# The mean ratio set for $ax + b$ valued cocycles

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**Abstract**

Let $X = \prod_{i=1}^{\infty} \mathbf{Z}_{\ell(i)}$ be acted upon by the group $\Gamma = \oplus_{i=1}^{\infty} \mathbf{Z}_{\ell(i)}$ of changes in finitely many coordinates and $\mu$ a $G$-measure on $X$ which is nonsingular for the $\Gamma$-action on $X$. We consider cocycles on $(X, \Gamma, \mu)$ taking values in the $ax + b$ group. We give a structure theorem for such cocycles, we define the mean ratio set which is a closed subgroup of the $ax + b$ group and we exhibit for each closed subgroup a cocycle whose mean ratio set is the given subgroup.

## 1 Introduction

The notion of essential range of real-valued cocycle was defined by Krieger [K] as a subset of $[-\infty, \infty]$. He showed that its intersection with $(-\infty, \infty)$ is a closed subgroup of the real line and that cohomologous cocycles have the same essential range. Parthasarathy and Schmidt [PS] extended this result to cocycles with values in locally compact abelian groups. The notion of essential range has been extended to cocycles with values in general nonabelian locally compact groups, but it is no longer cohomology invariant (see [S1]). In the case of a multiplicative cocycle with values in $\mathbb{R}^+$, the essential range is also called the ratio set.

In this article, we examine closely the example of cocycles with values in one of the simplest nonabelian groups, the $ax + b$ group. One motivation for this is to study the ways an additive and a multiplicative cocycle can interact.

In the next section, we produce a new type of essential range called the mean ratio set (MRS). In the case of a real-valued cocycle and a measure-preserving action our definition exactly coincides with the essential range. This closed subset of $[0,\infty]\times[-\infty,\infty]$ whilst not cohomology invariant, is close to being so. In fact, if the transfer function is integrable, mean ratio sets are conjugate in the $ax+b$ group by its integral – hence if the integral is the identity (1,0), the mean ratio set is preserved. Furthermore if $W_1$ and $W_2$ are cohomologous with integrable transfer function, then there is a constant transfer function under which $W_2$ is conjugate to $W_3$ where $MRS(W_1) = MRS(W_3)$.

An essential step in the proof, not without independent interest, is a structure theorem for $ax+b$-valued cocycles which generalizes theorems of Golodets [G] and Parthasarathy and Schmidt [PS].

The final section of the paper gives a classification of the closed subgroups of the $ax+b$-group. As a result we are able to classify the $ax+b$-valued cocycle in an $L^1$-cohomology invariant way.

This theory is the first step in a new approach to the study of nonabelian cocycles over $X$ (c.f. [Z]). We believe that it will lead to a new treatment of recurrence and skew products.

# 2 The Structure of $\mathbf{ax+b}$-valued cocycles

Let $X=\prod_{i=1}^{\infty} X_i$ with $X_i=\mathbb{Z}_{\ell(i)}$ for some integer $\ell(i)$ where $\mathbb{Z}_{l(i)}$ denotes the integers modulo $\ell(i)$. Let $\mathcal{B}$ be the $\sigma$-algebra generated by the cylinder sets. Let $\Gamma$ be the group of finite coordinate changes, that is

$$\Gamma=\{\gamma\in X:\gamma_i=0 \text{ for all but finitely many coordinates } i\}$$

$\Gamma$ acts on $X$ by coordinatewise addition, i.e., $(\gamma x)_i=\gamma_i+x_i$. For $k\geq 0$, let $\Gamma_k=\{\gamma\in\Gamma:\gamma_i=0 \text{ for all } i>k\}$.

**Motivation 2.1.** Before commencing our discussion of the $ax+b$-valued case, let us briefly recall from [PS] the real-valued case with a $\Gamma$-invariant measure $\mu$. Each $\mathbb{R}$-valued cocycle $W$ on $X$ for the action of $\Gamma$ can be written as

$$W(\gamma,x)=\sum_{n=1}^{\infty}\{\beta_n(\gamma x)-\beta_n(x)\}$$

where each $\beta_n$ is $\Gamma_n$-invariant.

Let $W^k(\gamma_0, x) = \frac{1}{|\Gamma_k|}\sum_{\gamma\in\Gamma_k} W(\gamma_0, \gamma x)$. We say that $r$ belongs to the **mean ratio set** of $W$, $MRS_\mu(W)$ if for every $\epsilon > 0$ and for every set $A$ of positive measure there is for each $k_0 \in \mathbb{N}$, a set of positive measure $B \subseteq A$ and $\gamma_0 \in \Gamma$ so that $k \geq k_0$ implies

$$\left|W^k(\gamma_0, x) - r\right| < \epsilon \text{ for all } x \in B.$$

It is readily seen that $r \in MRS_\mu(W)$ if and only if $r \in \bigcap_{k=0}^{\infty} \text{ess.range}(W^k)$.

This definition tries to capture the fact that the average value of $W$ is close to $r$. However, it turns out that we have achieved nothing new. A sufficient condition for $r$ to belong to $MRS(W)$ is that for each $\epsilon > 0$, for each $k$, and for each $\Gamma_k$-invariant set $A$ of positive measure there exists $\gamma_0 \in \Gamma_k$ and a $\Gamma_k$-invariant set $B$ of positive measure so that $|W(\gamma_0, x) - r| < \epsilon$ on $B$. Using this, one readily sees

**Proposition 2.1** *For an additive real-valued cocycle $W$ and a $\Gamma$-invariant measure $\mu$ one has*

$$MRS_\mu(W) = \text{ ess.range } (W).$$

**Proof:** The proof is left to the reader. □

The aim of this section is to extend the above structure theorem and definition to cocycles with values in the $ax + b$ group.

First, we recall some definition and notation concerning multiplicative cocycles [BD].

**Notation 2.1** In [BD], we considered a family of measurable functions $\{G_k\}$ satisfying the conditions of compatibility and normalization, that is, for any $k \leq n$ and any $\gamma \in \Gamma_k \subseteq \Gamma_n$

$$\frac{G_k(\gamma x)}{G_k(x)} = \frac{G_n(\gamma x)}{G_n(x)} \tag{C1}$$

and

$$\frac{1}{|\Gamma_k|}\sum_{\gamma\in\Gamma_k} G_k(\gamma x) = 1.$$

A nonsingular probability measure $\mu$ on $X$ was defined to be a $G-$measure if there is a compatible normalized family $\{G_k\}$

$$\frac{d\mu\circ\gamma}{d\mu}(x) = \frac{G_k(\gamma x)}{G_k(x)}$$

$\mu$ a.e. $x \in X$, and $\gamma \in \Gamma_k$.

In the case where there is a unique $G-$measure $\mu$, it is automatically ergodic, and we say that $\mu$ is uniquely ergodic. In [BD] proposition 3, we showed that $\mu$ is uniquely ergodic if and only if for every continuous function $f$ on $X$, the sequence

$$\frac{1}{|\Gamma_n|} \sum_{\gamma \in \Gamma_n} G_n(\gamma x) f(\gamma x)$$

converges uniformly to a constant.

Given a compatible family $\{G_k\}$ and a family of measurable functions $\{\beta_k\}$ on $X$ such that for all $\gamma \in \Gamma_k$, we have $\beta_k(\gamma x) = \beta_k(x)$. Define

$$W_k(\gamma, x) = \sum_{n=0}^{k-1} \Big( \frac{G_k(\gamma x)}{G_n(\gamma x)} \beta_n(\gamma x) - \frac{G_k(x)}{G_n(x)} \beta_n(x) \Big),$$

where $G_0(x) = 1$. Then, $W_k$ is well-defined, measurable and equals for $\gamma \in \Gamma_k$

$$W_k(\gamma, x) = \sum_{n=0}^{\infty} \Big( \frac{G_k(\gamma x)}{G_n(\gamma x)} \beta_n(\gamma x) - \frac{G_k(x)}{G_n(x)} \beta_n(x) \Big).$$

Furthermore, $W_k$ is an additive cocyle on $X$ for the $\Gamma_k$ action, in the sense that, for $\gamma_1, \gamma_2 \in \Gamma_k$ we have

$$W_k(\gamma_1\gamma_2, x) = W_k(\gamma_1, x) + W_k(\gamma_2, \gamma_1 x).$$

Moreover, the family $\{W_k\}$ satisfies the following compatibility condition

$$\frac{W_k(\gamma, x)}{G_k(x)} = \frac{W_n(\gamma, x)}{G_n(x)}, \text{ for all } \gamma \in \Gamma_k \subseteq \Gamma_n. \qquad (C2).$$

Equivalently,

$$\frac{W_{k+1}(\gamma, x)}{g_{k+1}(x)} = W_k(\gamma, x), \text{ for all } \gamma \in \Gamma_k. \qquad (C3),$$

with $g_{k+1}(x) = \dfrac{G_{k+1}(x)}{G_k(x)}$.

Let $\mathcal{A}$ denote the $ax + b$ group, that is the underlying space is $\mathbb{R}^+ \times \mathbb{R}$ and group operation defined by: $(a, b)(c, x) = (ac, ax + b)$. The identity is $(1, 0)$ and $(a, b)^{-1} = (a^{-1}, -a^{-1}b)$.

**Lemma 2.1** *Suppose $\{G_k\}$ is a compatible family, and $\{W_k\}$ a family of compatible additive cocycles. Define $\sigma : \Gamma \times X \to \mathcal{A}$ by*

$$\sigma(\gamma, x) = \Big(\frac{G_k(\gamma x)}{G_k(x)}, \frac{W_k(\gamma, x)}{G_k(x)}\Big)$$

*whenever $\gamma \in \Gamma_k$ and $x \in X$. Then $\sigma$ is an $ax + b$ valued cocycle on $X$ for the $\Gamma$ action.*

**Proof:** $\sigma$ is well-defined by the compatibility conditions, that is if $\gamma \in \Gamma_k \subseteq \Gamma_n$, then

$$\Big(\frac{G_k(\gamma x)}{G_k(x)}, \frac{W_k(\gamma, x)}{G_k(x)}\Big) = \Big(\frac{G_n(\gamma x)}{G_n(x)}, \frac{W_n(\gamma, x)}{G_n(x)}\Big).$$

Also, one can easily verify using the multiplication in $\mathcal{A}$ that

$$\sigma(\gamma_1\gamma_2, x) = \sigma(\gamma_1, x)\sigma(\gamma_2, \gamma_1 x).$$

**Notation:** For $k \geq 1$ let $X^k = \{x \in X : x_1 = x_2 = \ldots = x_k = 0\}$ and $\Gamma^k = \Gamma \cap X^k$. For $x \in X$, let $x_{(n)} = (x_1, \ldots, x_n, 0, 0, \ldots)$ and $x^{(n)} = (0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots)$, where $x_{(0)} = 0$ and $x^{(0)} = x$. Then, $x_{(n)} \in \Gamma_n$ and $x = x_{(n)}x^{(n)}$. Also, if $\{G_k\}$ satisfies condition (C1), then for each $k$, $g_{k+1}(x) = \frac{G_{k+1}(x)}{G_k(x)}$ is $\Gamma_k$ invariant (see [BD]).

**Lemma 2.2** *Given any $ax + b$ valued cocycle $\sigma$ on $X$ for the $\Gamma$ action, then there exists a compatible family of measurable functions $\{G_k\}$ and a compatible family of cocycles $\{W_k\}$ such that*

$$\sigma(\gamma, x) = \Big(\frac{G_k(\gamma x)}{G_k(x)}, \frac{W_k(\gamma, x)}{G_k(x)}\Big)$$

*whenever $\gamma \in \Gamma_k$ and $x \in X$.*

**Proof:** Let $\sigma(\gamma, x) = \Big(\sigma_1(\gamma, x), \sigma_2(\gamma, x)\Big)$. From the cocycle identity for $\sigma$ one gets that $\sigma_1$ is a multiplicative $\mathbf{R}^+$ valued cocycle, and $\sigma_2$ a $\sigma_1$ cocycle, in the sense that $\sigma_2(\gamma_1\gamma_2, x) = \sigma_2(\gamma_1, x) + \sigma_1(\gamma_1, x)\sigma_2(\gamma_2, \gamma_1 x)$. Set $G_0(x) = 1$ and for $k \geq 1$, let $G_k(x) = \sigma_1\Big(x_{(k)}, x^{(k)}\Big)$, then for $\gamma \in \Gamma_k$, we have $\frac{G_k(\gamma x)}{G_k(x)} = \sigma_1(\gamma, x)$. Also, for any $m \geq k$ and $\gamma \in \Gamma_k$, $\frac{G_m(\gamma x)}{G_m(x)} = \frac{G_k(\gamma x)}{G_k(x)}$. Now, for $\gamma \in \Gamma_k$ set $W_k(\gamma, x) = G_k(x)\sigma_2(\gamma, x)$. Using the fact that $\sigma_2$ is a $\sigma_1$ cocycle one can easily verify that $\{W_k\}$ is a family of cocycles satisfying condition (C2).

**Lemma 2.3** *Let $\{W_k\}$ be a compatible family. For $k \geq 0$ define*

$$\beta_k(x) = \frac{W_{k+1}(x_{(k+1)}, x^{(k+1)})}{g_{k+1}(x)} - W_k(x_{(k)}, x^{(k)}).$$

*Then $\beta_k$ is $\Gamma_k$ invariant. Also, for every $\gamma \in \Gamma_k$ and $x \in X$ we have*

$$W_k(\gamma, x) = \sum_{n=0}^{k-1} \Big( \frac{G_k(\gamma x)}{G_n(\gamma x)} \beta_n(\gamma x) - \frac{G_k(x)}{G_n(x)} \beta_n(x) \Big). \qquad (*)$$

**Proof:** Let $\gamma \in \Gamma_k$. Then,

$$\begin{aligned}
\beta_k(\gamma x) &= \frac{W_{k+1}(\gamma x_{(k+1)}, x^{(k+1)})}{g_{k+1}(x)} - W_k(\gamma x_{(k)}, x^{(k)}) \\
&= \frac{W_{k+1}(\gamma, x)}{g_{k+1}(x)} + \frac{W_{k+1}(\gamma x_{(k+1)}, x^{(k+1)})}{g_{k+1}(x)} - W_k(\gamma, x) - W_k(x_{(k)}, x^{(k)}) \\
&= \frac{W_{k+1}(x_{(k+1)}, x^{(k+1)})}{g_{k+1}(x)} - W_k(x_{(k)}, x^{(k)}) \\
&= \beta_k(x).
\end{aligned}$$

To verify (*) notice that both sides satisfy the cocycle identity, hence it is enough to prove only the case $\gamma$ is $x_{(k)}$ and $x$ is $x^{(k)}$. Then, $x = x_{(k)} x^{(k)}$ and for any $n < k$ we have $(x^{(k)})_{(n)} = 0$ and $(x^{(k)})^{(n)} = x^{(k)}$. The left hand side of (*) has then the form $W_k(x_{(k)}, x^{(k)})$. Now, the right hand side of (*) is

$$\begin{aligned}
&\sum_{n=0}^{k-1} \Big( \frac{G_k(x)}{G_n(x)} \beta_n(x) - \frac{G_k(x^{(k)})}{G_n(x^{(k)})} \beta_n(x^{(k)}) \Big) \\
&= \sum_{n=0}^{k-1} \Big( g_{n+1}(x) \ldots g_k(x) \frac{W_{n+1}(x_{(n+1)}, x^{(n+1)})}{g_{n+1}(x)} - g_{n+1}(x) \ldots g_k(x) W_n(x_{(n)}, x^{(n)}) \Big) \\
&= \sum_{n=0}^{k-2} \Big( g_{n+2}(x) \ldots g_k(x) W_{n+1}(x_{(n+1)}, x^{(n+1)}) - g_{n+1}(x) \ldots g_k(x) W_n(x_{(n)}, x^{(n)}) \Big) \\
&+ W_k(x_{(k)}, x^{(k)}) - g_k(x) W_{k-1}(x_{(k-1)}, x^{(k-1)}) \\
&= g_k(x) W_{k-1}(x_{(k-1)}, x^{(k-1)}) + W_k(x_{(k)}, x^{(k)}) - g_k(x) W_{k-1}(x_{(k-1)}, x^{(k-1)}) \\
&= W_k(x_{(k)}, x^{(k)}).
\end{aligned}$$

**Theorem 2.1** *There is a one-to-one correspondence between $ax + b$ valued cocycles on $X$ for the $\Gamma$ action and compatible families $\{G_k\}$ satisfying condition (C1) and $\{\beta_k\}$ with each $\beta_k$ a $\Gamma_k$ invariant function.*

## 3 The mean ratio set

**Definition 3.1** *Let $W$ be an additive cocyle on $X$ for the $\Gamma$ action. Define*

$$W^k(\gamma_0, x) = \frac{1}{|\Gamma_k|} \sum_{\gamma \in \Gamma_k} W(\gamma_0, \gamma x).$$

For $k \geq 1$, let $\mathcal{B}^k$ denote the tail $\sigma$- algebra generated by all cylinders of the form $\prod_{i=1}^{\infty} E_i$ where $E_i = X_i = \mathbf{Z}_{l(i)}$ for all $i < k$. If $\mu$ is $G$ measure on $X$ and $f$ a measurable function, we denote by $E_\mu(f|\mathcal{B}^k)$ the conditional expectation of $f$ given the sub-$\sigma$-algebra $\mathcal{B}^k$.

**Lemma 3.1** *Let $\mu$ be a $G$ measure on $X$, then*

**(i)** *$W^k$ is a cocycle on $X$ for the $\Gamma$ action,*

**(ii)** *For all $\gamma_0 \in \Gamma_k$, $W^k(\gamma_0, x) = 0$,*

**(iii)** *If $n \geq k$, we have $E_\mu\Big(\frac{W(\gamma_0,x)}{G_n(x)}|\mathcal{B}^k\Big) = \frac{G_k(x)}{G_n(x)} W^k(\gamma_0, x)$.*

**Proof:** (i) Clear since the sum of cocycles is a cocycle.
(ii) Follows from the cocycle identity; for $\gamma_0 \in \Gamma_k$ we have

$$\begin{aligned} W^k(\gamma_0, x) &= \frac{1}{|\Gamma_k|} \sum_{\gamma \in \Gamma_k} W(\gamma_0, \gamma x) \\ &= \frac{1}{|\Gamma_k|} \sum_{\gamma \in \Gamma_k} W(\gamma_0 \gamma, x) - W(\gamma, x) = 0. \end{aligned}$$

(iii) From [BD] one has

$$\begin{aligned} E_\mu\Big(\frac{W(\gamma_0, x)}{G_n(x)}|\mathcal{B}^k\Big) &= \frac{1}{|\Gamma_k|} \sum_{\gamma \in \Gamma_k} \frac{W(\gamma_0, \gamma x)}{G_n(\gamma x)} G_k(\gamma x) \\ &= \frac{1}{|\Gamma_k|} \sum_{\gamma \in \Gamma_k} W(\gamma_0, \gamma x) \frac{G_k(x)}{G_n(x)} \\ &= \frac{G_k(x)}{G_n(x)} W^k(\gamma_0, x). \end{aligned}$$

**Remarks 3.1**

**(a)** In each variable $W^k$ is independent of the first $k$ coordinates, in the sense that, if $\gamma_0 \in \Gamma_k$, then for any $\gamma \in \Gamma$ and $x \in X$ we have $W^k(\gamma\gamma_0, x) = W^k(\gamma, x) = W^k(\gamma, \gamma_0 x)$.

**(b)** If $W$ is a cocyle for the $\Gamma_n$ action, then for $\gamma_0 \in \Gamma_n$ we define $W^k(\gamma_0, x) = \frac{1}{|\Gamma_k|}\sum_{\gamma\in\Gamma_k} W(\gamma_0, \gamma x)$ if $k < n$, and 0 otherwise.

**(c)** In [BD1] we defined, for a quasi-invariant measure $\mu$ on $X$, $\mu^m = \frac{1}{|\Gamma_m|}\sum_{\gamma\in\Gamma_m} \mu\circ\gamma$, and noted that this is precisely $\mu$ conditioned on $\mathcal{B}^m$. The above notation is compatible with this.

Clearly the mean ratio sets of $\mu$ and $\mu^m$ coincide. Thus, by Proposition (2.1), for each $m$, the ratio sets of $\mu$ and of $\mu^m$ coincide.

**Definition 3.2** *Let $\mu$ be a nonsingular $G$ measure on $X$ and $\sigma$ an $ax+b$ valued cocycle for the $\Gamma$ action which has the form*

$$\sigma(\gamma, x) = \left(\frac{G_k(\gamma x)}{G_k(x)}, \frac{W_k(\gamma, x)}{G_k(x)}\right)$$

*whenever $\gamma \in \Gamma_k$ and $x \in X$. As before let $\mathcal{A}$ denote the $ax+b$ group. An element $(r, s) \in \mathcal{A}$ is said to belong to the mean ratio set of $\sigma$, denoted by $r_\mu(\sigma)$, if for every $\epsilon > 0$ there exists $m_0 \geq 1$ such that for every $A \in \mathcal{B}$ with $\mu(A) > 0$ and for every $m > m_0$, there exists a measurable subset $B \subseteq A$ with $\mu(B) > 0$ and there exist $n \geq 1$ and $\gamma_0 \in \Gamma_n$ such that the following hold*

**(i)** $\gamma_0 B \subseteq A$,

**(ii)** *For every $x \in B$,* $\left|\frac{G_n(\gamma_0 x)}{G_n(x)} - r\right| < \epsilon$,

**(iii)** *For every $x \in B$,* $\left|\frac{G_m(x)}{G_n(x)} W_n^m(\gamma_0, x) - s\right| < \epsilon$.

**Proposition 3.1** *The mean ratio set $r_\mu(\sigma)$ is a closed subgroup of $\mathcal{A}$.*

**Proof:** Let $(r_1, s_1), (r_2, s_2) \in r_\mu(\sigma)$. We want to show $(r_1 r_2, s_1 + r_1 s_2) \in r_\mu(\sigma)$. Let $\epsilon > 0$, there exists $m_0' \geq 1$ such that if $A \in \mathcal{B}$ with $\mu(A) > 0$ and $m > m_0$, there exists $B \subseteq A$ with $\mu(B) > 0$ and there exist a positive integer $n_1 \geq m$ and $\gamma_1 \in \Gamma_{n_1}$ such that

**(i)** $\gamma_1 B \subseteq A$, and for every $x \in B$, $\left|\frac{G_{n_1}(\gamma_1 x)}{G_{n_1}(x)} - r_1\right| < \epsilon$, and

$$\left|\frac{G_m(x)}{G_{n_1}(x)} W_{n_1}^m(\gamma_1, x) - s_1\right| < \epsilon.$$

Further, since $\mu(\gamma_1 B) > 0$ we can find $C \subseteq \gamma_1 B$ with $\mu(C) > 0$, and an integer $n_2 \geq m$ and $\gamma_2 \in \Gamma_{n_2}$ such that

**(ii)** $\gamma_2 C \subseteq \Gamma_1 B$, and for every $x \in C$,

$$\left|\frac{G_{n_2}(\gamma_2 x)}{G_{n_2}(x)} - r_2\right| < \epsilon,$$

and $\left|\frac{G_m(x)}{G_{n_2}(x)} W_{n_2}^m(\gamma_2, x) - s_2\right| < \epsilon$.

Let $n = n_1 + n_2$ and $D = \gamma_1^{-1} C \subseteq B$. Then, $\mu(D) > 0$ and $\gamma_2\gamma_1 D \subseteq A$. Now, for any $x \in D$ we have

$$\begin{aligned}
\left|\frac{G_n(\gamma_2\gamma_1 x)}{G_n(x)} - r_1 r_2\right| &= \left|\frac{G_n(\gamma_2\gamma_1 x)}{G_n(\gamma_1 x)}\frac{G_n(\gamma_1 x)}{G_n(x)} - r_1 r_2\right| \\
&= \left|\frac{G_{n_2}(\gamma_2\gamma_1 x)}{G_{n_2}(\gamma_1 x)}\frac{G_{n_1}(\gamma_1 x)}{G_{n_1}(x)} - r_1 r_2\right| \\
&\leq (r_1 + \epsilon)\epsilon + r_2\epsilon = (r_1 + r_2)\epsilon + \epsilon^2
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{G_m(x)}{G_n(x)} W_n^m(\gamma_2\gamma_1, x) - (s_1 + r_1 s_2)\right| \\
= \;& \left|\frac{G_m(x)}{G_{n_1}(x)} W_{n_1}^m(\gamma_1, x) + \frac{G_{n_1}(\gamma_1 x)}{G_{n_1}(x)}\frac{G_m(x)}{G_{n_2}(\gamma_1 x)} W_{n_2}^m(\gamma_2, \gamma_1 x) - (s_1 + r_1 s_2)\right| \\
\leq \;& \left|\frac{G_m(x)}{G_{n_1}(x)} W_{n_1}^m(\gamma_1, x) - s_1\right| + \frac{G_{n_1}(\gamma_1 x)}{G_{n_1}(x)}\left|\frac{G_m(x)}{G_{n_2}(\gamma_1 x)} W_{n_2}^m(\gamma_2, \gamma_1 x) - s_2\right| \\
+ \;& s_2\left|\frac{G_m(x)}{G_{n_1}(x)} - r_1\right| \\
< \;& (r_1 + s_2 + 1)\epsilon + \epsilon^2.
\end{aligned}$$

This shows $(r_1 r_2, s_1 + r_1 s_2) \in r_\mu(\sigma)$. Now, let $(r, s) \in r_\mu(\sigma)$. We want to show that $(r^{-1}, -r^{-1}s) \in r_\mu(\sigma)$. Let $\epsilon > 0$. For any measurable set $A$ with

$\mu(A) > 0$ and any integer $m$ choose a real number $N(m) > 0$ such that the set $A_m = \{x \in A : |G_m(x) - N(m)| < \epsilon\}$ has positive measure. Since $(r,s) \in r_\mu(\sigma)$, there exists $m_0 > 1$ such that for $m > m_0$ we can find $B \subseteq A_m$, an integer $n \geq m$ and $\gamma \in \Gamma_n$ such that

**(iii)** $\gamma B \subseteq A_m$, and for every $x \in B$, $\left|\frac{G_n(\gamma x)}{G_n(x)} - r\right| < \epsilon$, and

$$\left|\frac{G_m(x)}{G_n(x)} W_n^m(\gamma, x) - s\right| < \epsilon.$$

Let $C = \gamma B \subseteq A_m$, then $\gamma^{-1}C = B \subseteq A_m$. For $x \in C$, since $\gamma^{-1}x \in B$ we have $\left|\frac{G_n(x)}{G_n(\gamma^{-1}x)} - r\right| < \epsilon$, which implies that $\left|\frac{G_n(\gamma^{-1}x)}{G_n(x)} - r^{-1}\right| < \frac{\epsilon}{r(r-\epsilon)^2}$. By the cocycle identity we have $W_n^m(\gamma, \gamma^{-1}x) = -W_n^m(\gamma^{-1}, x)$ and for $x \in C$, $\left|\frac{G_m(x)}{G_m(\gamma^{-1}(x)} - 1\right| < \frac{2\epsilon}{N(m)-\epsilon}$ so that

$$\begin{aligned}
&\left|\frac{G_m(x)}{G_n(x)} W_n^m(\gamma^{-1}, x) + r^{-1}s\right| \\
&\quad = \left|\frac{G_m(x)}{G_m(\gamma^{-1}x)} \frac{G_n(\gamma^{-1}x)}{G_n(x)} \frac{G_m(\gamma^{-1}x)}{G_n(\gamma^{-1}x)} W_n^m(\gamma, \gamma^{-1}x) - r^{-1}s\right| \\
&\quad \leq \frac{G_m(x)}{G_m(\gamma^{-1}x)} \frac{G_n(\gamma^{-1}x)}{G_n(x)} \left|\frac{G_m(\gamma^{-1}x)}{G_n(\gamma^{-1}x)} W_n^m(\gamma, \gamma^{-1}x) - s\right| \\
&\quad + \frac{s}{r}\left|\frac{G_m(x)}{G_m(\gamma^{-1}(x)} - 1\right| + s\frac{G_m(x)}{G_m(\gamma^{-1}(x)}\left|\frac{G_n(\gamma^{-1}x)}{G_n(x)} - r\right| \\
&\quad < \frac{2\epsilon}{N(m)-\epsilon} + \frac{s}{r}\frac{2\epsilon}{N(m)-\epsilon} + \left(1 + \frac{2\epsilon}{N(m)-\epsilon}\right)s\epsilon
\end{aligned}$$

This proves that $(r^{-1}, -r^{-1}s) \in r_\mu(\sigma)$. The proof that $r_\mu(\sigma)$ is closed is straightforward since on $\mathcal{A}$ we have the product topology.

**Definition 3.3** *Two $ax+b$ valued cocycles $\sigma$ and $\tau$ are cohomologous if there exist measurable functions $\alpha$ and $\beta$ such that*

$$\sigma(\gamma, x) = (\alpha(x), \beta(x))\tau(\gamma, x)(\alpha(\gamma x), \beta(\gamma x))^{-1}.$$

*We call the function $(\alpha, \beta)$ a transfer function for $\sigma$ and $\tau$.*

**Lemma 3.2** *For $\gamma \in \Gamma_n$, let $\sigma(\gamma, x) = \left(\frac{G_n(\gamma x)}{G_n(x)}, \frac{W_n(\gamma,x)}{G_n(x)}\right)$ and $\tau(\gamma, x) = \left(\frac{F_n(\gamma x)}{F_n(x)}, \frac{V_n(\gamma,x)}{F_n(x)}\right)$. If $\sigma$ and $\tau$ are cohomologous, then*

$$\frac{G_n(\gamma x)}{G_n(x)} = \frac{\alpha(x)}{\alpha(\gamma x)} \frac{F_n(\gamma x)}{F_n(x)}$$

*and*

$$\frac{W_n(\gamma, x)}{G_n(x)} = \alpha(x)\frac{V_n(\gamma, x)}{F_n(x)} + \beta(x) - \frac{G_n(\gamma x)}{G_n(x)}\beta(\gamma x).$$

Let $\sigma$ and $\tau$ be two cohomologous $ax + b$ valued cocycles each having the form as given in Lemma 3.2, and with transfer function $(\alpha, \beta)$. Assume that the families $\{G_n\}$ and $\{F_n\}$ defining $\sigma$ and $\tau$ respectively are normalized. Set

$$F_n^o(x) = \frac{\alpha(x)G_n(x)}{\frac{1}{|\Gamma_n|}\sum_{\gamma_0\in\Gamma_n} \alpha(\gamma_0 x)G_n(\gamma_0 x)}$$

and for $\gamma \in \Gamma_n$

$$V_n^o(\gamma, x) = \frac{F_n^o(x)}{F_n(x)}V_n(\gamma, x) = \frac{\alpha(x)G_n(x)}{F_n(x)} \frac{V_n(\gamma, x)}{\frac{1}{|\Gamma_n|}\sum_{\gamma_0\in\Gamma_n} \alpha(\gamma_0 x)G_n(\gamma_0 x)}.$$

**Lemma 3.3** **(i)** *For each positive integer $n$, the functions $\frac{\alpha G_n}{F_n}$, $\frac{\alpha G_n}{F_n^o}$ and $\frac{F_n^o}{F_n}$ are $\Gamma_n$ invariant.*

**(ii)** *For each $m < n$ and $\gamma_0 \in \Gamma_n$ we have*

$$\frac{F_m^o(x)}{F_n^o(x)} V^{om}_{\ n}(\gamma_0, x) = \frac{F_m^o(x)}{F_m(x)} \frac{F_m(x)}{F_n(x)} V_n^m(\gamma_0, x).$$

**Lemma 3.4** *If $\alpha$ is $\mu$ integrable, then defining a measure $\nu$ on $X$ by*

$$\nu(A) = \frac{1}{\int_X \alpha(x)d\mu(x)} \int_A \alpha(x)d\mu(x),$$

*we have that $F^o$ is a normalized compatible family, $\nu$ is an $F^o$ measure and for $\gamma \in \Gamma_n$*

$$\tau(\gamma, x) = \left(\frac{F_n^o(\gamma x)}{F_n^o(x)}, \frac{V_n^o(\gamma, x)}{F_n^o(x)}\right).$$

**Theorem 3.1** *Let $\sigma$ and $\tau$ be cohomologous $ax+b$ valued cocycles having the form given in definition 3.3 and with transfer function $(\alpha,\beta)$. Suppose that $\mu$ is a uniquely ergodic $G$ measure and $\alpha$, $\beta$ are $\mu$ integrable. Define $\nu$ as given in lemma 3.4, then $(\int_X \alpha d\mu, \int_X \beta d\mu) r_\mu(\sigma)(\int_X \alpha d\mu, \int_X \beta d\mu)^{-1} = r_\nu(\tau)$. In particular, if $\int_X \alpha d\mu = 1$ and $\int_X \beta d\mu = 0$, then $r_\mu(\sigma) = r_\nu(\tau)$.*

**Proof:** Without loss of generality we assume that $\int \alpha d\mu = 1$, otherwise we normalize. Let $(r,s) \in r_\mu(\sigma)$ and let $\epsilon > 0$ be given. There exists a positive integer $N_1$ such that for all $m > N_1$,

$$\left| \frac{1}{|\Gamma_m|} \sum_{\gamma \in \Gamma_m} \alpha(\gamma x) G_m(\gamma x) - 1 \right| < \epsilon$$

and

$$\left| \frac{1}{|\Gamma_m|} \sum_{\gamma \in \Gamma_m} \beta(\gamma x) G_m(\gamma x) - \int_X \beta(x) d\mu(x) \right| < \epsilon,$$

uniformly in $x$. Let $\epsilon_0 = \frac{\epsilon}{|M-\epsilon|(r+\epsilon)(1+|\int_X \beta d\mu|)}$, and $m > N_1$ be sufficiently large. If $\nu(A) > 0$, then $\mu(A) > 0$. Choose sufficiently large real numbers $M_1$ and $M_2$ such that $A^o = \{x \in A : |G_m(x) - M_1| < \epsilon_0$ and $|\alpha(x) - M_2| < \epsilon_0\}$ has positive measure. There exist $B \subseteq A^o$, $n \geq m$ and $\gamma_0 \in \Gamma_n$ such that $\gamma_0 B \subseteq A^o$, and for every $x \in B$ we have $\left|\frac{G_n(\gamma x)}{G_n(x)} - r\right| < \frac{\epsilon}{(r+1)(|s|+1)}$ and $\left|\frac{G_m(x)}{G_n(x)} W_n^m(\gamma_0, x) - s\right| < \frac{\epsilon}{(r+1)(|s|+1)}$. Now,

$$\begin{aligned}
\frac{G_m(x)}{G_n(x)} W_n^m(\gamma_0, x) &= \frac{1}{|\Gamma_m|} \sum_{\gamma \in \Gamma_m} \beta(\gamma x) G_m(\gamma x) \\
&- \frac{1}{|\Gamma_m|} \sum_{\gamma \in \Gamma_m} \beta(\gamma\gamma_0 x) G_m(\gamma x) \frac{G_n(\gamma\gamma_0 x)}{G_n(\gamma x)} \\
&+ \frac{1}{|\Gamma_m|} \sum_{\gamma \in \Gamma_m} \frac{G_m(\gamma x)\alpha(\gamma x)}{F_n(\gamma x)} V_n(\gamma_0, \gamma x) \\
&= \frac{1}{|\Gamma_m|} \sum_{\gamma \in \Gamma_m} \beta(\gamma x) G_m(\gamma x) \\
&- \frac{G_m(x)}{G_m(\gamma_0 x)} \frac{G_n(\gamma_0 x)}{G_n(x)} \frac{1}{|\Gamma_m|} \sum_{\gamma \in \Gamma_m} \beta(\gamma\gamma_0 x) G_m(\gamma\gamma_0 x) \\
&+ \frac{1}{|\Gamma_m|} \sum_{\gamma \in \Gamma_m} \alpha(\gamma x) G_m(\gamma x) \frac{F_m^o(x)}{F_n^o(x)} V^{om}_{\ n}(\gamma_0, x).
\end{aligned}$$

Then,

$$\begin{aligned}\left|\frac{F_n^o(\gamma_0 x)}{F_n^o(x)} - r\right| &= \left|\frac{F_n(\gamma_0 x)}{F_n(x)} - r\right| \\ &\leq \frac{G_n(\gamma_0 x)}{G_n(x)}\left|\frac{\alpha(\gamma_0 x)}{\alpha(x)} - 1\right| \\ &+ \left|\frac{G_n(\gamma_0 x)}{G_n(x)} - r\right| < 2\epsilon.\end{aligned}$$

Also,

$$\begin{aligned}&\left|\frac{1}{|\Gamma_m|}\sum_{\gamma\in\Gamma_m}\alpha(\gamma x)G_m(\gamma x)\frac{F_m^o(x)}{F_n^o(x)}V^{om}_{\ n}(\gamma_0,x) - s - (1-r)\int_X \beta d\mu\right| \\ \leq\ &\left|\frac{G_m(x)}{G_n(x)}W_n^m(\gamma_0,x) - r\right| \\ +\ &\left|\frac{1}{|\Gamma_m|}\sum_{\gamma\in\Gamma_m}\beta(\gamma x)G_m(\gamma x) - \int_X \beta d\mu\right| \\ -\ &\left|\frac{G_m(x)}{G_m(\gamma_0 x)}\frac{G_n(\gamma_0 x)}{G_n(x)}\frac{1}{|\Gamma_m|}\sum_{\gamma\in\Gamma_m}\beta(\gamma\gamma_0 x)G_m(\gamma\gamma_0 x) - r\int_X \beta d\mu\right| \\ <\ &7\epsilon.\end{aligned}$$

Thus,

$$\left|\frac{F_m^o(x)}{F_n^o(x)}V^{om}_{\ n}(\gamma_0,x) - s - (1-r)\int_X \beta d\mu\right| < 8\epsilon.$$

This shows that $(1,\int_X \beta d\mu)(r,s)(1,-\int_X \beta d\mu) \in r_\nu(\tau)$. The other direction is proved similarly. Hence, $(1,\int_X \beta d\mu)r_\mu(\sigma)(1,-\int_X \beta d\mu) = r_\nu(\tau)$.

# 4 Classification and examples

In this section, we classify the closed subgroups of the $ax+b$ group and use the structure theorem from §2 to give examples of cocycles whose ratio sets correspond to the various possibilities.

The following theorem is perhaps well-known to experts, but we have not been able to find a convenient reference for it. We include a proof for completeness.

**Theorem 4.1** *Let $\mathcal{H}$ be a closed subgroup of the $ax+b$ group $\mathcal{A}$. Then $\mathcal{H}$ is one of the following*

**(i)** $\mathcal{A}$ *itself*

**(ii)** *The identity* $\{e\}$

**(iii)** *For each* $\mu \in (0,1), \{(1, n\mu) : n \in \mathbb{Z}\}$

**(iv)** $\mathbb{R} = \{(1,x) : x \in \mathbb{R}\}$

**(v)** *For each* $\lambda \in \mathbb{R}^+,\ \lambda \neq 1,\ \{(\lambda^n, x) : x \in \mathbb{R}\}$

**(vi)** *For each* $\mu \in \mathbb{R},\ \{(u, \mu(u-1)) : u \in \mathbb{R}^+\}$

**(vii)** *For each* $\mu \in \mathbb{R}$ *and for each* $\lambda \in \mathbb{R}^+,\ \lambda \neq 1,\ \{(\lambda^n, \mu(\lambda^n - 1)) : n \in \mathbb{Z}\}$

**Proof:** Let us realize $\mathcal{A}$ as the group of matrices of the form $\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0,\ b \in \mathbb{R} \right\}$. Its Lie algebra is then $\left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$, with exponential map

$$\exp \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \left( \begin{pmatrix} e^x & y(\frac{e^x - 1}{x}) \\ 0 & 1 \end{pmatrix} \right).$$

The component of the identity $\mathcal{H}_0$ of $\mathcal{H}$ is a connected closed subgroup of $\mathcal{A}$; hence we may identify three possibilities: $\mathcal{H}_0 = \mathcal{A}$, $\mathcal{H}_0 = \{e\}$, or $\mathcal{H}_0$ is a one-dimensional subgroup. In the first case, $\mathcal{H} = \mathcal{A}$ and we are in case (i). In the third case, $\mathcal{H}_0 = \{\exp t \begin{pmatrix} x & w \\ 0 & 0 \end{pmatrix} : t \in \mathbb{R}\}$ with $(x, w) \neq (0,0)$. If $x = 0$, $\mathcal{H}_0 = \{(1,x) : x \in \mathbb{R}\}$. One sees that $\mathcal{H}/\mathcal{H}_0$ is a discrete subgroup of $\mathbb{R}^+$ and we are either in case (iv) or (v). Otherwise, putting $\mu = \frac{w}{x}$, we have $\mathcal{H}_0 = \{(u, \mu(u-1)) : u \in \mathbb{R}^+\}$. We claim that $\mathcal{H} = \mathcal{H}_0$. In fact, since conjugation by $(1, \mu)$ maps $\mathcal{H}_0$ into $\{(u, 0) : u \in \mathbb{R}^+\}$, we may assume $\mu = 0$. Any subgroup containing $\{(u,0) : u \in \mathbb{R}^+\}$ and an element of the form $(u_0, s)$ with $s \neq 0$ is quickly seen to be all of $\mathcal{A}$. Thus $\mathcal{H} = \mathcal{H}_0$ and we are in case (vi).

Finally, let us consider the case when $\mathcal{H}_0 = \{e\}$. We claim that $\mathcal{H}$ is generated by a single element. Suppose first that every element of $\mathcal{H}$ is of the form $(1,x)$ with $x \in \mathbb{R}$. Then $\mathcal{H}$ is a discrete subgroup of $\mathbb{R}$ and we are in case (ii) or (iii). Otherwise, $\mathcal{H}$ contains an element $\gamma = (u,y)$ with $u > 1$. Conjugating as above by $(1,y)$ we may assume that $y = 0$. Thus $\mathcal{H}$ contains $\{(u^n,0) : n \in \mathbb{Z}\}$. Suppose that $\mathcal{H}$ contains also an element of the form $(u_0,y_0)$ with $\log u_0$ and $\log u$ rationally independent; we may suppose that $u_0 < 1$. Let $v \in \mathbb{R}_+$ be arbitrary and choose sequences $\{n_k\}$, $\{m_k\}$ so that $u^{n_k}u_0^{m_k} \to v$ as $k \to \infty$. Then $(u,0)^{n_k}(u_0,y_0)^{m_k} = (u^{n_k}u_0^{m_k},\ \left(\frac{1-u_0^{m_k}}{1-u_0}\right) y_0) \in \mathcal{H}$ for all $k$. Letting $k \to \infty$, we see that $\left(v,\ \frac{y_0}{1-u_0}\right) \in \mathcal{H}$ for all $v \in \mathbb{R}$. This contradicts our assumption that $\mathcal{H}_0 = \{e\}$. We conclude that $\log u_0$ and $\log u$ are rationally related, and so $\mathcal{H}$ contains both $\{(u^n,0) : n \in \mathbb{Z}\}$ and $\{(1,ky_0) : k \in \mathbb{Z}\}$. The set of all elements of $\mathcal{H}$ of the form $(1,w)$ is then a subgroup of $\mathbb{R}$ containing $u^n y_0$ for all $n \in \mathbb{Z}$. This is necessarily the whole of $\mathbb{R}$ except in the case $y_0 = 0$. We have proved that $\mathcal{H}$ is conjugate to $\{(u^n,0) : n \in \mathbb{Z}\}$ for some $u$ and we are therefore in case (vii). □

**(4.2)** Using Proposition 3.1 and Theorem 4.1 we now have a limited number of mutually exclusive possibilities for our mean ratio set, as a closed subset of $[0,\infty] \times [-\infty,\infty]$ whose intersection with $\mathcal{A}$ is a closed subgroup of $\mathcal{A}$.

Recall that the possible ratio sets for a cocycle with values in $\mathbb{R}^+$ are $\{1\}$ (type II), $\mathbb{R}^+$ (type III$_1$), for $0 < \lambda < 1$, $\{\lambda^n : n \in \mathbb{Z}\}$ (type III$_\lambda$) and $\{0,1,\infty\}$ (type III$_0$). For an additive cocycle, we have $\{0\}$ (type II), $\mathbb{R}$ (type III$_1$), for $0 < \mu < 1$, $\{m\mu : m \in \mathbb{Z}\}$ (type III$_\mu$) and $\{-\infty,0,\infty\}$ (type III$_0$.)

In fact, the closed subgroups of $\mathcal{A}$ listed in Theorem 4.1 lead to mean ratio sets of the form $R_1 \times R_2$ where $R_1 \subseteq [0,\infty]$ and $R_2 \subseteq [-\infty,\infty]$ are of the above type in all cases except types (vi) and (vii) with $\eta \neq 0$. On the other hand, as observed in the proof of Theorem 4.1, $(u,\eta(u-1))$ is conjugate to $(u,0)$ via $(1,\eta)$. This leads to

**Definition 4.1** *Let $\mu$ be a G-measure and $\sigma$ an $\mathcal{A}$-valued cocycle for the $\Gamma$ action. If the mean ratio set $r_\mu(\sigma)$ has the form $R_1 \times R_2$ where $R_1$ is of type $X$ for $\mathbb{R}^+$ and $R_2$ is of type $Y$ for $\mathbb{R}$ then we say that $\sigma$ is of type $X \times Y$. If $r_\mu(\sigma)$ can be conjugated into a set of this form by an element of the form $(1,\eta)$ we say that $\sigma$ is of the type $(X \times Y)^\eta$.*

The last possibility is realized only if $X = \mathrm{III}_1$ or $\mathrm{III}_\lambda$, and $Y = \mathrm{II}$ or $\mathrm{III}_0$.

Thus to say that $\sigma$ is of type $II \times III_0$ means that its mean ratio set is $\{1\} \times \{0, 1, \infty\}$, to say that $\sigma$ is of type $(\mathrm{III}_\lambda \times \mathrm{III}_0)^\eta$ means that its mean ratio set is $\{(\lambda^n, \eta(\lambda^n - 1)) : n \in \mathbb{Z}\} \cup \{-\infty, \infty\}$ and to say that $\sigma$ is of type $(\mathrm{III}_1 \times \mathrm{II})^\eta$ means that its mean ratio set is $\{(u, \eta(u - 1)) : u \in \mathbb{R}\}$.

The possible types are then $III_1 \times III_1$, $II \times II$, $III_0 \times II$, $II \times III_\lambda (0 \leq \lambda \leq 1)$, $III_0 \times III_\lambda (0 \leq \lambda \leq 1)$, $(III_\lambda \times I)^\eta$, $0 < \lambda \leq 1$, $\eta \in \mathbb{R}$ and $(III_\lambda \times III_0)^\mu$, $(0 < \lambda \leq 1)$, $\mu \in \mathbb{R}$.

Note that $III_\lambda \times III_\mu$ is not possible with $0 < \lambda, \mu < 1$. We denote by $R(\mu)$ the ratio set of $\mu$ with respect to the $\Gamma$ action (see [KW], [S1], [S2], [BDL]).

**Theorem 4.2** *Let $\{G_n\}$ be a normalized compatible family for which there exists a unique $G-$measure $\mu$. Let $\beta \in L^1(X, \mu)$ and define*

$$W_n(\gamma, x) = G_n(\gamma x)\beta(\gamma x) - G_n(x)\beta(x).$$

*Then $W_n$ is a compatible family of cocycles and*

$$\sigma(\gamma, x) = \left( \frac{G_n(\gamma x)}{G_n(x)}, \ \frac{W_n(\gamma, x)}{G_n(x)} \right)$$

*defines an $ax + b$-valued cocycle. Let $\eta = \int \beta d\mu$.*

**(i)** *If $R(\mu) = \mathbb{R}^+$, that is $\mu$ is of type $III_1$, then*

$$r_\mu(\sigma) = \{(r, (r - 1)\eta) : r \in \mathbb{R}^+\}$$

*so $\sigma$ is of type $(III_1 \times II)^\eta$.*

**(ii)** *If for some $0 < \lambda < 1$, $R(\mu) = \{\lambda^n : n \in \mathbb{Z}\}$, that is, $\mu$ is of type $III_\lambda$, then*

$$r_\mu(\sigma) = \{(\lambda^n, (\lambda^n - 1)\eta) : n \in \mathbb{Z}\},$$

*that is $\mu$ is of type $(III_\lambda \times II)^\eta$.*

**(iii)** *If $\mu$ is of type $III_0$, that is $R(\mu) = \{0, 1, \infty\}$ and $\int_X \beta d\mu = 0$ then $r_\mu(\sigma) = \{0, 1, \infty\} \times \{-\infty, 0, \infty\}$, that is $\sigma$ is of type $III_0 \times III_0$*

**Proof.** It is easily seen from the compatibility of the $G$'s that the $W_n$ satisfy condition (C3).

One calculates from the definition that

$$\frac{G_m(x)}{G_n(x)}W_n^m(\gamma_0, x) \quad = \quad \frac{1}{|\Gamma_m|}\sum_{\gamma\in\Gamma_m} G_m(\gamma x)\left\{\frac{G_n(\gamma_0\gamma x)}{G_n(\gamma x)}\beta(\gamma_0\gamma x) - \beta(\gamma x).\right\}$$

By unique ergodicity of $\mu$ we have $\frac{1}{|\Gamma_m|}\sum_{\gamma\in\Gamma_m} G_m(\gamma x)\beta(\gamma x) \to \eta$ uniformly in $x$ as $m \to \infty$. Now, for any $r \in R(\mu)$, any $\epsilon > 0$ and any $A$ of $X$ of positive $\mu$ measure, if $m_0$ is sufficienly large (so that $|\frac{1}{|\Gamma_m|}\sum_{\gamma\in\Gamma_m} G_m(\gamma x)\beta(\gamma x) - \eta| < \epsilon$ for any $m \geq m_0$ and any $x$ ), then for any $m \geq m_0$, there exist $n > m$, a $\gamma_0 \in \Gamma_m^n$ and a subset $B$ of $A$ of positive measure so that $\gamma_0 B \subset B$, $|\frac{G_m(\gamma_0 x)}{G_m(x)} - 1| < \epsilon$, and $|\frac{G_n(\gamma_0 x)}{G_n(x)} - r| < \epsilon$. From this it follows that

$$|\frac{G_m(x)}{G_n(x)}W_n^m(\gamma_0, x) - (r-1)\eta|$$

is dominated by a multiple of $\epsilon$, thus $(r, (r-1)\eta) \in r_\mu(\sigma)$.

□

Theorem 4.2 does not allow us to construct cocycles whose ratio sets have $\mathrm{III}_1$ in the second factor. The next theorem will allow this. Before giving the theorem, let us construct our cocycles. For the rest of this paper we assume that $\{G_n\}$ and $\mu$ satisfy the hypothesis of theorem 4.2.

**Lemma 4.1** *Suppose $u_n$ is a function on $X$ which depends only on the $(n+1)$st coordinate. Set $u_0 = 0$, and let*

$$\beta_n(x) = u_n(x) - \frac{u_{n+1}(x)}{g_{n+1}(x)} \text{ for } n = 0, 1, 2, 3, \cdots$$

*Define a compatible family of cocycles by*

$$W_k(\gamma, x) = \sum_{n=0}^{\infty}\left(\frac{G_k(\gamma x)}{G_n(\gamma x)}\beta_n(\gamma x) - \frac{G_k(x)}{G_n(x)}\beta_n(x)\right)$$

*for $\gamma \in \Gamma_k$.*

*Then*

$$\frac{G_m(x)}{G_k(x)} W_k^m(\gamma_0, x) = \begin{cases} \frac{G_m(x)}{G_k(x)} \{u_k(x) - u_k(\gamma_0 x)\} & \text{if } m \le k \\ 0 & \text{if } m > k \end{cases}.$$

**Proof.** This follows by an obvious telescoping sum argument. □

The following Theorem is based on example 3.3 of [PS] which corresponds to the case where $\mu$ is invariant.

**Theorem 4.3** *Let $G$ be a normalized compatible family, $\mu$ a uniquely ergodic $G$-measure of type $T = \{I, II, III_\lambda\}$. Let $\{s_k\}$ be a sequence of rational numbers in which each rational occurs infinitely often.*

*Let*

$$u_n(x) = \begin{cases} s_n & \text{if } x_n = 0 \\ 0 & \text{otherwise,} \end{cases}$$

*and define $W_k$ as in Lemma 4.2. Then the resulting cocycle is of type $T\times III_1$.*

**Proof.** Let $r \in R_\mu$, let $A$ be a set of positive measure and let $\epsilon > 0$. Choose $B \subseteq A$, $k > m$ and $\gamma_0 \in \Gamma_k^m$ so that $\gamma_0 B \subseteq A$,

$$\left| \frac{d\mu \circ \gamma_0}{d\mu}(x) - r \right| < \epsilon, \quad \left| \frac{d\mu^k \circ \gamma_0}{d\mu^k}(x) - r \right| < \epsilon$$

for all $x \in B$.

This is possible by comment following Remarks 3.1.

Choose $k_1 \ge k$ so large that there exists $\gamma \in \Gamma_{k_1}$ with $\mu(B \cap \gamma X^{k_1}) > (1-\epsilon)\mu(\gamma X^{k_1})$. (This is possible by Theorem 3.2 of [BDL]). The compatibility condition (C2) shows that

$$\frac{G_m(x)}{G_k(x)} W_k^m(\gamma_0, x) = \frac{G_m(x)}{G_{k_1}(x)} W_{k_1}^m(\gamma_0, x)$$

whenever $k_1 > k$.

Furthermore, by Lemma 4.1, the difference between the right hand side and

$$\frac{G_m(x)}{G_{k_1}(x)} u_{k_1}(x) - r \frac{G_m(\gamma_0 x)}{G_{k_1}(\gamma_0 x)} u_{k_1}(\gamma_0 x)$$

is dominated by a multiple of $\epsilon$. This expression equals

$$\begin{cases} s_{k_1} p^m_{k_1}(x) & \text{if } x_{k_1} = 0 \neq (\gamma_0)_{k_1} x_{k_1} \\ s_{k_1}(p^m_{k_1}(x) - r p^m_{k_1}(\gamma_0 x)) & \text{if } x_{k_1} = (\gamma_0)_{k_1} x_{k_1} = 0 \\ -r s_{k_1} p^m_{k_1}(\gamma_0 x) & \text{if } x_{k_1} \neq 0 = (\gamma_0)_{k_1} x_{k_1} \end{cases}$$

where $p^m_{k_1}(x) = \dfrac{G_m(x)}{G_{k_1}(x)}$.

Now, since $p^m_{k_1}(x)$ is a continuous function, we may choose a set $S^m_{k_1} \subseteq X^{k_1+1}$, of positive measure, and a number $q^m_{k_1}$ so that $|p^m_{k_1}(x) - q^m_{k_1}| < \epsilon$ for all $x \in S^m_{k_1}$. By the normalization condition, we may assume that $q^m_{k_1} \neq 0$. Since the sequence $s_{k_1} q^m_{k_1}$ may be chosen to approximate an arbitrary real number, we are done. $\square$

**Remarks.** It is an interesting issue to what extent one may generalise other familiar constructions of ergodic theory from $\mathbb{R}$-valued to $\mathcal{A}$-valued cocycles. Can one, for example, find a concrete realisation of some of the flows of Forrest [F] in this setting? We shall address these issues in future publications.

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