CENTRAL LIMIT THEOREM FOR THE EDWARDS MODEL

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Abstract

The Edwards model in one dimension is a transformed path measure for standard Brownian motion discouraging self-intersections. We prove a central limit theorem for the endpoint of the path, extending a law of large numbers proved by Westwater (1984). The scaled variance is characterized in terms of the largest eigenvalue of a one-parameter family of differential operators, introduced and analyzed in van der Hofstad and den Hollander (1995). Interestingly, the scaled variance turns out to be independent of the strength of self-repellence and to be strictly smaller than one (the value for free Brownian motion).

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0 Introduction and main result

0.1 The Edwards model

Let $(B_t)_{t\geq 0}$ be standard one-dimensional Brownian motion starting at 0. Let P denote its distribution on path space and E the corresponding expectation. The *Edwards model* is a transformed path measure discouraging self-intersections, defined by the intuitive formula

$$\frac{dP_T^\beta}{dP} = \frac{1}{Z_T^\beta} \exp\left[-\beta \int_0^T ds \int_0^T dt\ \delta(B_s - B_t)\right] \qquad (T \geq 0). \tag{0.1}$$

Here δ denotes Dirac's function, $\beta \in (0,\infty)$ is the *strength of self-repellence* and Z_T^β is the normalizing constant.

A rigorous definition of P_T^β is given in terms of Brownian local times as follows. It is well known (see Revuz and Yor (1991), Sect. VI.1) that there exists a jointly continuous version of the Brownian local time process $(L(t,x))_{t\geq 0, x\in\mathbf{R}}$ satisfying the occupation times formula

$$\int_0^t f(B_s)\,ds = \int_{\mathbf{R}} L(t,x) f(x)\,dx \quad P\text{-a.s.} \qquad (f : \mathbf{R} \to \mathbf{R}^+ \text{ Borel}, t \geq 0). \tag{0.2}$$

Think of $L(t,x)$ as the amount of time the Brownian motion spends in x until time t. The Edwards measure in (0.1) may now be defined by

$$\frac{dP_T^\beta}{dP} = \frac{1}{Z_T^\beta} \exp\left[-\beta \int_{\mathbf{R}} L(T,x)^2\,dx\right], \tag{0.3}$$

where $Z_T^\beta = E(\exp[-\beta \int_{\mathbf{R}} L(T,x)^2\,dx])$ is the normalizing constant. The random variable $\int_{\mathbf{R}} L(T,x)^2\,dx$ is called the *self-intersection local time*. Think of this as the amount of time the Brownian motion spends in self-intersection points until time T.

The path measure P_T^β is the continuous analogue of the self-repellent random walk (called the *Domb-Joyce model*), which is a transformed measure for the discrete simple random walk. The latter is used to study the long-time behavior of random polymer chains. The effect of the self-repellence is of particular interest. This effect is known to spread out the path on a linear scale (i.e., B_T is of order T under the law P_T^β as $T \to \infty$). It is the aim of this paper to study the fluctuations of B_T around the linear asymptotics. Our main result appears in Theorem 2 below.

0.2 Theorems

The starting point of our paper is the following law of large numbers:

Theorem 1 *(Westwater (1984)) For every $\beta \in (0,\infty)$ there exists $\theta^*(\beta) \in (0,\infty)$ such that* [2]

$$\lim_{T\to\infty} P_T^\beta\left(\left|\frac{B_T}{T} - \theta^*(\beta)\right| \leq \epsilon \,\middle|\, B_T > 0\right) = 1 \textit{ for every } \epsilon > 0. \tag{0.4}$$

[2] By symmetry, (0.4) says that the distribution of B_T/T under P_T^β converges weakly to $\frac{1}{2}(\delta_{\theta^*(\beta)} + \delta_{-\theta^*(\beta)})$ as $T \to \infty$, where δ_θ denotes the Dirac point measure at $\theta \in \mathbf{R}$.

Theorem 1 says that the self-repellence causes the path to have a ballistic behavior no matter how weak the interaction. Westwater proved this result by applying the Ray-Knight representation for the Brownian local times and using large deviation arguments.

The speed $\theta^*(\beta)$ was characterized by Westwater in terms of the smallest eigenvalue of a certain differential operator. In the present paper, however, we prefer to work with a different operator, introduced and analyzed in van der Hofstad and den Hollander (1995). For $a \in \mathbf{R}$, define $\mathcal{K}^a : L^2(\mathbf{R}_0^+) \cap C^2(\mathbf{R}_0^+) \to C(\mathbf{R}_0^+)$ by

$$\Big(\mathcal{K}^a x\Big)(u) = 2ux''(u) + 2x'(u) + (au - u^2)x(u) \tag{0.5}$$

for $u \in \mathbf{R}_0^+ = [0,\infty)$. The Sturm-Liouville operator $\mathcal{K}^a$ will play a key role in the present paper.[3] It is symmetric and has a largest eigenvalue $\rho(a)$ with multiplicity 1. The map $a \mapsto \rho(a)$ is real-analytic, strictly convex and strictly increasing, with $\rho(0) < 0$, $\lim_{a\to-\infty} \rho(a) = -\infty$ and $\lim_{a\to\infty} \rho(a) = \infty$.

Define a^*, b^*, $c^* \in (0,\infty)$ by

$$\rho(a^*) = 0, \qquad b^* = \frac{1}{\rho'(a^*)}, \qquad c^{*2} = \frac{\rho''(a^*)}{\rho'(a^*)^3}. \tag{0.6}$$

Our main result is the following central limit theorem:

Theorem 2 *For every $\beta \in (0,\infty)$ there exists $\sigma^*(\beta) \in (0,\infty)$ such that*

$$\lim_{T\to\infty} P_T^\beta\left(\frac{B_T - \theta^*(\beta)T}{\sigma^*(\beta)\sqrt{T}} \le C \,\middle|\, B_T > 0\right) = \mathcal{N}((-\infty, C]) \text{ for all } C \in \mathbf{R}, \tag{0.7}$$

where $\mathcal{N}$ denotes the normal distribution with mean 0 and variance 1. The scaled mean and variance are given by

$$\theta^*(\beta) = b^*\beta^{\frac{1}{3}}, \qquad \sigma^*(\beta) = c^*. \tag{0.8}$$

Theorem 2 says that the fluctuations around the asymptotic mean have the classical order $\sqrt{T}$, are symmetric, and even do not depend on the interaction strength.

The numerical values of the constants in (0.6) are

$$a^* = 2.189 \pm 0.001, \qquad b^* = 1.11 \pm 0.01, \qquad c^* = 0.7 \pm 0.1. \tag{0.9}$$

The values for a^* and b^* were obtained in van der Hofstad and den Hollander (1995), Sect. 0.5, by estimating $\rho(a)$ for a range of a-values. This can be done very accurately via a discretization procedure. (A rigorous upper bound for a^* is given in Lemma 6 in Subsection 4.1.) The same data produce the value for c^*. Note that $c^* < 1$. Apparently, as the path is pushed out to infinity its fluctuations are squeezed compared to those of the free motion with $\theta^*(0) = 0$, $\sigma^*(0) = 1$.

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[3] The operator $\mathcal{K}^a$ is a scaled version of the operator $\mathcal{L}^a$ originally analyzed in van der Hofstad and den Hollander (1995), Sect. 5, namely $(\mathcal{K}^a x)(u) = (\mathcal{L}^a \overline{x})(u/2)$ where $\overline{x}(u) = x(2u)$.

0.3 Scaling in β

It is noteworthy that the scaled mean depends on β in such a simple manner and that the scaled variance does not depend on β at all. These facts are direct consequences of the Brownian scaling property. Namely, we shall deduce from (0.7) that for every $\beta \in (0,\infty)$

$$\theta^*(\beta) = \theta^*(1)\beta^{\frac{1}{3}}, \qquad \sigma^*(\beta) = \sigma^*(1). \tag{0.10}$$

Indeed, for $a, T > 0$

$$\Big(B_T, (L(T,x))_{x\in\mathbf{R}}\Big) \stackrel{\mathcal{D}}{=} \Big(a^{-\frac{1}{2}}B_{aT}, (a^{-\frac{1}{2}}L(aT, a^{\frac{1}{2}}x))_{x\in\mathbf{R}}\Big) \tag{0.11}$$

where $\stackrel{\mathcal{D}}{=}$ means equality in distribution (see Revuz and Yor (1991), Ch. VI, Ex. (2.11), 1°)). Apply this to $a = \beta^{\frac{2}{3}}$ to obtain, via (0.3), that

$$P_T^\beta(B_T)^{-1} = P^1_{\beta^{\frac{2}{3}}T}\Big(\beta^{-\frac{1}{3}}B_{\beta^{\frac{2}{3}}T}\Big)^{-1}, \tag{0.12}$$

where we write $\mu(X)^{-1}$ for the distribution of a random variable X under a measure μ. In particular, we have for all $C \in \mathbf{R}$

$$\begin{aligned} &P_T^\beta\Big(\frac{B_T - \theta^*(1)\beta^{\frac{1}{3}}T}{\sigma^*(1)\sqrt{T}} \le C \,\Big|\, B_T > 0\Big) \\ &= P^1_{\beta^{\frac{2}{3}}T}\Big(\frac{B_{\beta^{\frac{2}{3}}T} - \theta^*(1)\beta^{\frac{2}{3}}T}{\sigma^*(1)\sqrt{\beta^{\frac{2}{3}}T}} \le C \,\Big|\, B_{\beta^{\frac{2}{3}}T} > 0\Big). \end{aligned} \tag{0.13}$$

The r.h.s. tends to $\mathcal{N}((-\infty, C])$ as $T \to \infty$ (in (0.7) pick $\beta = 1$ and replace T by $\beta^{\frac{2}{3}}T$). Since the pair $(\theta^*(\beta), \sigma^*(\beta))$ is uniquely determined by (0.7), we arrive at (0.10).

0.4 Outline of the proof

Theorem 2 is the continuous analogue of the central limit theorem for the Domb-Joyce model proved by König (preprint 1994). We shall be able to use the skeleton of that paper, but the Brownian context will require new ideas and methods. The remaining sections are devoted to the proof of Theorem 2. We give a short outline.

In Section 1, we use the well-known Ray-Knight theorems for the local times of Brownian motion to express the l.h.s. of (0.7) in terms of two- and zero-dimensional squared Bessel processes. The former describes the local times in the area $[0, B_T]$, the latter describe the local times in $(-\infty, 0]$ resp. $[B_T, \infty)$.

In Section 2, with the help of some analytical properties of the operator $\mathcal{K}^a$ proved in van der Hofstad and den Hollander (1995), we introduce a Girsanov transformation of the two-dimensional squared Bessel process. The goal of this transformation is to absorb the random variable $\exp\Big(-\beta\int_0^{B_T} L(T,x)^2\,dx\Big)$ into the transition probabilities. The transformed process turns out to have strong recurrence properties. The Gaussian behavior of $(B_T - \theta^*(\beta)T)/\sqrt{T}$ is traced back to the asymptotic normality of the *inverse* of a certain additive functional of this transformed process. Thus, the central limit behavior is determined by those parts of the Brownian path that fall in the area $[0, B_T]$.

In Section 3, we prove a central limit theorem for the inverse process. Furthermore, as a second important ingredient in the proof, we derive a limit law and a rate of convergence result for the composition of the transformed process with the inverse process.

In Section 4, we finish the proof of Theorem 2 by showing that the contribution of the local times in $(-\infty, 0] \cup [B_T, \infty)$ remains bounded as $T \to \infty$ and is therefore cancelled by the normalization in the definition of the transformed path measure in (0.3).

1 Brownian local times

Since the dependence on β has already been isolated (see (0.13)), we may and shall restrict to the case $\beta = 1$.

Throughout the sequel we shall frequently refer to Revuz and Yor (1991), Karatzas and Shreve (1991) and van der Hofstad and den Hollander (1995). We shall therefore adopt the abbreviations RY resp. KS resp. HH for these references.

The remainder of this paper is devoted to the proof of the following key proposition:

Proposition 1 *There exists an $S \in (0, \infty)$ such that for all $C \in \overline{\mathbf{R}}$*

$$\lim_{T\to\infty} e^{a^*T} E\Big(e^{-\int_{\mathbf{R}} L(T,x)^2\,dx} 1_{0<B_T\le b^*T+C\sqrt{T}}\Big) = S\mathcal{N}_{c^{*2}}((-\infty, C]), \tag{1.1}$$

where a^*, b^* and c^* are defined in (0.6), and $\mathcal{N}_{\sigma^2}$ denotes the normal distribution with mean 0 and variance σ^2.

Theorem 2 follows from Proposition 1, since it implies that the distribution of $(B_T - b^*T)/\sqrt{T}$ converges to $\mathcal{N}_{c^{*2}}$ (divide the l.h.s. of (1.1) by the same expression with $C = \infty$ and recall (0.3)).

Subsections 1.1 and 1.2 contain preparatory material. Subsection 1.3 contains the key representation in terms of squared Bessel processes on which the proof of Proposition 1 will be based.

1.1 Ray-Knight theorems

This subsection contains a description of the *time-changed* local time process in terms of squared Bessel processes. The material being fairly standard, our main purpose is to introduce appropriate notations and to prepare for Lemma 1 in Subsection 1.2 and Lemma 2 in Subsection 1.3.

For $u \in \mathbf{R}$ and $h \ge 0$, let τ_h^u denote the time change associated with $L(t, u)$, i.e.,

$$\tau_h^u = \inf\{\, t > 0 : L(t, u) > h \,\}. \tag{1.2}$$

Obviously, the map $h \mapsto \tau_h^u$ is right-continuous and increasing, and therefore makes at most countably many jumps for each $u \in \mathbf{R}$. Moreover, $P(L(\tau_h^u, u) = h$ for all $u \geq 0) = 1$ (see RY, Ch. VI). The following lemma contains the well-known Ray-Knight theorems. It identifies the distribution of the local times at the random time τ_h^u as a process in the spatial variable running forwards resp. backwards from u. We write $C_c^2(\mathbf{R}^+)$ to denote the set of twice continuously differentiable functions on $\mathbf{R}^+ = (0, \infty)$ with compact support.

RK theorems *Fix $u, h \geq 0$. The random processes $(L(\tau_h^u, u+v))_{v\geq 0}$ and $(L(\tau_h^u, u-v))_{v\geq 0}$ are independent Markov processes, both starting at h.*

(i) $(L(\tau_h^u, u+v))_{v\geq 0}$ is a zero-dimensional squared Bessel process (BESQ^0) with generator

$$(G^\star f)(v) = 2vf''(v) \qquad (f \in C_c^2(\mathbf{R}^+)). \tag{1.3}$$

(ii) $(L(\tau_h^u, u-v))_{v\in[0,u]}$ is the restriction to the interval $[0, u]$ of a two-dimensional squared Bessel process (BESQ^2) with generator

$$(Gf)(v) = 2vf''(v) + 2f'(v) \qquad (f \in C_c^2(\mathbf{R}^+)). \tag{1.4}$$

(iii) $(L(\tau_h^u, -v))_{v\geq 0}$ has the same transition probabilities as the process in (i).

Proof. See RY, Sects. XI.1-2 and KS, Sects. 6.3-4. □

1.2 The distribution of $((L(T, x))_{x\in\mathbf{R}}, B_T)$

The RK theorems give us a nice description of the local time process at certain stopping times. In order to apply them to (0.3), we need to go back to the fixed time T. This causes some complications (e.g. we must handle the global restriction $\int_{\mathbf{R}} L(T, x)\,dx = T$), but these may be overcome by an appropriate conditioning.

This subsection contains a formal description of the joint distribution of the three random processes

$$(L(T, B_T + x))_{x\geq 0}, \quad (L(T, B_T - x))_{x\in[0,B_T]}, \quad (L(T, -x))_{x\geq 0}, \tag{1.5}$$

in terms of the squared Bessel processes. The main intuitive idea is that, up to a P-nullset (recall (1.2)),

$$\{\, \tau_h^u = T \,\} = \{\, B_T = u, L(T, B_T) = h \,\} \text{ for all } u, h \geq 0. \tag{1.6}$$

This has two consequences:

(i) Conditioned on $\{\, B_T = u, L(T, B_T) = h \,\}$, the three processes in (1.5) are the squared Bessel processes from the RK theorems conditioned on having total integral equal to T.

(ii) The distribution of $(B_T, L(T, B_T))$ can be expressed in terms of the squared Bessel processes.

We shall make this precise in Lemma 1 below.

Let us first mention some earlier works on the distribution of $(L(T,x))_{x\in\mathbf{R}}$ where $T\geq 0$ is independent of the motion. Perkins (1982) proves that $(L(1,x))_{x\in\mathbf{R}}$ is a semimartingale. Jeulin (1985) uses stochastic calculus, in particular Tanaka's formula, to recover the RK theorems and Perkins' result and to prove the conditioned Markov property of the triple $(L(1,x), x\wedge B_1, \int_{-\infty}^x L(1,u)\,du)$ in x, given $\inf_{s\leq 1} B_s$. In Biane and Yor (1988), the RK theorems are extended to the case where T is an exponential time, independent of the Brownian motion, under $P(\cdot\,|L(T,0)=s, B_T=a)$ for any fixed $s,a>0$. Finally, Biane, Le Gall and Yor (1987) also deal with the intuitive idea (1.6) when identifying the law of the process $(\frac{1}{\sqrt{\tau_h^0}} B_{u\tau_h^0})_{u\in[0,1]}$.

Let us return to our identification of the law of the process $((L(T,x))_{x\in\mathbf{R}}, B_T)$. In order to formulate the details, we must first introduce some notation. For the remainder of this paper, let

$$(X_v)_{v\geq 0} = \mathrm{BESQ}^2, \qquad\qquad (X_v^\star)_{v\geq 0} = \mathrm{BESQ}^0. \tag{1.7}$$

Note that $(X_v)_{v\geq 0}$ is recurrent and has 0 as an entrance boundary, while $(X_v^\star)_{v\geq 0}$ is transient and has 0 as an absorbing boundary (see RY, Sect. XI.1). Denote by $\mathbf{P}_h$ and $\mathbf{P}_h^\star$ the distributions of the respective processes conditioned on starting at $h\geq 0$. Denote the corresponding expectations by $\mathbf{E}_h$ resp. $\mathbf{E}_h^\star$. Furthermore, define the following additive functional of BESQ^2 and its time change:

$$\begin{aligned} A(u) &= \textstyle\int_0^u X_v\,dv & &(u\geq 0),\\ A^{-1}(t) &= \inf\{\,u>0 : A(u)>t\,\} & &(t\geq 0). \end{aligned} \tag{1.8}$$

Note that both $u\mapsto A(u)$ and $t\mapsto A^{-1}(t)$ are continuous and strictly increasing towards infinity, $\mathbf{P}_h$-a.s. So A and A^{-1} are in fact inverse functions of each other. We also need the analogous functional for BESQ^0:

$$\begin{aligned} A^\star(u) &= \textstyle\int_0^u X_v^\star\,dv & &(u\in[0,\infty]),\\ A^{\star-1}(t) &= \inf\{\,u\geq 0 : A^\star(u)>t\,\} & &(t\geq 0). \end{aligned} \tag{1.9}$$

Note that, $\mathbf{P}_h^\star$-a.s., $u\mapsto A^\star(u)$ is strictly increasing on the time interval $[0,\xi_0]$, where $\xi_0=\inf\{\,v\geq 0 : X_v^\star = 0\,\}<\infty$ denotes the absorption time of BESQ^0. Define Lebesgue densities φ_h and $\psi_{h_1,t}$ by

$$\begin{aligned} \varphi_h(t)\,dt &= \mathbf{P}_h^\star\left(A^\star(\infty)\in dt\right),\\ \psi_{h_1,t}(u,h_2)\,du\,dh_2 &= \mathbf{P}_{h_1}(A^{-1}(t)\in du, X_u\in dh_2) \end{aligned} \tag{1.10}$$

for a.e. $h,t,h_1,u,h_2\geq 0$. (The function φ_h is explicitly identified in Lemma 7 in Subsection 4.2.) Put the quantities defined in (1.8) – (1.10) equal to zero if any of the variables is negative. Now the joint distribution of the three processes in (1.5) can be described as follows:

Lemma 1 *Fix $T > 0$. For all nonnegative Borel functions Φ_1, Φ_2 and Φ_3 on $C(\mathbf{R}_0^+)$ and for any interval $I \subset [0, \infty)$,*

$$
\begin{aligned}
&E\Big(\Phi_1\Big((L(T, B_T + x))_{x\geq 0}\Big)\Phi_2\Big((L(T, -x))_{x\geq 0}\Big)\Phi_3\Big((L(T, B_T - x))_{x\in[0,B_T]}\Big)\, 1_{B_T\in I}\Big) \\
&\quad = \int_I du \int_{[0,\infty)^4} dt_1\, dh_1\, dt_2\, dh_2 \prod_{i=1}^{2} \mathbf{E}^\star_{h_i}\Big(\Phi_i\Big((X^\star_v)_{v\geq 0}\Big)\,\Big|\, A^\star(\infty) = t_i\Big)\, \varphi_{h_i}(t_i) \\
&\qquad \times \mathbf{E}_{h_1}\Big(\Phi_3\Big((X_v)_{v\in[0,u]}\Big)\,\Big|\, A^{-1}(T - t_1 - t_2) = u, X_u = h_2\Big)\psi_{h_1,T-t_1-t_2}(u, h_2).
\end{aligned}
\tag{1.11}
$$

Proof. Essentially, Lemma 1 is a formal rewrite using (1.8), (1.10) and the RK-theorems, which say that under $\mathbf{P}_h$ resp. $\mathbf{P}^\star_h$

$$
\begin{aligned}
(X_v)_{v\in[0,u]} &\stackrel{\mathcal{D}}{=} (L(\tau^u_h, u - v))_{v\in[0,u]} \\
(X^\star_v)_{v\geq 0} &\stackrel{\mathcal{D}}{=} (L(\tau^u_h, u + v))_{v\geq 0}.
\end{aligned}
\tag{1.12}
$$

However, the details are far from trivial.

We proceed in four steps, the first of which makes (1.6) precise and is the most technical.

STEP 1 *$P(\tau^u_h \in dT)\, du\, dh = P(B_T \in du, L(T, B_T) \in dh)\, dT$ for a.e. $u, h, T \geq 0$.*

Proof. From the occupation times formula (0.2) we get for every $t \geq 0$

$$
\int_0^t 1_{B_s\in du}\, ds = L(t, u)\, du. \tag{1.13}
$$

Thus, we obtain for every bounded and measurable functions $f : (\mathbf{R}^+)^2 \to \mathbf{R}$ and $g : \mathbf{R}^+ \to \mathbf{R}$ with compact support:

$$
\begin{aligned}
&\int_0^\infty du \int_0^\infty dh\, f(u, h)\, E(g(\tau^u_h)) \\
&= \int_0^\infty du\, E\left(\int_0^\infty d_t(L(t, u))\, f(u, L(t, u))g(t)\right) \\
&= \int_0^\infty du\, E\left(\int_0^\infty d_t(L(t, u))\, g(t)E[f(u, L(t, u)) \mid B_t = u]\right) \\
&= \int_0^\infty du \int_0^\infty dt\, \frac{dE(L(t, u))}{dt} g(t)E[f(u, L(t, u)) \mid B_t = u] \\
&\stackrel{(1.13)}{=} \int_0^\infty du \int_0^\infty dt\, \frac{P(B_t \in du)}{du}\, g(t)E[f(u, L(t, u)) \mid B_t = u] \\
&= \int_0^\infty dt\, g(t)E[f(B_t, L(t, B_t))].
\end{aligned}
\tag{1.14}
$$

(The second equality follows from Prop. 3 in Fitzsimmons, Pitman and Yor (1993).) $\square$

Next, abbreviate for $u, h \geq 0$,

$$\mathcal{Z}_h^u = \Big(\tau_h^u, \int_0^\infty L(\tau_h^u, u+v)\,dv, L(\tau_h^u, 0), \int_0^\infty L(\tau_h^u, -v)\,dv\Big). \tag{1.15}$$

Then the distribution of $\mathcal{Z}_h^u$ is identified as:

STEP 2 *For every $u, h \geq 0$ and a.e. T, t_1, h_2, t_2,*

$$P(\mathcal{Z}_h^u \in d(T, t_1, h_2, t_2)) = \varphi_h(t_1)\psi_{h,T-t_1-t_2}(u, h_2)\varphi_{h_2}(t_2)\,dT\,dt_1\,dh_2\,dt_2. \tag{1.16}$$

Proof. According to the RK theorems, $(L(\tau_h^u, -x))_{x\geq 0}$ is BESQ^0 starting at $L(\tau_h^u, 0)$. Moreover, $L(\tau_h^u, 0)$ itself has distribution $\mathbf{P}_h(X_u)^{-1}$. Furthermore, from (0.2) we have

$$\tau_h^u = \int_0^\infty L(\tau_h^u, u+v)\,dv + \int_0^u L(\tau_h^u, u-v)\,dv + \int_0^\infty L(\tau_h^u, -v)\,dv. \tag{1.17}$$

Combining these statements with the RK theorems and (1.12), we obtain

$$\begin{aligned} P(\mathcal{Z}_h^u \in d(T, t_1, h_2, t_2)) = {} & \mathbf{P}_h^\star\Big(\int_0^\infty X_v^\star\,dv \in dt_1\Big)\,\mathbf{P}_{h_2}^\star\Big(\int_0^\infty X_v^\star\,dv \in dt_2\Big) \\ & \times \mathbf{P}_h\Big(\int_0^u X_v\,dv \in d(T-t_1-t_2), X_u \in dh_2\Big). \end{aligned} \tag{1.18}$$

But the r.h.s. of (1.18) equals the r.h.s. of (1.16), because of (1.10) and the identity $\{A(u) < T - t_1 - t_2\} = \{A^{-1}(T - t_1 - t_2) > u\}$ implied by (1.8). $\square$

STEP 3 $P(\tau_{L(T,B_T)}^{B_T} = T) = 1$.

Proof. Simply note that $\tau_{L(T,B_T)}^{B_T} - T$ is distributed as the time change τ_0^0 for the process $(B_{T+t} - B_T)_{t\geq 0}$ (recall (1.2)). But $P(\tau_0^0 = 0) = 1$ (see RY, Remark 1°) following Prop. VI.2.5). $\square$

STEP 4 *Proof of Lemma 1.*

Proof. First condition and integrate the l.h.s. of (1.11) w.r.t. the distribution of $(B_T, L(T, B_T))$, which is identified in Step 1. According to Step 3, we may then replace T by $\tau_{h_1}^u$ on $\{B_T = u, L(T, B_T) = h_1\}$. Next, condition and integrate w.r.t. the conditional distribution of $\mathcal{Z}_{h_1}^u$ given $\{\,\tau_{h_1}^u = T\,\}$. Then the l.h.s. of (1.11) becomes

$$\begin{aligned} \int_I du \int_0^\infty dh_1\, \frac{P(\tau_{h_1}^u \in dT)}{dT} \int_{[0,\infty)^3} & \frac{P(\mathcal{Z}_{h_1}^u \in d(T, t_1, h_2, t_2))}{P(\tau_{h_1}^u \in dT)} \\ & \times E\Big(\Phi_1\Big((L(\tau_{h_1}^u, u+x))_{x\geq 0}\Big)\Phi_2\Big((L(\tau_{h_1}^u, -x))_{x\geq 0}\Big) \\ & \times \Phi_3\Big((L(\tau_{h_1}^u, u-x))_{x\in[0,u]}\Big)\,\Big|\,\mathcal{Z}_{h_1}^u = (T, t_1, h_2, t_2)\Big). \end{aligned} \tag{1.19}$$

Now use Step 2, apply the description of the local time processes provided by the RK theorems in combination with (1.12) and (1.15), and again use the elementary relation between A and A^{-1} stated at the end of the proof of Step 2. Then we obtain that (1.19) is equal to the r.h.s. of (1.11). $\square$

$\square$

In Lemma 1, note that $A^{*}(\infty) = t_1$ resp. t_2 corresponds to the Brownian motion spending t_1 resp. t_2 time units is the boundary areas $[B_T, \infty)$ resp. $(-\infty, 0]$, while $A^{-1}(T - t_1 - t_2)$ corresponds to the size of the middle area $[0, B_T]$ when the Brownian motion spends $T - t_1 - t_2$ time units there.

1.3 Application to the Edwards model

We are now ready to formulate the key representation of the expectation appearing in the l.h.s. of (1.1). This representation will be the starting point for the proof of Proposition 1 in Sections 2-4. Abbreviate

$$C_T = b^* T + C\sqrt{T}. \tag{1.20}$$

Lemma 2 *For all $T > 0$,*

$$\begin{aligned} E\Big(e^{-\int_{\mathbf{R}} L(T,x)^2\,dx} 1_{0 < B_T \le C_T}\Big) &= \int_0^{C_T} du \int_{[0,\infty)^4} dt_1\, dh_1\, dt_2\, dh_2 \prod_{i=1}^{2} \mathbf{E}^{\star}_{h_i}\Big(e^{-\int_0^\infty X_v^{\star 2}\,dv} \,\Big|\, A^{\star}(\infty) = t_i\Big) \varphi_{h_i}(t_i) \\ &\quad \times \mathbf{E}_{h_1}\Big(e^{-\int_0^u X_v^2\,dv} \,\Big|\, A^{-1}(T - t_1 - t_2) = u, X_u = h_2\Big) \psi_{h_1, T-t_1-t_2}(u, h_2). \end{aligned} \tag{1.21}$$

Proof. This follows from Lemma 1. $\square$

Thus, we have expressed the expectation in the l.h.s. of (1.1) in terms of integrals over BESQ^0 and BESQ^2 and their additive functionals. Henceforth we can forget about the underlying Brownian motion and focus on these processes using their generators given in (1.3) and (1.4).

The importance of Lemma 2 is the decomposition into a *product* of three expectations. The main reason to introduce the densities φ_h and $\psi_{h_1,t}$ is the fact that the last factor in (1.21) depends on t_1 and t_2. This dependence will vanish in the limit as $T \to \infty$, as we shall see in the sequel. After that the densities φ_h and $\psi_{h_1,t}$ can again be absorbed into the expectations (recall (1.10)). Thus, we shall need little about these densities other than their existence.

2 A transformed Markov process

All we have done so far is to rewrite the key object of Proposition 1 in terms of expectations involving squared Bessel processes. We are now ready for our main attack.

In Subsection 2.1 we use Girsanov's formula to transform BESQ^2 into a new Markov process. The purpose of this transformation is to absorb the exponential factor appearing under the expectation in the last line of (1.21) into the transition probabilities of the new process. In Subsection 2.2 we list some properties of the transformed process. These are used in Subsection 2.3 to obtain a final reformulation of (1.21) on which the proof of Proposition 1 will be based. In Subsection 2.4 we formulate three main propositions, the proof of which is deferred to Sections 3-4. In Subsection 2.5 the proof of Proposition 1 is completed subject to these propositions.

2.1 Construction of the transformed process

Fix $a \in \mathbf{R}$ (later we shall pick $a = a^*$). Recall from Subsection 0.2 that $\rho(a) \in \mathbf{R}$ is the largest eigenvalue of the operator $\mathcal{K}^a$ defined in (0.5). We denote the corresponding strictly positive and L^2-normalized eigenvector by x_a. From HH, Lemmas 20 and 22, we know that $x_a : \mathbf{R}_0^+ \to \mathbf{R}^+$ is real-analytic with $\lim_{u\to\infty} u^{-\frac{3}{2}} \log x_a(u) < 0$, and that $a \mapsto x_a \in L^2(\mathbf{R}_0^+)$ is real-analytic. Define

$$F_a(u) = u^2 - au + \rho(a) \qquad (u \in \mathbf{R}_0^+). \tag{2.1}$$

The following lemma defines the Girsanov transformation of BESQ^2 that we shall need later:

Lemma 3 *For $t, h_1, h_2 \geq 0$, let $P_t(h_1, dh_2)$ denote the transition probability function of BESQ^2. Then*

$$\widehat{P}_t^a(h_1, dh_2) = \frac{x_a(h_2)}{x_a(h_1)} \mathbf{E}_{h_1}\left(e^{-\int_0^t F_a(X_v)\,dv} \,\middle|\, X_t = h_2\right) P_t(h_1, dh_2) \tag{2.2}$$

defines the transition probability function of a diffusion $(X_v)_{v\geq 0}$ on $\mathbf{R}_0^+$.

Proof. Recall the definition of the generator G of BESQ^2 given in (1.4). According to RY, Sect. VIII.3, if $f \in C^2(\mathbf{R}_0^+)$ satisfies the equation

$$G(f) + \frac{1}{2}G(f^2) - f\,G(f) = F_a, \tag{2.3}$$

then

$$(D_t^{f,a})_{t\geq 0} = \left(e^{f(X_t) - f(X_0) - \int_0^t F_a(X_s)\,ds}\right)_{t\geq 0} \tag{2.4}$$

is a local martingale under $\mathbf{P}_h$ for any $h \geq 0$. Substitute $f = \log x$ in the l.h.s. of (2.3). Then an elementary calculation yields that for all $u \geq 0$

$$\begin{aligned} \Big(G(f) + \frac{1}{2}G(f^2) - f\,G(f)\Big)(u) &= 2uf''(u) + 2f'(u) + 2uf'(u)^2 \\ &= \frac{2ux''(u) + 2x'(u)}{x(u)}. \end{aligned} \tag{2.5}$$

We now easily derive from the eigenvalue relation $\mathcal{K}_a x_a = \rho(a) x_a$ (recall (0.5)) that (2.3) is satisfied for $f = f_a = \log x_a$. Hence, $(D_t^{f_a,a})_{t\geq 0}$ is a local martingale under $\mathbf{P}_h$. Since F_a is bounded from below and x_a is bounded from above, each $D_t^{f_a,a}$ is bounded $\mathbf{P}_h$-a.s. Hence $(D_t^{f_a,a})_{t\geq 0}$ is a martingale under $\mathbf{P}_h$. The lemma now follows from RY, Prop. VIII.3.1. $\square$

We shall denote the distribution of the transformed process, conditioned on starting at $h \geq 0$, by $\widehat{\mathbf{P}}_h^a$ and the corresponding expectation by $\widehat{\mathbf{E}}_h^a$. Note that we have

$$\widehat{\mathbf{E}}_h^a(g(X_t)) = \mathbf{E}_h(D_t^{f_a,a} g(X_t)) \qquad (t \geq 0, g : \mathbf{R}_0^+ \to \mathbf{R} \text{ measurable}). \tag{2.6}$$

2.2 Properties of the transformed process

We are going to list some properties of the process constructed in the preceding subsection.

1. The process introduced in Lemma 3 is a Feller process. According to RY, Prop. VIII.3.4, its generator is given by (recall $f_a = \log x_a$)

$$\begin{aligned} (\widehat{G}^a f)(u) &= (Gf)(u) + \Big(G(f_a f) - f_a G(f) - f G(f_a)\Big)(u) \\ &= (Gf)(u) + 4u f_a'(u) f'(u) \\ &= 2u f''(u) + 2f'(u)\Big(1 + 2u\,\frac{x_a'(u)}{x_a(u)}\Big) \qquad (f \in C_c^2(\mathbf{R}^+)). \end{aligned} \tag{2.7}$$

2. According to KS, Ch. 5, Eq. (5.42), the scale function for the process is given (up to an affine transformation) by

$$s_a(u) = \int_c^u \frac{dv}{v\, x_a^2(v)} \qquad (c > 0 \text{ arbitrary}). \tag{2.8}$$

Since x_a does not vanish at zero and has a subexponential tail at infinity (see the remarks at the beginning of Subsection 2.1), the scale function satisfies

$$\lim_{u\downarrow 0} s_a(u) = -\infty \qquad \text{and} \qquad \lim_{u\to\infty} s_a(u) = \infty. \tag{2.9}$$

3. The probability measure on $\mathbf{R}_0^+$ given by

$$\mu_a(du) = x_a(u)^2\, du \tag{2.10}$$

is the normalized speed measure for the process (see KS, Ch. 5, Eq. (5.51)). Since it has finite mass, and because (2.9) holds, the process converges weakly towards μ_a from any starting point $h > 0$ (see KS, Ch. 5, Ex. 5.40), i.e.,

$$\lim_{t\to\infty} \widehat{\mathbf{E}}_h^a(f(X_t)) = \int_0^\infty f(u)\, \mu_a(du) \text{ for all bounded } f \in C(\mathbf{R}_0^+). \tag{2.11}$$

Using this convergence and the Feller property, one derives in a standard way that μ_a is the invariant distribution for the process. We write

$$\widehat{\mathbf{P}}^a = \int_0^\infty \widehat{\mathbf{P}}_h^a \, \mu_a(dh) \tag{2.12}$$

to denote the distribution of the process starting in the invariant distribution and write $\widehat{\mathbf{E}}^a$ for the corresponding expectation.

4. According to Ethier and Kurtz (1986), Th. 6.1.4, the process $(Y_t)_{t\geq 0}$ given by

$$Y_t = X_{A^{-1}(t)} \qquad (t \geq 0) \tag{2.13}$$

is a diffusion under $\widehat{\mathbf{P}}^a$ with generator

$$\left(\widetilde{G}^a f\right)(u) = \frac{1}{u}\left(\widehat{G}^a f\right)(u) \qquad (u > 0, f \in C_c^2(\mathbf{R}^+)) \tag{2.14}$$

(see (2.7)). This process has the same scale function s_a as $(X_t)_{t\geq 0}$ (see (2.8)), and its normalized speed measure is given by

$$\nu_a(du) = \frac{u}{\rho'(a)} x_a^2(u)\, du. \tag{2.15}$$

(In order to see that $\nu_a(\mathbf{R}^+) = 1$, differentiate the relation $\rho(a) = \langle x_a, \mathcal{K}^a x_a\rangle_{L^2}$ w.r.t. a. Use (0.5) and the relation $\frac{d}{da}\langle x_a, x_a\rangle_{L^2} = 0$.) Similarly as in (2.11), for any starting point $h > 0$

$$\lim_{t\to\infty} \widehat{\mathbf{E}}_h^a(f(Y_t)) = \int_0^\infty f(u)\, \nu_a(du) \text{ for all bounded } f \in C(\mathbf{R}_0^+) \tag{2.16}$$

and hence ν_a is the invariant distribution of the process $(Y_t)_{t\geq 0}$. We write

$$\widetilde{\mathbf{P}}^a = \int_0^\infty \widehat{\mathbf{P}}_h^a \, \nu_a(dh) \tag{2.17}$$

to denote the distribution of the process $(X_t)_{t\geq 0}$ starting in the invariant distribution ν_a of the process $(Y_t)_{t\geq 0}$ and we write $\widetilde{\mathbf{E}}^a$ for the corresponding expectation.

2.3 Final reformulation

Using the representation in Lemma 2, we shall rewrite the l.h.s. of (1.1) in terms of the transformed process introduced in Lemma 3. This will be the final reformulation in terms of which the proof of Proposition 1 will be finished in Subsections 2.4-2.5.

For $h, t \geq 0$ and $a \in \mathbf{R}$, introduce the abbreviation (recall (1.9) and (1.10))

$$\begin{aligned} F_a^\star(u) \quad &= -u^2 + au \qquad (u \in \mathbf{R}_0^+) \\ w_a(h,t) \quad &= \mathbf{E}_h^\star\Big(e^{-\int_0^\infty F_a^\star(X_v^\star)\, dv} \,\Big|\, A^\star(\infty) = t\Big)\varphi_h(t) = e^{at} w_0(h,t). \end{aligned} \tag{2.18}$$

Recall that $\widehat{\mathbf{E}}^a$ denotes the expectation for the transformed process $(X_t)_{t\geq 0}$ starting in the invariant starting distribution μ_a given by (2.10).

Lemma 4 *For every $T > 0$,*

$$e^{a^*T} E\Big(e^{-\int_{\mathbf{R}} L(T,x)^2\,dx} 1_{0<B_T\le C_T}\Big)$$
$$= \int_0^\infty dt_1 \int_0^\infty dt_2\, \widehat{\mathbf{E}}^{a^*}\Big(\frac{w_{a^*}(X_0,t_1)}{x_{a^*}(X_0)}\, 1_{A^{-1}(T-t_1-t_2)\le C_T}\, \frac{w_{a^*}(X_{A^{-1}(T-t_1-t_2)},t_2)}{x_{a^*}(X_{A^{-1}(T-t_1-t_2)})}\Big). \tag{2.19}$$

Proof. First, from (1.8), (2.1) and from $\rho(a^*) = 0$ it follows that on $\{\, A^{-1}(t) = u \,\}$

$$a^*t - \int_0^u X_v^2\,dv = -\int_0^u F_{a^*}(X_v)\,dv \qquad (t,u\ge 0). \tag{2.20}$$

By an absolute continuous transformation from $\mathbf{P}_h$ to $\widehat{\mathbf{P}}_h^{a^*}$, we therefore obtain via (2.2) the identity (recall (1.10))

$$\begin{aligned} &e^{a^*t}\,\mathbf{E}_{h_1}\Big(e^{-\int_0^u X_v^2\,dv}\,\Big|\, A^{-1}(t) = u, X_u = h_2\Big)\psi_{h_1,t}(u,h_2)\,du\,dh_2 \\ &\quad = \widehat{\mathbf{P}}_{h_1}^{a^*}\Big(A^{-1}(t)\in du, X_u\in dh_2\Big)\frac{x_{a^*}(h_1)}{x_{a^*}(h_2)} \end{aligned} \tag{2.21}$$

for a.e. $u, h_1, h_2, t \ge 0$. Similarly to (2.20), we have on $\{\, \int_0^\infty X_v^\star\,dv = t \,\}$

$$a^*t - \int_0^\infty (X_v^\star)^2\,dv = -\int_0^\infty F_{a^*}(X_v^\star)\,dv \qquad (t\ge 0) \tag{2.22}$$

and hence

$$e^{a^*t_i}\mathbf{E}_{h_i}^\star\Big(e^{-\int_0^\infty (X_v^\star)^2\,dv}\,\Big|\, A^\star(\infty) = t_i\Big)\varphi_{h_i}(t_i) = w_{a^*}(h_i,t_i) \qquad (i=1,2). \tag{2.23}$$

Next, note that the l.h.s. of (2.19) is equal to the l.h.s. of (1.21) times the factor e^{a^*T}. We divide this factor into three parts, according to the identity $T = t_1 + (T - t_1 - t_2) + t_2$, and assign them to each of the three expectations in the r.h.s. of (1.21). Substitute (2.21) with $t = T - t_1 - t_2$ and (2.23) into (1.21). Then we obtain that the l.h.s. of (2.19) is equal to

$$\begin{aligned} &\int_{[0,\infty)^4} dh_1\,dh_2\,dt_1\,dt_2\, w_{a^*}(h_1,t_1) w_{a^*}(h_2,t_2)\frac{x_{a^*}(h_1)}{x_{a^*}(h_2)} \\ &\qquad\times \widehat{\mathbf{P}}_{h_1}^{a^*}\Big(A^{-1}(T-t_1-t_2)\le C_T, X_{A^{-1}(T-t_1-t_2)}\in dh_2\Big). \end{aligned} \tag{2.24}$$

Now formally carry out the integration over h_1, h_2, recalling (2.10) and (2.12), to arrive at the r.h.s. of (2.19). □

Roughly speaking, the function w_{a^*} in the r.h.s. of (2.19) describes the contribution to the random variable $\exp[-\int_{\mathbf{R}} L(T,x)^2\,dx]$ coming from the boundary pieces (i.e., the parts of the path in $(-\infty,0]\cup[B_T,\infty)$), while A^{-1} gives the size of the area over which the middle piece (i.e., the parts of the path in $[0,B_T]$) spreads out.

2.4 Key steps in the proof of Proposition 1

The proof of Proposition 1 now basically requires the following three ingredients:

(1) A CLT for $(A^{-1}(t))_{t\geq 0}$ under $\widehat{\mathbf{P}}^{a^*}$.

(2) An extension of the weak convergence of $(Y_t)_{t\geq 0} = (X_{A^{-1}(t)})_{t\geq 0}$ stated in (2.16).

(3) Some integrability properties of w_{a^*}.

The precise statements that we shall need are formulated in Propositions 2-4 below. The proof of these propositions is deferred to Sections 3 and 4.

We need some more notation. Let $\langle \cdot , \cdot \rangle_{L^2}$ denote the standard inner product on $L^2(\mathbf{R}_0^+)$. Let $\langle \cdot , \cdot \rangle_{L^2}^\circ$ denote the weighted inner product

$$\langle f, g\rangle_{L^2}^\circ = \int_0^\infty dh\ hf(h)g(h) \tag{2.25}$$

on $L^{2,\circ}(\mathbf{R}_0^+) = \{ f : \mathbf{R}_0^+ \to \mathbf{R} \text{ measurable} \,|\, \int_0^\infty dh\ hf^2(h) < \infty \}$. We write $|| \cdot ||_{L^2}$ resp. $|| \cdot ||_{L^2}^\circ$ for the corresponding norms.

For bounded and measurable $f, g : \mathbf{R}_0^+ \to \mathbf{R}$, $T \geq 0$ and $a \in \mathbf{R}$, abbreviate (recall Lemma 3, (2.10) and (2.12))

$$N_{T,a}^{f,g} = \widehat{\mathbf{E}}^a \left(\frac{f}{x_a}(Y_0)\, \frac{g}{x_a}(Y_T) \right) = \int_0^\infty dh\ f(h)\mathbf{E}_h \left(e^{-\int_0^{A^{-1}(T)} F_a(X_s)\,ds} g(X_{A^{-1}(T)}) \right). \tag{2.26}$$

Furthermore, define

$$\sigma^2(a) = \frac{\rho''(a)}{\rho'(a)^3} \tag{2.27}$$

and note that $\sigma^2(a^*) = c^{*2}$ defined in (0.6). Denote by $\rho^{-1} : \mathbf{R} \to \mathbf{R}$ the inverse function of $\rho : \mathbf{R} \to \mathbf{R}$.

Proposition 2 *For all bounded and measurable $f, g : \mathbf{R}_0^+ \to \mathbf{R}$ and for every $a, \lambda \in \mathbf{R}$ and all T, $T' \geq 0$,*

$$\widehat{\mathbf{E}}^a \left(\frac{f}{x_a}(Y_0) e^{\frac{\lambda}{\sqrt{T}}\left(A^{-1}(T') - \frac{T}{\rho'(a)}\right)} \frac{g}{x_a}(Y_{T'}) \right) = e^{\frac{\lambda^2}{2}\sigma^2(\xi_T)} N_{T',a_{\lambda,T}}^{f,g} e^{(T-T')(a_{\lambda,T}-a)}, \tag{2.28}$$

where

$$a_{\lambda,T} = \rho^{-1}\left(\rho(a) - \frac{\lambda}{\sqrt{T}}\right) \tag{2.29}$$

and $\xi_T \in [a, a_{\lambda,T}] \cup [a_{\lambda,T}, a]$.

Proposition 3 *Let $f, g : \mathbf{R}_0^+ \to \mathbf{R}$ be measurable such that $f/\mathrm{id}, g \in L^{2,\circ}$. Then for every $a \in \mathbf{R}$ and $a_T \to a$,*

$$\lim_{T\to\infty} N^{f,g}_{T,a_T} = \frac{1}{\rho'(a)} \langle f, x_a\rangle_{L^2} \langle g, x_a\rangle^\circ_{L^2}. \tag{2.30}$$

Recall (2.18). For $a \in \mathbf{R}$, define $y_a : \mathbf{R}_0^+ \to [0,\infty]$ by

$$y_a(h) = \int_0^\infty w_a(h,t)\,dt = \mathbf{E}^\star_h \left(e^{-\int_0^\infty F^\star_a(X^\star_v)\,dv} \right). \tag{2.31}$$

Furthermore, define, for $p \in (1,2)$ resp. $q \in (2,\infty)$, and $t \geq 0$,

$$\begin{aligned} W_p^{(1)}(t) &= \left(\textstyle\int_0^\infty h^{1-p} x_{a^*}(h)^{2-p} w_{a^*}(h,t)^p\,dh\right)^{\frac{1}{p}}, \\ W_q^{(2)}(t) &= \left(\textstyle\int_0^\infty h x_{a^*}(h)^{2-q} w_{a^*}(h,t)^q\,dh\right)^{\frac{1}{q}}. \end{aligned} \tag{2.32}$$

Proposition 4

(i) y_{a^*} is bounded and measurable.

(ii) For any $p \in (1,2)$, $W_p^{(1)}$ is integrable on $\mathbf{R}^+$.

(iii) For any $q \in (2,\infty)$ that is sufficiently close to 2, $W_q^{(2)}$ is integrable on $\mathbf{R}^+$.

2.5 Proof of Proposition 1

In this subsection we finish the proof of Proposition 1 subject to Propositions 2–4. We shall show that (1.1) follows from (2.19), with S identified as

$$S = b^* \langle y_{a^*}, x_{a^*}\rangle_{L^2} \langle y_{a^*}, x_{a^*}\rangle^\circ_{L^2}. \tag{2.33}$$

STEP 1 *For all $t_1, t_2 > 0$, as $T \to \infty$ the integrand on the r.h.s. of (2.19) tends to*

$$b^* \langle w_{a^*}(\cdot,t_1), x_{a^*}\rangle_{L^2} \langle w_{a^*}(\cdot,t_2), x_{a^*}\rangle^\circ_{L^2} \,\mathcal{N}_{c^{*2}}((-\infty,C]).$$

Proof. By Proposition 4(ii), for all $t_1, t_2 > 0$ the functions $f = w_{a^*}(\cdot,t_1)$ and $g = w_{a^*}(\cdot,t_2)$ satisfy the assumptions of Proposition 3. Define a (non-Markovian) path measure $\mathbf{P}^{f,g}_{T,a}$ by

$$\frac{d\mathbf{P}^{f,g}_{T,a}}{d\widehat{\mathbf{P}}^a} = \frac{1}{N^{f,g}_{T,a}} \frac{f}{x_a}(Y_0) \frac{g}{x_a}(Y_T). \tag{2.34}$$

Write $\mathbf{E}^{f,g}_{T,a}$ for the corresponding expectation. Apply Proposition 2 for $a = a^*$ and $T' = T - t_1 - t_2$ to obtain that for every $\lambda \in \mathbf{R}$ and $T \geq t_1 + t_2$,

$$\mathbf{E}^{f,g}_{T-t_1-t_2,a^*} \left(e^{\frac{\lambda}{\sqrt{T}}[A^{-1}(T-t_1-t_2) - b^* T]} \right) = e^{\frac{\lambda^2}{2}\sigma^2(\xi^*_T)} \frac{N^{f,g}_{T-t_1-t_2,a^*_{\lambda,T}}}{N^{f,g}_{T-t_1-t_2,a^*}} e^{(t_1+t_2)(a^*_{\lambda,T} - a^*)}, \tag{2.35}$$

where $\rho(a^*) = 0$, $b^* = \frac{1}{\rho'(a^*)}$ (recall (0.6)), $a^*_{\lambda,T} = \rho^{-1}(-\frac{\lambda}{\sqrt{T}})$ and $\xi^*_T \in [a^*, a^*_{\lambda,T}] \cup [a^*_{\lambda,T}, a^*]$. Since ρ', ρ'' and ρ^{-1} are continuous, we have $a^*_{\lambda,T} \to a^*$ and $\sigma^2(\xi^*_T) \to c^{*2}$ as $T \to \infty$. Therefore, by Proposition 3, the r.h.s. of (2.35) tends to $e^{\frac{\lambda^2}{2}c^{*2}}$ as $T \to \infty$. Thus, the distribution of $\frac{1}{\sqrt{T}}[A^{-1}(T-t_1-t_2)-b^*T]$ under $\mathbf{P}^{f,g}_{T-t_1-t_2,a^*}$ converges weakly towards $\mathcal{N}_{c^{*2}}$. Via (2.34), this in turn implies that (recall (1.20)

$$\begin{aligned}\lim_{T\to\infty} \widehat{\mathbf{E}}^{a^*}\left(\frac{w_{a^*}(X_0,t_1)}{x_{a^*}(X_0)}\, 1_{A^{-1}(T-t_1-t_2)\leq C_T}\, \frac{w_{a^*}(X_{A^{-1}(T-t_1-t_2)},t_2)}{x_{a^*}(X_{A^{-1}(T-t_1-t_2)})}\right)& \\ = \lim_{T\to\infty} N^{f,g}_{T-t_1-t_2,a^*}\mathbf{P}^{f,g}_{T-t_1-t_2,a^*}\left(A^{-1}(T-t_1-t_2) - b^*T \leq C\sqrt{T}\right)& \\ = b^*\langle f, x_{a^*}\rangle_{L^2}\langle g, x_{a^*}\rangle^{\circ}_{L^2}\,\mathcal{N}_{c^{*2}}((-\infty, C]),& \qquad (2.36)\end{aligned}$$

again according to Proposition 3. □

STEP 2 *For all $t_1, t_2 > 0$, and any $p, q > 1$ satisfying $\frac{1}{p}+\frac{1}{q} = 1$, the integrand on the r.h.s. of (2.19) is bounded uniformly in $T > 0$ by $W^{(1)}_p(t_1)W^{(2)}_q(t_2)$ defined in (2.32).*

Proof. Recall 3. and 4. in Subsection 2.2. Make a change of measure from $\widehat{\mathbf{E}}^{a^*}$ to $\widetilde{\mathbf{E}}^{a^*}$, use the Hoelder inequality and the stationarity of $(Y_t)_{t\geq 0}$ under $\widetilde{\mathbf{P}}^{a^*}$ (recall (2.15) and (2.17)), to obtain

$$\begin{aligned}&\widehat{\mathbf{E}}^{a^*}\left(\tfrac{w_{a^*}(X_0,t_1)}{x_{a^*}(X_0)}\, 1_{A^{-1}(T-t_1-t_2)\leq C_T}\, \tfrac{w_{a^*}(X_{A^{-1}(T-t_1-t_2)},t_2)}{x_{a^*}(X_{A^{-1}(T-t_1-t_2)})}\right)\\ &\leq \rho'(a^*)\widetilde{\mathbf{E}}^{a^*}\left(\tfrac{w_{a^*}(Y_0,t_1)}{Y_0 x_{a^*}(Y_0)}\, \tfrac{w_{a^*}(Y_{T-t_1-t_2},t_2)}{x_{a^*}(Y_{T-t_1-t_2})}\right)\\ &\leq \rho'(a^*)\left(\widetilde{\mathbf{E}}^{a^*}\left(\left[\tfrac{w_{a^*}(Y_0,t_1)}{Y_0 x_{a^*}(Y_0)}\right]^p\right)\right)^{\frac{1}{p}}\left(\widetilde{\mathbf{E}}^{a^*}\left(\left[\tfrac{w_{a^*}(Y_{T-t_1-t_2},t_2)}{x_{a^*}(Y_{T-t_1-t_2})}\right]^q\right)\right)^{\frac{1}{q}} \qquad (2.37)\\ &= W^{(1)}_p(t_1)\, W^{(2)}_q(t_2).\end{aligned}$$

□

STEP 3 *Conclusion of the proof.*

Proof. Let $T \to \infty$ in (2.19) and note that, for some $p, q > 1$ satisfying $\frac{1}{p}+\frac{1}{q} = 1$, the bound in Step 2 is integrable in $(t_1,t_2) \in (\mathbf{R}^+)^2$ by Proposition 4(ii) and (iii). By Steps 1-2 and the dominated convergence theorem we may interchange $T \to \infty$ and $\int_0^\infty dt_1\ \int_0^\infty dt_2$, to obtain

$$\begin{aligned}&\lim_{T\to\infty} \text{l.h.s. of (2.19)}\\ &= b^* \int_0^\infty dt_1\ \int_0^\infty dt_2\ \langle w_{a^*}(\cdot,t_1), x_{a^*}\rangle_{L^2}\langle w_{a^*}(\cdot,t_2), x_{a^*}\rangle^{\circ}_{L^2}\mathcal{N}_{c^{*2}}((-\infty, C]). \qquad (2.38)\end{aligned}$$

Use (2.31), Proposition 4(i) and Fubini's theorem to identify the r.h.s. of (2.38) as $S\mathcal{N}_{c^{*2}}((-\infty, C])$, with S given in (2.33). □

3 CLT for the middle piece

This section contains the proofs of Propositions 2 and 3.

3.1 Proof of Proposition 2

Recall Lemma 3 and (2.26) to see that the l.h.s. of (2.28) is equal to

$$e^{-\frac{\lambda\sqrt{T}}{\rho'(a)}}\int_0^\infty dh\ f(h)\mathbf{E}_h\left(e^{-\int_0^{A^{-1}(T')}\left(F_a(X_s)-\frac{\lambda}{\sqrt{T}}\right)ds}g(X_{A^{-1}(T')})\right). \tag{3.1}$$

According to (2.29), $\rho(a_{\lambda,T})=\rho(a)-\frac{\lambda}{\sqrt{T}}$. Since $T'=\int_0^{A^{-1}(T')}X_s\,ds$ (see (1.8)) and $F_a(u)=u^2-au+\rho(a)$ (see (2.1)), we may write the exponents in (3.1) as

$$\begin{aligned}&-\int_0^{A^{-1}(T')}F_{a_{\lambda,T}}(X_s)\,ds+(a-a_{\lambda,T})\int_0^{A^{-1}(T')}X_s\,ds-\frac{\lambda\sqrt{T}}{\rho'(a)}\\ &\quad=-\int_0^{A^{-1}(T')}F_{a_{\lambda,T}}(X_s)\,ds+T\left(a-a_{\lambda,T}-\frac{\lambda}{\sqrt{T}\rho'(a)}\right)+(T-T')(a_{\lambda,T}-a).\end{aligned} \tag{3.2}$$

Substitute this into (3.1) and use (2.26) to get that

$$\text{l.h.s. of (2.28)}=e^{T\left(a-a_{\lambda,T}-\frac{\lambda}{\sqrt{T}\rho'(a)}\right)}N^{f,g}_{T',a_{\lambda,T}}e^{(T-T')(a_{\lambda,T}-a)}. \tag{3.3}$$

Next, expand the inverse function ρ^{-1} of ρ as a Taylor series around $\rho(a)$ up to second order. It follows that there is an r_T inbetween $\rho(a)$ and $\rho(a)-\frac{\lambda}{\sqrt{T}}$ such that

$$\begin{aligned}a_{\lambda,T}&=\rho^{-1}(\rho(a)-\tfrac{\lambda}{\sqrt{T}})=\rho^{-1}(\rho(a))-\tfrac{\lambda}{\sqrt{T}}(\rho^{-1})'(\rho(a))+\tfrac{\lambda^2}{2T}(\rho^{-1})''(r_T)\\ &=a-\tfrac{\lambda}{\sqrt{T}\rho'(a)}-\tfrac{\lambda^2}{2T}\,\tfrac{\rho''}{(\rho')^3}\,(\rho^{-1}(r_T))=a-\tfrac{\lambda}{\sqrt{T}\rho'(a)}-\tfrac{\lambda^2}{2T}\,\sigma^2(\xi_T)\end{aligned} \tag{3.4}$$

(see (2.27)) with $\xi_T=\rho^{-1}(r_T)$. Observe that ξ_T is inbetween a and $a_{\lambda,T}$ by monotonicity of ρ. Now substitute (3.4) into (3.3) to arrive at (2.28).

3.2 Proof of Proposition 3

We shall use an expansion in terms of the eigenfunctions of the operator $\mathcal{M}^a$: $L^{2,\circ}(\mathbf{R}_0^+)\cap C^2(\mathbf{R}_0^+)\to C(\mathbf{R}_0^+)$ defined by

$$(\mathcal{M}^a x)(u)=\frac{(\mathcal{K}^a x)(u)-\rho(a)x(u)}{u} \tag{3.5}$$

(recall (0.5)). Obviously, $\mathcal{M}^a$ is a symmetric operator w.r.t. $\langle\cdot,\cdot\rangle^\circ_{L^2}$ because $\mathcal{K}^a$ is a symmetric operator w.r.t. $\langle\cdot,\cdot\rangle_{L^2}$. It is also a Sturm-Liouville operator. We are going to identify its eigenvalues and eigenvectors in terms of the ones of $\mathcal{K}^a$.

For $l \in \mathbf{N}_0$, let $\rho^{(l)}(a)$ denote the l-th largest eigenvalue of $\mathcal{K}^a$ and $x_a^{(l)} \in L^2(\mathbf{R}^+)$ the corresponding eigenfunction, normalized such that $||x_a^{(l)}||_{L^2} = 1$ (all eigenspaces are one-dimensional by HH, Lemma 20). Then $\rho^{(0)} = \rho$, and each $\rho^{(l)}$ is continuous and strictly increasing (differentiate the formula $\rho^{(l)}(a) = \langle x_a^{(l)}, \mathcal{K}^a x_a^{(l)} \rangle_{L^2}$ to obtain $\frac{d}{da}\rho^{(l)}(a) = ||x_a^{(l)}||_{L^2}^{\circ 2}$ via (0.5)). Moreover, $\lim_{a\to\pm\infty} \rho^{(l)}(a) = \pm\infty$. Since $x_a^{(l)}$ has a subexponentially small tail at infinity (see HH, Lemma 20), it is also an element of $L^{2,\circ}(\mathbf{R}_0^+)$.

Next, define $\alpha^{(l)}(a) \in \mathbf{R}$ and $y_a^{(l)} \in L^{2,\circ}(\mathbf{R}_0^+)$ by

$$\rho^{(l)}(a - \alpha^{(l)}(a)) = \rho(a) \qquad \text{and} \qquad y_a^{(l)} = \frac{x^{(l)}_{a-\alpha^{(l)}(a)}}{||x^{(l)}_{a-\alpha^{(l)}(a)}||^{\circ}_{L^2}} \qquad (l \in \mathbf{N}_0). \tag{3.6}$$

Note that $\alpha^{(0)}(a) = 0$, $y_a^{(0)} = x_a/\sqrt{\rho'(a)}$, and $\alpha^{(l+1)}(a) < \alpha^{(l)}(a)$ for all $l \in \mathbf{N}_0$ since $\rho^{(l)}(a)$ is strictly decreasing in l and strictly increasing in a.

STEP 1 *For each $a \in \mathbf{R}$, the sequence $(y_a^{(l)})_{l\in\mathbf{N}_0}$ is an orthonormal basis in $L^{2,\circ}(\mathbf{R}^+)$.*

Proof. Since $\mathcal{M}^a$ is a symmetric Sturm-Liouville operator, all the eigenspaces are orthogonal to each other and one-dimensional, and they span the space $L^{2,\circ}(\mathbf{R}^+)$. Thus, it suffices to show that the functions $y_a^{(0)}, y_a^{(1)}, \dots$ are all the eigenfunctions of $\mathcal{M}^a$. Now, from (0.5) and (3.5) we easily derive the equivalence

$$\mathcal{M}^a x = \alpha x \qquad \Longleftrightarrow \qquad \mathcal{K}^{a-\alpha} x = \rho(a) x, \tag{3.7}$$

which is valid for every $a, \alpha \in \mathbf{R}$ and $x \in C^2(\mathbf{R}_0^+)$. From (3.6) and (3.7) we see that $(\alpha^{(l)}(a))_{l\in\mathbf{N}_0}$ is the sequence of all the eigenvalues of $\mathcal{M}^a$ with corresponding eigenfunctions $(y_a^{(l)})_{l\in\mathbf{N}_0}$, since (3.7) implies that for every eigenvalue α of $\mathcal{M}^a$, there is an $l \in \mathbf{N}_0$ such that $\rho^{(l)}(a - \alpha) = \rho(a)$. $\square$

STEP 2 *For every $h, T \geq 0$, $l \in \mathbf{N}_0$ and $a \in \mathbf{R}$,*

$$\widehat{\mathbf{E}}_h^a \left(\frac{y_a^{(l)}}{x_a}(Y_T) \right) = e^{\alpha^{(l)}(a)T} \, \frac{y_a^{(l)}}{x_a}(h). \tag{3.8}$$

Proof. Use (2.7) and (2.14) to compute, for $f \in C^2(\mathbf{R}^+)$,

$$\left(\widetilde{G}^a \left(\frac{f}{x_a} \right) \right)(u) = \frac{f(u)}{u x_a(u)} \left(\frac{2uf''(u) + 2f'(u)}{f(u)} - \frac{2ux_a''(u) + 2x_a'(u)}{x_a(u)} \right). \tag{3.9}$$

Apply this for $f = y_a^{(l)}$, use (0.5) and the eigenvalue relation $\mathcal{K}^{a'} x_{a'}^{(l)} = \rho^{(l)}(a') x_{a'}^{(l)}$ for $(a', l) = (a, 0)$ and for $(a', l) = (a - \alpha^{(l)}(a), l)$ to obtain

$$\widetilde{G}^a \left(\frac{y_a^{(l)}}{x_a} \right) = \alpha^{(l)}(a) \frac{y_a^{(l)}}{x_a}. \tag{3.10}$$

Thus, $\widetilde{G}^a$ being the generator of the process $(Y_t)_{t\geq 0}$, the function $f(T) = \widehat{\mathbf{E}}_h^a \left(\frac{y_a^{(l)}}{x_a}(Y_T) \right)$ satisfies the differential equation $f' = \alpha^{(l)}(a) f$. Therefore $f(T) = e^{\alpha^{(l)}(a)T} f(0)$, which is our assertion. $\square$

STEP 3 *Conclusion of the proof.*

Proof. According to Step 1, we may expand $g \in L^{2,\circ}(\mathbf{R}_0^+)$ as

$$g = \sum_{l=0}^{\infty} y_{a_T}^{(l)} \langle g, y_{a_T}^{(l)} \rangle_{L^2}^{\circ} = \frac{x_{a_T}}{\rho'(a_T)} \langle g, x_{a_T} \rangle_{L^2}^{\circ} + \sum_{l=1}^{\infty} y_{a_T}^{(l)} \langle g, y_{a_T}^{(l)} \rangle_{L^2}^{\circ} \qquad (T \geq 0). \tag{3.11}$$

Substitute this into (2.26) to obtain (recall (2.10) and (2.12))

$$\begin{aligned} &\left| N_{T,a_T}^{f,g} - \tfrac{1}{\rho'(a)} \langle f, x_a \rangle_{L^2} \langle g, x_a \rangle_{L^2}^{\circ} \right| \\ &\leq \left| \tfrac{1}{\rho'(a_T)} \langle f, x_{a_T} \rangle_{L^2} \langle g, x_{a_T} \rangle_{L^2}^{\circ} - \tfrac{1}{\rho'(a)} \langle f, x_a \rangle_{L^2} \langle g, x_a \rangle_{L^2}^{\circ} \right| \\ &\qquad + \textstyle\sum_{l=1}^{\infty} \left| \left(\int_0^{\infty} dh \; f(h) x_{a_T}(h) \widehat{\mathbf{E}}_h^a \left(\frac{y_{a_T}^{(l)}}{x_{a_T}}(Y_T) \right) \right) \langle g, y_{a_T}^{(l)} \rangle_{L^2}^{\circ} \right|. \end{aligned} \tag{3.12}$$

With the help of Step 2, the second term on the r.h.s. of (3.12) equals

$$\begin{aligned} &\textstyle\sum_{l=1}^{\infty} e^{\alpha^{(l)}(a_T)T} \left| \left(\int_0^{\infty} dh \; f(h) x_{a_T}(h) \frac{y_{a_T}^{(l)}}{x_{a_T}}(h) \right) \langle g, y_{a_T}^{(l)} \rangle_{L^2}^{\circ} \right| \\ &\leq e^{\alpha^{(1)}(a_T)T} \textstyle\sum_{l=0}^{\infty} \left| \langle \frac{f}{\mathrm{id}}, y_{a_T}^{(l)} \rangle_{L^2}^{\circ} \langle g, y_{a_T}^{(l)} \rangle_{L^2}^{\circ} \right| \\ &\leq e^{\alpha^{(1)}(a_T)T} \sqrt{\textstyle\sum_{l=0}^{\infty} \left(\langle \frac{f}{\mathrm{id}}, y_{a_T}^{(l)} \rangle_{L^2}^{\circ} \right)^2} \sqrt{\textstyle\sum_{l=0}^{\infty} \left(\langle g, y_{a_T}^{(l)} \rangle_{L^2}^{\circ} \right)^2} \\ &= e^{\alpha^{(1)}(a_T)T} \| \tfrac{f}{\mathrm{id}} \|_{L^2}^{\circ} \| g \|_{L^2}^{\circ}. \end{aligned} \tag{3.13}$$

This tends to zero as $T \to \infty$ since $\lim_{T\to\infty} \alpha^{(1)}(a_T) = \alpha^{(1)}(a) < 0$. The first term on the r.h.s. of (3.12) vanishes as $T \to \infty$ because of the continuity of $a \mapsto x_a \in L^2(\mathbf{R}^+)$ and $a \mapsto \rho'(a)$ (see HH, Lemma 22). $\square$

4 Integrability for the boundary pieces

This section contains the proof of Proposition 4. It turns out that the functions w_a (in (2.18)) and y_a (in (2.31)) have a nice representation in terms of standard one-dimensional Brownian motion, and that y_a is a transformation of the Airy function. This will be explored in Subsection 4.2. Subsection 4.1 contains some preparations.

4.1 Preparations

Let $\mathrm{Ai} : \mathbf{R} \to \mathbf{R}$ denote the Airy function, i.e., the unique (modulo a constant multiple) solution of the Airy equation

$$x''(u) - ux(u) = 0 \qquad (u \in \mathbf{R}) \tag{4.1}$$

that is bounded on $\mathbf{R}_0^+$. Let $u_1 = \sup\{u \in \mathbf{R} \mid \mathrm{Ai}(u) = 0\}$ be its largest zero. From Abramowitz and Stegun (1970), Table 10.13 and p. 450, it is known that $u_1 = -2,3381\dots$. For $a < -2^{\frac{1}{3}}u_1$, define $z_a : \mathbf{R}_0^+ \to \mathbf{R}^+$ by

$$z_a(u) = \frac{\mathrm{Ai}\left(2^{-\frac{1}{3}}(u-a)\right)}{\mathrm{Ai}\left(-2^{-\frac{1}{3}}a\right)} \qquad (u \geq 0). \tag{4.2}$$

In Lemma 8 in Subsection 4.2, z_a will turn out to be equal to y_a. Some of its properties are given in the following lemma.

Lemma 5 *For all $a < -2^{\frac{1}{3}}u_1$, the function z_a is real-analytic, strictly positive on $\mathbf{R}_0^+$ with $z_a(0) = 1$, and satisfies*

$$2z_a''(u) + (a-u)\,z_a(u) = 0 \qquad (u \geq 0). \tag{4.3}$$

Moreover,

$$\lim_{u\to\infty} u^{-\frac{3}{2}} \log z_a(u) < 0. \tag{4.4}$$

Proof. It is well known that Ai is analytic. From (4.2) and the definition of u_1 it is clear that $z_a(0) = 1$ and that $z_a(u) > 0$ for $u \geq 0$. Equation (4.3) follows easily from (4.1). The asymptotics in (4.4) follows from Abramowitz and Stegun (1970), 10.4.59. □

The following lemma shows in particular that Lemma 5 can be used for $a = a^*$.

Lemma 6 $a^* \leq \frac{3}{2}\pi^{\frac{1}{3}} < -u_1$.

Proof. The first inequality is proved via the variational representation

$$a^* = \inf_{x\in L^2(\mathbf{R}_0^+)\cap C^2(\mathbf{R}_0^+):\, x\neq 0} \frac{\int_0^\infty [u^2x^2(u) + 2ux'(u)^2]\,du}{\int_0^\infty ux^2(u)\,du}. \tag{4.5}$$

This representation stems from the relation (see HH, Sect. 5.1)

$$0 = \rho(a^*) = \max_{x\in L^2(\mathbf{R}_0^+)\cap C^2(\mathbf{R}_0^+):||x||_{L^2}=1} \langle x, \mathcal{K}^{a^*}x\rangle_{L^2}, \tag{4.6}$$

in which, by (0.5),

$$\langle x, \mathcal{K}^{a^*}x\rangle_{L^2} = \int_0^\infty \left[(a^*u - u^2)x(u)^2 - 2ux'(u)^2\right] du. \tag{4.7}$$

In (4.5), we choose the test function

$$x(u) = \exp\left(-u^2\frac{\pi^{\frac{1}{3}}}{8}\right). \tag{4.8}$$

Elementary computations give that $\int_0^\infty ux^2(u)\,du = 2\pi^{-\frac{1}{3}}$ and $\int_0^\infty u^2x^2(u)\,du = 2$ and $\int_0^\infty ux'(u)^2\,du = \frac{1}{2}$. Substituting this into (4.5), we obtain the bound $a^* \leq \frac{3}{2}\pi^{\frac{1}{3}} = 2.1968\dots$. □

4.2 Proof of Proposition 4

Let P_h be the distribution of standard one-dimensional Brownian motion $(B_t)_{t\geq 0}$ conditioned on starting at h and let E_h be the corresponding expectation. Define

$$T_u = \inf\{\, t \geq 0 : B_t = u \,\} \qquad (u \in \mathbf{R}). \tag{4.9}$$

Lemma 7 *For every $a \in \mathbf{R}$ and $h, t > 0$,*

$$\begin{aligned} w_a(h,t) &= e^{at} E_{\frac{h}{2}} \left(e^{-\int_0^t 2B_s\, ds} \Big| \, T_0 = t \right) \varphi_h(t), \\ \varphi_h(t) &= \frac{P_{\frac{h}{2}}(T_0 \in dt)}{dt} = \frac{h}{2\sqrt{2\pi t^3}}\, e^{-\frac{h^2}{8t}}. \end{aligned} \tag{4.10}$$

Consequently,

$$y_a(h) = E_{\frac{h}{2}} \left(e^{\int_0^{T_0} (a - 2B_s)\, ds} \right) \in [0, \infty]. \tag{4.11}$$

Proof. Recall (1.9). According to Ethier and Kurtz (1986), Th. 6.1.4, the process $(Y_t^\star)_{t\geq 0} = (X^\star_{A^{\star -1}(t)})_{t \geq 0}$ is a diffusion with generator (see (1.4))

$$\left(\widetilde{G}^\star f\right)(u) = \frac{1}{u}\left(G^\star f\right)(u) = 2 f''(u) \qquad (f \in C_c^2(\mathbf{R}^+)). \tag{4.12}$$

In other words, the distribution of $(Y_t^\star)_{t\geq 0}$ under $\mathbf{P}_h^\star$ is equal to that of $(B_{4t\wedge T_0})_{t\geq 0}$ under P_h, which in turn is equal to that of $(2B_{t\wedge T_0})_{t \geq 0}$ under $P_{\frac{h}{2}}$. Thus, noting that $\frac{d}{dt} A^{\star -1}(t) = 1/X^\star_{A^{\star-1}(t)}$ and hence $\int_0^{A^{\star -1}(t)} X_v^{\star 2}\, dv = \int_0^t X^\star_{A^{\star -1}(s)}\, ds$, we have

$$\begin{aligned} \mathbf{E}_h^\star \left(e^{-\int_0^\infty X_v^{\star 2}\, dv} \,\Big|\, A^\star(\infty) = t \right) &= \mathbf{E}_h^\star \left(e^{-\int_0^{\xi_0} X_v^{\star 2}\, dv} \,\Big|\, A^\star(\xi_0) = t \right) \\ = \mathbf{E}_h^\star \left(e^{-\int_0^{A^{\star -1}(t)} X_v^{\star 2}\, dv} \,\Big|\, A^{\star -1}(t) = \xi_0 \right) &= E_{\frac{h}{2}} \left(e^{-\int_0^t 2B_s\, ds} \,\Big|\, T_0 = t \right), \end{aligned} \tag{4.13}$$

which proves the first formula in (4.10) (see (2.18)). In the same way, we see that φ_h defined in (1.10) equals the Lebesgue density of T_0 under $P_{\frac{h}{2}}$, and its explicit shape is found in RY, p. 102. Finally, the representation (4.11) is a direct consequence of (2.31). $\square$

Proof of Proposition 4(i). In view of Lemmas 5-6, the following lemma implies Proposition 4(i).

Lemma 8 *$z_a = y_a$ for all $a < -2^{\frac{1}{3}} u_1$.*

Proof. Since $y_a(0) = z_a(0) = 1$ and since z_a is bounded on $\mathbf{R}_0^+$, it suffices to show that y_a satisfies the same differential equation as z_a (see (4.3)). But this easily follows from the argument in the proof of KS, Th. 4.6.4.3, picking (in the notation used there) $\alpha = a < -2^{\frac{1}{3}} u_1$, $k(u) = u$, $\gamma_l = 0$, $b = 0$, and $c = \infty$. $\square$

□

Proof of Proposition 4(ii) and (iii). Fix $p \in (1,2)$ and $q \in (2,\infty)$. In the following, we use c as a generic positive constant, possibly varying from line to line.

STEP 1 $W_p^{(1)}$ *is integrable at zero.*

Proof. Use (4.10) to estimate $w_{a^*}(h,t) \le ct^{-\frac{3}{2}} h e^{-\frac{h^2}{8t}}$ for any $h \ge 0$ and $t \in (0,1]$. Using the boundedness of $x_{a^*}^{2-p}$ on $\mathbf{R}^+$, this gives

$$W_p^{(1)}(t) \le c \left(\int_0^\infty h^{1-p} h^p t^{-\frac{3p}{2}} e^{-\frac{ph^2}{8t}}\, dh \right)^{\frac{1}{p}}$$

$$= ct^{-\frac{3}{2}} \left(\int_0^\infty h e^{-\frac{ph^2}{8t}}\, dh \right)^{\frac{1}{p}}$$

$$= ct^{\frac{1}{p}-\frac{3}{2}}, \tag{4.14}$$

which is integrable at zero. □

STEP 2 $W_q^{(2)}$ *is integrable at zero.*

Proof. Use $h^{1+q} e^{-\frac{qh^2}{16t}} \le ct^{\frac{1+q}{2}}$ for $t \in (0,1]$ and (as in Step 1) use (4.10) to estimate $w_{a^*}(h,t) \le ct^{-\frac{3}{2}} h e^{-\frac{h^2}{8t}}$ for any $h \ge 0$ and $t \in (0,1]$. This gives

$$W_q^{(2)}(t) \le ct^{-\frac{3}{2}} \left(\int_0^\infty h x_{a^*}(h)^{2-q} h^q e^{-\frac{qh^2}{8t}}\, dh \right)^{\frac{1}{q}}$$

$$= ct^{-\frac{3}{2}} \left(\int_0^\infty x_{a^*}(h)^{2-q} t^{\frac{1+q}{2}} e^{-\frac{qh^2}{16t}}\, dh \right)^{\frac{1}{q}}$$

$$\le ct^{\frac{1}{2q}-1} \left(\int_0^\infty x_{a^*}(h)^{2-q} e^{-\frac{qh^2}{16}}\, dh \right)^{\frac{1}{q}}. \tag{4.15}$$

The integral is finite for any $q > 2$ since $\lim_{h\to\infty} h^{-\frac{3}{2}} \log x_{a^*}(h)$ is finite (see the beginning of Subsection 2.1). Thus, the r.h.s. of (4.15) is integrable in t at zero. □

STEP 3 $W_p^{(1)}$ *is integrable at infinity.*

Proof. Since $t \mapsto t^{-\frac{3}{2}}$ is a probability density on $[4,\infty)$, Jensen's inequality (and the boundedness of $x_{a^*}^{2-p}$ on $\mathbf{R}^+$) give

$$\int_4^\infty W_p^{(1)}(t)\, dt \le c \int_4^\infty \left(\int_0^\infty h^{1-p} t^{\frac{3p}{2}} w_{a^*}(h,t)^p\, dh \right)^{\frac{1}{p}} t^{-\frac{3}{2}}\, dt$$

$$\le c \left(\int_0^\infty \int_0^\infty h^{1-p} t^{\frac{3}{2}(p-1)} w_{a^*}(h,t)^p\, dt\, dh \right)^{\frac{1}{p}}. \tag{4.16}$$

Use (4.10), Jensen's inequality for the conditioned expectation, and the Brownian scaling property to estimate

$$w_{a^*}(h,t)^p \le \varphi_h(t)^{p-1}\varphi_h(t)\, E_{\frac{h}{2}}\left(e^{a^*pt - p\int_0^t 2B_s\,ds} \,\middle|\, T_0 = t\right)$$

$$\le ch^{p-1}t^{\frac{3}{2}(1-p)}\varphi_{hp^{\frac{1}{3}}}(tp^{\frac{2}{3}})\, E_{\frac{hp^{\frac{1}{3}}}{2}}\left(e^{a^*p^{\frac{1}{3}}tp^{\frac{2}{3}} - \int_0^{tp^{\frac{2}{3}}} 2B_s\,ds} \,\middle|\, T_0 = tp^{\frac{2}{3}}\right)$$

$$= ch^{p-1}t^{\frac{3}{2}(1-p)}w_{a^*p^{\frac{1}{3}}}(hp^{\frac{1}{3}}, tp^{\frac{2}{3}}). \tag{4.17}$$

Substitute this into (4.16) to get

$$\left(\int_4^\infty W_p^{(1)}(t)\,dt\right)^p \le c\int_0^\infty z_{a^*p^{\frac{1}{3}}}(hp^{\frac{1}{3}})\,dh. \tag{4.18}$$

This is finite by (4.4) (note Lemma 6). □

STEP 4 $W_q^{(2)}$ *is integrable at infinity if* $q \in (2,\infty)$ *is sufficiently close to 2.*

Proof. If we estimate in the same way as in (4.16) and in (4.17), but do not estimate $x_{a^*}(h)^{2-q}$ then we end up with

$$\left(\int_4^\infty W_q^{(2)}(t)\,dt\right)^q \le c\int_0^\infty h^q x_{a^*}(h)^{2-q} z_{a^*q^{\frac{1}{3}}}(hq^{\frac{1}{3}})\,dh. \tag{4.19}$$

For q sufficiently close to 2, we have $a^*q^{\frac{1}{3}} < -2^{\frac{1}{3}}u_1$ (see Lemma 6) and may apply (4.4). Now use that $\lim_{h\to\infty} h^{-\frac{3}{2}}\log x_{a^*}(h)$ is finite to deduce that the r.h.s. of (4.19) is finite for q sufficiently close to 2. □

□

References

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (9th edition), Dover Publications, Inc., New York 1970.

[2] P. Biane, J.-F. Le Gall and M. Yor, Un processus qui ressemble au pont Brownien, Sém. Prob. XXI, LNM 1247, Springer, 270–275 (1987).

[3] P. Biane and M. Yor, Sur la loi des temps locaux Browniens pris en un temps exponential, Sém. Prob, XXII, LNM 1321, Springer, 454–466 (1988).

[4] S.N. Ethier and T.G. Kurtz, *Markov processes*, Wiley, New York 1986.

[5] P. Fitzsimmons, J. Pitman and M. Yor, Markovian Bridges: Construction, Palm intersections, and Splicing. In: Seminar on Stochastic Processes, 102–133 (1992). Basel Boston Berlin: Birkhäuser (1993).

[6] R. van der Hofstad and F. den Hollander, Scaling for a random polymer, Commun. Math. Phys. 169, 397–440 (1995).

[7] T. Jeulin, Application de la théorie du grossisement à l'étude du temps locaux Browniens, in: Grossisements de filtrations, LNM 1118, Springer, 197–304 (1985).

[8] I. Karatzas and S.E. Shreve, *Brownian Motion and Stochastic Calculus* (2nd edition), Springer, New York 1991.

[9] W. König, A central limit theorem for a one-dimensional polymer measure. Preprint 1994. To appear in The Annals of Probability.

[10] E. Perkins, Local time is a semimartingale, Z. f. W. 60, 79–117 (1982).

[11] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer, Berlin 1991.

[12] J. Westwater, On Edwards model for polymer chains, in: *Trends and Developments in the Eighties* (S. Albeverio and P. Blanchard, eds.), Bielefeld Encounters in Math. Phys. 4/5, World Scientific, Singapore 1984.