A restriction algebra related to Fourier algebras

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1 Introduction

In this paper we consider the Fourier algebra $A(\mathbb{R}^2) = \mathcal{F}L^1(\mathbb{R}^2)$, Fourier transformation $\mathcal{F} : \varphi \mapsto \widehat{\varphi}$ for $\varphi \in L^1(\mathbb{R}^2)$ being defined by

$$\widehat{\varphi}(y) = \int_{\mathbb{R}^2} e^{ix\cdot y}\varphi(x)\,dx\,, \tag{1}$$

where, in the usual way, for $x = (x_1, x_2)$, $y = (y_1, y_2)$ we write $x \cdot y = x_1y_1 + x_2y_2$ and $|x| = \sqrt{x \cdot x}$. The injectivity of $\mathcal{F}$ implies that by $\|\widehat{\varphi}\|_{A(\mathbb{R}^2)} = \|\varphi\|_1$ a norm is defined on $A(\mathbb{R}^2)$, and in this way $A(\mathbb{R}^2)$, equipped with pointwise multiplication and complex conjugation, is made into a Banach function $*$-algebra.
We further consider the circle group $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$. Functions on $\mathbb{T}$ will be freely identified with 2π-periodic functions on $\mathbb{R}$. As a topological space $\mathbb{T}$ will be identified with the circle $\{y \mid |y| = 1\} \subset \mathbb{R}^2$ by means of the mapping

$$t \mapsto (\cos t, \sin t) \qquad (t \in \mathbb{R}). \tag{2}$$

By restriction of $A(\mathbb{R}^2)$ to the subset $\mathbb{T}$ we obtain the restriction algebra

$$\widetilde{A}(\mathbb{T}) = \left\{ f \in C(\mathbb{T}) \,\middle|\, f = \widehat{\varphi}|_{\mathbb{T}} \text{ for some } \varphi \in L^1(\mathbb{R}^2) \right\} . \tag{3}$$

Of course, $\widetilde{A}(\mathbb{T})$ is just the quotient algebra $A(\mathbb{R}^2)/I_{\mathbb{T}}$, where $I_{\mathbb{T}}$ is the closed ideal of all $\psi \in A(\mathbb{R}^2)$ vanishing on $\mathbb{T}$. The restriction –or quotient– norm is

$$\|f\|_{\widetilde{A}(\mathbb{T})} = \inf \left\{ \|\varphi\|_1 \,\middle|\, \varphi \in L^1(\mathbb{R}^2)\,, \widehat{\varphi}|_{\mathbb{T}} = f \right\} , \tag{4}$$

and with this norm $\widetilde{A}(\mathbb{T})$ is a Banach function algebra. Observe that

$$\|f\|_{\widetilde{A}(\mathbb{T})} \geq \|f\|_\infty \qquad (f \in \widetilde{A}(\mathbb{T}))\,, \tag{5}$$

because $|\widehat{\varphi}(y)| \leq \|\varphi\|_1$ for all $\varphi \in L^1(\mathbb{R}^2)$ and $y \in \mathbb{R}^2$.

Neither the algebra $A(\mathbb{R}^2) = \mathcal{F}L^1(\mathbb{R}^2)$ nor the algebra $A(\mathbb{T}) = \mathcal{F}(\ell^1(\mathbb{Z}))$ admits of a simple description in terms of smoothness conditions, and so it is natural to try to compare the algebras $A(\mathbb{T})$ and $\widetilde{A}(\mathbb{T})$. It is precisely this comparison, and comparison with related function algebras on the circle, which is the object of this paper.

Recall that $A(\mathbb{T})$ consists of all continuous functions f on $\mathbb{T}$ of the form

$$f = \sum_{n=-\infty}^{\infty} c_n \chi_n\,, \tag{6}$$

where $\chi_n(t) = e^{int}$ $(n \in \mathbb{Z},\, t \in \mathbb{R})$, and the complex numbers c_n, the Fourier coefficients of f, satisfy $\sum_{n=-\infty}^{\infty} |c_n| < \infty$. $A(\mathbb{T})$ is equipped with the norm

$$\|f\|_{A(\mathbb{T})} = \sum_{n=-\infty}^{\infty} |c_n|\,, \tag{7}$$

and for f as in (6) one has

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt \qquad (n \in \mathbb{Z}). \tag{8}$$

More generally we will consider the Beurling algebras $A_\alpha(\mathbb{T})$ $(\alpha \geq 0)$ of all $f \in A(\mathbb{T})$ as in (6) for which $\sum_{n=-\infty}^{\infty} |c_n|\,|n|^\alpha < \infty$, with the norm

$$\|f\|_{A_\alpha(\mathbb{T})} = \sum_{n=-\infty}^{\infty} |c_n|(1+|n|)^\alpha\,. \tag{9}$$

The $A_\alpha(\mathbb{T})$ $(\alpha \geq 0)$ form a shrinking family of Banach algebras with $A_0(\mathbb{T}) = A(\mathbb{T})$.
We can now announce our main results:

Theorem 1 *$\widetilde{A}(\mathbb{T})$ is not contained in $A(\mathbb{T})$.*

Theorem 2 *$A_\alpha(\mathbb{T})$ is contained in $\widetilde{A}(\mathbb{T})$ if and only if $\alpha \geq \frac{1}{3}$.*

Generalizations of these theorems for the case that $L^1(\mathbb{R}^2)$ is replaced by certain Beurling algebras with rotation-invariant weight will be obtained in the last section. We further obtain, as a consequence of Theorem 1, a seemingly new result on Bessel functions (Corollary 1). We would like to thank A. Olde Daalhuis and M. Gatesoupe for useful conversations on the topic of Theorem 1, and O. Berndt for carefully checking the manuscript.

Although we have not emphasized this, the methods used are in part derived from the harmonic analysis of the non-abelian group of plane Euclidean motions, the semi-direct product of $\mathbb{R}^2$ and the circle group, and they are susceptible of generalization to higher dimensional spaces.

2 Basic elements of the restriction algebra

For $\varphi \in L^1(\mathbb{R}^2)$ let us denote the function $\widehat{\varphi}|_{\mathbb{T}}$ by $\widetilde{\varphi}$. The mapping $\varphi \mapsto \widetilde{\varphi}$ is a norm-decreasing algebra homomorphism of $L^1(\mathbb{R}^2)$ onto $\widetilde{A}(\mathbb{T})$.
For the study of $\widetilde{A}(\mathbb{T})$ it will be useful to consider the Fourier–Stieltjes algebra $B(\mathbb{R}^2) = \mathcal{F}M(\mathbb{R}^2)$ as well. This algebra consists of all (bounded, continuous) functions ψ on $\mathbb{R}^2$ of the form

$$\psi(y) = \widehat{\mu}(y) = \int_{\mathbb{R}^2} e^{ix\cdot y}d\mu(x) \tag{10}$$

with μ belonging to $M(\mathbb{R}^2)$, the Banach algebra of all bounded Radon measures on $\mathbb{R}^2$. The norm on $B(\mathbb{R}^2)$ is given by $\|\widehat{\mu}\|_{B(\mathbb{R}^2)} = \|\mu\|$. The algebra $A(\mathbb{R}^2)$ is, of course, a closed, even isometric, subalgebra of $B(\mathbb{R}^2)$. It is well known (and can be seen by regularizing with functions $\tau \in L^1(\mathbb{R}^2)$ whose Fourier transforms equal 1 on K and whose L^1-norms tend to 1) that for a compact set $K \subset \mathbb{R}^2$ the restriction algebras $\widetilde{A}(K)$ and $\widetilde{B}(K)$ (defined analogous to (3) and with norm analogous to (4)), are the same, even with the same norm. Adopting, as before, the notation $\widehat{\mu}|_{\mathbb{T}} = \widetilde{\mu}$, we therefore have for $f \in \widetilde{A}(\mathbb{T})$:

$$\|f\|_{\widetilde{A}(\mathbb{T})} = \inf\left\{\|\mu\| \,\Big|\, \mu \in M(\mathbb{R}^2)\,,\ \widetilde{\mu} = f\right\}. \tag{11}$$

Now let x be an arbitrary element of $\mathbb{R}^2$. Take $r \geq 0$ and $\theta \in \mathbb{R}$ such that

$$x = (r\cos\theta, r\sin\theta)\,. \tag{12}$$

If $y = (\cos t, \sin t) \in \mathbb{T} \subset \mathbb{R}^2$, then

$$x \cdot y = r\cos(t-\theta)\,. \tag{13}$$

Let δ_x be the Dirac measure at the point x. Then $\widehat{\delta}_x(y) = e^{ix\cdot y}$, and hence, by (13), we have for the function $\widetilde{\delta}_x \in \widetilde{A}(\mathbb{T})$:

$$\widetilde{\delta}_x(t) = e^{ir\cos(t-\theta)}\,. \tag{14}$$

Because $\|\delta_x\| = 1$, it follows from (5) that

$$\|\widetilde{\delta}_x\|_{\widetilde{A}(\mathbb{T})} = 1\,. \tag{15}$$

Before proving our first theorem, let us fix some notation for asymptotic equality. For positive functions f and g we shall write:

- $f(x) \sim g(x)\ \ (x \to \infty)$ if $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$;
- $f(x) \approx g(x)\ \ (x \to \infty)$ if $c \leq \dfrac{f(x)}{g(x)} \leq C$ for some positive constants c and C, and x sufficiently large.

Proof of Theorem 1

It is known (cf. [2, page 77]) that for every C^2-function $F : \mathbb{T} \to \mathbb{R}$ one has $\|e^{irF}\|_{A(\mathbb{T})} \approx \sqrt{|r|}$ $(|r| \to \infty)$. Applying this to $F(t) = \cos(t-\theta)$ we obtain

$$\|\widetilde{\delta}_x\|_{A(\mathbb{T})} \approx \sqrt{|r|} \qquad (|r| \to \infty)\,. \tag{16}$$

As follows from a familiar theorem on Banach function algebras (cf. e.g. [3, Chapter 2, Corollary 3.6]), the formulas (15) and (16) together imply that $\widetilde{A}(\mathbb{T}) \not\subset A(\mathbb{T})$. $\square$

The Fourier series of the functions $\widetilde{\delta}_x$ can be expressed in terms of Bessel functions. Indeed, from the formula

$$J_n(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(r\sin t - nt)}\,dt \qquad (r \in \mathbb{R}, n \in \mathbb{Z}) \tag{17}$$

(cf. e.g. [5, p. 20]) we see (cf. (8)) that for a fixed $r \in \mathbb{R}$ the numbers $J_n(r)$ ($n \in \mathbb{Z}$) are the Fourier coefficients of the function $t \mapsto e^{ir\sin t}$. Thus

$$e^{ir\sin t} = \sum_{n=-\infty}^{\infty} J_n(r)e^{int} . \tag{18}$$

Substituting $\cos(t-\theta) = \sin(t-\theta+\frac{\pi}{2})$ one obtains from (18), using (14):

$$\widetilde{\delta}_x(t) = \sum_{n=-\infty}^{\infty} J_n(r)e^{in(t-\theta+\frac{\pi}{2})} = \sum_{n=-\infty}^{\infty} i^n e^{-in\theta} J_n(r)e^{int} ,$$

thus

$$\widetilde{\delta}_x = \sum_{n=-\infty}^{\infty} i^n e^{-in\theta} J_n(r)\,\chi_n \tag{19}$$

and (cf. (7)):

$$\|\widetilde{\delta}_x\|_{A(\mathbb{T})} = \sum_{n=-\infty}^{\infty} |J_n(r)| . \tag{20}$$

Combining (16) and (20) we obtain:

Corollary 1 $\displaystyle\sum_{n=-\infty}^{\infty} |J_n(r)| \approx \sqrt{|r|} \qquad (|r| \to \infty)$.
In particular, the sum on the left is unbounded, as a function of r.

The functions $\widetilde{\delta}_x$ ($x \in \mathbb{R}^2$) form a collection of "basic" functions for the algebra $\widetilde{A}(\mathbb{T})$. It is convenient to reduce the collection by considering only elements x for which $\theta = 0$, thus of the form $x = (r, 0)$ with $r \geq 0$; the others are mere translates of these functions. Let us denote $\widetilde{\delta}_{(r,0)}$ by d_r. We then have (cf. (19)):

$$d_r(t) = e^{ir\cos t} = \sum_{n=-\infty}^{\infty} i^n J_n(r)\chi_n(t) \text{ and } \|d_r\|_{\widetilde{A}(\mathbb{T})} = 1 . \tag{21}$$

3 Rotational convolution

So far we have considered algebras of continuous functions on $\mathbb{T}$. We now turn to the Banach algebra $L^1(\mathbb{T})$ with the norm

$$\|F\|_1 = \frac{1}{2\pi}\int_0^{2\pi} |F(\theta)|\, d\theta . \tag{22}$$

The Fourier coefficients of $F \in L^1(\mathbb{T})$ are given by

$$\widehat{F}(n) = \frac{1}{2\pi}\int_0^{2\pi} F(\theta)e^{-in\theta} d\theta , \tag{23}$$

and

$$\sum_{n=-\infty}^{\infty} \widehat{F}(n)e^{in\theta} . \tag{24}$$

is the Fourier series of F. In what follows $L^1(\mathbb{T})$ will play a dual rôle. On the one hand, by embedding $\mathbb{T}$ into $\mathbb{R}^2$ as a circle of radius $r > 0$, the elements of $L^1(\mathbb{T})$ become elements of $M(\mathbb{R}^2)$, and hence provide elements of $\widetilde{A}(\mathbb{T})$. On the other hand, the elements of $L^1(\mathbb{T})$ will act as rotational convolution operators on $L^1(\mathbb{R}^2)$.
First of all, we emphasize that when in our paper functions φ defined on $\mathbb{R}^2$ are considered, these may be pointwise defined functions, but more often they are equivalence classes of almost everywhere equal functions, even when the notation suggests pointwise definition; this is common practice. Statements about equality etc. of functions should always be interpreted accordingly.

For $F \in L^1(\mathbb{T})$ and $r > 0$ we define an element $\mu_{F,r} \in M(\mathbb{R}^2)$ by

$$\int_{\mathbb{R}^2} \varphi(x) d\mu_{F,r}(x) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(r\cos\theta, r\sin\theta) F(\theta) d\theta \quad (\varphi \in C(\mathbb{R}^2)). \tag{25}$$

We clearly have

$$\|\mu_{F,r}\| = \|F\|_1 . \tag{26}$$

We will denote the corresponding function $(\mu_{F,r})^{\sim}$ by $\widetilde{F}_r$. Thus

$$\widetilde{F}_r = \widehat{\mu}_{F,r}|_{\mathbb{T}} \quad (F \in L^1(\mathbb{T}),\, r > 0). \tag{27}$$

We have then

Proposition 1 *For each $r > 0$ the mapping $F \mapsto \widetilde{F}_r$ (cf. (27)) is injective from $L^1(\mathbb{T})$ into $\widetilde{A}(\mathbb{T})$ with the following properties:*

(i) $\widetilde{F}_r = d_r * F$;

(ii) $\widetilde{F}_r = \sum_{n=-\infty}^{\infty} i^n J_n(r) \widehat{F}(n) \chi_n$;

(iii) $\|\widetilde{F}_r\|_{\widetilde{A}(\mathbb{T})} \leq \|F\|_1$.

Proof

Let $F \in L^1(\mathbb{T})$. Then

$$\widetilde{F}_r(t) = \widehat{\mu}_{F,r}(\cos t, \sin t) = \int_{\mathbb{R}^2} e^{ix\cdot y} d\mu_{F,r}(x)$$

with $y = (\cos t, \sin t)$; cf. (10). Thus

$$\widetilde{F}_r(t) = \frac{1}{2\pi} \int_0^{2\pi} e^{ir\cos(t-\theta)} F(\theta) d\theta = d_r * F(t)$$

(cf. (25), (13), (21)). This proves (i); (ii) then follows from (21) and (iii) from (11) and (26). $\square$

Let us apply Proposition 1 to the functions $F = \chi_n$. From (ii) we get that

$$\widetilde{\chi}_{n,r} = i^n J_n(r) \chi_n \quad (r > 0,\ n \in \mathbb{Z}), \tag{28}$$

and then (iii) implies that $|J_n(r)|\|\chi_n\|_{\widetilde{A}(\mathbb{T})} \leq 1$. By choosing for r the value where $|J_n(r)|$ is maximal (respectively, for the case $n = 0$, by letting r tend to 0), we get

$$\|\chi_n\|_{\widetilde{A}(\mathbb{T})} \leq \|J_n\|_\infty^{-1} . \tag{29}$$

In Proposition 3 it will be shown that we actually have equality.

We now turn to the other rôle of $L^1(\mathbb{T})$. Rotation of the points of $\mathbb{R}^2$ over an angle $\theta \in \mathbb{R}$, or, equivalently, rotation by an element $\theta \in \mathbb{T}$, will be denoted by R_θ. Thus

$$R_\theta(r \cos \eta, r \sin \eta) = (r \cos(\eta + \theta), r \sin(\eta + \theta)) . \tag{30}$$

If φ is a function defined on $\mathbb{R}^2$, then the rotated function $R_\theta \varphi$ $(\theta \in \mathbb{R})$ is defined by :

$$R_\theta \varphi(x) = \varphi(R_{-\theta} x) \qquad (x \in \mathbb{R}^2) . \tag{31}$$

It is easily seen (and well known) that rotation commutes with Fourier transformation:

$$\widehat{R_\theta \varphi} = R_\theta \widehat{\varphi} \quad (\theta \in \mathbb{R},\ \varphi \in L^1(\mathbb{R}^2)). \tag{32}$$

Through approximation by continuous functions with compact supports one further sees that rotation acts continuously on $L^1(\mathbb{R}^2)$, i.e., for each $\varphi \in L^1(\mathbb{R}^2)$ the mapping $\theta \mapsto R_\theta \varphi$ is continuous. Moreover R_θ is an isometry on $L^1(\mathbb{R}^2)$, by the rotation invariance of Lebesgue measure. From (32) it follows that the same two properties then hold for $A(\mathbb{R}^2)$ as well. These observations imply that the following definition can be applied to the cases $\mathcal{B}(\mathbb{R}^2) = L^1(\mathbb{R}^2)$ and $\mathcal{B}(\mathbb{R}^2) = A(\mathbb{R}^2)$.

Definition 1 *Let $\mathcal{B}(\mathbb{R}^2)$ be a Banach space of functions defined on $\mathbb{R}^2$ which is invariant under rotations and on which rotation acts continuously and isometrically (boundedly would actually suffice). Then for $\varphi \in \mathcal{B}(\mathbb{R}^2)$ the* rotational convolution *of φ by F, denoted by $F *' \varphi$, is defined by vector valued integration:*

$$F *' \varphi = \frac{1}{2\pi} \int_0^{2\pi} F(\theta) R_\theta \varphi d\theta . \tag{33}$$

We collect some properties of rotational convolution in the following lemma.

Lemma 1 *For $\varphi \in L^1(\mathbb{R}^2)$ and $F \in L^1(\mathbb{T})$ we have:*

(i) *$F *' \varphi \in L^1(\mathbb{R}^2)$ and $\|F *' \varphi\|_1 \leq \|F\|_1 \|\varphi\|_1$.*

(ii) *$(F *' \varphi)^\frown = F *' \widehat{\varphi}$.*

(iii) *$(F *' \varphi)^\sim = F * \widetilde{\varphi}$.*

Proof

For φ and F as in the lemma we have

$$\|F *' \varphi\|_1 \leq \frac{1}{2\pi} \int_0^{2\pi} |F(\theta)| \, \|R_\theta \varphi\|_1 d\theta = \|F\|_1 \, \|\varphi\|_1 \, ,$$

hence (i). Similarly, (ii) follows from

$$(F *' \varphi)\widehat{\ } = \frac{1}{2\pi} \int_0^{2\pi} F(\theta) \widehat{R_\theta \varphi} d\theta = \frac{1}{2\pi} \int_0^{2\pi} F(\theta) R_\theta \widehat{\varphi} d\theta = F *' \widehat{\varphi} \, .$$

Finally, (iii) follows from (ii), because, as is easily seen,

$$(F *' \widehat{\varphi}) \, |_{\mathbb{T}} = F * \widetilde{\varphi} \, .$$

□

Proposition 2 *If $F \in L^1(\mathbb{T})$ and $f \in \widetilde{A}(\mathbb{T})$, then $F * f \in \widetilde{A}(\mathbb{T})$ and*

$$\|F * f\|_{\widetilde{A}(\mathbb{T})} \leq \|F\|_1 \|f\|_{\widetilde{A}(\mathbb{T})} \tag{34}$$

Proof

Take $\varphi \in L^1(\mathbb{R}^2)$ such that $\widetilde{\varphi} = f$. Then $F * f = (F *' \varphi)\widetilde{\ }$, by (iii) of Lemma 1, and we have $\|F * f\|_{\widetilde{A}(\mathbb{T})} \leq \|F *' \varphi\|_1 \leq \|F\|_1 \|\varphi\|_1$, by (i) of the same lemma. Formula (34) follows by considering all possible φ (cf. (4)). □

A function φ defined on $\mathbb{R}^2$ is commonly called a *radial* function if $R_\theta \varphi = \varphi$ for all $\theta \in \mathbb{R}$. Clearly, φ is radial if and only if it is of the form $\varphi(r \cos\theta, r \sin\theta) = \Phi(r)$ for some function Φ defined on $\mathbb{R}_+ = [0, \infty)$. We now define, more generally:

Definition 2 *For $n \in \mathbb{Z}$ a function φ defined on $\mathbb{R}^2$ will be called n–*radial *if*

$$R_\theta \varphi = \overline{\chi_n(\theta)} \varphi$$

for all $\theta \in \mathbb{R}$.

The 0–radial functions are just the radial functions.

In what follows we will consider the characters χ_n, defined on $\mathbb{T}$, also as functions defined on $\mathbb{R}^2 \setminus \{(0,0)\}$, as follows:

$$\chi_n(r \cos\theta, r \sin\theta) = \chi_n(\theta) \qquad (r > 0) \, . \tag{35}$$

We have then:

Lemma 2 *Let φ be a function defined on $\mathbb{R}^2$. Then the following statements are equivalent:*

(i) *φ is n–radial.*

(ii) *$\overline{\chi_n} \varphi$ is radial.*

(iii) *$\varphi(r \cos\theta, r \sin\theta) = \Phi(r) \chi_n(\theta)$ for some Φ defined on $\mathbb{R}_+$.*

Proof

If φ is n–radial, then

$$R_\theta(\overline{\chi_n}\varphi) = R_\theta\overline{\chi_n} \cdot R_\theta\varphi = \chi_n(\theta)\overline{\chi_n}\overline{\chi_n(\theta)}\varphi = \overline{\chi_n}\varphi\,,$$

thus *(ii)* holds. Next, if $\overline{\chi_n}\varphi$ is radial, then $(\overline{\chi_n}\varphi)(r\cos\theta, r\sin\theta) = \Phi(r)$ for some Φ, thus

$$\overline{\chi_n(\theta)}\varphi(r\cos\theta, r\sin\theta) = \Phi(r)\,,$$

whence *(iii)*. Finally, if *(iii)* holds then

$$R_\theta\varphi(r\cos\theta, r\sin\theta) = \Phi(r)\chi_n(\theta-\theta) = \overline{\chi_n(\theta)}\Phi(r)\chi_n(\theta)\,,$$

thus φ is n–radial. □

We can now obtain the following crucial improvement of the inequality (29).

Proposition 3 $\quad \|\chi_n\|_{\widetilde{A}(\mathbb{T})} = \|J_n\|_\infty^{-1}\,.$

Proof

Let $\varphi \in L^1(\mathbb{R}^2)$ be any function satisfying $\widetilde{\varphi} = \chi_n$. Then $\chi_n *' \varphi$ is another such function, because, by (iii) of Lemma 1:

$$(\chi_n *' \varphi)^{\sim} = \chi_n * \widetilde{\varphi} = \chi_n * \chi_n = \chi_n.$$

Now $\chi_n *' \varphi$ is an n-radial function (e.g. because $R_\theta(\chi_n *' \varphi) = (R_\theta\chi_n) *' \varphi = \overline{\chi_n(\theta)} \cdot \chi_n *' \varphi$). And because $\|\chi_n *' \varphi\|_1 \le \|\varphi\|_1$ (cf. (i) of Lemma 1), it follows that among the functions $\varphi \in L^1(\mathbb{R}^2)$ satisfying $\widetilde{\varphi} = \chi_n$ we can restrict our attention to n-radial ones. In other words (cf. Lemma 2), we can take φ of the form

$$\varphi(r\cos\theta, r\sin\theta) = \Phi(r)\chi_n(\theta) \tag{36}$$

and have then:

$$\|\chi_n\|_{\widetilde{A}(\mathbb{T})} = \inf\{\|\varphi\|_1 \mid \varphi \text{ as in (36)}, \widetilde{\varphi} = \chi_n\}\,. \tag{37}$$

For φ as in (36) we have

$$\|\varphi\|_1 = \int_0^\infty 2\pi r|\Phi(r)|dr \tag{38}$$

and (cf. (28)):

$$\widetilde{\varphi} = \int_0^\infty \Phi(r)\widetilde{\chi}_{n,r}2\pi r dr = \int_0^\infty 2\pi r\Phi(r)i^n J_n(r)dr \cdot \chi_n\,.$$

It follows that $\widetilde{\varphi} = \chi_n$ will hold if and only if Φ satisfies

$$\int_0^\infty 2\pi r\Phi(r)i^n J_n(r)dr = 1\,. \tag{39}$$

The proof is now concluded by a duality argument. Consider $L^1(\mathbb{R}_+)$ and its dual space $L^\infty(\mathbb{R}_+)$ with respect to the ordinary Lebesgue measure on $\mathbb{R}_+$. For shortness we shall denote these spaces by L^1 and L^∞, respectively. Duality is defined by

$$\langle u, v\rangle = \int_0^\infty u(r)v(r)dr \qquad (u \in L^1, v \in L^\infty)\,.$$

For $v \in L^\infty$ we have

$$\begin{aligned} \|v\|_\infty &= \sup\left\{ \frac{|\langle u, v\rangle|}{\|u\|_1} \;\middle|\; u \in L^1, u \neq 0\right\} \\ &= \sup\left\{ \frac{1}{\|u\|_1} \;\middle|\; u \in L^1, \langle u, v\rangle = 1\right\} \\ &= \left(\inf\left\{ \|u\|_1 |\; u \in L^1, \langle u, v\rangle = 1\right\}\right)^{-1}\,. \end{aligned}$$

Writing u in the form $u(r) = 2\pi r\Phi(r)$, and taking $v = i^n J_n$ we obtain:

$$\|J_n\|_\infty^{-1} = \inf\left\{ \int_0^\infty 2\pi r|\Phi(r)|dr \;\middle|\; \int_0^\infty 2\pi r\Phi(r).i^n J_n(r)dr = 1\right\}\,.$$

By (38), (39) and (37) this gives

$$\|J_n\|_\infty^{-1} = \inf\{\|\varphi\|_1 \mid \varphi \text{ as in (36), } \tilde{\varphi} = \chi_n\} = \|\chi_n\|_{\widetilde{A}(\mathbb{T})}\,,$$

as desired. □

It is well known that $\|J_0\|_\infty = 1$, that $\|J_n\|_\infty = \|J_{-n}\|_\infty$ and that for $n \geq 1$ $\|J_n\|_\infty$ is equal to the first maximum of J_n on the positive half-axis. Let us denote the point where this maximum is taken by r_n (the usual notation is $j'_{n,1}$, the first zero of the derivative of J_n). It is known that r_n is slightly larger than n (see (41) below for a more precise statement). In [5, section 8.2, formula (1)] the following asymptotic formula is given:

$$J_n(n) \sim \frac{\Gamma(\frac{1}{3})}{2^{2/3}3^{1/6}\pi}\, n^{-1/3} = 0.4473\ldots n^{-1/3} \qquad (n \to \infty)\,. \tag{40}$$

We remark in passing that formula (17) suggests why there is a special interest for the value of J_n at the point n.

Asymptotic expressions for $J_n(r_n)$ seem to be less readily available in the literature. In [1, formula 9.5.25] an asymptotic expression for $J_n(r_n)$ is given using Airy functions. We see there that the same asymptotic order of magnitude as for $J_n(n)$ holds. In [4, page xviii] the first few terms of an asymptotic expansion for r_n as well as for $J_n(r_n)$ are given. We infer from these:

$$r_n - n \sim 0.8086165\ldots n^{1/3} \quad (n \to \infty)\,, \tag{41}$$

$$J_n(r_n) = \|J_n\|_\infty \sim 0.6748851\ldots n^{-1/3} \quad (n \to \infty)\,. \tag{42}$$

Proof of Theorem 2

For arbitrary $\alpha \geq 0$ we have

$$\|\chi_n\|_{A_\alpha(\mathbb{T})} = (1+|n|)^\alpha .$$

Now Proposition 3 and formula (42) entail that

$$\|\chi_n\|_{\widetilde{A}(\mathbb{T})} \approx |n|^{1/3} \qquad (|n| \to \infty) .$$

From this one derives, precisely as in the proof of Theorem 1, that for $\alpha < \frac{1}{3}$ one has

$$A_\alpha(\mathbb{T}) \not\subset \widetilde{A}(\mathbb{T}) .$$

Conversely, if a function f as in (6) belongs to $A_{1/3}(\mathbb{T})$, then

$$\sum_{n=-\infty}^{\infty} |c_n| \|\chi_n\|_{\widetilde{A}(\mathbb{T})} \leq |c_0| + C \sum_{n=-\infty}^{\infty} |c_n||n|^{1/3} < \infty \quad \text{for some } C > 0 ,$$

and this implies that $f \in \widetilde{A}(\mathbb{T})$, because $\widetilde{A}(\mathbb{T})$ is complete. □

4 Some additional results

The methods of proof above enable us to say a little more about functions belonging to $\widetilde{A}(\mathbb{T})$. Consider a function $F \in L^1(\mathbb{T})$. The following proposition gives a sufficient condition on the Fourier coefficients of F to have $F \in \widetilde{A}(\mathbb{T})$.

Proposition 4 *Let a function $F \in L^1(\mathbb{T})$ be given. Consider its Fourier series, as in (24). Suppose there are functions $F_m \in L^1(\mathbb{T})$ and positive real numbers r_m $(m \geq 1)$, such that*

(i) $\displaystyle\sum_{m=1}^{\infty} \|F_m\|_1 = M < \infty,$

(ii) $\displaystyle i^n \sum_{m=1}^{\infty} J_n(r_m)\widehat{F}_m(n) = \widehat{F}(n)$ *for all* $n \in \mathbb{Z}$.

Then $F \in \widetilde{A}(\mathbb{T})$ (in particular, F is continuous), and $\|F\|_{\widetilde{A}(\mathbb{T})} \leq M$.

Proof

Consider the measure $\mu = \displaystyle\sum_{m=1}^{\infty} \mu_{F_m,r_m}$ (cf. (25) for the notation). From (26) it follows that $\|\mu\| \leq \displaystyle\sum_{m=1}^{\infty} \|F_m\|_1$. Notice that we actually have equality here if all r_m are different. From condition (i) above we see that $\mu \in M(\mathbb{R}^2)$. Now we derive:

$$\widetilde{F}_{m,r_m} = \sum_{n=-\infty}^{\infty} i^n J_n(r_m)\widehat{F}_m(n)\chi_n$$

(cf. (ii) of Proposition 1), and then it follows from condition (ii) above that

$$\widetilde{\mu} = \sum_{m=1}^{\infty} \widetilde{F}_{m,r_m} = \sum_{n=-\infty}^{\infty} \widehat{F}(n)\chi_n \,,$$

showing that indeed $F \in \widetilde{A}(\mathbb{T})$. □

At first sight this criterion may not seem very useful. But take for instance for F a trigonometric polynomial, say

$$F = \sum_{n=-N}^{N} c_n \chi_n \,.$$

Then certainly $F \in \widetilde{A}(\mathbb{T})$ and (cf. Proposition 3)

$$\|F\|_{\widetilde{A}(\mathbb{T})} \leq \sum_{n=-N}^{N} |c_n| \|\chi_n\|_{\widetilde{A}(\mathbb{T})} = \sum_{n=-N}^{N} \frac{|c_n|}{\|J_n\|_\infty} \,.$$

This estimate for $\|F\|_{\widetilde{A}(\mathbb{T})}$ is obtained by considering each term $c_n\chi_n$ individually. However, a better estimate of the $\widetilde{A}(\mathbb{T})$-norm of F may well be obtained when, instead of representing each $c_n\chi_n$ on its proper circle with radius $r_{|n|}$ ($r_0 = 0$ with obvious interpretation), some terms are taken together and the radius is chosen that is optimal for that specific group of terms. For each individual character this will cause a loss, the character not being represented on the circle with optimal radius. But this loss will be more than compensated for by the fact that several terms are taken together. This is the principle described in Proposition 4. A closer analysis of this idea of finding suitable decompositions of F might reveal some interesting structure.

We end this paper by discussing a generalization. Instead of considering the ordinary Fourier algebra $A(\mathbb{R}^2)$, one can take a positive real number β and consider the weight function w_β defined by $w_\beta(x) = (1 + |x|)^\beta$ ($x \in \mathbb{R}^2$). The Beurling algebra $L^1_\beta(\mathbb{R}^2)$ consists of all φ with $\varphi w_\beta \in L^1(\mathbb{R}^2)$. The norm is given by

$$\|\varphi\|_{1,\beta} = \int_{\mathbb{R}^2} |\varphi(x)|(1 + |x|)^\beta dx < \infty \,. \tag{43}$$

The corresponding Fourier–Beurling algebra is

$$A_\beta(\mathbb{R}^2) = \left\{ \widehat{\varphi} \,\middle|\, \varphi \in L^1_\beta(\mathbb{R}^2) \right\}. \tag{44}$$

By restriction to the circle $\mathbb{T}$ we obtain the function algebra $\widetilde{A}_\beta(\mathbb{T})$, again with the restriction norm. As before, we can use measures. Explicitly we have

$$B_\beta(\mathbb{R}^2) = \{\widehat{\mu} \mid w_\beta \mu \in M(\mathbb{R}^2)\} \,. \tag{45}$$

The restriction algebra $\widetilde{B}_\beta(\mathbb{T})$ will again be isometrically isomorphic to $\widetilde{A}_\beta(\mathbb{T})$; this is seen as in section 2, because for suitably chosen functions $\tau \in L^1_\beta(\mathbb{R}^2)$ as described there, the L^1_β-norms will tend to 1 as well. Now one can ask questions analogous to those answered in the theorems 1 and 2. First we obtain the following analogues of Theorem 1.

Theorem 3

(i) *If* $0 \leq \beta < \frac{1}{2}$, *then* $\widetilde{A}_\beta(\mathbb{T}) \not\subset A(\mathbb{T})$.

(ii) *If* $\beta > \frac{1}{2}$, *then* $\widetilde{A}_\beta(\mathbb{T}) \subset A(\mathbb{T})$.

Proof

(i) The β-weighted norm of the Dirac measure δ_x is equal to $(1+|x|)^\beta$. Therefore $\|\widetilde{\delta}_x\|_{\widetilde{A}_\beta(\mathbb{T})} \leq (1+|x|)^\beta$, and for $\beta < \frac{1}{2}$ the proof of Theorem 1 still works.
(ii) If $0 < \beta \leq 1$ and $\varphi \in L^1_\beta(\mathbb{R}^2)$ (or, for that matter, $\varphi \in L^1_\beta(\mathbb{R}^n)$ for arbitrary n), then $\widehat{\varphi}$ is a Lipschitz function of order β. This well-known fact is seen as follows. If $0 \neq h \in \mathbb{R}^2$, then

$$\widehat{\varphi}(y+h) - \widehat{\varphi}(y) = \int_{\mathbb{R}^2} \varphi(x) e^{ix\cdot y}(e^{ix\cdot h} - 1)dx\,.$$

Using the two estimates $|e^{it} - 1| \leq |t|$ and $|e^{it} - 1| \leq 2$ ($t \in \mathbb{R}$), we have

$$|e^{it} - 1| \leq 2^{1-\beta}|t|^\beta\,,$$

and hence

$$|\widehat{\varphi}(y+h) - \widehat{\varphi}(y)| \leq 2^{1-\beta}|h|^\beta \int_{\mathbb{R}^2} |\varphi(x)||x|^\beta dx \leq 2^{1-\beta}\|\varphi\|_{1,\beta}|h|^\beta\,.$$

Thus $\widehat{\varphi}$ is Lipschitz-β. The theorem now follows from Bernstein's theorem (cf. [6, Chapter VI, (3.1)] or [2, page 13]), which states that if a function $f \in C(\mathbb{T})$ is Lipschitz of order $\beta > \frac{1}{2}$, then $f \in A(\mathbb{T})$. □

Unfortunately, we do not know how to settle the critical case $\beta = \frac{1}{2}$.

Remark

The value $\beta = \frac{1}{2}$ is critical for the circle in yet another respect. It is a famous result of C.S. Herz (for the case $\beta = 0$) and H. Reiter (for general β), that the circle is a set of spectral synthesis for the Beurling algebra $A_\beta(\mathbb{R}^2)$ if and only if $\beta < \frac{1}{2}$ (cf. [3, Chapter 7, section 3.9]). In the case at hand the question now arises whether $\beta = \frac{1}{2}$ behaves like the smaller β's, as in Bernstein's theorem, or like the larger β's, as in the matter of spectral synthesis.

We can, however, obtain a complete analogue of Theorem 2. Let us write $w_\beta(r) = (1+|r|)^\beta$ ($r \in \mathbb{R}$). A suitable modification of the proof of Proposition 3 gives:

Proposition 5 $\qquad \|\chi_n\|_{\widetilde{A}_\beta(\mathbb{T})} = \left\| \dfrac{J_n}{w_\beta} \right\|_\infty^{-1}.$

To determine the value of the above supremum norm it suffices to find the maximum of the function

$$r \mapsto \frac{J_n(r)}{(1+r)^\beta} = \frac{J_n}{w_\beta}(r) \tag{46}$$

for positive values of n and r. Let $r_{n,\beta}$ be the point where this maximum is attained. For small r the n-th Bessel function grows like $(\frac{r}{2})^n$ (cf. e.g. [5, 2.11, formula (2)]). Therefore for n large compared to β the function (46) is increasing as well for small values of r. On the other hand it is clear, because of the increasing denominator, that $r_{n,\beta} < r_n$. These qualitative considerations are made more precise in the following lemma.

Lemma 3 *Let $\beta > 0$ be given. Then for n sufficiently large one has $n < r_{n,\beta} < r_n$.*

Proof

Taking the derivative in (46) we see that the point $r_{n,\beta}$ must satisfy the equality

$$J_n'(r)(1+r) = \beta J_n(r)\,.$$

On the interval $[n, r_n]$ one has $J_n''(r) < 0$ (as follows from Bessel's differential equation $x^2 J_n'' + xJ_n' + (x^2 - n^2)J_n = 0$, cf. e.g. [5, 2.13, formula (1)]), and from this it is easily derived that the function

$$r \mapsto \frac{J_n'(r)}{J_n(r)}(1+r) \tag{47}$$

is decreasing on this interval. The following approximation, analogous to (40), is known (cf. [5, 8.2, formula (3)]):

$$J_n'(n) \sim \frac{3^{1/6}\Gamma(\frac{2}{3})}{2^{1/3}\pi} n^{-2/3} = 0.41085\ldots n^{-2/3} \quad (n \to \infty) \tag{48}$$

Using (40) and (48) we get:

$$\frac{J_n'(n)}{J_n(n)} \sim 6^{1/3}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} n^{-1/3} = 0.91850\ldots n^{-1/3} \quad (n \to \infty) \tag{49}$$

From (49) it follows that the function (47) asymptotically has a value of order $n^{2/3}$ at $r = n$, while at $r = r_n$ it takes the value 0. This entails that for n sufficiently large and $n > 2\beta^{3/2}$ the value at the point n is larger than β. Therefore, for any fixed $\beta > 0$ the function (47) will take the desired value β at an internal point of $[n, r_n]$ for n large enough. Actually, $r_{n,\beta}$ will asymptotically be close to r_n rather than to n. □

We now obtain the following generalization of Theorem 2:

Theorem 4 *For arbitrary $\alpha \geq 0$ and $\beta \geq 0$ the Beurling algebra $A_\alpha(\mathbb{T})$ is contained in the restriction algebra $\widetilde{A}_\beta(\mathbb{T})$ if and only if $\alpha \geq \frac{1}{3} + \beta$.*

Proof

From (40) and (42) it follows that $J_n(r)$ is of the order $n^{-1/3}$ on the entire interval $[n, r_n]$. Therefore the value of the function (46) at the point $r_{n,\beta}$ is of the order $n^{-\beta-1/3}$. Hence Proposition 5 yields

$$\|\chi_n\|_{\widetilde{A}_\beta(\mathbb{T})} \approx n^{\beta+1/3} \quad (|n| \to \infty)\,.$$

Both the "if" part and the "only if" part of the theorem now follow as in the proof of Theorem 2. □

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