proof. Let P_j have the Langlands decomposition $P_j = MAN_j$ for $j = 1, 2$. If $C > 0$, we write $\mathcal{A}(P_2, P_1, C)$ for the set of $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$ such that $\langle \operatorname{Re}\lambda, \alpha\rangle > C$ for all $\alpha \in \Sigma$ with $\mathfrak{g}_\alpha \subset \bar{\mathfrak{n}}_2 \cap \mathfrak{n}_1$. Then by [2, Prop. 4.1] there exists a constant $C_1 > 0$ such that for all $\lambda \in \mathcal{A}(P_2, P_1, C_1)$ the operator $A(P_2{:}P_1{:}\xi{:}\lambda)$ is given by an absolutely convergent integral. In fact, if $f \in C^\infty(P_1{:}\xi{:}\lambda)$, then

$$A(P_2{:}P_1{:}\xi{:}\lambda)f(x) = \int_{N_2 \cap \bar{N}_1} f(nx)\, dn, \qquad (x \in G).$$

Similarly if we write $\mathcal{A}_Q(P_2, P_1, C)$ for the set of $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$ such that $\langle \operatorname{Re}\lambda, \alpha\rangle > C$ for every root α of $\mathfrak{a}_{\mathrm{q}}$ in ${}^*\mathfrak{n}_2 \cap {}^*\bar{\mathfrak{n}}_1$, then there exists a constant $C_2 > 0$ such that for $\lambda \in \mathcal{A}_Q(P_2, P_1, C_2)$ the operator $A({}^*P_2{:}{}^*P_1{:}\xi{:}\lambda)$ is given by the absolutely convergent integral

$$A({}^*P_2{:}{}^*P_1{:}\xi{:}\lambda)g(m) = \int_{{}^*N_2 \cap {}^*\bar{N}_1} g(nm)\, dn, \qquad (m \in M_{1Q}),$$

for every $g \in C^\infty({}^*P_1{:}\xi{:}\lambda)$. Since P_1 and P_2 are both contained in Q, the intersection $P_2 \cap \bar{P}_1$ is contained in $Q \cap \bar{Q} = M_{1Q}$, and we conclude that in fact

$$N_2 \cap \bar{N}_1 = {}^*N_2 \cap {}^*\bar{N}_1. \tag{79}$$

Hence the two integrals above are over the same set. Moreover, the Haar measures dn in both integrals are the same (cf. [2, definition below (4.1)]), and from (79) one also sees that $\mathcal{A}(P_2, P_1, C) = \mathcal{A}_Q(P_2, P_1, C)$ for all $C > 0$. Hence if $C = \max(C_1, C_2)$, then for $\lambda \in \mathcal{A}(P_2, P_1, C)$ the two integrals with f, g replaced by $f, f|_{M_{1Q}}$ converge absolutely and are equal for $x = m \in M_{1Q}$. This establishes the result for λ contained in the non-empty open subset $\mathcal{A}(P_2, P_1, C)$ of $\mathfrak{a}_{\mathrm{qc}}^*$. Now apply meromorphic continuation. □

Combining (77) for P_1, P_2 and (78) with the diagram (8) and its analogue for M_Q, it is plausible to expect that we have

$$B({}^*P_2{:}{}^*P_1{:}\xi{:}\lambda) \circ \mathrm{pr}_Q = \mathrm{pr}_Q \circ B(P_2{:}P_1{:}\xi{:}\lambda)\colon V(\xi) \to V_Q(\xi) \tag{80}$$

for P_1, P_2, Q as in Lemma 8. However, since (77) was only valid for λ in a certain region depending on P, in general with no overlap to the region for a different parabolic subgroup, it seems difficult to derive (80) this way. We shall now derive it in another way.

Lemma 9. *Let $Q \in \mathcal{P}_\sigma$ and let $P_1, P_2 \in \mathcal{P}_{\sigma,Q}^{\min}$. Then the endomorphism $B(P_2{:}P_1{:}\xi{:}\lambda)$ of $V(\xi)$ preserves the subspace $V_Q(\xi)$, and we have (80) for all generic $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$.*

Proof. We will prove this proposition by a σ–split rank one reduction. The following lemma paves the way. Recall from [2, Sect. 7], that two parabolic subgroups $P_1, P_2 \in \mathcal{P}_\sigma^{\min}$ are called σ–adjacent if $P_1 \neq P_2$ and all $\mathfrak{a}_{\mathrm{q}}$-roots in $\bar{\mathfrak{n}}_1 \cap \mathfrak{n}_2$ are proportional.

Lemma 10. *Let $Q \in \mathcal{P}_\sigma$ and let $P_1, P_2 \in \mathcal{P}_{\sigma,Q}^{\min}$. Then P_1 and P_2 are σ–adjacent if and only if ${}^*P_1$ and ${}^*P_2$ are $\sigma|_{M_{1Q}}$–adjacent parabolic subgroups of M_{1Q}.*

Proof. This is immediate from (79) and the definition of adjacency. □

We continue the proof of Lemma 9. There exists a sequence of parabolic subgroups P_j', $1 \leq j \leq n$, contained in $\mathcal{P}_{\sigma,Q}^{\min}$, such that $P_1' = P_1$, $P_n' = P_2$, and ${}^*P_j'$ and ${}^*P_{j+1}'$ are $(\sigma|M_{1Q})$–adjacent for all $1 \leq j < n$. By Lemma 10 the parabolic subgroups P_j' and P_{j+1}' are σ–adjacent, and by the product formula for the B–endomorphism in [2, Prop. 6.2], applied for G as well as for M_{1Q}, we see that it suffices to prove the result in the case that P_1 and P_2 are adjacent.

Thus assume that $P_1, P_2 \in \mathcal{P}_{\sigma,Q}^{\min}$ are σ–adjacent. Then the $\mathfrak{a}_{\mathrm{q}}$-roots in $\mathfrak{n}_2 \cap \bar{\mathfrak{n}}_1$ are proportional, and belong to the root system of $\mathfrak{a}_{\mathrm{q}}$ in $\mathfrak{m}_{Q1}$. Let α be the smallest $\mathfrak{a}_{\mathrm{q}}$–root in $\mathfrak{n}_2 \cap \bar{\mathfrak{n}}_1$, and let s_α denote the associated reflection in W. Then $s_\alpha \in W_Q$. Recall from the previous section the action $(w, v) \mapsto w \cdot v$ of W on $\mathcal{W}$, defined via transference of the multiplication action under the natural bijection $\mathcal{W} \xrightarrow{\sim} W/W_{K\cap H}$. In particular, if $v \in W_Q$ and $w \in \mathcal{W}_Q$, then $v \cdot w \in \mathcal{W}_Q$. Hence the multiplication by s_α maps $\mathcal{W}_Q$ to itself. Moreover, it follows from [2, Lemma 7.2] that for every $w \in W$ the space $V(\xi, w) + V(\xi, s_\alpha \cdot w)$ is invariant under $B(P_2{:}P_1{:}\xi{:}\lambda)$. Here $V(\xi, w)$ denotes the direct summand $\mathcal{H}_\xi^{wH_Mw^{-1}}$ of $V(\xi)$. In particular this shows that $V_Q(\xi)$, as well as it orthocomplement, is invariant under $B(P_2{:}P_1{:}\xi{:}\lambda)$.

In order to verify (80) it now suffices to prove the meromorphic identity

$$B(P_2{:}P_1{:}\xi{:}\lambda)\eta = B({}^*P_2{:}{}^*P_1{:}\xi{:}\lambda)\eta \tag{81}$$

for $\eta \in V(\xi, w)$, for all $w \in \mathcal{W}_Q$. Without loss of generality we may assume that $1 \in \mathcal{W}$. For $\eta \in V(\xi, 1)$ the identity (81) is a consequence of the fact that the σ-split rank one reduction given in [2, Lemma 7.4] gives identical results for G and for M_{1Q}.

To verify the identity in general we fix an element $w \in \mathcal{W}_Q$. Pick $v \in N_{K_Q}(\mathfrak{a}_{\mathrm{q}})$ such that $v \cdot 1 = w$, and observe that conjugation by v preserves σ–adjacency and that ${}^*(vPv^{-1}) = v{}^*Pv^{-1}$ for all $P \in \mathcal{P}_{\sigma,Q}^{\min}$.

Recall from the previous section the endomorphism $L(\xi, v)$ of $V(\xi)$. It maps $V(\xi, u)$ isomorphically onto $V(v\xi, v \cdot u)$, for all $u \in \mathcal{W}$, hence in particular $V(\xi, 1)$ onto $V(v\xi, w)$. It is easily seen that the restriction of $L(\xi, v)$ to $V_Q(\xi)$ coincides with its analogue $L_Q(\xi, v)$ for M_{1Q} (one can for example use (77) and the diagram (61)). Let $\eta \in V(\xi, 1)$, then (81) has been established above. Applying $L(\xi, v)$ to it and using (62) (for B instead of A, and for G as well as for M_{1Q}), we obtain

$$B(vP_2v^{-1}{:}vP_1v^{-1}{:}v\xi{:}v\lambda)\eta' = B(v{}^*P_2v^{-1}{:}v{}^*P_1v^{-1}{:}v\xi{:}v\lambda)\eta'$$

for all $\eta' = L(\xi, v)\eta \in L(\xi, v)(V(\xi, 1)) = V(v\xi, w)$. Since the σ-adjacent pair $P_1, P_2 \in \mathcal{P}_{\sigma,Q}^{\min}$, as well as ξ and λ were arbitrary, the proof is complete. $\square$

In particular, it follows from (77), (11) and (80) that the relation (77) holds as well with j replaced by j°, for λ in the same region as before.

The relation between the Eisenstein integral $E(P{:}\psi{:}\lambda)$ for G and its analogue for M_{1Q} is much more subtle than these relations between the H-fixed distribution vectors (see Theorem 4 below). However, a simple relation between the c-functions can be derived from (47) and Lemmas 8 and 9. In order to discuss this we let ${}^\circ\mathcal{C}_Q(\tau) = {}^\circ\mathcal{C}(M_{1Q}{:}\tau|_{K_Q})$ be the subspace of ${}^\circ\mathcal{C}(\tau) = \oplus_{w\in\mathcal{W}} C^\infty(M/wH_Mw^{-1}{:}\tau)$ corresponding to the direct summands labeled by $w \in \mathcal{W}_Q$, and denote again by $\mathrm{pr}_Q\colon {}^\circ\mathcal{C}(\tau) \to {}^\circ\mathcal{C}_Q(\tau)$ the (orthogonal) projection along the remaining components. Recall from Lemma 3 the map $T \mapsto \psi_T$ from $C(K{:}\xi{:}\tau)\otimes \bar{V}(\xi)$ to ${}^\circ\mathcal{C}(\tau)$, and let $T \mapsto \psi_T$ denote also the similar map (for M_{1Q}) from $C(K_Q{:}\xi{:}\tau|_{K_Q})\otimes \bar{V}(\xi)$ to ${}^\circ\mathcal{C}_Q(\tau)$. Let $\mathbf{e}\colon C(K{:}\xi{:}\tau) \to [\mathcal{H}_\xi \otimes V_\tau]^{K_M}$ and $\mathbf{e}_Q\colon C(K_Q{:}\xi{:}\tau|_{K_Q}) \to [\mathcal{H}_\xi \otimes V_\tau]^{K_M}$ denote the Frobenius reciprocity maps given by evaluation at the identity, then clearly the restriction r_Q maps $C(K{:}\xi{:}\tau)$ into $C(K_Q{:}\xi{:}\tau|_{K_Q})$, and we have $\mathbf{e}_Q \circ \mathrm{r}_Q = \mathbf{e}$. By the bijectivity of the Frobenius maps we have that r_Q maps $C(K{:}\xi{:}\tau)$ isomorphically onto $C(K_Q{:}\xi{:}\tau|_{K_Q})$. It is now obvious from (23) that we have

$$\mathrm{pr}_Q\,\psi_{f\otimes\eta} = \psi_{\mathrm{r}_Q f\otimes \mathrm{pr}_Q\,\eta} \in {}^\circ\mathcal{C}_Q(\tau) \tag{82}$$

for all $f \in C(K{:}\xi{:}\tau)$, $\eta \in V(\xi)$. The following result is a generalization of a result of Harish-Chandra in the group case (cf. [14, p. 153, Lemma 4]), but its proof is quite different.

Proposition 6. *Let $Q \in \mathcal{P}_\sigma$, and let $P_1, P_2 \in \mathcal{P}_{\sigma,Q}^{\min}$ and $s \in W_Q$. Then the endomorphism $C^\circ_{P_2|P_1}(s{:}\lambda)$ of ${}^\circ\mathcal{C}(\tau)$ preserves the subspace ${}^\circ\mathcal{C}_Q(\tau)$, and we have*

$$\mathrm{pr}_Q \circ C^\circ_{P_2|P_1}(s{:}\lambda) = C^\circ_{{}^*P_2|{}^*P_1}(s{:}\lambda) \circ \mathrm{pr}_Q,$$

for generic $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbf{c}}$.

Proof. For $s = 1$ this follows immediately from (57), (82), and Lemmas 8 and 9. In order to obtain it in general we shall use (68) and its analogue for M_{1Q}. We need the analogue of $\mathcal{L}(s)$ for M_{1Q}. As in the proof of Lemma 9 we have, for $v \in N_{K_Q}(\mathfrak{a}_\mathfrak{q})$, that the endomorphism $L(\xi, v)$ of $V(\xi)$ maps $V_Q(\xi)$ into $V_Q(v\xi)$, and that its restriction to $V_Q(\xi)$ coincides with its analogue $L_Q(\xi, v)$ for M_{1Q}. From (65) and (82) we conclude that the map $\mathcal{L}(s)$ for $s \in W_Q$ preserves ${}^\circ\mathcal{C}_Q(\tau)$, and its restriction to this space coincides with its analogue $\mathcal{L}_Q(s)$ for M_{1Q}. The result follows easily. $\square$

Finally in this section we shall relate the endomorphisms $\mu(D{:}\xi{:}\lambda)$ and $\mu(D{:}\tau{:}\lambda)$ of $V(\xi)$ and ${}^\circ\mathcal{C}(\tau)$, respectively, with their analogues for M_{1Q}. We denote these analogues by $\mu^Q(D{:}\xi{:}\lambda)$ and $\mu^Q(D{:}\tau|_{K_Q}{:}\lambda)$, respectively, where now $D \in \mathbb{D}(M_{1Q}/H_{M_{1Q}})$. Recall from Section 3 the injective homomorphism μ from $\mathbb{D}(G/H)$ to $\mathbb{D}(M_1/H_{M_1})$. When defining this homomorphism we assumed that $P = M_1N = MAN \in \mathcal{P}_\sigma^{\min}$, but actually the minimality of P was not essential. Repeating the steps of this definition we get an injective homomorphism

$$\mu_Q = T_Q \circ {}'\mu_Q\colon \mathbb{D}(G/H) \to \mathbb{D}(M_{1Q}/H_{M_{1Q}})$$

determined by

$$D - {}'\mu_Q(D) \in \mathfrak{n}_Q U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}$$

and $T_Q(D) = d_Q^{-1} \circ D \circ d_Q$ for $D \in \mathbb{D}(M_{1Q}/H_{M_{1Q}})$, where $d_Q(m) = \left|\det(\mathrm{Ad}(m)|_{\mathfrak{n}_Q})\right|^{1/2}$ for $m \in M_{1Q}$. In analogy with the map $\mu'_P = {}'\mu_{\bar{P}}$ (cf. (36)) we also let $\mu'_Q = {}'\mu_{\bar{Q}}: \mathbb{D}(G/H) \to \mathbb{D}(M_{1Q}/H_{M_{1Q}})$. Then

$$\mu'_Q = T_Q \circ \mu_Q. \tag{83}$$

Furthermore, we also have the analoguous maps $\mathbb{D}(M_{1Q}/H_{M_{1Q}}) \to \mathbb{D}(M_1/H_{M_1})$ of μ, ${}'\mu_P$, and μ'_P for the parabolic subgroup ${}^*P$ in M_{1Q}. Denoting these by μ^Q, ${}'\mu^Q_{*P}$, and μ'^Q_{*P}, respectively, we have

$$\mu^Q = T_{*P} \circ {}'\mu^Q_{*P}, \quad \mu'^Q_{*P} = T_{*P} \circ \mu^Q,$$

where

$$D - {}'\mu^Q_{*P}(D) \in {}^*\mathfrak{n}\, U(\mathfrak{m}_{Q1}) + U(\mathfrak{m}_{Q1})(\mathfrak{m}_{Q1} \cap \mathfrak{h}).$$

Since $\mathfrak{n} = {}^*\mathfrak{n} \oplus \mathfrak{n}_Q$ it follows from the above that ${}'\mu_P = {}'\mu^Q_{*P} \circ {}'\mu_Q$, and that $d_P = d_{*P} d_Q$ on M_1. Using that $d_Q = 1$ on M_Q one now easily sees that

$$\mu = \mu^Q \circ \mu_Q. \tag{84}$$

In particular this shows that μ_Q actually only depends on the Levi component M_{1Q} of Q. Furthermore, by inspection of the definitions of $\mu(D{:}\xi{:}\lambda)$ and $\mu(D{:}\tau{:}\lambda)$ we see that these endomorphisms preserve $V_Q(\xi)$ and ${}^\circ\mathcal{C}_Q(\tau)$, respectively, and that

$$\mathrm{pr}_Q \circ \mu(D{:}\xi{:}\lambda) = \mu^Q(\mu_Q(D){:}\xi{:}\lambda) \circ \mathrm{pr}_Q \tag{85}$$

and

$$\mathrm{pr}_Q \circ \mu(D{:}\tau{:}\lambda) = \mu^Q(\mu_Q(D){:}\tau|_{K_Q}{:}\lambda) \circ \mathrm{pr}_Q, \tag{86}$$

for all $D \in \mathbb{D}(G/H)$.

9. The asymptotic behavior of eigenfunctions. In this section we collect some definitions and results from [3, Sect. 12] and [1, Sect. 5].

Let $\|\cdot\|_\sigma: G \to [1, \infty[$ be the distance function defined as follows. Let $\mathfrak{a}_{\mathrm{q}\Sigma}$ be the intersection of the root hyperplanes $\ker\alpha$ ($\alpha \in \Sigma$) in $\mathfrak{a}_{\mathrm{q}}$, and let ${}^\circ\mathfrak{a}_{\mathrm{q}}$ be its orthocomplement in $\mathfrak{a}_{\mathrm{q}}$. Moreover, put $A_{\mathrm{q}\Sigma} = \exp \mathfrak{a}_{\mathrm{q}\Sigma}$ and ${}^\circ A_{\mathrm{q}} = \exp {}^\circ\mathfrak{a}_{\mathrm{q}}$. Then $A_{\mathrm{q}} \simeq {}^\circ A_{\mathrm{q}} \times A_{\mathrm{q}\Sigma}$, and for $a \in {}^\circ A_{\mathrm{q}}$, $b \in A_{\mathrm{q}\Sigma}$ we define:

$$\|ab\|_\sigma = \max_{\alpha \in \Sigma} a^\alpha \, e^{|\log b|}.$$

In view of the Cartan decomposition $G = KA_{\mathrm{q}}H$ the distance function is now completely determined by:

$$\|kah\|_\sigma = \|a\|_\sigma, \qquad (k \in K,\ a \in A_{\mathrm{q}},\ h \in H).$$

We define

$$\|f\|_r = \sup_{x \in G} \|x\|_\sigma^{-r} |f(x)|$$

for every $r \in \mathbb{R}$ and any function $f\colon G/H \to \mathbb{C}$. Moreover, we define $C_r(G/H)$ to be the space of continuous functions $f\colon G/H \to \mathbb{C}$ with $\|f\|_r < \infty$. Equipped with the norm $\|\cdot\|_r$ this space is a Banach space. It is invariant under the left regular representation of G. The associated space $C_r^\infty(G/H)$ of smooth vectors is a Fréchet space. In analogy with [6, (2.7)] it is seen that given $D \in \mathbb{D}(G/H)$ there exists a constant $s \geq 0$ such that D maps $C_r^\infty(G/H)$ continuously into $C_{r+s}^\infty(G/H)$, for all $r \in \mathbb{R}$.

Let $\mathfrak{b} \subset \mathfrak{q}$ be a Cartan subspace containing $\mathfrak{a}_\mathrm{q}$, and let $\gamma = \gamma_{G/H}\colon \mathbb{D}(G/H) \xrightarrow{\sim} S(\mathfrak{b})^{W(\mathfrak{b})}$ be the Harish-Chandra isomorphism, as in Section 3. If $\nu \in \mathfrak{b}_\mathbf{c}^*$, then we write $\mathcal{E}_\nu^\infty(G/H)$ for the space of smooth functions $f\colon G/H \to \mathbb{C}$ satisfying the system of differential equations:

$$Df = \gamma(D{:}\nu)f, \qquad (D \in \mathbb{D}(G/H)).$$

If $r \in \mathbb{R}$, then the space $\mathcal{E}_{\nu,r}^\infty(G/H) := \mathcal{E}_\nu^\infty(G/H) \cap C_r^\infty(G/H)$ is a closed subspace of $C_r^\infty(G/H)$, hence a Fréchet space. We define

$$\mathcal{E}_{\nu,*}^\infty(G/H) = \cup_{r>0}\, \mathcal{E}_{\nu,r}^\infty(G/H).$$

It follows from [3, Lemma 12.3] that the K-finite elements of $\mathcal{E}_\nu^\infty(G/H)$ belong to this space. Notice that $\mathcal{E}_{\nu,*}^\infty(G/H)$ is a $\mathbb{D}(G/H)$-invariant subspace of $C^\infty(G/H)$.

Recall that $\mathfrak{b} = \mathfrak{b}_\mathrm{k} \oplus \mathfrak{a}_\mathrm{q}$ is the decomposition of $\mathfrak{b}$ in ± 1-eigenspaces for θ. According to this decomposition we view $\mathfrak{b}_{\mathrm{k}\mathbf{c}}^*$ and $\mathfrak{a}_{\mathrm{q}\mathbf{c}}^*$ as subspaces of $\mathfrak{b}_\mathbf{c}^*$. Let $\Lambda \in \mathfrak{b}_{\mathrm{k}\mathbf{c}}^*$ be fixed from now on, and let λ denote a parameter in $\mathfrak{a}_{\mathrm{q}\mathbf{c}}^*$. Let $Q = M_Q A_Q N_Q \in \mathcal{P}_\sigma$ be fixed, let $\Sigma(Q)$ denote the set of roots $\alpha \in \Sigma$ with $\mathfrak{g}_\alpha \subset \mathfrak{n}_Q$, let $\mathfrak{a}_{Q\mathrm{q}} = \mathfrak{a}_Q \cap \mathfrak{a}_\mathrm{q}$, and put

$$\mathfrak{a}_{Q\mathrm{q}}^+ = \{X \in \mathfrak{a}_{Q\mathrm{q}} \mid \alpha(X) > 0 \text{ for all } \alpha \in \Sigma(Q)\}.$$

The purpose of this section is to study the asymptotic behavior along $A_{Q\mathrm{q}}^+ = \exp \mathfrak{a}_{Q\mathrm{q}}^+$ of functions $f \in \mathcal{E}_{\Lambda+\lambda,*}^\infty(G/H)$. Our starting point is the following result. If V is a finite dimensional real linear space, then by $P_m(V)$ we denote the space of polynomial functions $f\colon V \to \mathbb{C}$ of degree at most m. Let the set $X_Q(\Lambda, \lambda) \subset \mathfrak{a}_{Q\mathrm{q}\mathbf{c}}^*$ be given by

$$X_Q(\Lambda,\lambda) = \{0\} \cup \big\{[w(\Lambda+\lambda) - \rho_Q - \mu]|_{\mathfrak{a}_{Q\mathrm{q}}} \bigm| w \in W(\mathfrak{b}), \mu \in \mathbb{N}\Sigma(Q)\big\},$$

where $\mathbb{N}\Sigma(Q)$ is the set of linear combinations of elements from $\Sigma(Q)$ with non-negative integral coefficients. Finally, let $d\colon [0,\infty[\to \mathbb{N}$ be the locally bounded function of [3, Prop. 12.4]. From [3, Thm. 12.8] we have:

Proposition 7. *Let* $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbf{c}}^*$.

(a) *Let* $f \in \mathcal{E}_{\Lambda+\lambda,*}^\infty(G/H)$ *and* $x \in G$. *Then there exist unique polynomials* $p_{\lambda,\xi}(Q|f,x)$ *on* $\mathfrak{a}_{Q\mathrm{q}}$ *of degree at most* $d(|\operatorname{Re}\lambda| + |\operatorname{Re}\xi|)$, *for* $\xi \in X_Q(\Lambda,\lambda)$, *such that*

$$f(x \exp tX) \sim \sum_{\xi \in X_Q(\Lambda,\lambda)} p_{\lambda,\xi}(Q|f,x,tX)\, e^{t\xi(X)} \quad (t \to \infty) \tag{87}$$

at every $X_0 \in \mathfrak{a}_{Q\mathrm{q}}^+$.

(b) *Let* $r \in \mathbb{R}$, $\xi \in X_Q(\Lambda, \lambda)$, *and put* $d = d(|\operatorname{Re}\lambda| + |\operatorname{Re}\xi|)$. *Then there exists a number* $r' \in \mathbb{R}$ *such that* $f \mapsto p_{\lambda,\xi}(Q|f)$ *is a continuous linear map from* $\mathcal{E}^\infty_{\Lambda+\lambda,r}(G/H)$ *into* $C^\infty_{r'}(G) \otimes P_d(\mathfrak{a}_{Q\mathrm{q}})$, *equivariant for the left regular actions of* G *on* $\mathcal{E}^\infty_{\Lambda+\lambda,r}(G/H)$ *and* $C^\infty_{r'}(G)$.

Remark. The asymptotic symbol $\sim$, and the phrase 'at X_0', means the following (cf. [6, Sect. 3]). There exist, for each real number N, a neighborhood U of X_0 in $\mathfrak{a}_{Q\mathrm{q}}^+$ and constants $\epsilon > 0$, $C > 0$ such that

$$|f(x \exp tX) - \sum_{\substack{\xi \in X_Q(\Lambda,\lambda) \\ \operatorname{Re}\xi(X_0) \geq N}} p_{\lambda,\xi}(Q|f, x, tX)\, e^{t\xi(X)}| \leq C e^{(N-\epsilon)t} \tag{88}$$

for all $X \in U$, $t \geq 0$.

Before proceeding we list some properties of the coefficients in the expansion which will be needed in the sequel. Fix $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbf{c}}$ and $f \in \mathcal{E}^\infty_{\Lambda+\lambda,*}(G/H)$. An element $\xi \in \mathfrak{a}^*_{Q\mathrm{q}\mathbf{c}}$ will be called an exponent along Q of f if $\xi \in X_Q(\Lambda, \lambda)$ and $p_{\lambda,\xi}(Q|f, \cdot)$ is not identically zero. The set of exponents along Q of f is denoted by $\mathrm{E}(Q|f)$. By [3, Lemma 13.1] we have

$$p_{\lambda,\xi}(Q|f, xma, X) = p_{\lambda,\xi}(Q|f, x, X + \log a) a^\xi \tag{89}$$

for all $\xi \in \mathrm{E}(Q|f)$, $x \in G$, $m \in H_{M_{1Q}}$, $X \in \mathfrak{a}_{Q\mathrm{q}}$, and $a \in A_{Q\mathrm{q}}$.

It will be convenient to use the following notations. For a given $x \in G$ we denote by $\mathrm{E}(Q|f, x)$ the set of $\xi \in \mathrm{E}(Q|f)$ for which $p_{\lambda,\xi}(Q|f, x) \neq 0$. Then obviously $\mathrm{E}(Q|f)$ is the union over $x \in G$ of the sets $\mathrm{E}(Q|f, x)$.

We define the partial ordering $\preceq_Q$ on $\mathfrak{a}^*_{Q\mathrm{q}\mathbf{c}}$ by

$$\eta_1 \preceq_Q \eta_2 \iff \eta_2 - \eta_1 \in \mathbb{N}\Sigma(Q)|_{\mathfrak{a}_{Q\mathrm{q}}}.$$

The $\preceq_Q$-maximal elements of $\mathrm{E}(Q|f)$ are called the leading exponents along Q of f; the set of these is denoted by $\mathrm{E}_L(Q|f)$. Let $\xi \in \mathrm{E}_L(Q|f)$ and let $x \in G$, $X \in \mathfrak{a}_{Q\mathrm{q}}$. By (89) the function $\varphi \in C^\infty(M_{1Q})$ defined by $\varphi(m) = p_{\lambda,\xi}(Q|f, xm, X)$ is right $H_{M_{1Q}}$-invariant, and by [3, Cor. 13.3] it satisfies the system of differential equations

$$\mu'_Q(D)\varphi = \gamma(D{:}\Lambda + \lambda)\varphi, \qquad (D \in \mathbb{D}(G/H)). \tag{90}$$

Here the map $\mu'_Q : \mathbb{D}(G/H) \to \mathbb{D}(M_{1Q}/H_{M_{1Q}})$ is defined by (83).

For general functions $f \in \mathcal{E}^\infty_{\Lambda+\lambda,*}(G/H)$ the expansion (87) holds asymptotically in the sense of (88). However, if f is K-finite it follows from the asymptotic theory in [1] that it actually converges absolutely and locally uniformly in $X \in \mathfrak{a}_{Q\mathrm{q}}^+$. Since we shall need also this theory, we recall the basic properties.

It will be convenient to use the following notations. As above, let $Q \in \mathcal{P}_\sigma$ be fixed, and fix a basis Γ for $\mathfrak{a}^*_{Q\mathrm{q}}$. If $m \in \mathbb{N}^\Gamma$ then we use the usual multi-index notation $|m| = \sum_{\gamma\in\Gamma} m_\gamma$. Moreover, if $H \in \mathfrak{a}^*_{Q\mathrm{q}}$, then we put $H^m = \prod_{\gamma\in\Gamma} \gamma(H)^{m_\gamma}$. Let Σ^+ be a system of positive roots for Σ containing $\Sigma(Q)$. Let Δ denote the set of simple roots in Σ^+, and put $\Delta(Q) = \Delta \cap \Sigma(Q)$. Then restriction to $\mathfrak{a}_{Q\mathrm{q}}$ maps $\Delta(Q)$ bijectively onto a set $\Delta_r(Q)$ of linearly independent elements in $\mathfrak{a}_{Q\mathrm{q}}^*$. If $H \in \mathfrak{a}_{Q\mathrm{q}}$ then we define the element $z(H) \in \mathbb{C}^{\Delta(Q)}$ by

$$z(H)_\alpha = e^{-\alpha(H)} \quad \text{for} \quad \alpha \in \Delta(Q).$$

Let D be the open unit disk in $\mathbb{C}$. Then the map $H \mapsto z(H)$ maps $\mathfrak{a}_{Q\mathrm{q}}^+$ into $D^{\Delta(Q)}$.

For the moment let V be a finite dimensional complex vector space, and let f be an arbitrary K-finite V-valued function on G/H which is also $\mathbb{D}(G/H)$-finite. Then according to [1, Thm. 5.3], there exists a finite subset $S \subset \mathfrak{a}_{Q\mathrm{qc}}^*$ such that the natural map $S \to \mathfrak{a}_{Q\mathrm{qc}}^*/\mathbb{Z}\Delta_r(Q)$ is injective, and moreover a positive integer d and for each $s \in S$, $m \in \mathbb{N}^\Gamma$, $|m| \leq d$, a holomorphic function $f_{s,m}\colon D^{\Delta(Q)} \to V$, such that for all $H \in \mathfrak{a}_{Q\mathrm{q}}^+$ we have:

$$f(\exp H) = \sum_{s \in S, |m| \leq d} H^m e^{s(H)} f_{s,m}(z(H)). \tag{91}$$

Being holomorphic the functions $f_{s,m}$ have (V-valued) Taylor expansions

$$f_{s,m}(z) = \sum_{\mu \in \mathbb{N}\Delta_r(Q)} c_{s-\mu,m}\, z^\mu, \qquad (z \in D^{\Delta(Q)}).$$

Here we have written $z^\mu = \prod_{\alpha \in \Delta(Q)} z_\alpha^{\mu_\alpha}$ if $\mu = \sum_{\alpha \in \Delta(Q)} \mu_\alpha \alpha|_{\mathfrak{a}_{Q\mathrm{q}}}$. Substituting these Taylor expansions in (91) we obtain the following converging expansion when $H \in \mathfrak{a}_{Q\mathrm{q}}^+$:

$$f(\exp H) = \sum_{\substack{\xi \in S - \mathbb{N}\Delta_r(Q) \\ |m| \leq d}} c_{\xi,m} H^m e^{\xi(H)}. \tag{92}$$

Let $\mathbf{E}(Q|f,e)$ denote the set of elements $\xi \in S - \mathbb{N}\Delta_r(Q)$ for which $c_{\xi,m} \neq 0$ for some m.

Let now $f \in \mathcal{E}^\infty_{\Lambda+\lambda,*}(G/H)$ be K-finite. The asymptotic theory for K- and $\mathbb{D}(G/H)$-finite functions just outlined applies to f and thus in addition to (87) we have the converging expansion (92). By holomorphy of the $f_{s,m}$ the latter expansion is an asymptotic expansion if H tends radially to infinity in $\mathfrak{a}_{Q\mathrm{q}}^+$, hence by uniqueness of asymptotics it coincides with the expansion (87) at $x = e$. We conclude that

$$\mathrm{E}(Q|f,e) = \mathbf{E}(Q|f,e) \subset S - \mathbb{N}\Delta_r(Q)$$

and moreover that

$$p_{\lambda,\xi}(Q|f,e,H) = \sum_{|m| \leq d} c_{\xi,m} H^m, \qquad (\xi \in \mathrm{E}(Q|f,e),\ H \in \mathfrak{a}_{Q\mathrm{q}}).$$

10. Transitivity of asymptotics. If $P, Q \in \mathcal{P}_\sigma$ and $P \subset Q$, then the expansions along P and Q of a function $f \in \mathcal{E}^\infty_{\Lambda+\lambda,*}(G/H)$ are related. The following theorem gives this relation. As in Section 8 let ${}^*\!P = P \cap M_{1Q}$, then ${}^*\!P$ has the Langlands decomposition ${}^*\!P = M_P A_P {}^*\!N$, where ${}^*\!N = N_P \cap M_{1Q}$. Let $\mathfrak{a}_{{}^*\!P\mathrm{q}}^+$ denote the set of elements $H \in \mathfrak{a}_{P\mathrm{q}}$ with $\alpha(H) > 0$ for all $\alpha \in \Sigma({}^*\!P) = \Sigma(P) \setminus \Sigma(Q)$, and put $A_{{}^*\!P\mathrm{q}}^+ = \exp(\mathfrak{a}_{{}^*\!P\mathrm{q}}^+)$. In particular we have $\mathfrak{a}_{P\mathrm{q}}^+ \subset \mathfrak{a}_{{}^*\!P\mathrm{q}}^+$.

Theorem 3. *Let two $\sigma\theta$-stable parabolic subgroups P and Q be given such that $A_{\mathrm{q}} \subset P \subset Q$. Let $\Lambda \in \mathfrak{b}_{\mathrm{kc}}^*$, $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$, and $f \in \mathcal{E}^\infty_{\Lambda+\lambda,*}(G/H)$. Then*

$$\mathrm{E}(Q|f) \subset \{\eta|_{\mathfrak{a}_{Q\mathrm{q}}} \mid \eta \in \mathrm{E}(P|f)\}.$$

Moreover, if f is K–finite then we have:

$$p_{\lambda,\xi}(Q|f,a,X) = \sum_{\substack{\eta \in \mathrm{E}(P|f) \\ \eta|_{\mathfrak{a}_{Q\mathrm{q}}}=\xi}} p_{\lambda,\eta}(P|f,a,X) \tag{93}$$

for all $\xi \in \mathrm{E}(Q|f)$, $X \in \mathfrak{a}_{Q\mathrm{q}}$, and $a \in A^+_{{}^*P\mathrm{q}}$. The series is absolutely convergent.

Remark. For the Riemannian case (i.e. when H is compact) and if P is minimal, this result is a consequence of [7, Thm. 3.1]. Notice that in *loc. cit.* it is not required that f be K-finite, but then the expansions (87) and (93) are asymptotic and need not converge. We expect that an analogous result should hold in the present case. However, since as mentioned our applications will be to K-finite functions we do not need such a more general result. In the proof of Theorem 3 we follow [7].

Proof. We first prove (93). Let $f \in \mathcal{E}^\infty_{\Lambda+\lambda,*}(G/H)$ be K-finite. Then the asymptotic theory outlined at the end of the previous section applies to f, and we have

$$f(\exp H) = \sum_{\xi \in \mathrm{E}(P|f,e)} p_{\lambda,\xi}(P|f,e,H)e^{\xi(H)}, \qquad (H \in \mathfrak{a}^+_{P\mathrm{q}}).$$

In the following, let H_0 always denote an element of $\mathfrak{a}^+_{{}^*P\mathrm{q}}$. Moreover, H_1 will always denote an element of $\mathfrak{a}_{Q\mathrm{q}}$. Given R we write $\mathfrak{a}^+_{Q\mathrm{q}}(R)$ for the set of $X \in \mathfrak{a}_{Q\mathrm{q}}$ with $\alpha(X) > R$ for all $\alpha \in \Delta(Q)$.

Let H_0 be fixed for the moment. We fix $R > 0$ such that for $H_1 \in \mathfrak{a}^+_{Q\mathrm{q}}(R)$ we have $H_0 + H_1 \in \mathfrak{a}^+_{P\mathrm{q}}$ (here we have used that the roots of $\Delta \setminus \Delta(Q)$ vanish on $\mathfrak{a}_{Q\mathrm{q}}$).

Let $H_1 \in \mathfrak{a}^+_{Q\mathrm{q}}(R)$ and $t \geq 1$. Then substituting $H = H_0 + tH_1 \in \mathfrak{a}^+_{P\mathrm{q}}$ in the above expansion we obtain:

$$f(\exp H_0 \exp tH_1) = \sum_{\eta \in \mathrm{E}(P|f,e)|_{\mathfrak{a}_{Q\mathrm{q}}}} \Big[\sum_{\substack{\xi \in \mathrm{E}(P|f,e) \\ \xi|_{\mathfrak{a}_{Q\mathrm{q}}}=\eta}} p_{\lambda,\xi}(P|f,e,H_0+tH_1)e^{\xi(H_0)}\Big] e^{t\eta(H_1)}.$$

Notice that the series between square brackets converges absolutely by the holomorphy of the $f_{s,m}$ in (91). Moreover, again by holomorphy of these functions, the above expansion is an asymptotic expansion as $t \to \infty$. By uniqueness of asymptotics we conclude that $\mathrm{E}(Q|f, \exp H_0) \subset \mathrm{E}(P|f,e)|_{\mathfrak{a}_{Q\mathrm{q}}}$, and that for all $\eta \in \mathrm{E}(P|f,e)|_{\mathfrak{a}_{Q\mathrm{q}}}$ we have:

$$p_{\lambda,\eta}(Q|f,\exp H_0, H_1) = \sum_{\substack{\xi \in \mathrm{E}(P|f,e) \\ \xi|_{\mathfrak{a}_{Q\mathrm{q}}}=\eta}} p_{\lambda,\xi}(P|f,e,H_0+H_1)e^{\xi(H_0)}.$$

This expansion converges absolutely and holds for all $H_1 \in \mathfrak{a}_{Q\mathrm{q}}^+(R)$. Since it is polynomial in H_1, it holds in fact for all $H_1 \in \mathfrak{a}_{Q\mathrm{q}}$. By using the transformation rule (89) we get (93).

To establish the assertion about the set of exponents, notice that for K-finite f we have proved that $\mathrm{E}(Q|f, \exp H_0) \subset \mathrm{E}(P|f)|_{\mathfrak{a}_{Q\mathrm{q}}}$. By density of the K-finite functions in $\mathcal{E}^\infty_{\Lambda+\lambda,r}(G/H)$, for all $r \in \mathbb{R}$, and continuity of the maps $f \mapsto p_{\lambda,\eta}(Q|f)$ and $f \mapsto p_{\lambda,\xi}(P|f)$, we see that for general f we also have $\mathrm{E}(Q|f, \exp H_0) \subset \mathrm{E}(P|f)|_{\mathfrak{a}_{Q\mathrm{q}}}$. By equivariance of the maps $f \mapsto p_{\lambda,\eta}(Q|f)$ we conclude from this that $\mathrm{E}(Q|f,x) \subset \mathrm{E}(P|f)|_{\mathfrak{a}_{Q\mathrm{q}}}$ for all $x \in G$. $\square$

11. Asymptotic expansions of holomorphic families. The set of exponents occurring in the expansion (87) can be limited drastically if f is part of an analytic family. Let $\Omega_0 \subset \mathfrak{a}^*_{\mathrm{q}\mathbf{c}}$ be an open subset. If f is a function $\Omega_0 \times G/H \to \mathbb{C}$ then if $\lambda \in \Omega_0$, we shall write f_λ for the function $G/H \to \mathbb{C}$, $x \mapsto f(\lambda, x)$. Let $\Lambda \in \mathfrak{b}^*_{\mathrm{k}\mathbf{c}}$ be fixed.

Definition 1. *We define $\mathcal{E}_*(G/H, \Lambda, \Omega_0)$ to be the space of functions $f\colon \Omega_0 \times G/H \to \mathbb{C}$ satisfying the following two conditions:*

(a) *for every $\lambda \in \Omega_0$ the function f_λ belongs to $\mathcal{E}^\infty_{\Lambda+\lambda,*}(G/H)$;*
(b) *for every $\lambda_0 \in \Omega_0$ there exists a constant $r \in \mathbb{R}$ such that $\lambda \mapsto f_\lambda$ maps a neighborhood of λ_0 holomorphically into $C^\infty_r(G/H)$.*

Notice that, as mentioned earlier, each element $D \in \mathbb{D}(G/H)$ maps $\mathcal{E}^\infty_{\nu,*}(G/H)$ to itself for all $\nu \in \mathfrak{a}^*_{\mathrm{q}\mathbf{c}}$, and $C^\infty_r(G/H)$ continuously to $C^\infty_{r+s}(G/H)$ for all $r \in \mathbb{R}$ and some $s \geq 0$. Consequently, D maps the space $\mathcal{E}_*(G/H, \Lambda, \Omega_0)$ to itself.

The above mentioned limitation on the set of exponents is expressed in the following proposition. Let $Q \in \mathcal{P}_\sigma$ be arbitrary. We denote by $\Sigma_r(Q)$ the set of elements in $\mathfrak{a}^*_{Q\mathrm{q}}$ obtained by restriction of an element from $\Sigma(Q)$. For $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbf{c}}$ we define the set

$$X(Q,\lambda) = \left\{ (s\lambda - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}} - \mu \;\middle|\; s \in W,\ \mu \in \mathbb{N}\Sigma_r(Q) \right\} \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbf{c}}.$$

Proposition 8. *Let Ω_0 be an open subset of $\mathfrak{a}^*_{\mathrm{q}\mathbf{c}}$ and let $f \in \mathcal{E}_*(G/H, \Lambda, \Omega_0)$. Then for every $\lambda \in \Omega_0$ we have:*

$$\mathrm{E}(Q|f_\lambda) \subset X(Q,\lambda).$$

Proof. For $Q \in \mathcal{P}^{\min}_\sigma$ this was established in the proof of [3, Thm. 13.7]. (In particular this means that eqn. (107) in *loc. cit.* holds for all $\lambda \in \Omega_0$, and not just for $\lambda \in \mathfrak{a}^{*\prime}_{\mathrm{q}\mathbf{c}}$.)

Let now Q be arbitrary, and fix $P \in \mathcal{P}^{\min}_\sigma$ such that $P \subset Q$. Let $\lambda \in \Omega_0$, and suppose that $\xi \in \mathrm{E}(Q|f_\lambda)$. Then by Theorem 3 we have that $\xi = \eta|_{\mathfrak{a}_{Q\mathrm{q}}}$ for some $\eta \in \mathrm{E}(P|f)$. Moreover, by the first part of this proof we have that $\eta \in X(P,\lambda)$, hence there exist $s \in W$ and $\mu \in \mathbb{N}\Sigma(P)$ such that $\eta = s\lambda - \rho_P - \mu$. It follows that $\xi = (s\lambda - \rho_P - \mu)|_{\mathfrak{a}_{Q\mathrm{q}}}$. Since the roots in $\Sigma(P) \setminus \Sigma(Q)$ restrict to zero on $\mathfrak{a}_{Q\mathrm{q}}$ we have that $\rho_P|_{\mathfrak{a}_{Q\mathrm{q}}} = \rho_Q|_{\mathfrak{a}_{Q\mathrm{q}}}$ and $\mu|_{\mathfrak{a}_{Q\mathrm{q}}} \in \mathbb{N}\Sigma_r(Q)$, and hence $\xi \in X(Q,\lambda)$. $\square$

According to the above result, the asymptotic expansion (87) holds with f replaced by f_λ and with $X_Q(\Lambda,\lambda)$ replaced by the smaller set $X(Q,\lambda)$. The following result asserts that the asymptotic expansion for f_λ obtained in this way depends holomorphically on λ in a suitable sense.

Proposition 9. *Let $f \in \mathcal{E}_*(G/H, \Lambda, \Omega_0)$, and fix $\lambda_0 \in \Omega_0$ and $\xi_0 \in X(Q, \lambda_0)$. For $\lambda \in \Omega_0$, let $\Xi(\lambda)$ be the the set of elements $\xi \in X(Q, \lambda)$ of the form*

$$\xi = (s\lambda - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}} - \mu,$$

where $s \in W$ and $\mu \in \mathbb{N}\Sigma_r(Q)$ satisfy the equation

$$\xi_0 = (s\lambda_0 - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}} - \mu.$$

Then there exists an open neighborhood $\Omega \subset \Omega_0$ of λ_0 and a constant $r' \in \mathbb{R}$ such that the map:

$$(\lambda, X) \mapsto \sum_{\xi \in \Xi(\lambda)} p_{\lambda,\xi}(Q|f_\lambda, \cdot, X)$$

is continuous from $\Omega \times \mathfrak{a}_{Q\mathrm{q}}$ to $C^\infty_{r'}(G)$, and in addition holomorphic in λ.

Proof. For $\lambda \in \Omega_0$, let $\Xi_0(\lambda)$ be the union of the set $\{0\} \cap \{\xi_0\}$ with the set of elements of the form

$$\xi = [w(\Lambda + \lambda) - \rho_Q]|_{\mathfrak{a}_{Q\mathrm{q}}} - \mu \in X_Q(\Lambda, \lambda),$$

where $w \in W(\mathfrak{b})$ and $\mu \in \mathbb{N}\Sigma_r(Q))$ satisfy

$$\xi_0 = [w(\Lambda + \lambda_0) - \rho_Q]|_{\mathfrak{a}_{Q\mathrm{q}}} - \mu.$$

Then according to [3, Thm. 12.9], there exists an open neighborhood $\Omega \subset \Omega_0$ of λ_0 and a constant $r' \in \mathbb{R}$ such that the above assertion holds with $\Xi_0(\lambda)$ instead of $\Xi(\lambda)$. (Notice that in that theorem there is a slight error in the definition of the set $\Xi(\lambda)$, denoted $\Xi_0(\lambda)$ in the present notation.) In view of Proposition 8 it suffices to show that Ω may be chosen so that

$$\Xi(\lambda) = \Xi_0(\lambda) \cap X(Q, \lambda) \qquad \text{for} \quad \lambda \in \Omega. \tag{94}$$

Obviously the inclusion '$\subset$' holds in (94). It therefore remains to prove the converse inclusion.

Fix a bounded open neighborhood V of ξ_0 in $\mathfrak{a}^*_{Q\mathrm{q}\mathbf{c}}$ such that $\bar{V} \cap X(Q, \lambda_0) = \{\xi_0\}$. Then there exists an open neighborhood U of λ_0 in Ω such that $V \cap X(Q, \lambda) \subset \Xi(\lambda)$ for $\lambda \in U$. Shrinking U if necessary, we may also assume that $\Xi_0(\lambda) \subset V$ for $\lambda \in \Omega$, from which the inclusion '$\supset$' in (94) then follows. ☐

Following [3, p. 399] we define

$$'\mathfrak{a}^*_{\mathrm{q}\mathbf{c}} = \{\, \lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbf{c}} \mid \langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}\, (\forall \alpha \in \Sigma) \,\},$$

where $\alpha^\vee = 2\langle \alpha, \alpha \rangle^{-1}\alpha$ as usual. We recall that $'\mathfrak{a}^*_{\mathrm{q}\mathbf{c}}$ is the complement of a locally finite union of hyperplanes. Moreover, if $\lambda \in {}'\mathfrak{a}^*_{\mathrm{q}\mathbf{c}}$, and $s, t \in W$, then $s\lambda - t\lambda \in \mathbb{Z}\Sigma \Rightarrow s = t$. Analogously we have the following result.

Lemma 11. *There exists a subset* $''\mathfrak{a}_{\mathrm{qc}}^* \subset \mathfrak{a}_{\mathrm{qc}}^*$ *with the following properties:*

(a) *The set* $''\mathfrak{a}_{\mathrm{qc}}^*$ *is the complement of a locally finite union of proper affine subspaces in* $\mathfrak{a}_{\mathrm{qc}}^*$.

(b) *Let* $\lambda \in {}''\mathfrak{a}_{\mathrm{qc}}^*$ *and* $Q \in \mathcal{P}_\sigma$ *be arbitrary, and suppose that* $s, t \in W$ *are such that* $(s\lambda - t\lambda)|_{\mathfrak{a}_{Q\mathrm{q}}} \in \mathbb{Z}\Sigma_r(Q)$. *Then* $W_Q s = W_Q t$.

Proof. Fix Q as in (b) and let $s, t \in W$. Define $V(Q, s, t)$ to be the set of $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$ for which $(s\lambda - t\lambda)|_{\mathfrak{a}_{Q\mathrm{q}}} \in \mathbb{Z}\Sigma_r(Q)$. We claim that for $W_Q s \neq W_Q t$ the set $V(Q, s, t)$ is a locally finite union of proper affine subspaces. It suffices to establish this claim, for its validity implies that the desired result holds with $''\mathfrak{a}_{\mathrm{qc}}^*$ equal to the complement of the union of the finitely many sets $V(Q, s, t)$ where Q is arbitrary and $s, t \in W$, $W_Q s \neq W_Q t$.

Fix $P \in \mathcal{P}_\sigma^{\min}$ such that $P \subset Q$. Then $\Sigma(P)$ is a positive system for Σ which contains the set $\Sigma(Q)$. Let Δ be the set of simple roots in $\Sigma(P)$, and put $\Delta(Q) = \Delta \cap \Sigma(Q)$ and $\Delta_Q = \Delta \setminus \Delta(Q)$. Then $\mathfrak{a}_{Q\mathrm{q}}$ is the intersection of the root hyperplanes $\ker \alpha$, $\alpha \in \Delta_Q$. If $\alpha \in \Delta$, we write ω_α for the element of the real linear span $\mathbb{R}\Sigma$ of Σ satisfying $\langle \omega_\alpha, \beta\rangle = \delta_{\alpha\beta}$.

If $\nu \in \mathfrak{a}_{\mathrm{qc}}^*$, then the condition that $\nu|_{\mathfrak{a}_{Q\mathrm{q}}} \in \mathbb{Z}\Sigma_r(Q)$ is equivalent to the condition $\nu \in \mathbb{C}\Delta_Q + \mathbb{Z}\Delta(Q)$, which in turn is equivalent to the condition that $\langle \nu, \omega_\alpha\rangle \in \mathbb{Z}$ for every $\alpha \in \Delta(Q)$. From this we see that the set $V(Q, s, t)$ equals the union of the following sets, parametrized by $n \in \mathbb{Z}^{\Delta(Q)}$:

$$V(Q, s, t, n) = \{\lambda \in \mathfrak{a}_{\mathrm{qc}}^* \mid \langle s\lambda - t\lambda, \omega_\alpha\rangle = n_\alpha \quad (\forall \alpha \in \Delta(Q))\ \}.$$

Suppose $V(Q, s, t, n) = \mathfrak{a}_{\mathrm{qc}}^*$. Then it follows that $n = 0$ and that for all $\alpha \in \Delta(Q)$ we have $s^{-1}\omega_\alpha = t^{-1}\omega_\alpha$. This implies that ts^{-1} centralizes the fundamental weights orthogonal to Δ_Q, hence belongs to the subgroup of W generated by the reflections s_α, $\alpha \in \Delta_Q$, i.e. to W_Q. Thus we see that for $s, t \in W$ with $W_Q s \neq W_Q t$ the set $V(Q, s, t, n)$ is a proper affine subspace of $\mathfrak{a}_{\mathrm{qc}}^*$, for any n. Since it is clear that the collection of $V(Q, s, t, n)$ is locally finite this establishes the claim. □

Let Ω_0 be an open subset of $\mathfrak{a}_{\mathrm{qc}}^*$, and suppose that $\Lambda \in \mathfrak{b}_{\mathrm{kc}}^*$, $f \in \mathcal{E}_*(G/H, \Lambda, \Omega_0)$. If $\lambda \in \Omega_0$, $s \in W$ and $\mu \in \mathbb{N}\Sigma_r(Q)$, then we denote the value at zero of the $C^\infty(G)$-valued polynomial function $X \mapsto p_{\lambda, (s\lambda - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}} - \mu}(Q|f_\lambda, \cdot, X)$ by

$$p_{Q,\mu}(f{:}s{:}\lambda) := p_{\lambda, (s\lambda - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}} - \mu}(Q|f_\lambda, \cdot, 0) \in C^\infty(G). \tag{95}$$

Obviously (95) remains unchanged if we replace s by any element from the coset $W_Q s$. Therefore we shall also use the notation $p_{Q,\mu}(f{:}s{:}\lambda)$ for left cosets $s \in W_Q\backslash W$.

Proposition 10. *Let* $\Lambda \in \mathfrak{b}_{\mathrm{kc}}^*$, *let* Ω_0 *be an open subset of* $\mathfrak{a}_{\mathrm{qc}}^*$, *and assume that* $f \in \mathcal{E}_*(G/H, \Lambda, \Omega_0)$. *Let* $s \in W$, *and* $\mu \in \mathbb{N}\Sigma_r(Q)$. *Then for every* $\lambda \in \Omega_0 \cap {}'\mathfrak{a}_{\mathrm{qc}}^*$ *we have*

$$p_{\lambda, (s\lambda - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}} - \mu}(Q|f_\lambda, \cdot, X) = p_{Q,\mu}(f{:}s{:}\lambda), \qquad (X \in \mathfrak{a}_{Q\mathrm{q}}). \tag{96}$$

Moreover, $p_{Q,\mu}(f{:}s{:}\lambda)$ *is holomorphic as a* $C^\infty(G)$*-valued function of* λ *on* $\Omega_0 \cap {}''\mathfrak{a}_{\mathrm{qc}}^*$ *and allows a meromorphic extension to* Ω_0.

If $\lambda_0 \in \Omega_0$, then there exists an open neighborhood Ω of λ_0 in Ω_0 and a constant $r' \in \mathbb{R}$ such that $p_{Q,\mu}(f{:}s{:}\lambda)$ defines a meromorphic $C^\infty_{r'}(G)$-valued function of λ on Ω.

Proof. If $Q \in \mathcal{P}_\sigma^{\min}$ this is just [3, Thm. 13.10]. In the general case we fix $P \in \mathcal{P}_\sigma^{\min}$ such that $P \subset Q$. We define a linear representation L of G in $\mathcal{E}_*(G/H, \Lambda, \Omega_0)$ by $L_g(f)_\lambda = L_g(f_\lambda)$, $g \in G$. The subspace of K-finite elements for this action is denoted by $\mathcal{E}_*(G/H, \Lambda, \Omega_0)_K$. Let $\hat{K}$ denote the set of equivalence classes of finite dimensional unitary irreducible representations of K, and for $\delta \in \hat{K}$ let χ_δ be its character and put $\alpha_\delta = \dim(\delta)\bar{\chi}_\delta$. Furthermore, let

$$(f^\delta)_\lambda(x) = \int_K \alpha_\delta(k)(L_k f)_\lambda(x) dk$$

for $f \in \mathcal{E}_*(G/H, \Lambda, \Omega_0)$, $\lambda \in \Omega_0$ and $x \in G/H$. Then $f^\delta \in \mathcal{E}_*(G/H, \Lambda, \Omega_0)_K$ and we have

$$f_\lambda = \sum_{\delta \in \hat{K}} (f^\delta)_\lambda, \tag{97}$$

in the topology of $C^\infty_r(G/H)$, for all $\lambda \in \Omega_0$, with r locally independent of λ (as in item (b) of Definition 1).

Fix $\lambda \in \Omega_0 \cap {}'\mathfrak{a}^*_{\mathrm{qc}}$ and let $\xi = (s\lambda - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}} - \mu \in X(Q, \lambda)$. Fix $a \in A^+_{{}^*P\mathrm{q}}$. Let $f \in \mathcal{E}_*(G/H, \Lambda, \Omega_0)_K$. If $\xi \notin \mathrm{E}(Q|f_\lambda)$, then $p_{\lambda,\xi}(Q|f) = 0$ and (96) follows. Thus assume that $\xi \in \mathrm{E}(Q|f_\lambda)$. Then the expansion (93) of Theorem 3 holds. By the first sentence of the proof this implies that

$$p_{\lambda,\xi}(Q|f_\lambda, a, X) = \sum_{\substack{\eta \in \mathrm{E}(P|f_\lambda) \\ \eta|_{\mathfrak{a}_{Q\mathrm{q}}} = \xi}} p_{\lambda,\eta}(P|f_\lambda, e, 0) a^\eta, \qquad (a \in A^+_{{}^*P\mathrm{q}}, X \in \mathfrak{a}_{Q\mathrm{q}}).$$

In particular it follows that $X \mapsto p_{\lambda,\xi}(Q|f_\lambda, a, X)$ is a constant function. By continuity of the map $g \mapsto p_{\lambda,\xi}(Q|g, a, X)$ and density of K-finite functions (cf. (97)) we now infer that the polynomial $X \mapsto p_{\lambda,\xi}(Q|f_\lambda, a, X)$ is constant for all $f \in \mathcal{E}_*(G/H, \Lambda, \Omega_0)$. By equivariance of the map $f \mapsto p_{\lambda,\xi}(Q|f_\lambda, \cdot, X)$ it finally follows that the polynomial function $X \mapsto p_{\lambda,\xi}(Q|f_\lambda, x, X)$ is constant for every f and all $x \in G$. This establishes (96).

The assertion about holomorphy is proved as follows. Let $\lambda_0 \in \Omega_0 \cap {}''\mathfrak{a}^*_{\mathrm{qc}}$, fix $\xi_0 \in X(Q, \lambda_0)$, and for $\lambda \in \Omega_0 \cap {}''\mathfrak{a}^*_{\mathrm{qc}}$ define $\Xi(\lambda)$ as in Proposition 9. There exist $s \in W$ and $\mu \in \mathbb{N}\Sigma_r(Q)$ such that $\xi_0 = (s\lambda_0 - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}} - \mu$. If $t \in W$, $\nu \in \mathbb{N}\Sigma_r(Q)$ and $(t\lambda_0 - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}} - \nu = \xi_0$, then it follows that $(s\lambda_0 - t\lambda_0)|_{\mathfrak{a}_{Q\mathrm{q}}} - \mu + \nu = 0$. Since $\lambda_0 \in {}''\mathfrak{a}^*_{\mathrm{qc}}$ this implies that $\mu = \nu$ and $W_Q s = W_Q t$. Hence $s\lambda|_{\mathfrak{a}_{Q\mathrm{q}}} = t\lambda|_{\mathfrak{a}_{Q\mathrm{q}}}$ for all $\lambda \in \Omega$, and thus we see that $\Xi(\lambda)$ has only one element:

$$\Xi(\lambda) = \{\, (s\lambda - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}} - \mu \,\}.$$

From Proposition 9 it now follows that (96) depends holomorphically on λ when this variable is restricted to a suitable neighborhood of λ_0.

Finally it remains to prove the assertions about meromorphy. Fix $s \in W$, $\mu \in \mathbb{N}\Sigma_r(Q)$ and $\lambda_0 \in \Omega_0$. Put $\xi_0 = (s\lambda_0 - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}} - \mu$ and let Π be the set of pairs $(t, \nu) \in W_Q\backslash W \times$

$\mathbb{N}\Sigma_r(Q)$ such that $(t\lambda_0 - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}} - \nu = \xi_0$. For $X \in \mathfrak{a}_{Q\mathrm{q}}$ and $\lambda \in \Omega_0 \cap {}''\mathfrak{a}^*_{\mathrm{qc}}$ we define the following function in $C^\infty(G)$:

$$\psi(X,\lambda) = \sum_{(t,\nu)\in\Pi} p_{Q,\nu}(f{:}t{:}\lambda)e^{(t\lambda-\rho_Q-\nu)(X)}. \tag{98}$$

Define $\Xi(\lambda)$ as in Proposition 9. Then for $\lambda \in \Omega_0 \cap {}''\mathfrak{a}^*_{\mathrm{qc}}$ the map $(t,\nu) \mapsto (t\lambda - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}} - \nu$ is a bijection from Π onto $\Xi(\lambda)$. Thus, taking (96) into account we see that

$$\psi(X,\lambda) = \sum_{\xi\in\Xi(\lambda)} p_{\lambda,\xi}(Q|f_\lambda,\cdot,X)e^{\xi(X)}. \tag{99}$$

We now see that by Proposition 9 there exists an open neighborhood Ω of λ in Ω_0 and a constant $r' \in \mathbb{R}$ such that for every $X \in \mathfrak{a}_{Q\mathrm{q}}$ the map $\lambda \mapsto \psi(X,\lambda)$ extends to a holomorphic $C^\infty_{r'}(G)$-valued map on Ω.

For $\lambda \in {}''\mathfrak{a}^*_{\mathrm{qc}}$ the elements $(t\lambda - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}} - \nu$, $(t,\nu) \in \Pi$ are mutually different. Therefore the exponential functions $e^{t\lambda-\rho_Q-\nu}$ on $\mathfrak{a}_{Q\mathrm{q}}$ are linearly independent. Thus we may fix elements $X_l \in \mathfrak{a}_{Q\mathrm{q}}$, $l \in \Pi$ such that the determinant

$$\det(\, e^{(t\lambda-\rho_Q-\nu)(X_l)}\,;\ (t,\nu)\in\Pi,\ l\in\Pi\,)$$

is a non-trivial holomorphic function of λ. By Cramer's rule this implies that the functions $p_{Q,\nu}(f{:}t{:}\lambda)$, $(t,\nu) \in \Pi$ may be solved as $C^\infty_{r'}(G)$-valued meromorphic functions of λ from the system which arises if in (98) one substitutes for X the values X_l, $l \in \Pi$. $\square$

In the final part of the above proof we have seen that for a holomorphic family of eigenfunctions the coefficients in the expansion (87) can be retrieved from the coefficients $p_{Q,\mu}(f,s,\lambda)$, $s \in W_Q\backslash W$, $\mu \in \mathbb{N}\Sigma_r(Q)$ introduced in (95). We formulate this result as a separate lemma.

Lemma 12. *Let $\Lambda \in \mathfrak{b}^*_{\mathrm{kc}}$, let Ω_0 be an open subset of $\mathfrak{a}^*_{\mathrm{qc}}$, and let $f \in \mathcal{E}_*(G/H,\Lambda,\Omega_0)$. Moreover, let $\lambda_0 \in \Omega_0$, $\xi_0 \in X(Q,\lambda_0)$. Then the meromorphic function*

$$\lambda \mapsto \sum_{\substack{s\in W_Q\backslash W,\ \nu\in\mathbb{N}\Sigma_r(Q)\\ (s\lambda_0-\rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}}-\nu=\xi_0}} p_{Q,\nu}(f{:}s{:}\lambda)(x)\, e^{(s\lambda-\rho_Q-\nu)(X)} \tag{100}$$

has a removable singularity at $\lambda = \lambda_0$ for every $x \in G$ and $X \in \mathfrak{a}_{Q\mathrm{q}}$. Moreover, it has the limit value $p_{\lambda_0,\xi_0}(Q|f_{\lambda_0},x,X)e^{\xi_0(X)}$ at λ_0.

Proof. Fix $x \in G$ and $X \in \mathfrak{a}_{Q\mathrm{q}}$ and let $\varphi(\lambda)$ be the function given in (100). Define $\Pi, \Xi(\lambda)$ and $\psi(X,\lambda)$ as in the final part of the above proof. Then for $\lambda \in \Omega_0 \cap {}''\mathfrak{a}^*_{\mathrm{qc}}$ we have $\varphi(\lambda) = \psi(X,\lambda)(x)$. The result now follows from (99) by application of Proposition 9. $\square$

12. Principal parts of families of eigenfunctions. Let Ω be a non-empty open subset of $\mathfrak{a}_{\mathrm{qc}}^*$. Then $\mathcal{E}_*(G/H, \Omega)$ will denote the space of functions $f\colon \Omega \times G/H \to \mathbb{C}$ which may be expressed as finite sums $f = \sum_\Lambda f_\Lambda$, where Λ ranges over a finite subset of $\mathfrak{b}_{\mathrm{kc}}^*$, and where $f_\Lambda \in \mathcal{E}_*(G/H, \Lambda, \Omega)$. If $\Lambda_1, \Lambda_2 \in \mathfrak{b}_{\mathrm{kc}}^*$ are conjugate under the centralizer $W_{M_1}(\mathfrak{b})$ of $\mathfrak{a}_{\mathrm{q}}$ in $W(\mathfrak{b})$, then one readily checks that $\mathcal{E}_*(G/H, \Lambda_1, \Omega) = \mathcal{E}_*(G/H, \Lambda_2, \Omega)$. On the other hand if $\{\Lambda_1, \dots, \Lambda_m\}$ is a finite set of mutually non-$W_{M_1}(\mathfrak{b})$-conjugate elements of $\mathfrak{b}_{\mathrm{kc}}^*$, and if $f_i \in \mathcal{E}_*(G/H, \Lambda_i, \Omega)$ $(i = 1, \dots, m)$, then we claim that $\sum_{i=1}^m f_i = 0$ (identically in λ) only if $f_1 = \cdots = f_m = 0$. Indeed assume that $\sum_{i=1}^m f_i = 0$, then applying the operator $D - \gamma(D{:}\Lambda_m + \lambda)$ we obtain $\sum_{i=1}^{m-1} [\gamma(D{:}\Lambda_i + \lambda) - \gamma(D{:}\Lambda_m + \lambda)] f_{i,\lambda} = 0$, for all $D \in \mathbb{D}(G/H)$, $\lambda \in \Omega$. Invoking induction on m we see that for each $i = 1, \dots m-1$ we have $f_i = 0$ or $\gamma(D{:}\Lambda_i + \lambda) = \gamma(D{:}\Lambda_m + \lambda)$ for all $D \in \mathbb{D}(G/H)$, $\lambda \in \Omega$. However, the latter possibility is excluded by the non-$W_{M_1}(\mathfrak{b})$-conjugacy of Λ_i and Λ_m. Hence $f_1 = \cdots = f_{m-1} = 0$, and then also $f_m = 0$. This establishes our claim.

Thus, abusing notations slightly, we have the direct sum of linear spaces:

$$\mathcal{E}_*(G/H, \Omega) = \bigoplus_{\Lambda \in \mathfrak{b}_{\mathrm{kc}}^*/W_{M_1}(\mathfrak{b})} \mathcal{E}_*(G/H, \Lambda, \Omega).$$

Hence by linearity all definitions and results of the previous section extend to families $f \in \mathcal{E}_*(G/H, \Omega)$.

If V is a finite dimensional complex linear space, then by $\mathcal{E}_*(G/H, V, \Omega)$ we denote the space of functions $f\colon \Omega \times G/H \to V$ all of whose vector components belong to $\mathcal{E}_*(G/H, \Omega)$. Thus

$$\mathcal{E}_*(G/H, V, \Omega) \simeq \mathcal{E}_*(G/H, \Omega) \otimes_{\mathbb{C}} V,$$

and again we see that all definitions and results of the previous section extend to families $f \in \mathcal{E}_*(G/H, V, \Omega)$ by identity on the second tensor component. This will be used from now on.

From now on let (τ, V_τ) be a finite dimensional unitary representation of K. Then by $\mathcal{E}_*(G/H, \tau, \Omega)$ we denote the space of $f \in \mathcal{E}_*(G/H, V_\tau, \Omega)$ which are τ-spherical in the sense that

$$f_\lambda(kx) = \tau(k) f_\lambda(x), \qquad (x \in G/H,\ k \in K,\ \lambda \in \Omega).$$

Let $f \in \mathcal{E}_*(G/H, \tau, \Omega)$, and let $Q \in \mathcal{P}_\sigma$. Then we define the *Q-principal part* of f, denoted f_Q, by

$$f_Q(\lambda{:}m) = d_Q(m) \sum_{s \in W_Q \backslash W} p_{Q,0}(f{:}s{:}\lambda)(m) \tag{101}$$

for $m \in M_{1Q}, \lambda \in \Omega$. Clearly $f_Q(\lambda{:}\,\cdot)$ is a smooth V_τ-valued function on $M_{1Q}/H_{M_{1Q}}$, depending meromorphically on λ. Furthermore, using the equivariance of the map $f \mapsto p_{Q,0}(f{:}s{:}\lambda)$ one readily verifies that $f_Q(\lambda{:}\,\cdot)$ is $\tau|_{K_Q}$-spherical. Finally we notice that $d_Q(ma) = d_Q(m) a^{\rho_Q}$ for $m \in M_{1Q}$, $a \in A_{Q\mathrm{q}}$, and hence it follows from the transformation rule (89) that

$$f_Q(\lambda{:}ma) = d_Q(m) \sum_{s \in W_Q \backslash W} a^{s\lambda} p_{Q,0}(f{:}s{:}\lambda)(m), \qquad (m \in M_{1Q}, a \in A_{Q\mathrm{q}}).$$

The following property of the Q-principal part shows that it is closely related to Harish-Chandra's notion of the constant term (see [13, p. 153]). Recall that for $\epsilon > 0$ we have defined the set $\mathfrak{a}_{\mathrm{q}}^*(\epsilon)$ by (3).

Lemma 13. *There exists a constant $\epsilon_0 > 0$ such that if $0 < \epsilon \le \epsilon_0$ then the function $\lambda \mapsto f_Q(\lambda{:}m)$ is regular on $\mathfrak{a}_{\mathrm{q}}^*(\epsilon)$ for every $f \in \mathcal{E}_*(G/H, \tau, \mathfrak{a}_{\mathrm{q}}^*(\epsilon))$ and $m \in M_{1Q}$. Moreover, given a compact subset $\mathcal{K} \subset \mathfrak{a}_{Q\mathrm{q}}^+$ there exists $\epsilon > 0$ such that for all $f \in \mathcal{E}_*(G/H, \tau, \mathfrak{a}_{\mathrm{q}}^*(\epsilon))$, $m \in M_{1Q}$, and $\lambda \in \mathfrak{a}_{\mathrm{q}}^*(\epsilon)$ we have*

$$e^{t\rho_Q(X)} d_Q(m)\, f(m \exp tX) - f_Q(\lambda{:}m \exp tX) \to 0 \tag{102}$$

as $t \to \infty$, uniformly in $X \in \mathcal{K}$.

Proof. Let $\delta = \min_{\nu \in \mathbb{N}\Sigma_r(Q)\setminus\{0\}} |\nu|$, then $\epsilon_0 > 0$ can be chosen such that $|\operatorname{Re} \lambda|_{\mathfrak{a}_{Q\mathrm{q}}}| < \delta/2$ for all $\lambda \in \mathfrak{a}_{\mathrm{q}}^*(\epsilon_0)$. Let $0 < \epsilon \le \epsilon_0$, fix $\lambda_0 \in \mathfrak{a}_{\mathrm{q}}^*(\epsilon)$ and $w \in W$, and put $\xi_0 = (w\lambda_0 - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}}$. Let Π be the set of pairs $(s, \nu) \in W_Q\backslash W \times \mathbb{N}\Sigma_r(Q)$ for which $(s\lambda_0 - \rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}} - \nu = \xi_0$, then it follows easily that $\Pi \subset W_Q\backslash W \times \{0\}$. Hence Lemma 12 shows that the function

$$\lambda \mapsto \sum_{\substack{s \in W_Q\backslash W \\ (s\lambda_0)|_{\mathfrak{a}_{Q\mathrm{q}}} = (w\lambda_0)|_{\mathfrak{a}_{Q\mathrm{q}}}}} p_{Q,0}(f{:}s{:}\lambda)(m)$$

is regular near λ_0, with the limit value $p_{\lambda_0,\xi_0}(Q|f_{\lambda_0}, m, 0)$ at λ_0. Since w was arbitrary we obtain the asserted regularity. Moreover, let a compact subset $\mathcal{K} \subset \mathfrak{a}_{Q\mathrm{q}}^+$ be given. Then if $\epsilon > 0$ is sufficiently small we have $(\operatorname{Re} s\lambda - \nu)(X) < 0$ for all $X \in \mathcal{K}$, $\lambda \in \mathfrak{a}_{\mathrm{q}}^*(\epsilon)$, $s \in W$, and $\nu \in \mathbb{N}\Sigma_r(Q) \setminus \{0\}$. The property (102) is now a consequence of (88). $\square$

The notion of principal part can be extended to meromorphic families of eigenfunctions as follows. Given a complex manifold U, we write $\mathcal{O}(U)$ for the algebra of holomorphic functions $U \to \mathbb{C}$, and $\mathcal{M}(U)$ for the algebra of meromorphic functions $U \to \mathbb{C}$.

Let $\mathcal{M}_*(G/H, \tau, \Omega)$ be the space of maps $\lambda \mapsto f_\lambda$, $\Omega \to C^\infty(G/H) \otimes V$ such that for every $\lambda_0 \in \Omega$ there exists an open neighborhood Ω_0 of λ_0 in Ω and a holomorphic function $\varphi \in \mathcal{O}(\Omega_0)$, not identically zero, such that $\lambda \mapsto \varphi(\lambda) f_\lambda$ belongs to $\mathcal{E}_*(G/H, \tau, \Omega_0)$.

Then $\Omega \rightsquigarrow \mathcal{M}_*(G/H, \tau, \Omega)$ defines a sheaf on $\mathfrak{a}_{\mathrm{q}\mathbf{c}}^*$ which is isomorphic to the tensor product of the $\mathcal{O}$-module sheaves $\mathcal{M}$ and $\Omega \rightsquigarrow \mathcal{E}_*(G/H, \tau, \Omega)$. The principal part map $f \mapsto f_Q$ is a morphism of sheaves of $\mathcal{O}$-modules and therefore has a unique $\mathcal{M}$-linear extension to the sheaf $\mathcal{M}_*(G/H, \tau, \cdot)$.

13. The principal part of the Eisenstein integrals. It is easily seen that the Eisenstein integrals belong to the space of meromorphic families of eigenfunctions just defined, with $\Omega = \mathfrak{a}_{\mathrm{q}\mathbf{c}}^*$. In fact we have the following stronger result. Recall that for $R \in \mathbb{R}$ the set $\mathfrak{a}_{\mathrm{q}}^*(P, R)$ is defined by (7), and that the set $\Pi_\Sigma(\mathfrak{a}_{\mathrm{q}}) \subset S(\mathfrak{a}_{\mathrm{q}}^*)$ has been defined above Lemma 2.

Lemma 14. *Let $P \in \mathcal{P}_\sigma^{\min}$, $\psi \in {}^\circ\mathcal{C}(\tau)$, and $R \in \mathbb{R}$. Then there exists a polynomial $p \in \Pi_\Sigma(\mathfrak{a}_{\mathrm{q}})$ such that $\lambda \mapsto p(\lambda) E(P{:}\psi{:}\lambda)$ is regular on $\mathfrak{a}_{\mathrm{q}}^*(P, R)$. Moreover, if p is any polynomial in $\Pi_\Sigma(\mathfrak{a}_{\mathrm{q}})$ with this property, then the family*

$$E_p(P{:}\psi){:}\ (\lambda, x) \mapsto p(\lambda) E(P{:}\psi{:}\lambda)(x)$$

belongs to $\mathcal{E}_*(G/H, V, \mathfrak{a}_{\mathrm{q}}^*(P,R))$. *In particular it follows that the family* $\lambda \mapsto E(P{:}\psi{:}\lambda)$ *belongs to* $\mathcal{M}_*(G/H, \tau, \mathfrak{a}_{\mathrm{qc}}^*)$.

The above statements hold as well (with a possibly different polynomial p) when the Eisenstein integral E is replaced by the normalized Eisenstein integral E° *and the set* $\mathfrak{a}_{\mathrm{q}}^*(P,R)$ *is replaced by* $\mathfrak{a}_{\mathrm{q}}^*(\bar{P},R)$.

Proof. See [3, Prop. 10.3, Lemma 14.1, and Cor. 16.2] (cf. (52); use functoriality to generalize from τ_ϑ to arbitrary τ). $\square$

Corollary 1. *Let* $P \in \mathcal{P}_\sigma^{\min}$, $\psi \in {}^\circ\mathcal{C}(\tau)$, $x \in G$, *and* $Q \in \mathcal{P}_\sigma$. *Then for* $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$ *generic we have the expansion*

$$E(P{:}\psi{:}\lambda)(x \exp tX) \sim \sum_{\substack{s \in W_Q\backslash W,\\ \nu \in \mathbb{N}\Sigma_r(Q)}} p_{Q,\nu}(E(P{:}\psi{:}\lambda){:}\ s{:}\lambda)(x) e^{(s\lambda - \rho_Q - \nu)(tX)} \qquad (t \to \infty)$$

at every $X_0 \in \mathfrak{a}_{Q\mathrm{q}}^+$, *as well as the similar expansion for* $E^\circ(P{:}\psi{:}\lambda)$.

Proof. By Lemma 14 and the remarks of the previous section we can apply Propositions 7, 9, and 10. $\square$

The corollary allows us to define the Q-principal parts $E_Q(P{:}\psi{:}\lambda)$ and $E_Q^\circ(P{:}\psi{:}\lambda)$ of the Eisenstein integrals. If $Q \in \mathcal{P}_\sigma^{\min}$ then this notion coincides with the notions introduced in Sections 4-5, see (46) and (56), with $w = 1$. From Lemma 13 we now obtain:

Corollary 2. *Let* $\epsilon > 0$ *and let* $p \in \Pi_\Sigma(\mathfrak{a}_{\mathrm{q}})$ *be a polynomial such that the meromorphic function* $\lambda \mapsto p(\lambda)E(P{:}\psi{:}\lambda)$ *is regular on* $\mathfrak{a}_{\mathrm{q}}^*(\epsilon)$. *Let* ϵ *be sufficiently small. Then the function*

$$\lambda \mapsto E_{p,Q}(P{:}\psi{:}\lambda) := p(\lambda)E_Q(P{:}\psi{:}\lambda)$$

is regular on $\mathfrak{a}_{\mathrm{q}}^*(\epsilon)$. *Moreover, given a compact subset* $\mathcal{K} \subset \mathfrak{a}_{Q\mathrm{q}}^+$ *there exists* $\epsilon > 0$ *such that for all* $m \in M_{1Q}$ *and* $\lambda \in \mathfrak{a}_{\mathrm{q}}^*(\epsilon)$ *we have*

$$e^{t\rho_Q(X)} d_Q(m) E_p(P{:}\psi{:}\lambda)(m \exp tX) - E_{p,Q}(P{:}\psi{:}\lambda)(m \exp tX) \to 0,$$

as $t \to \infty$, *uniformly in* $X \in \mathcal{K}$.

The above statements hold as well with the Eisenstein integral E replaced by the normalized Eisenstein integral E°.

Our next goal is to determine the Q-principal part of the normalized Eisenstein integral for all $Q \in \mathcal{P}_\sigma$. Notice that it is an immediate consequence of the functional equation (58) that

$$E_Q^\circ(P'{:}\psi{:}\lambda) = E_Q^\circ(P{:}C_{P|P'}^\circ(1{:}\lambda)\psi{:}\lambda). \tag{103}$$

This will allow us to reduce the problem to the case that Q contains P.

Lemma 15. *Let $P_1, P_2 \in \mathcal{P}_\sigma^{\min}$ and assume $P_1, P_2 \subset Q$. Then for $s \in W_Q$ and $t \in W$ we have the following identity of meromorphic functions on $\mathfrak{a}_{\mathrm{qc}}^*$:*

$$\mathrm{pr}_Q \circ C^\circ_{P_2|P_1}(st{:}\lambda) = C^\circ_{{}^*P_2|{}^*P_1}(s{:}t\lambda) \circ \mathrm{pr}_Q \circ C^\circ_{P_1|P_1}(t{:}\lambda). \tag{104}$$

Moreover we have

$$E^\circ({}^*P_2{:}\ \mathrm{pr}_Q[C^\circ_{P_2|P_1}(st{:}\lambda)\psi]{:}\ st\lambda) = E^\circ({}^*P_1{:}\ \mathrm{pr}_Q[C^\circ_{P_1|P_1}(t{:}\lambda)\psi]{:}\ t\lambda) \tag{105}$$

for all $\psi \in {}^\circ\mathcal{C}(\tau)$.

Proof. Using (71) we may rewrite the left-hand side of (104) as

$$\mathrm{pr}_Q \circ C^\circ_{P_2|P_1}(s{:}t\lambda) \circ C^\circ_{P_1|P_1}(t{:}\lambda).$$

In view of Proposition 6 we may rewrite this in turn as the expression on the right-hand side. This proves (104). Inserting this expression in (105) and applying Proposition 4 we obtain the proof of (105). □

In particular we see from (105) with $P_1 = P_2 = P \subset Q$ that the Eisenstein integral

$$E^\circ({}^*P{:}\ \mathrm{pr}_Q[C^\circ_{P|P}(t{:}\lambda)\psi]{:}\ t\lambda)$$

only depends on the coset $[t] = W_Q t$, for any $t \in W$.

Theorem 4. *Let $P \in \mathcal{P}_\sigma^{\min}$ and $P \subset Q$. Then the Q–principal part of the normalized Eisenstein integral is given by*

$$E^\circ_Q(P{:}\psi{:}\lambda) = \sum_{[t] \in W_Q \backslash W} E^\circ({}^*P{:}\ \mathrm{pr}_Q[C^\circ_{P|P}(t{:}\lambda)\psi]{:}\ t\lambda) \tag{106}$$

for all $\psi \in {}^\circ\mathcal{C}(\tau)$, as a meromorphic identity in $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$.

In the group case this result is given in [12, Thm. 7]. In the Riemannian case it is then obtained by specializing to the trivial $K \times K$-type (see also [10, Thm. 5.9.4]). The proof will be given in the next three sections. The idea is to show that both members of the equation are functions in $C^\infty(M_{1Q}/H_{M_{1Q}}{:}\tau|_{K_Q})$ which are annihilated by the same cofinite ideal in $\mathbb{D}(M_{1Q}/H_{M_{1Q}})$. From this it follows that the two members of the equation allow converging expansions of polynomial exponential type. We will then determine the possible leading exponents and the associated leading coefficients, and show that they are the same for the functions in both sides of the equation. This will finally allow us to conclude the equality.

14. Asymptotic expansions on M_{1Q}. In this section we will study the system of differential equations on M_{1Q} satisfied by the principal parts of the Eisenstein integrals. We shall see that the solutions to this system have asymptotic expansions, and we shall determine the possible leading exponents.

Let (τ, V_τ) be a finite dimensional unitary representation of K, and let Ω be an open subset of $\mathfrak{a}_{\mathrm{qc}}^*$. If $D \in \mathbb{D}(G/H)$ then it is easily verified that if $f \in \mathcal{M}_*(G/H, \tau, \Omega)$, then the family $Df{:}\ \lambda \mapsto D(f_\lambda)$ belongs to $\mathcal{M}_*(G/H, \tau, \Omega)$ as well. Hence it makes sense to form its Q-principal part $(Df)_Q$, for any $Q \in \mathcal{P}_\sigma$. We now have:

Lemma 16. *Let $f \in \mathcal{M}_*(G/H, \tau, \Omega)$. Then for all $D \in \mathbb{D}(G/H)$ we have:*

$$\mu_Q(D)\, f_Q = (Df)_Q.$$

Proof. This follows easily from (101), (90) and (83). $\square$

In particular it follows from the differential equations (44) and (54) that we have

$$\mu_Q(D)\, E_Q(P{:}\psi{:}\lambda) = E_Q(P{:}\mu(D{:}\tau{:}\lambda)\psi{:}\lambda) \tag{107}$$

for all $\psi \in {}^\circ\mathcal{C}(\tau)$, $D \in \mathbb{D}(G/H)$, as well as the same relation with E replaced by E°.

Since $\mathbb{D}(M_{1Q}/H_{M_{1Q}})$ is a finite $\mu_Q(\mathbb{D}(G/H))$-module, it follows in particular that $E_Q(P{:}\psi{:}\lambda)$ and $E_Q^\circ(P{:}\psi{:}\lambda)$ are $\mathbb{D}(M_{1Q}/H_{M_{1Q}})$-finite functions in $C^\infty(M_{1Q}/H_{M_{1Q}}{:}\tau|_{K_Q})$. Therefore the theory of converging expansions of [1] (see Section 9) is applicable.

For the moment assume that $F \in C^\infty(M_{1Q}/H_{M_{1Q}}{:}\tau|_{K_Q})$ is a $\mathbb{D}(M_{1Q}/H_{M_{1Q}})$-finite function. If $P \in \mathcal{P}_\sigma^{\min}$ is contained in Q, then ${}^*\!P = M_{1Q} \cap P$ is a minimal $\sigma\theta$-stable parabolic subgroup of M_{1Q} containing A_{q}. Let $\Sigma({}^*\!P) = \Sigma(P) \setminus \Sigma(Q)$ be the associated system of positive roots, and let $A^+_{{}^*\!P\mathrm{q}}$ be the associated positive chamber in A_{q}. Then according to (92) the function F has a converging series expansion of the form

$$F(a) = \sum_{\xi, m} c_{\xi,m} \log^m a\; a^\xi, \qquad (a \in A^+_{{}^*\!P\mathrm{q}}). \tag{108}$$

Here ξ ranges over a set of the form $S - \mathbb{N}\Sigma({}^*\!P)$, where $S \subset \mathfrak{a}^*_{\mathrm{qc}}$ is a finite subset, $m \in \mathbb{N}^\Gamma$ is a multiindex with $|m| \leq d$, and the coefficients $c_{\xi,m}$ belong to V_τ. In fact, since the restriction of F to A_{q} has values in the space $V_\tau^{M\cap K\cap H}$, the coefficients $c_{\xi,m}$ belong to that space as well.

Notice that F is determined by the expansion (108), in the sense that if all the coefficients $c_{\xi,m}$ vanish, then $F = 0$. Indeed, since the expansion converges, the vanishing of the coefficients implies that F vanishes on $A^+_{{}^*\!P\mathrm{q}}$. By sphericality this implies that F vanishes on $K_Q A^+_{{}^*\!P\mathrm{q}} H_{M_{1Q}}$, which is open in M_{1Q}. Being K_Q-spherical and $\mathbb{D}(M_{1Q}/H_{M_{1Q}})$-finite the function F is real analytic, and we conclude that $F = 0$.

As before the set of elements $\xi \in S - \mathbb{N}\Sigma({}^*\!P)$, for which there exists $m \in \mathbb{N}^\Gamma$ such that $c_{\xi,m} \neq 0$, is denoted by $\mathbf{E}({}^*\!P|F, e)$. The set of $\preceq_{{}^*\!P}$-maximal elements in $\mathbf{E}({}^*\!P|F, e)$ is denoted by $\mathbf{E}_L({}^*\!P|F, e)$. Clearly this is a finite set. The function

$$a \mapsto \sum_{\xi \in \mathbf{E}_L({}^*\!P|F,e), m} c_{\xi,m} \log^m a\; a^\xi \in V_\tau^{M\cap K\cap H}$$

on A_{q} is called the leading part along ${}^*\!P$ of F. Notice that if $\mathbf{E}_L({}^*\!P|F, e)$ is empty then $F = 0$ according to the discussion above.

We recall from Lemma 4 that the space ${}^\circ\mathcal{C}(\tau)$ has a finite direct sum decomposition in simultaneous eigenspaces for the endomorphisms $\mu(D{:}\tau{:}\lambda)$, and that every simultaneous eigenvalue is of the form $\gamma(D{:}\Lambda + \lambda)$, with $\Lambda \in \mathfrak{b}^*_{\mathrm{kc}}$. Therefore the following result is of particular interest to us.

If $\Lambda \in \mathfrak{b}^*_{\mathrm{kc}}$, let $\mathfrak{a}^{*\prime}_{\mathrm{qc}}(\Lambda) \subset \mathfrak{a}^*_{\mathrm{qc}}$ be the set defined in [3, Eqn. (99)]. Then $\mathfrak{a}^{*\prime}_{\mathrm{qc}}(\Lambda)$ is the complement of a locally finite union of hyperplanes. In particular it is an open dense subset of $\mathfrak{a}^*_{\mathrm{qc}}$.

Lemma 17. *Let $P \in \mathcal{P}_\sigma^{\min}$, $Q \in \mathcal{P}_\sigma$ and assume $P \subset Q$. Fix $\Lambda \in \mathfrak{b}_{\mathrm{kc}}^*$. Let $\lambda \in \mathfrak{a}_{\mathrm{qc}}^{*\prime}(\Lambda)$, and suppose that $F \in C^\infty(M_{1Q}/H_{M_{1Q}}{:}\tau|_{K_Q})$ satisfies the system*

$$\mu_Q(D)F = \gamma(D{:}\Lambda + \lambda)F, \qquad (D \in \mathbb{D}(G/H)). \tag{109}$$

Then $\mathbf{E}_L({}^*P|F, e) \subset W\lambda - \rho_{{}^*P}$.

Proof. Suppose that $\xi \in \mathbf{E}_L({}^*P|F, e)$, and let $c_{\xi,m} \in V_\tau$ be the coefficients of the expansion (108). Fix $m_0 \in \mathbb{N}^\Gamma$ such that $c_{\xi,m_0} \neq 0$ and $c_{\xi,m} = 0$ for all $m \in \mathbb{N}^\Gamma$ with $|m| > |m_0|$. Define $\varphi\colon M_1/H_{M_1} \to V_\tau$ by

$$\varphi(kaH_{M_1}) = a^\xi \tau(k) c_{\xi,m_0}, \qquad (k \in K_M,\, a \in A_{\mathrm{q}}).$$

Note that this definition makes sense because $M_1/H_{M_1} \simeq (K_M/K_M \cap H_M) \times A_{\mathrm{q}}$ (cf. Lemma 1) and $c_{\xi,m_0} \in V_\tau^{K_M \cap H_M}$.

Let μ^Q and $\mu'^Q_{{}^*P}$ be the maps $\mathbb{D}(M_{1Q}/H_{M_{1Q}}) \to \mathbb{D}(M_1/H_{M_1})$ corresponding to ${}^*P$ (cf. Section 8). Notice that $\rho_{{}^*P} = \rho_P - \rho_Q$. As in [3, Cor. 13.3] one shows that (109) implies:

$$\mu'^Q_{{}^*P}(\mu_Q(D))\varphi = \gamma(D{:}\Lambda + \lambda)\varphi, \qquad (D \in \mathbb{D}(G/H)).$$

Using (84) we obtain $\mu'^Q_{{}^*P} \circ \mu_Q = T_{-\rho_Q} \circ \mu'_P$. Hence $\tilde\varphi := (d_Q|_{M_1})^{-1}\varphi$ satisfies the system

$$\mu'_P(D)\tilde\varphi = \gamma(D{:}\Lambda + \lambda)\tilde\varphi, \qquad (D \in \mathbb{D}(G/H)).$$

According to [3, Prop. 13.5] this implies that $\tilde\varphi$ has A_{q}-exponents contained in $W\lambda - \rho_P$. Hence φ has exponents contained in $W\lambda - \rho_{{}^*P}$, and it follows that ξ belongs to this set. $\square$

15. The leading part of the principal part of the Eisenstein integral. Let Q be a $\sigma\theta$-stable parabolic subgroup containing $P \in \mathcal{P}_\sigma^{\min}$. Then from (107) and the theory of the previous section we see that it makes sense to speak of the leading part along ${}^*P$ of the Q-principal part $E_Q^\circ(P{:}\psi{:}\lambda)$ of the normalized Eisenstein integral. This leading part can be determined using transitivity of asymptotics.

Proposition 11. *Let $P \in \mathcal{P}_\sigma^{\min}$, $Q \in \mathcal{P}_\sigma$, and assume $P \subset Q$. Then there exists an open dense subset $\Omega \subset \mathfrak{a}_{\mathrm{qc}}^*$ such that for all $\psi \in {}^\circ\mathcal{C}(\tau)$, $\lambda \in \Omega$ the function $E_Q^\circ(P{:}\psi{:}\lambda)$ has the leading part*

$$a \mapsto \sum_{u \in W} a^{u\lambda - \rho_{{}^*P}} [C^\circ_{P|P}(u{:}\lambda)\psi]_1(e) \in V_\tau^{M \cap K \cap H}, \qquad (a \in A_{\mathrm{q}}) \tag{110}$$

along ${}^*P$.

Proof. By Lemma 4 and linearity we may fix $\psi \in {}^\circ\mathcal{C}(\tau)$ so that there exists a $\Lambda \in \mathfrak{b}_{\mathrm{kc}}^*$ such that $\mu(D{:}\tau{:}\lambda)\psi = \gamma(D{:}\Lambda + \lambda)\psi$ for all $D \in \mathbb{D}(G/H)$, $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$. Write

$$F_\lambda = E_Q^\circ(P{:}\psi{:}\lambda).$$

Then F_λ is a function in $C^\infty(M_{1Q}/H_{M_{1Q}}:\tau|_{K_Q})$, which depends meromorphically on $\lambda \in \mathfrak{a}^*_{\mathrm{qc}}$. Moreover, from (107) we see that F_λ satisfies the system (109).

Let Ω be any non-empty dense open subset of ${}''\mathfrak{a}^*_{\mathrm{qc}}$ such that $\lambda \mapsto F_\lambda$ is regular on Ω, and assume moreover that $\Omega \subset \mathfrak{a}^{*\prime}_{\mathrm{qc}}(\Lambda)$. From now on we will always assume that $\lambda \in \Omega$. By Lemma 17 we know that

$$\mathbf{E}_L({}^*P|F_\lambda, e) \subset W\lambda - \rho_{{}^*P}. \tag{111}$$

Put $f_\lambda = E^\circ(P:\psi:\lambda)$. Then F_λ is the Q-principal part of f_λ. Shrinking Ω if necessary we may assume that also $\lambda \mapsto f_\lambda$ is regular on Ω. According to (101) we have

$$F_\lambda = d_Q \sum_{t \in W_Q\backslash W} p_{Q,0}(f:t:\lambda)|_{M_{1Q}}. \tag{112}$$

Of course we may replace the set of summation $W_Q\backslash W$ by any set of representatives in W.

We will now use transitivity of asymptotics to expand the right-hand side of (112) along ${}^*P$. Let $a \in A^+_{{}^*P\mathrm{q}}$. By definition we have for $t \in W$ that

$$p_{Q,0}(f:t:\lambda)(a) = p_{\lambda,(t\lambda-\rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}}}(Q|f_\lambda, a, 0).$$

By Theorem 3 the latter expression equals

$$\sum_{\substack{\eta \in \mathrm{E}(P|f_\lambda) \\ \eta|_{\mathfrak{a}_{Q\mathrm{q}}} = (t\lambda-\rho_Q)|_{\mathfrak{a}_{Q\mathrm{q}}}}} p_{\lambda,\eta}(P|f_\lambda, a, 0). \tag{113}$$

In view of Lemma 14 we can apply Proposition 8 to f, and hence

$$\mathrm{E}(P|f_\lambda) \subset \{\, v\lambda - \rho_P - \nu \mid v \in W,\ \nu \in \mathbb{N}\Sigma(P)\,\}.$$

Let $\eta = v\lambda - \rho_P - \nu$ ($v \in W$, $\nu \in \mathbb{N}\Sigma(P)$). If η has the same restriction to $\mathfrak{a}_{Q\mathrm{q}}$ as $t\lambda - \rho_Q$, then since $\Omega \subset {}''\mathfrak{a}^*_{\mathrm{qc}}$ it follows from Lemma 11 that $v \in W_Q t$. Moreover we must then have that $\nu \in \mathbb{N}\Delta_Q$, where $\Delta_Q = \Delta \setminus \Sigma(Q)$. Thus we see that (113) equals

$$\sum_{s \in W_Q, \nu \in \mathbb{N}\Delta_Q} p_{\lambda, st\lambda-\rho_P-\nu}(P|f_\lambda, a, 0) = \sum_{s \in W_Q, \nu \in \mathbb{N}\Delta_Q} p_{P,\nu}(f:st:\lambda)(e)a^{st\lambda-\rho_P-\nu}.$$

Since $d_Q(a) = a^{\rho_Q}$ and $\rho_P = \rho_Q + \rho_{{}^*P}$ we finally obtain

$$d_Q(a)p_{Q,0}(f:t:\lambda)(a) = \sum_{s \in W_Q, \nu \in \mathbb{N}\Delta_Q} p_{P,\nu}(f:st:\lambda)(e)a^{st\lambda-\rho_{{}^*P}-\nu}.$$

Inserting these expansions for $t \in W_Q\backslash W$ in (112) we now obtain

$$F_\lambda(a) = \sum_{u \in W, \nu \in \mathbb{N}\Delta_Q} p_{P,\nu}(f:u:\lambda)(e)a^{u\lambda-\rho_{{}^*P}-\nu}.$$

By uniqueness of asymptotics this must be the same as the expansion (108) along $A^+_{{}^*P\mathrm{q}}$ for F_λ. By (111) the leading part is

$$\sum_{u \in W} p_{P,0}(f:u:\lambda)(e)a^{u\lambda-\rho_{{}^*P}},$$

and by (56) (with $w = 1$) this is identical to (110). $\square$

16. Proof of Theorem 4. We first determine the differential equations satisfied by the right-hand side of Equation (106). Essentially these differential equations are identical to the differential equation satisfied by the left-hand side (cf. (107)).

Lemma 18. *Let $P \in \mathcal{P}_\sigma^{\min}$, $Q \in \mathcal{P}_\sigma$ and assume $P \subset Q$. Then for every $t \in W$ and all $\psi \in {}^\circ\mathcal{C}(\tau)$ we have:*

$$\mu_Q(D)\, E^\circ({}^*P\colon \mathrm{pr}_Q[C^\circ_{P|P}(t{:}\lambda)\psi]\colon t\lambda) = E^\circ({}^*P\colon \mathrm{pr}_Q[C^\circ_{P|P}(t{:}\lambda)\mu(D{:}\tau{:}\lambda)\psi]\colon t\lambda)$$

for all $D \in \mathbb{D}(G/H)$.

Proof. This is a straightforward consequence of the differential equations (54), applied to the Eisenstein integrals for M_{1Q}, and of (86), (73), the latter with C replaced by C°. ☐

Because of this lemma the theory of Section 14 can be applied to the Eisenstein integral $E^\circ({}^*P\colon \mathrm{pr}_Q[C^\circ_{P|P}(t{:}\lambda)\psi]\colon t\lambda)$ on M_{1Q}, and we may speak of the leading part along ${}^*P$ of this function.

Lemma 19. *Let $P \in \mathcal{P}_\sigma^{\min}$, $Q \in \mathcal{P}_\sigma$ and assume $P \subset Q$. Then there exists an open dense subset $\Omega \subset \mathfrak{a}^*_{\mathrm{qc}}$ such that for all $\psi \in {}^\circ\mathcal{C}(\tau)$, $\lambda \in \Omega$, $t \in W$ the function*

$$E^\circ({}^*P\colon \mathrm{pr}_Q[C^\circ_{P|P}(t{:}\lambda)\psi]\colon t\lambda)$$

has the leading part

$$a \mapsto \sum_{u \in W_Q t} a^{u\lambda - \rho_{*P}} [C^\circ_{P|P}(u{:}\lambda)\psi]_1(e), \qquad (a \in A_{\mathrm{q}}) \tag{114}$$

along ${}^*P$.

Proof. By (56) the Eisenstein integral $E^\circ({}^*P{:}\psi^*{:}t\lambda)$ has the principal part

$$ma \mapsto \sum_{s \in W_Q} a^{st\lambda}[C^\circ_{{}^*P|{}^*P}(s{:}t\lambda)\psi^*]_1(m)$$

along ${}^*P$, for all $\psi^* \in {}^\circ\mathcal{C}_Q(\tau)$. If we insert $\psi^* = \mathrm{pr}_Q[C^\circ_{P|P}(t{:}\lambda)\psi]$ and use Lemma 15 we easily obtain (114). ☐

By summation over $W_Q\backslash W$ it follows from this lemma that the function on the right-hand side of (106) has the leading part

$$a \mapsto \sum_{u \in W} a^{u\lambda - \rho_{*P}} [C^\circ_{P|P}(u{:}\lambda)\psi]_1(e) \in V_\tau^{M\cap K\cap H}, \qquad (a \in A_{\mathrm{q}})$$

along ${}^*P$, that is, exactly the same as that of the left-hand side (cf. (110)).

Proof of Theorem 4. Let F_λ and G_λ denote the left- and right-hand side of (106), respectively. By Lemma 4 we may fix $\psi \in {}^\circ\mathcal{C}(\tau)$ so that there exists a $\Lambda \in \mathfrak{b}^*_{\mathrm{kc}}$ such that $\mu(D{:}\tau{:}\lambda)\psi = \gamma(D{:}\Lambda + \lambda)\psi$ for all $D \in \mathbb{D}(G/H)$, $\lambda \in \mathfrak{a}^*_{\mathrm{qc}}$. Let Ω be any non-empty open dense subset of $\mathfrak{a}^{*\prime}_{\mathrm{q}\mathbf{c}}(\Lambda)$ such that $\lambda \mapsto F_\lambda, G_\lambda$ are regular on Ω. Then F_λ and G_λ satisfy the differential equation (109) for $\lambda \in \Omega$, and hence by Lemma 17

$$\mathbf{E}_L({}^*\!P|F_\lambda - G_\lambda, e) \subset W\lambda - \rho_{{}^*\!P}.$$

Shrinking Ω if necessary, we may assume that it allows the conclusions of Proposition 11 and Lemma 19. Hence, as mentioned above, F_λ and G_λ have the same leading part, and we conclude that $\mathbf{E}_L({}^*\!P|F_\lambda - G_\lambda, e)$ is empty. This implies that $F_\lambda - G_\lambda = 0$ for $\lambda \in \Omega$, and hence F_λ and G_λ are identical as meromorphic functions in λ. $\square$

17. Proof of Theorem 2. Recall that this theorem asserts the regularity of the normalized Eisenstein integrals on a neighborhood of $i\mathfrak{a}^*_{\mathrm{q}}$. We shall prove this by induction on the split rank $\dim \mathfrak{a}_{\mathrm{q}}$ of G/H.

Let $\mathfrak{a}_{\mathrm{q}\Sigma} = \cap_{\alpha\in\Sigma} \ker\alpha$ and ${}^\circ G = \cap_{\chi\in X(G)} \ker|\chi|$ (cf. [1, Sect. 1]), then $A_{\mathrm{q}\Sigma} := \exp(\mathfrak{a}_{\mathrm{q}\Sigma})$ is central in G, and $G/H \simeq A_{\mathrm{q}\Sigma} \times {}^\circ G/({}^\circ G \cap H)$. For this reason we call $A_{\mathrm{q}\Sigma}$ the vectorial part of G/H. One readily checks from the definition of the Eisenstein integral that $E^\circ(P{:}\psi{:}\lambda)(ax) = a^\lambda E^\circ({}^\circ G \cap P{:}\psi{:}\lambda|_{{}^\circ\mathfrak{g}\cap\mathfrak{a}_{\mathrm{q}}})(x)$, for $x \in {}^\circ G/({}^\circ G \cap H)$, $a \in A_{\mathrm{q}\Sigma}$. We thus see that the assertion of Theorem 2 holds for the symmetric space G/H if and only if it holds for ${}^\circ G/({}^\circ G \cap H)$.

In the course of the proof we shall be using the Schwartz functions on G/H (see [3, Sect. 17] for the notion of Schwartz functions on G/H, and for the topology on the Schwartz space). Let $\mathcal{C}(G/H{:}\tau)$ denote the Fréchet space consisting of the τ-spherical L^2-Schwartz functions $f\colon G/H \to V_\tau$. The space $C^\infty_c(G/H{:}\tau)$ is a dense subspace. We need the following result.

Proposition 12. *Assume that the vectorial part $A_{\mathrm{q}\Sigma}$ of G/H is trivial, and let $f \in C^\infty(G/H{:}\tau)$ be a $\mathbb{D}(G/H)$–finite function such that for all maximal parabolic subgroups $Q \in \mathcal{P}_\sigma$ containing A_{q} and all $m \in M_Q$ and $X \in \mathfrak{a}^+_{Q\mathrm{q}}$ we have:*

$$\lim_{t\to\infty} e^{t\rho_Q(X)} f(m\,\exp tX) = 0.$$

Then f belongs to the Schwartz space $\mathcal{C}(G/H{:}\tau)$.

Proof. We start by recalling some further results from [1]. Let $P \in \mathcal{P}^{\min}_\sigma$ be arbitrary and let $\Sigma^+ = \Sigma(P)$. Let $\mathfrak{a}^+_{\mathrm{q}}$ be the associated positive Weyl chamber.

Let $Q \in \mathcal{P}_\sigma$ with $Q \supset P$. Then Q is standard with respect to Σ^+, so that $\mathfrak{a}^+_{Q\mathrm{q}}$ is a wall for $\mathfrak{a}^+_{\mathrm{q}}$. According to [1, Thm. 5.3] the asymptotic behavior of a $\mathbb{D}(G/H)$–finite function $f \in C^\infty(G/H{:}\tau)$ along this wall may be described as follows.

Let ${}^*\bar{\mathfrak{a}}^+_{\mathrm{q}}$ be the intersection of the closure of $\mathfrak{a}^+_{\mathrm{q}}$ with $\mathfrak{m}_Q$, and put ${}^*\!\bar{A}^+_{\mathrm{q}} = \exp({}^*\bar{\mathfrak{a}}^+_{\mathrm{q}})$. Then there exist analytic functions $q_\xi : {}^*\!\bar{A}^+_{\mathrm{q}} \times \mathfrak{a}_{Q\mathrm{q}} \to V_\tau$, polynomial in the second component, such that for all ${}^*a \in {}^*\!\bar{A}^+_{\mathrm{q}}$ and all $a \in A^+_{Q\mathrm{q}}$ with ${}^*aa \in A^+_{\mathrm{q}}$ we have:

$$f({}^*aa) = \sum_\xi q_\xi({}^*a, \log a)\, a^\xi.$$

Here the summation extends over a subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbf{c}}$ of the form $S - \mathbb{N}\Sigma_r(Q)$, with S a finite subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbf{c}}$. The convergence is absolute. Notice that this expansion is a refinement of (92); the latter is obtained at ${}^*a = e$. By uniqueness of asymptotics the functions q_ξ are uniquely determined, and therefore so is the set $\mathbf{E}(Q|f)$ of $\xi \in \mathfrak{a}^*_{Q\mathrm{qc}}$ such that $q_\xi \neq 0$. Arguing as in the proof of Theorem 3 (or inspecting [1, proof of Thm. 5.3]) we see that $\mathbf{E}(Q|f) = \mathbf{E}(P|f)|_{\mathfrak{a}_{Q\mathrm{q}}}$ for $Q \supset P$.

If Q is maximal the hypothesis of the lemma implies that for every $\xi \in \mathbf{E}(Q|f)$ and $X \in \mathfrak{a}^+_{Q\mathrm{q}}$ we have $(\mathrm{Re}\,\xi + \rho_Q)(X) < 0$. Let now $\xi \in \mathbf{E}(P|f)$. Then $\xi|_{\mathfrak{a}_{Q\mathrm{q}}} \in \mathbf{E}(Q|f)$ and since ρ_P has restriction ρ_Q on $\mathfrak{a}_{Q\mathrm{q}}$ it follows that

$$\mathrm{Re}\,\xi + \rho_P < 0 \quad \text{on} \quad \mathfrak{a}^+_{Q\mathrm{q}}. \tag{115}$$

If α is a simple root in Σ^+, let L_α be the set of points $X \in \mathfrak{a}_\mathrm{q}$ such that $\alpha(X) > 0$ and such that $\beta(X) = 0$ for all simple roots $\beta \neq \alpha$. Then $L_\alpha = \mathfrak{a}^+_{Q\mathrm{q}}$ for a suitable maximal $\sigma\theta$-stable parabolic subgroup $Q \supset P$; in fact $Q = Z_G(L_\alpha)P$. Thus by (115) we see that $\mathrm{Re}\,\xi + \rho_P$ is strictly negative on L_α for all $\xi \in \mathbf{E}(Q|f)$. This being valid for every simple root it follows that

$$\mathrm{Re}\,\xi + \rho_P < 0 \quad \text{on} \quad \mathrm{cl}(\mathfrak{a}^+_\mathrm{q}) \setminus \{0\} \tag{116}$$

(here we have used the assumption that $A_{\mathrm{q}\Sigma} = \{1\}$). The estimate (116) is valid for each $P \in \mathcal{P}^{\min}_\sigma$ and all $\xi \in \mathbf{E}(P|f)$. It now follows from [1, Thm. 6.4] that f is square integrable, and from [1, Thm. 7.3] that f belongs to the Schwartz space. □

Proof of Theorem 2. The main steps of the proof are summarized in the following four lemmas. If the split rank of G/H is zero then $\mathfrak{a}^*_{\mathrm{qc}} = \{0\}$, $G = M_1$, hence $G/H \simeq M/H_M$ is compact (cf. Lemma 1), and one readily sees that $E^\circ(P{:}\psi{:}0) = \psi$. Thus the statement is trivially verified for spaces of split rank 0.

Let now G/H be a space of split rank $r \geq 1$, and assume that the result has been proved already for all Harish-Chandra class reductive symmetric spaces of split rank strictly smaller than r. As we have seen above, we may assume that the vectorial part of G/H is trivial.

The basic idea of the proof is that the regularity of the Eisenstein integral is governed by the principal parts of its asymptotic expansions along (maximal) parabolic subgroups Q. The following lemma asserts that these principal parts are regular for imaginary values of the spectral parameter.

Lemma 20. *Let $P \in \mathcal{P}^{\min}_\sigma$, $Q \in \mathcal{P}_\sigma \setminus \{G\}$, and let $\psi \in {}^\circ\mathcal{C}(\tau)$. Then as a function of λ the Q-principal part $E^\circ_Q(P{:}\psi{:}\lambda)$ of $E^\circ(P{:}\psi{:}\lambda)$ is regular on $i\mathfrak{a}^*_\mathrm{q}$.*

Proof. Using (103) and the unitarity of the c–function occurring there (cf. Proposition 5) we see that it suffices to establish the lemma in the case that $Q \supset P$. But then the principal part is given by (106). The Eisenstein integrals occurring in the right hand side of (106) are regular functions of λ on $i\mathfrak{a}^*_\mathrm{q}$, by the induction hypothesis, since the split rank of ${}^\circ M_{1Q}/({}^\circ M_{1Q} \cap H) = M_Q/H_{M_Q}$ is less than r when Q is proper. Moreover, the c-functions occurring in (106) are also regular, by the unitarity of the c-functions (Proposition 5). □

It is convenient to introduce the set $\mathbf{P}$ consisting of all $p \in \Pi_\Sigma(\mathfrak{a}_\mathrm{q})$ with the property that the function

$$\lambda \mapsto E^\circ_p(P{:}\psi{:}\lambda) := p(\lambda) E^\circ(P{:}\psi{:}\lambda)$$

is regular on $\mathfrak{a}_{\mathrm{q}}^*(\bar{P},1)$, for each $\psi \in {}^\circ\mathcal{C}(\tau)$. It follows from Lemma 14 that $\mathbf{P}$ is non-empty. To establish the regularity of $E^\circ(P{:}\psi{:}\lambda)$ on $\mathfrak{a}_{\mathrm{q}}^*(\epsilon)$ for some $\epsilon > 0$ it obviously suffices to show that

$$p^{-1}(0) \cap i\mathfrak{a}_{\mathrm{q}}^* = \emptyset. \tag{117}$$

for some $p \in \mathbf{P}$.

Lemma 21. *Suppose that $p \in \mathbf{P}$ vanishes at $\lambda_0 \in i\mathfrak{a}_{\mathrm{q}}^*$. Then for every $\psi \in {}^\circ\mathcal{C}(\tau)$ the function $E_p^\circ(P{:}\psi{:}\lambda_0)$ is $\mathbb{D}(G/H)$-finite and belongs to the Schwartz space $\mathcal{C}(G/H{:}\tau)$.*

Proof. Put $E_{p,Q}^\circ(P{:}\psi{:}\lambda) = p(\lambda)E_Q^\circ(P{:}\psi{:}\lambda)$. Then it follows from the lemma above that $E_{p,Q}^\circ(P{:}\psi{:}\lambda_0) = 0$ for $Q \in \mathcal{P}_\sigma$ proper. Using Corollary 2 we obtain

$$\lim_{t\to\infty} e^{t\rho_Q(X)}\, E_p^\circ(P{:}\psi{:}\lambda_0)(m\, \exp tX) = 0.$$

for all $m \in M_Q$, $X \in \mathfrak{a}_{Q\mathrm{q}}^+$. The lemma is now an immediate consequence of Proposition 12. □

To complete the proof of Theorem 2 we need the following lemma. Given $p \in \mathbf{P}$, we define, as in [3, Sect. 19], a continuous linear map $\mathcal{F}_p : \mathcal{C}(G/H{:}\tau) \to \mathcal{S}(i\mathfrak{a}_{\mathrm{q}}^*) \otimes {}^\circ\mathcal{C}(\tau)$ by

$$\langle \mathcal{F}_p f(\lambda) | \psi\rangle = \langle f | E_p^\circ(P{:}\psi{:}\lambda)\rangle, \qquad (\psi \in {}^\circ\mathcal{C}(\tau)),$$

for $f \in \mathcal{C}(G/H{:}\tau)$, $\lambda \in i\mathfrak{a}_{\mathrm{q}}^*$.

Lemma 22. *Let $p \in \mathbf{P}$, and let $f \in \mathcal{C}(G/H{:}\tau)$ be a $\mathbb{D}(G/H)$-finite function. Then $\mathcal{F}_p f = 0$.*

Proof. Let $\triangle \in \mathbb{D}(G/H)$ denote the canonical image of the Casimir element (it is a constant multiple of the Laplace-Beltrami operator associated with the pseudo-Riemannian structure on G/H). Then by $\mathbb{D}(G/H)$-finiteness there exist a positive integer m and constants $a_0, \dots, a_{m-1} \in \mathbb{C}$ such that

$$D := \triangle^m + a_{m-1}\triangle^{m-1} + \cdots + a_0$$

annihilates f. In view of (60), which is valid also for Schwartz functions f by the density of $C_c^\infty(G/H{:}\tau)$ in $\mathcal{C}(G/H{:}\tau)$, this implies that

$$\mu(D{:}\tau{:}\lambda)\, \mathcal{F}_p f(\lambda) = \mathcal{F}_p(Df)(\lambda) = 0, \tag{118}$$

for all $\lambda \in i\mathfrak{a}_{\mathrm{q}}^*$. The $\mathrm{End}({}^\circ\mathcal{C}(\tau))$-valued polynomial function $\lambda \mapsto \mu(D{:}\tau{:}\lambda)$ on $i\mathfrak{a}_{\mathrm{q}}^*$ has highest degree homogeneous part equal to $\langle\lambda,\lambda\rangle^m$ times the identity operator. Hence $\det \mu(D{:}\tau{:}\lambda)$ is not identically zero, and therefore (118) implies that $\mathcal{F}_p f = 0$. □

Lemma 23. *Suppose that $p \in \mathbf{P}$ vanishes at $\lambda_0 \in i\mathfrak{a}_{\mathrm{q}}^*$. Then $E_p^\circ(P{:}\psi{:}\lambda_0) = 0$ for all $\psi \in {}^\circ\mathcal{C}(\tau)$.*

Proof. Fix $\psi \in {}^\circ\mathcal{C}(\tau)$. Then by Lemma 21 the function $f := E_p^\circ(P{:}\psi{:}\lambda_0)$ is a $\mathbb{D}(G/H)$-finite τ-spherical Schwartz function. From Lemma 22 we then obtain that:

$$\langle f|f\rangle = \langle f|E_p^\circ(P{:}\psi{:}\lambda)\rangle_{\lambda=\lambda_0} = \langle \mathcal{F}_P f(\lambda_0)|\psi\rangle = 0,$$

and it follows that $f = 0$. $\square$

We can now complete the proof of Theorem 2 by an argument similar to the one used in the proof of Theorem 1 (see Section 5). Choose some $p \in \mathbf{P}$, and suppose that (117) does not hold. Then p has a linear factor $l \in \mathbf{P}$ vanishing at a point of $i\mathfrak{a}_{\mathrm{q}}^*$. This factor must be of the form $l(\lambda) = \langle \lambda, \alpha\rangle - c$, with $\alpha \in \Sigma$ and c a purely imaginary number. Thus $\mathrm{H} := l^{-1}(0) \cap i\mathfrak{a}_{\mathrm{q}}^*$ is a codimension 1 hyperplane in $i\mathfrak{a}_{\mathrm{q}}^*$. Let $\psi \in {}^\circ\mathcal{C}(\tau)$. Then it follows from Lemma 23 that $E_p^\circ(P{:}\psi{:}\,\cdot)$ vanishes on H, and hence on the connected set $l^{-1}(0) \cap \mathfrak{a}_{\mathrm{q}}^*(\bar{P}, 1)$, by analytic continuation. Therefore l is a factor of the holomorphic function $\lambda \mapsto E_p^\circ(P{:}\psi{:}\lambda)$ on $\mathfrak{a}_{\mathrm{q}}^*(\bar{P}, 1)$, and by definition it follows that $l^{-1}p \in \mathbf{P}$. Using this argument repeatedly we arrive in a finite number of steps at a $p \in \mathbf{P}$ such that (117) holds. $\square$

It follows from Theorem 2 that the results of [3, Sect. 19] are valid with $\pi = 1$ (see *loc. cit.* for the meaning of this notation). In particular we get the following result (cf. *loc. cit.*, Thm. 19.1).

Corollary 4. *The Fourier transform $\mathcal{F}$ defines a continuous linear map from $\mathcal{C}(G/H{:}\tau)$ into $\mathcal{S}(i\mathfrak{a}_{\mathrm{q}}^*) \otimes {}^\circ\mathcal{C}(\tau)$.*

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