The natural extension of the β-transformation

Karma Dajani, Cor Kraaikamp and Boris Solomyak*

Abstract

For each real number $\beta > 1$ the β-transformation is defined by $T_\beta x = \beta x (\text{mod} 1)$. In this paper the natural extension $\mathcal{T}_\beta$ of the ergodic system underlying T_β is explicitly given. Furthermore, it is shown that a certain induced system of this natural extension is Bernoulli. Since T_β is weakly mixing, due to W. Parry, it follows from a deep result of A. Saleski that the natural extension is also Bernoulli, a result previously obtained by M. Smorodinsky.

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1 Introduction

Let $\beta > 1$ be a real number, and let T_β be the transformation of the unit interval [0,1) given by $T_\beta x = \beta x (\text{mod} 1)$. This transformation gives rise to the β-expansion introduced by A. Renyi [9] : for any $0 \le x \le 1$,

$$x = \sum_{k=1}^{\infty} d_k \beta^{-k} , \tag{1}$$

where $d_k = d_k(x, \beta) = \lfloor \beta T_\beta^{k-1} x \rfloor$, $k \ge 1$ (here $\lfloor \xi \rfloor$ denotes the greatest integer not exceeding ξ). We will denote (1) by $x = .d_1 d_2 \dots d_n \dots$ or by $d(x, \beta) = (d_k(x, \beta))_{k \ge 1}$ (we will write d_k instead of $d_k(x, \beta)$ whenever there is no risk of confusion).

When β is an integer, (1) is the usual digit expansion with base β. In this case T_β is clearly related to the Bernoulli-shift on β symbols; however, for $\beta \notin \mathbf{Z}$ the situation is more complicated. Renyi showed that for each $\beta > 1$ the β-transformation T_β is ergodic, and that there exists a unique probability measure ν_β, equivalent to Lebesgue measure and invariant under T_β, such that for each Borel measurable set $B \in \mathcal{B}$ one has

$$\nu_\beta(B) = \int_B h_\beta(x) dx ,$$

where $h_\beta(x)$ is a measurable function satisfying

$$1 - \frac{1}{\beta} \le h_\beta(x) \le \frac{1}{1 - \frac{1}{\beta}} .$$

Shortly afterwards, W. Parry [8] noticed that $T_\beta \times T_\beta$ can easily be shown to be ergodic, and therefore T_β is weakly-mixing. Parry showed that

$$h_\beta(x) = \frac{1}{F(\beta)} \sum_{x < T^n(1)} \frac{1}{\beta^n} , \;\; x \in [0, 1) ,$$

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where $F(\beta) = \int_0^1 (\sum_{x<T^n(1)} \frac{1}{\beta^n})dx$ is a normalizing constant.

The "gap" between the ergodic properties of T_β for $\beta \in \mathbf{Z}$ and $\beta \notin \mathbf{Z}$ (as is well-known, Bernoullicity is a much stronger property than weakly-mixing) was bridged by M. Smorodinsky in [12]. Smorodinsky showed that for each $\beta > 1$ the system

$$([0,1), \mathcal{B}, \nu_\beta, T_\beta) \ ,$$

where $\mathcal{B}$ is the collection of Borel sets of [0,1), is weakly Bernoulli. A deep result by N. Friedman and D. S. Ornstein [5] then yields that the natural extension of $([0,1), \mathcal{B}, \nu_\beta, T_\beta)$ is a Bernoulli automorphism.

In this paper we will obtain Smorodinsky's result by explicitly constructing the natural extension of $([0,1), \mathcal{B}, \nu_\beta, T_\beta)$ (section 2), and showing that a certain induced transformation of this natural extension is isomorphic to a Bernoulli automorphism, which is completely understood (see also section 3 or [1]). In section 4 we then show that Smorodinsky's result follows from Parry's observation that T_β is weakly-mixing, and a theorem by A. Saleski [11].

2 Natural Extension of the β-transformation

2.1 An easy example

In case $\beta > 1$ is the positive root of the polynomial $X^m - X^{m-1} - \ldots - X - 1$, $m \geq 2$, one easily sees[1] that the natural extension of $([0,1), \mathcal{B}, \nu_\beta, T_\beta)$ is given by $(X_\beta, \mathcal{B}, \mu_\beta, \mathcal{T}_\beta)$, where

$$X_\beta \ := \bigcup_{k=0}^{m-1} (T_\beta^{m-k}1, T_\beta^{m-k-1}1] \times [0, T_\beta^k 1] \ , \tag{2}$$

and $\mathcal{T}_\beta : X_\beta \to X_\beta$ is defined by $\mathcal{T}_\beta(x,y) := (T_\beta x, \frac{1}{\beta}(\lfloor \beta x \rfloor + y))$. Furthermore, μ is normalized Lebesgue measure on X_β with $\left(\frac{1}{\beta^2} + \frac{2}{\beta^3} + \ldots + \frac{m}{\beta^{m+1}}\right)^{-1}$ as a normalizing constant. See also figure 1 (for the case $m = 4$).

Figure 1

[1] See also [1]. For $m = 2$ one has that $\beta = \frac{\sqrt{5}+1}{2}$, which is the golden mean. For $m \geq 3$ we call these β's *(pseudo) golden mean* numbers. See [6] for a different approach.

2.2 Definition of the natural extension

¿From the example in the previous section it is obvious that for (pseudo) golden mean numbers β the orbit of 1 under T_β :

$$1 > T_\beta 1 > T_\beta^2 1 > \ldots > T_\beta^{m-1} 1 > T_\beta^m 1 = 0\,,$$

plays a crucial role in the definition of the natural extension. In order to generalize the above example to any $\beta > 1$ we recall the following well-known result, which describes the admissible sequences of digits from $\{0, 1, 2, \ldots, \lfloor\beta\rfloor\}$, see also [2, Prop. 2.3].

Proposition 1 *Let* $\mathtt{W} := \{0, 1, 2, \ldots, \lfloor\beta\rfloor\}^{\mathbf{N}}$ *and let* $\mathcal{D}_\beta$ *be the set of* β*-expansions of the real numbers* $x \in [0, 1)$*. Furthermore, let* σ *be the shift and* $<_{lex}$ *be the lexicographical ordering on* $\mathtt{W}$*.*

1. *If* $d(1,\beta) = .b_1 b_2 \ldots b_\ell 0000\ldots$*, that is, the* β*-expansion* $d(1,\beta)$ *of* 1 *is finite, then* $s \in \mathtt{W}$ *belongs to* $\mathcal{D}_\beta$ *if and only if for all* $n \geq 0$*,* $\sigma^n(s)$ *is lexicographically less than the periodic sequence*

   $$d^*(\beta) = .b_1 b_2 \ldots b_{\ell-1}(b_\ell - 1) b_1 b_2 \ldots b_{\ell-1}(b_\ell - 1) b_1 b_2 \ldots b_{\ell-1}(b_\ell - 1)\ldots\ .$$

   In this case we put $\ell := \sup\{k \in \mathbf{N} : b_k \neq 0,\, b_{k+i} = 0\,, \forall i \geq 1\}$*.*

2. *When the* β*-expansion* $d(1,\beta)$ *of* 1 *is not finite, the condition*

   $$\sigma^n(s) <_{lex} d(1,\beta)\ ,\ \ \textit{for all } n \geq 0$$

   is necessary and sufficient for the sequence $s \in \mathtt{W}$ *to belong to* $\mathcal{D}_\beta$*. In this case we write* $d^*(\beta) = d(1,\beta)$ *and* $\ell = \infty$*.*

Using the procedure described in [10] (see also [4], p. 239), the natural extension $(\mathcal{H}_\beta, \mathcal{F}, \mu_\beta, \mathcal{T}_\beta)$ of $([0,1), \mathcal{B}, \nu_\beta, T_\beta)$ is constructed in the following way. Let

$$R_0 \;=\; [0,1]^2 \text{ and } R_i \;=\; [0, T_\beta^i 1] \times [0, \frac{1}{\beta^i}]\,,\ i \geq 1,$$

the underlying space $\mathcal{H}_\beta$ is obtained by stacking (as pages in a book) R_{i+1} on top of R_i, for each[2] $i \geq 0$ (in case 1 has a finite β-expansion of length n, only n R_i's are stacked). Let $\mathcal{B}_i$ be the collection of Borel sets of R_i, and let the $\sigma-$algebra $\mathcal{F}$ on $\mathcal{H}_\beta$ be the direct sum of the $\mathcal{B}_i$'s, i.e. $\mathcal{F} = \oplus\mathcal{B}_i$. Furthermore, the measure on $\mathcal{H}_\beta$ which is Lebesgue measure on each rectangle R_i is denoted by η, and we put $\mu = \frac{1}{\eta(\mathcal{H}_\beta)}\eta$. Finally $\mathcal{T}_\beta : \mathcal{H}_\beta \to \mathcal{H}_\beta$ is defined as follows. Let $d^*(\beta) = .b_1 b_2 \ldots$ and let $(x,y) \in R_i$, $i \geq 0$, where $x = .d_1 d_2 \ldots$ is the β-expansion of x and $y = .\underbrace{00\ldots0}_{i-\text{times}} c_{i+1} c_{i+2}\ldots$ is the β-expansion of y (notice that $(x,y) \in R_i$ implies that $d_1 \leq b_{i+1}$). In view of proposition 1, we now define

$$\mathcal{T}_\beta(x,y) \;:=\; (T_\beta x, y^*) \in \begin{cases} R_0 & \text{, if } d_1 < b_{i+1}\,, \\ R_{i+1} & \text{, if } d_1 = b_{i+1}\,, \end{cases} \tag{3}$$

[2] The index i indicates at what height one is in the stack.

where

$$y^* = \begin{cases} \frac{b_1}{\beta} + \ldots + \frac{b_i}{\beta^i} + \frac{d_1}{\beta^{i+1}} + \frac{y}{\beta} = .b_1 \ldots b_i d_i c_{i+1} c_{i+2} \ldots & \text{, if } d_1 < b_{i+1}\,, \\ \\ \frac{y}{\beta} = \underbrace{.000\ldots00}_{i+1-\text{times}} c_{i+1}c_{i+2}\ldots & \text{, if } d_1 = b_{i+1}\,. \end{cases}$$

Notice that in case $i = 0$ one has

$$y^* = \begin{cases} \frac{1}{\beta}(y + d_1) & \text{, if } d_1 < b_1\,, \\ \\ \frac{y}{\beta} & \text{, if } d_1 = b_1\,. \end{cases}$$

Remark At first glance it seems that for a (pseudo) golden mean number $\beta > 1$ the natural extension $(\, X_\beta, \mathcal{B}, \mu_\beta, \mathcal{T}_\beta\,)$ from section 2.1 is a far cry from the above mentioned system $(\, \mathcal{H}_\beta, \mathcal{F}, \mu_\beta, \mathcal{T}_\beta\,)$. E.g. each point in the space X_β is 2-dimensional, while each point in $\mathcal{H}_\beta$ is 3-dimensional (the third dimension being the height in the stack). We will discuss here briefly for each $\beta > 1$ a system $(\, \tilde{\mathcal{H}}_\beta, \mathcal{B}, \tilde{\mu}_\beta, \tilde{\mathcal{T}}_\beta\,)$, metrically isomorphic to $(\, \mathcal{H}_\beta, \mathcal{F}, \mu_\beta, \mathcal{T}_\beta\,)$ (and therefore also a natural extension of T_β), which coincides with $(\, X_\beta, \mathcal{B}, \mu_\beta, \mathcal{T}_\beta\,)$ in case β is a (pseudo) golden mean number.

Put

$$\tilde{\mathcal{H}}_\beta := \bigcup_{i=0}^{\ell-1} [0,\, T_\beta^i 1) \times [\, \frac{1}{\mathtt{S}} \sum_{j=0}^{i-1} \frac{1}{\beta^j},\, \frac{1}{\mathtt{S}} \sum_{j=0}^{i} \frac{1}{\beta^j}\,)\,,$$

where ℓ is defined as in proposition 1 and

$$\mathtt{S} := \sum_{j=0}^{\ell-1} \frac{1}{\beta^j}\,.$$

For $(x, y) \in \tilde{\mathcal{H}}_\beta$, let $i = i(y) \in \mathbf{N} \cup \{0\}$ be such, that

$$y \in [\, \frac{1}{\mathtt{S}} \sum_{j=0}^{i-1} \frac{1}{\beta^j},\, \frac{1}{\mathtt{S}} \sum_{j=0}^{i} \frac{1}{\beta^j}\,)\,,$$

and define the map $\Psi : \tilde{\mathcal{H}}_\beta \to \mathcal{H}_\beta$ by

$$\Psi(x, y) := \left(x,\, \mathtt{S}y - \sum_{j=0}^{i(y)-1} \frac{1}{\beta^j},\, i(y) \right)\,,$$

where $i(y)$ indicates at what height one is in the stack $\mathcal{H}_\beta$. Clearly the map $\Psi : \tilde{\mathcal{H}}_\beta \to \mathcal{H}_\beta$ is a bijection. Let $\mathcal{B}$ be the collection of Borel sets of $\tilde{\mathcal{H}}_\beta$ and let $\tilde{\mu}_\beta$ be defined by

$$\tilde{\mu}_\beta(B) := \mu_\beta(\Psi(B)) \text{ , for } B \in \mathcal{B}\,.$$

Notice that $\tilde{\mu}_\beta$ is normalized Lebesgue measure on $(\tilde{\mathcal{H}}_\beta,\, \mathcal{B})$ and that Parry's invariant measure ν_β for T_β is the projection of $\tilde{\mu}_\beta$ on the first axis. Finally, define $\tilde{\mathcal{T}}_\beta : \tilde{\mathcal{H}}_\beta \to \tilde{\mathcal{H}}_\beta$ by

$$\tilde{\mathcal{T}}_\beta(x, y) := \Psi^{-1}(\mathcal{T}_\beta(\Psi(x, y))) \text{ , for } (x, y) \in \tilde{\mathcal{H}}_\beta\,.$$

We have the following proposition.

Proposition 2 *For each $\beta > 1$ the systems $(\, \mathcal{H}_\beta, \mathcal{F}, \mu_\beta, \mathcal{T}_\beta\,)$ and $(\, \tilde{\mathcal{H}}_\beta, \mathcal{B}, \tilde{\mu}_\beta, \tilde{\mathcal{T}}_\beta\,)$ are metrically isomorphic. In case β is a (pseudo) golden mean number the systems $(\, \mathcal{H}_\beta, \mathcal{F}, \mu_\beta, \mathcal{T}_\beta\,)$ and $(\, X_\beta, \mathcal{B}, \mu_\beta, \mathcal{T}_\beta\,)$ are identical.*

3 Generalized Lüroth Series and β-expansions

In [1] the so-called Generalized Lüroth Series (shortly GLS) were introduced and their ergodic properties studied. We recall here some of the basic definitions and properties of GLS-expansions, and then we will show how GLS-expansions can be used to obtain a deeper understanding of the natural extension of the $\beta-$transformation T_β.

3.1 Generalized Lüroth Series

Let $I_n = (\ell_n, r_n]$, $n \in \mathcal{D} \subset \mathbf{N} = \{0, 1, 2, \ldots\}$, be a finite or infinite collection of disjoint intervals of length $L_n := r_n - \ell_n$, such that

$$\overline{\bigcup_{n\in\mathcal{D}} I_n} = [0,1] \text{ and } \sum_{n\in\mathcal{D}} L_n \;=\; 1$$

and

$$0 < L_i \le L_j < 1\,, \quad \text{for all } i, j \in \mathcal{D},\ i > j\ .$$

Furthermore, let $I_\infty := [0,1] \setminus \bigcup_{n\in\mathcal{D}} I_n$, $L_\infty := 0$ and define the maps $T_\mathcal{D}$, $S_\mathcal{D}\ : [0,1] \to [0,1]$ by

$$T_\mathcal{D} x \;:=\; \begin{cases} \frac{x-\ell_n}{r_n-\ell_n} & , x \in I_n,\, n \in \mathcal{D}\,, \\ 0 & , x \in I_\infty \end{cases} \qquad \text{and } S_\mathcal{D} x \;:=\; \begin{cases} \frac{r_n-x}{r_n-\ell_n} & , x \in I_n,\, n \in \mathcal{D}\,, \\ 0 & , x \in I_\infty \end{cases}$$

Now let $\varepsilon = (\varepsilon(n))_{n\in\mathcal{D}}$ be an arbitrary, fixed sequence of zeroes and ones. We define the map $T_\varepsilon\ : [0,1] \to [0,1]$ by

$$T_\varepsilon x \;:=\; \varepsilon(x) S_\mathcal{D} x + (1-\varepsilon(x)) T_\mathcal{D} x\,,\quad x \in [0,1] \tag{4}$$

where

$$\varepsilon(x) \;:=\; \begin{cases} \varepsilon(n) & , x \in I_n\,,\ n \in \mathcal{D} \\ 0 & , x \in I_\infty\,. \end{cases}$$

Let $\varepsilon_n \;:=\; \varepsilon(T_\varepsilon^{n-1} x)$ and define for $x \in \Omega := [0,1] \setminus I_\infty = \cup_{n\in\mathcal{D}} I_n$

$$s(x) := \frac{1}{r_n - \ell_n} \quad \text{and} \quad h(x) := \frac{\ell_n}{r_n - \ell_n} \quad , \text{ in case } x \in I_n,\, n \in \mathcal{D}\ .$$

Moreover we define

$$s_n = s_n(x) \;:=\; \begin{cases} s(T_\varepsilon^{n-1} x) & , T_\varepsilon^{n-1} x \notin I_\infty\,, \\ \infty & , T_\varepsilon^{n-1} x \in I_\infty\,, \end{cases}$$

and h_n is defined in a similar way. One finds that

$$x = \frac{h_1+\varepsilon_1}{s_1} + \frac{(-1)^{\varepsilon_1}}{s_1} T_\varepsilon x = \frac{h_1+\varepsilon_1}{s_1} + (-1)^{\varepsilon_1}\frac{h_2+\varepsilon_2}{s_1 s_2} + \ldots \tag{5}$$

$$\ldots\ldots + (-1)^{\varepsilon_1+\ldots+\varepsilon_{n-1}}\frac{h_n+\varepsilon_n}{s_1 s_2\ldots s_n} + \frac{(-1)^{\varepsilon_1+\ldots+\varepsilon_n}}{s_1 s_2 \ldots s_n} T_\varepsilon^n x\ \ .$$

For each $k \ge 1$ and $1 \le i \le k$ one has $s_i \ge 1/L > 1$, where $L = \max_{n\in\mathcal{D}} L_n$, and $|T_\varepsilon^k x| \le 1$. Thus,

$$\left| x - \frac{p_k}{q_k} \right| = \frac{T_\varepsilon^k x}{s_1 s_2 \ldots s_k} \le L^k \to 0 \text{ as } k \to \infty\ ,$$

where

$$\frac{p_k}{q_k} = \frac{h_1+\varepsilon_1}{s_1} + (-1)^{\varepsilon_1}\frac{h_2+\varepsilon_2}{s_1 s_2} + (-1)^{\varepsilon_1+\varepsilon_2}\frac{h_3+\varepsilon_3}{s_1 s_2 s_3} + \ldots + (-1)^{\varepsilon_1+\ldots+\varepsilon_{k-1}}\frac{h_k+\varepsilon_k}{s_1 s_2 \ldots s_k}\,.$$

Let $\varepsilon_0 := 0$, then for each $x \in [0,1]$ one has

$$x = \sum_{n=1}^{\infty} (-1)^{\varepsilon_0+\ldots+\varepsilon_{n-1}} \frac{h_n+\varepsilon_n}{s_1 s_2 \ldots s_n}\,. \tag{6}$$

For each $x \in [0,1]$ we define its sequence of DIGITS $a_n = a_n(x)$, $n \geq 1$, as follows :

$$a_n = k \iff_{\mathrm{DEF}} T_\varepsilon^{n-1} x \in I_k\,,$$

for $k \in \mathcal{D} \cup \{\infty\}$. The expansion (6) is called the (I, ε)−GENERALIZED LÜROTH SERIES (GLS) of x. Notice that for each $x \in [0,1] \backslash I_\infty$ one finds a unique expansion (6), and therefore a unique sequence of digits a_n, $n \geq 1$. Conversely, each sequence of digits a_n, $n \geq 1$, with $a_n \in \mathcal{D} \cup \{\infty\}$, $a_1 \neq \infty$ defines a unique series expansion (6). We denote (6) by

$$x = \begin{bmatrix} \varepsilon_1, & \varepsilon_2, & \varepsilon_3, & \ldots & \varepsilon_n, & \ldots \\ a_1, & a_2, & a_3, & \ldots & a_n, & \ldots \end{bmatrix}.$$

Examples

1. Let $I_n := (\frac{1}{n}, \frac{1}{n-1}]$, $n \geq 2$. In case $\varepsilon_n = 0$, $n \geq 2$, one gets the classical Lüroth Series, while $\varepsilon_n = 1$, $n \geq 2$ yields the alternating Lüroth Series.
2. For $n \in \mathbf{N}$, $n \geq 2$, put $I_i = (\frac{i}{n}, \frac{i+1}{n}]$, $i = 0, 1, \ldots, n-1$. In case $\varepsilon(i) = 0$ for all i, T_ε yields the n-adic expansion. In case $n = 2$ and $\varepsilon(0) = 0$, $\varepsilon(1) = 1$ T_ε is the tent map.

The ergodic properties of the GLS-transformation are given by the following theorem.

Theorem 1 *1. The stochastic variables $a_1(x), a_2(x), \ldots, a_n(x), \ldots$ corresponding to the (I, ε)-GLS operator T_ε from (4) are i.i.d with respect to the Lebesgue measure λ, and*

$$\lambda(a_n = k) = L_k \quad \text{for } k \in \mathcal{D} \cup \{\infty\}\,.$$

Furthermore, $(I_n)_{n\in\mathcal{D}}$ is a generating partition.

2. The $(I, \varepsilon)-$GLS operator T_ε from (4) is measure preserving and Bernoulli.

3. Let $\mathcal{T}_\varepsilon : [0,1] \times [0,1] \to [0,1] \times [0,1]$ be given by :

$$\mathcal{T}_\varepsilon(x,y) := \left(T_\varepsilon x, \frac{h(x)+\varepsilon(x)}{s(x)} + \frac{(-1)^{\varepsilon(x)} y}{s(x)}\right) \quad , (x,y) \in [0,1] \times [0,1]\,,$$

then the system

$$([0,1] \times [0,1], \mathcal{B} \times \mathcal{B}, \lambda \times \lambda, \mathcal{T}_\varepsilon)$$

is the natural extension of $([0,1], \mathcal{B}, \lambda, T_\varepsilon)$. Furthermore, $([0,1] \times [0,1], \mathcal{B} \times \mathcal{B}, \lambda \times \lambda, \mathcal{T}_\varepsilon)$ is Bernoulli.

Remark The proof of theorem 1 in [1] is straightforward; e.g. *1.* follows from (5) and from $L_k = 1/s_k$, $0 \leq T_\varepsilon^n x < 1$. It is also easy to see that T_ε is Bernoulli, and that $\mathcal{T}_\varepsilon$ is the natural extension of T_ε. Since T_ε is Bernoulli, it certainly is weakly Bernoulli, hence [5] yields the desired result that $\mathcal{T}_\varepsilon$ is Bernoulli. In fact it is an exercise to show that T_ε is Bernoulli implies that $\mathcal{T}_\varepsilon$ is Bernoulli, see also [3].

3.2 Inducing on the Natural Extension of the β−transformation

Let us return for a moment to the example of section 2.1. Again, let $\beta > 1$ be a (pseudo) golden mean number and let X_β be as in (2). Let $Y_\beta := [0,1] \times [0, 1/\beta]$ and $\mathcal{W}_\beta : Y_\beta \to Y_\beta$ the corresponding induced transformation under $\mathcal{T}_\beta$, that is

$$\mathcal{W}_\beta(x,y) \;=\; \mathcal{T}_\beta^{k+1}(x,y)\,,$$

where $k = k(x) \in \{0, 1, \ldots, m-1\}$ is such, that $x \in (T_\beta^{m-k}1, T_\beta^{m-k-1}1]$. The orbit of 1 under T_β yields a natural partition of [0,1). Let T_ε be the (I, ε)-GLS operator with this partition, i.e. the partition I is given by $(T_\beta^{m-i}1, T_\beta^{m-i-1}1]$, $0 \le i \le m-1$ (see also figure 1), and let $\varepsilon(n) := 0$ for each digit n. Notice that for $x \in (T_\beta^{m-i}1, T_\beta^{m-(i+1)}1]$, $0 \le i \le m-1$, one has that

$$T_\varepsilon x \;=\; T_\beta^{i+1} x\ .$$

In [1] the following theorem was obtained for the special case $\beta > 1$ is a (pseudo) golden mean number.

Theorem 2 *Let* $\Psi : [0,1] \times [0,1] \to Y_\beta$ *be defined by*

$$\Psi(x,y) := (x, \frac{y}{\beta})$$

and let the measure ρ *on* Y_β *be defined by* $\rho(A) := (\lambda \times \lambda)(\Psi^{-1}(A))$ *for each Borel set* $A \subset Y_\beta$. *Then one has*

1. Ψ *is a measurable bijection which satisfies* $\Psi \circ \mathcal{T}_\varepsilon = \mathcal{W}_\beta \circ \Psi$.
2. *The dynamical systems* $([0,1]\times[0,1], \lambda\times\lambda, \mathcal{T}_\varepsilon)$ *and* $(Y_\beta, \rho, \mathcal{W}_\beta)$ *are isomorphic.*

In the remaining part of this section we give a generalization of theorem 2 to all $\beta > 1$.

Let $\mathcal{W}_\beta : R_0 \to R_0$ be the induced map of $\mathcal{T}_\beta$ on R_0. That is, $\mathcal{W}_\beta$ is given as follows; let $(x,y) \in R_0$, where $x = .d_1 d_2 \ldots$ is the β-expansion of x and $d^*(\beta) = .b_1 b_2 \ldots$, then

$$\mathcal{W}_\beta(x,y) \;=\; \begin{cases} \mathcal{T}_\beta(x,y) & \text{if } d_1 < b_1\,, \\ \mathcal{T}_\beta^n(x,y) & \text{if } d_i = b_i \text{ for } 1 \le i \le n-1 \text{ and } d_n < b_n\,. \end{cases}$$

This follows from definition (3) of $\mathcal{T}_\beta$ on R_0, for

- if $d_1 < b_1$ then $\mathcal{T}_\beta(x,y) \in R_0$.
- if $d_i = b_i$ for $1 \le i \le n-1$ and $d_n < b_n$ one has $\mathcal{T}_\beta^i(x,y) \in R_i$ for $1 \le i \le n-1$ and $\mathcal{T}_\beta^n(x,y) \in R_0$.

In other words

$$\mathcal{W}_\beta(x,y) \;=\; \begin{cases} \Big(T_\beta x,\ \frac{1}{\beta}(y+d_1)\Big) & ,\ \text{if } (x,y) \in R_0^1\,, \\[2ex] \Big(T_\beta^k x,\ \frac{b_1}{\beta} + \cdots + \frac{b_{k-1}}{\beta^{k-1}} + \frac{d_k}{\beta^k} + \frac{y}{\beta^k}\Big) & ,\ \text{if } (x,y) \in R_0^k\,,\ k \ge 2. \end{cases}$$

For $(x,y)\in R_0$, let $n(x,y) := \inf\{\, k\geq 1\,;\, \mathcal{T}_\beta^k(x,y)\in R_0\,\}$ and $R_0^k := \{\,(x,y)\in R_0\,;\, n(x,y)=k\,\}$. Notice that $n(x,y)$ is the first return time of the map $\mathcal{T}_\beta$ to R_0, where $n\in\mathbf{N}$. The above shows that

$$R_0^1 \;=\; [\,0,\, .b_1\,]\times[0,1]\,,\;\; R_0^k \;=\; [\,.b_1\ldots b_{k-1},\, .b_1\ldots b_{k-1}b_k\,]\times[0,1]\;, k\geq 2$$

and

$$R_0 \;=\; \bigcup_{k=1}^{\infty} R_0^k\,.$$

Let $\mathcal{I} \;=\; \{I_n\,;\, n\;\geq\; 1\}$ be the partition on [0,1) defined as follows : for each $n\;\geq\;1$, there exist unique integers $k\geq 0$ and $1\leq i\leq b_{k+1}$ such that

$$n \;=\; b_0+b_1+\cdots+b_k+(i-1)\,,$$

where by definition we put $b_0 := 0$. Set

$$I_n \;=\; \left(b_0+\frac{b_1}{\beta}+\cdots+\frac{b_k}{\beta^k}+\frac{i-1}{\beta^{k+1}}\,,\; b_0+\frac{b_1}{\beta}+\cdots+\frac{b_k}{\beta^k}+\frac{i}{\beta^{k+1}}\right]\;. \tag{7}$$

Note that for $(x,y)\in I_n\times[0,1]$,

$$\mathcal{W}_\beta(x,y) \;=\; \mathcal{T}_\beta^{k+1}(x,y) \;=\; \left(T_\beta^{k+1}x\;,\; b_0+\frac{b_1}{\beta}+\cdots+\frac{b_k}{\beta^k}+\frac{i-1}{\beta^{k+1}}+\frac{y}{\beta^{k+1}}\right)\;.$$

We have the following proposition.

Proposition 3 *Let $\mathcal{I}=(I_n)_{n\in\mathcal{D}}$ be the partition from (7). Then for $\varepsilon_n=0$ for all $n\geq 1$ we have that the natural extension $\mathcal{T}_\varepsilon$ of the $(\mathcal{I},\,\varepsilon)$-GLS expansion T_ε is identical to $\mathcal{W}_\beta$.*

Proof. The proof follows from the fact that $\mathcal{T}_\varepsilon$ on $I_{b_0+\cdots+b_k+(i-1)}\times[0,1]$ for $1\leq i\leq b_{k+1}$ is given by

$$\mathcal{T}_\varepsilon(x,y) \;=\; (T_\varepsilon x,\; b_0+\frac{b_1}{\beta}+\cdots+\frac{b_k}{\beta^k}+\frac{i-1}{\beta^{k+1}}+\frac{y}{\beta^{k+1}})$$

and that for $x\in(b_0+\frac{b_1}{\beta}+\cdots+\frac{b_k}{\beta^k},\, b_0+\frac{b_1}{\beta}+\cdots+\frac{b_{k+1}}{\beta^{k+1}}]$, $T_\varepsilon x \;=\; T_\beta^{k+1}x$. Comparing with (7), one sees that $\mathcal{W}_\beta=\mathcal{T}_\varepsilon$, and hence by theorem 1 it follows that $(R_0,\mathcal{F}_{R_0},\mu_{R_0},\mathcal{W}_\beta)$, with $\mathcal{F}_{R_0} \;=\; \mathcal{F}\cap R_0$ and $\mu_{R_0} \;=\; \mu_\beta|_{R_0}$, is a Bernoulli system.$\Box$

4 The natural extension of the β-transformation is Bernoulli

In this section we will show that from proposition 3 and a result by Saleski we at once have Smorodinsky's result that the natural extension of the β-transformation is Bernoulli. We first recall Saleski's result, see also [11].

Theorem 3 *Let $(X,\mathcal{B},\mu,T)$ be a non-atomic Lebesgue space with an automorphism T. Let $A\in\mathcal{B}$ be a subset of X of positive measure and denote by T_A the induced transformation of T on A. Moreover, suppose we have that T_A is Bernoulli, T is weakly mixing and*

$$H_{\mu_A}\left(\vee_{i=1}^{\infty}\vee_{j=1}^{\infty}\,T_A^iY_j\mid\vee_{i=0}^{\infty}T_A^iP\right)\;<\;\infty\,,$$

where P is a Bernoulli partition of (A,T_A) and $Y_j \;=\; \{A-\cup_{i=1}^{j}T^{-i}A,\, A\cap\cup_{i=1}^{j}T^{-i}A\}$. Then T is a Bernoulli automorphism.

We have the following theorem.

Theorem 4 *The system* $(\mathcal{H}_\beta, \mathcal{F}, \mu_\beta, \mathcal{T}_\beta)$ *is Bernoulli.*

Proof. The system $(\mathcal{H}_\beta, \mathcal{F}, \mu_\beta, \mathcal{T}_\beta)$ is weakly-mixing, due to W. Parry, and for which the induced transformation $\mathcal{W}_\beta$ is Bernoulli with Bernoulli partition $P = \mathcal{I}^\sharp := \mathcal{I} \times [0,1]$ where $\mathcal{I}$ is as given in (7). The partitions Y_j of R_0 as defined in Saleski's theorem 3 have the following form :

$$Y_j \;=\; \left\{ \left(0\,,\, b_0 + \frac{b_1}{\beta} + \cdots + \frac{b_j}{\beta^j}\right] \times [0,1]\,,\; \left(b_0 + \frac{b_1}{\beta} + \cdots + \frac{b_j}{\beta^j}\,,\, 1\right] \times [0,1] \right\}\,.$$

Clearly, $Y_j \,<\, \mathcal{I}^\sharp$ for each $j \geq 1$, i.e. $\mathcal{I}^\sharp$ is a refinement of each Y_j, so that

$$\vee_{i=1}^{\infty} \vee_{j=1}^{\infty} \mathcal{W}_\beta^i Y_j \;<\; \vee_{i=0}^{\infty} \mathcal{W}_\beta^i \mathcal{I}^\sharp$$

and hence

$$H_{\mu_{R_0}} \left(\vee_{i=1}^{\infty} \vee_{j=1}^{\infty} \mathcal{W}_\beta^i Y_j \mid \vee_{i=0}^{\infty} \mathcal{W}_\beta^i \mathcal{I}^\sharp \right) \;=\; 0.$$

Thus, by theorem 3, we have that $(\mathcal{H}_\beta, \mathcal{F}, \mu_\beta, \mathcal{T}_\beta)$ is Bernoulli.□

Final remark After reading a first draft of this paper, H. Nakada [7] remarked that the above described method to prove Bernoullicity of the β-transformation can easily be generalized to a class of functions $\mathcal{L}$, introduced and studied by K. M. Wilkinson [13]. This class $\mathcal{L}$ of piecewise linear transformations of the unit interval onto itself contains among others the β-transformation and the linear mod one transformation. Let I_n, $n \in \mathcal{D}$, be a finite or infinite collection of intervals as in section 3.1, and let $T : [0,1) \to [0,1)$ for $x \in I_n$, $n \in \mathcal{D}$, be defined by

$$Tx \;:=\; \beta_n(x - \ell_n) + \alpha_n\;, \tag{8}$$

where $\beta = \inf_{n \in \mathcal{D}} \beta_n > 1$ and for each $n \in \mathcal{D}$, $0 \leq \alpha_n < 1$, $\beta_n r_n + \alpha_n \leq 1$. Put

$$\Delta(j_1, j_2, \cdots, j_n) \;:=\; I_{j_1} \cap T^{-1} I_{j_2} \cap \cdots \cap T^{-(n-1)} I_{j_n}\,,$$

where $j_k \in \mathcal{D}$, $1 \leq k \leq n$. We say that $\Delta_n = \Delta(j_1, j_2, \cdots, j_n)$ is a full-interval of rank n if $\lambda(T^n(\Delta_n)) = 1$, and non-full otherwise.
Let B_n be the union of those intervals $\Delta(j_1, j_2, \cdots, j_n)$ which are full, but none of $\Delta(j_1, j_2, \cdots, j_k)$, $1 \leq k \leq n-1$, is full, and suppose that

$$\sum_{n=1}^{\infty} \lambda(B_n) \;=\; 1. \tag{9}$$

Denote by $\mathcal{N}$ the class of transformations of the form (8), satisfying (9). Then $\mathcal{N}$ contains Wilkinson's class $\mathcal{L}$ (see [13], corollary 3.5). Clearly each $T \in \mathcal{N}$ has a GLS as an induced transformation, with GLS-partition $\mathcal{P}$, consisting of all full intervals $\Delta(j_1, j_2, \cdots, j_n)$ for which none of $\Delta(j_1, j_2, \cdots, j_k)$, $1 \leq k \leq n-1$, is full.

References

[1] Barrionuevo, Jose, Robert M. Burton, Karma Dajani and Cor Kraaikamp – *Ergodic Properties of Generalized Lüroth Series,* TU Delft Report 94-105 (1994), 1-16.

[2] Blanchard, F. – *β-Expansions and Symbolic Dynamics,* Theoretical Comp. Sc., **65** (1989), 131-141.

[3] Brown, J. R. – *Ergodic Theory and Topological Dynamics,* Academic Press, New York, 1976.

[4] Cornfeld, I. P., S. V. Fomin and Ya. G. Sinai – *Ergodic Theory,* Grundlehren der math. Wiss. **245**, Springer-Verlag New York, Heidelberg, Berlin (1982).

[5] Friedman, N. A. and D. S. Ornstein – *On isomorphisms of weak Bernoulli transformations,* Adv. in Math., **5** (1970), 365-390.

[6] Ito, Shunji – *A Construction of Transversal Flows for Maximal Markov Automorphisms,* Tokyo J. Math., **1** (1978), 305-324.

[7] Nakada, Hitoshi. Personal communication with the second author, Delft. March 20, 1995.

[8] Parry, W. – *On the β-expansion of real numbers,* Acta Math. Acad. Sci. Hung., **11** (1960), 401-416.

[9] Rényi, A. – *Representations for real numbers and their ergodic properties,* Acta Math. Acad. Sci. Hung., **8** (1957), 472-493.

[10] Rohlin, V. A. – *Exact endomorphisms of a Lebesgue space,* Izv. Akad. Naik SSSR, Ser. Mat., **24** (1960); English AMS translation, Series **2**, **39** (1969), 1-36.

[11] Saleski, Alan – *On induced Transformations of Bernoulli Shifts,* Math. Systems Th. **7** (1973), 83-93.

[12] Smorodinsky, Meir – *β-Automorphisms are Bernoulli Shifts,* Acta Math. Acad. Sci. Hung., **24** (1973), 273-278.

[13] Wilkinson, Keith M. – *Ergodic Properties of a Class of Piecewise Linear Transformations,* Z. Wahrscheinlichkeitstheorie verw. Gebiete, **31** (1975), 303-328.

Karma Dajani
Universiteit Utrecht
Dept. of Math.
Budapestlaan 6
P.O.Box 80.010
3508 TA Utrecht
the Netherlands
dajani@math.ruu.nl

Cor Kraaikamp
Technische Universiteit Delft
and Thomas Stieltjes Institute for Mathematics
TWI (SSOR)
Mekelweg 4
2628 CD Delft
the Netherlands
cork@dutiosd.twi.tudelft.nl

Boris Solomyak
Dept. of Mathematics, GN-50
University of Washington
Seattle, Washington 98195
solomyak@math.washington.edu