On L^p estimates for the Laplacian on a class of symmetric spaces of noncompact type

Michael Ruzhansky

Abstract

The domination properties of the Laplace operator on a class of symmetric spaces of noncompact type are investigated. Using algebraic methods we will show that derivatives of a function can be uniformly estimated by function and its Laplacian in L^p spaces for all $1 \leq p \leq \infty$. We will also discuss some relative aspects of the theory of convolutions.

1991 *Mathematics subject classification (Amer.Math.Soc)*: 58G30, 43A15, 53C35

1 Introduction

In this paper we will investigate the question of the domination of the first order derivatives by the Laplace operator for a class of symmetric spaces. We are interested in the estimates of the derivative of a function by the function and its Laplacian in L^p spaces for $1 \leq p \leq \infty$. Similar estimates are known for $p = \infty$ for manifolds with bounded curvature ([1]). On the other hand estimates for $1 < p < \infty$ are well known for Euclidean spaces ([7]) and are closely related to the question of the continuity of pseudo-differential operators of order zero, which hold only locally on general manifolds. We will give a unifying algebraic proof for global estimates in L^p-space for all $1 \leq p \leq \infty$ in a class of symmetric spaces of the noncompact type.

Let M be a Riemannian manifold, which we assume to be a Riemannian symmetric space of the noncompact type. This means, that it can be viewed as a quotient $M = G/H$, where G is a connected semisimple Lie group with trivial center and H is its maximal compact subgroup. The Riemannian structure on M will be supposed to be invariant under the left action of G. We also assume G to be a complex Lie group. One of the important examples of such spaces is three dimensional hyperbolic geometry.

Let p be the origin of the manifold M and H an isotropy subgroup (stabilizer) of p in G. The Laplace-Beltrami operator associated to the Riemannian structure of M will be denoted by Δ.

Let $X \in \Gamma^\infty(TM)$ be a smooth vector field on M. X is called *bounded* if there exist a constant C such that $||X_x|| \leq C$ for every point $x \in M$, where $|| \cdot || =< \cdot, \cdot >^{1/2}$ is the Riemannian norm on T_xM, corresponding to the Riemannian structure. All the distributions we consider are complex, which means that their test functions are complex valued.

We will prove the following domination property of the Laplacian.

Theorem 1 *Let X be a smooth bounded vector field on M and let $1 \leq p \leq \infty$. Then there exist constants A, B such that for every $u \in L^p(M)$ satisfying $\Delta u \in L^p(M)$, the derivative Xu of u with respect to X is an element of $L^p(M)$ and*

$$||Xu||_p \leq A||u||_p + B||\Delta u||_p. \tag{1}$$

Moreover, A and B can be chosen uniformly over the set of vector fields X bounded by 1: $||X|| \leq 1$. If $p = \infty$, then Xu is a bounded continuous function.

In the next section we discuss invariant Riemannian structures on G. In particular we will construct one for which the canonical projection $\pi : G \to M$ becomes a Riemannian submersion. In Section 3 we discuss the actions of π and several definitions of the convolution on a symmetric space. One usually starts with convolution on a Lie group and can define the convolution of two distributions directly either by using their tensor product ([5]) or by "integrating" one with respect to another ([2]). We intend to use more "practical" definition, gradually defining the convolution of two test functions, a test function and a distribution, two distributions. Then we make use of the so called "flat" and "sharp" mappings to define the convolution on a symmetric space. Finally, we review Young type inequalities and establish several auxiliary results. The details concerning convolutions on general locally compact groups can be found in ([4]) and its relation to the convolutions on symmetric spaces in ([2]).

In Section 4 we construct H-invariant fundamental solution for the Laplace operator (Lemma 6) and give H-invariant representation of delta function (Lemma 8). Section 5 consists of a proof of Theorem 1. First we establish the convolutive representation of the identity mapping on a subspace of the space of distributions both on G and M and then reduce the whole problem to the integrability properties of the fundamental solution of the Laplace operator. The last is shown using Helgason formula ([3]) and its relation to the Neumann functions. For that we will also establish some useful facts concerning analiticity of the Bessel functions (Lemma 10).

Acknowledgements
I would like to thank E.G.F.Thomas for drawing my attention to the subject and many valuable discussions. I would also like to thank J.J.Duistermaat for broadening my understanding of the topic and J.A.C.Kolk for his remarks about contents and presentation of the material.

2 Riemannian structures on G

In this section we will fix our geometrical notation and make some preliminary constructions. We will need a special Riemannian structure on G, namely an invariant Riemannian structure, which is H–right invariant and such that the canonical projection $\pi : G \to M$ is a Riemannian submersion, which means that it is a submersion, for which horizontal lift of vector fields preserves Riemannian norms.

From now on the left (right) group action on G considered as a homogeneous space will be denoted by l_g (r_g) : $l_g h = gh$, $r_g h = hg$, $g, h \in G$. One can consider the diffeomorphism $l_g r_{g^{-1}}$ of G for $g \in G$. Its differential at the identity e of G is usually denoted by $\mathrm{Ad}(g)$ and is an automorphism of T_eG, the tangent space to G at e, which is isomorphic to the Lie algebra $\mathfrak{g}$ of G. The mapping $\mathrm{Ad} : G \to GL(T_eG)$ is usually called *the adjoint representation* of G and is a homomorphism of groups.

The induced action of G on M will be also denoted by l_g : if $g \in G$ and $x = \pi(g') \in M$, then $l_g(x) = \pi(gg')$. One readily checks that this definition is independent of the choice of g'. We will also make occasional use of the following notation. We notice that $\pi(g) = \pi(ge) = l_g(\pi(e)) = l_g(p)$, p being the origin of M, and we will denote it also by gp, namely $\pi(g) = gp$.

Due to the compactness of H the set $\mathrm{Ad}(H)$ is compact in $GL(T_eG)$. Let μ be left Haar measure on $\mathrm{Ad}(H)$ which is automatically bi-invariant (modular function is 1 identically). Let $\langle\cdot,\cdot\rangle_0$ be arbitrary inner product on T_eG. Then one can define $\mathrm{Ad}(H)$–invariant inner product $(\cdot,\cdot)_0$ on T_eG by taking

$$(X, Y)_0 := \int_{\mathrm{Ad}(H)} \langle A(X), A(Y)\rangle_0 d\mu(A) \tag{2}$$

for $X, Y \in T_eG$. Applying $d_e l_g$ to $(\cdot,\cdot)_0$, one defines an invariant Riemannian structure on G. This Riemannian structure is H–right invariant as well because of formula (2).

This shows the existence of G–left and H–right invariant structure on G. The tangent space T_pM to M at $p = \pi(e)$ can be identified with the quotient T_eG/T_eH via an isomorphism $i([X]) = d\pi(X)$, where $[X] \in T_eG/T_eH$ is an equivalence class of $X \in T_eG$. For every $g \in G$ let $K_g = \mathrm{Ker}\ d_g\pi$ (this $K_g \cong T_gH \cong T_eH$). Let N_g be the orthogonal complement to K_g with respect to $(\cdot,\cdot)_0$: $T_gG = K_g \bigoplus N_g$. Define an inner product $(\cdot,\cdot)_{N_e}$ on N_e for vectors $\bar{X}, \bar{Y} \in N_e$ by

$$(\bar{X}, \bar{Y})_{N_e} :=< X, Y >_{M_p}, \tag{3}$$

where $< \cdot,\cdot >_{M_p}$ is restriction to T_pM of the given invariant Riemannian structure on M and $X = d\pi(\bar{X}), Y = d\pi(\bar{Y}) \in T_pM$. Vectors $\bar{X}$ and $\bar{Y}$ are uniquely defined, $d\pi$ being an isomorphism of N_e and T_pM, and they are called the *horizontal lifts* of X and Y. Note, that $(\cdot,\cdot)_{N_e}$ can differ from the restriction of $(\cdot,\cdot)_0$ to N_e, because we started with an arbitrary inner product $\langle\cdot,\cdot\rangle_0$ on G. Applying the differential $d_e l_g$ of the left regular action of G, we can define a left invariant Riemannian structure on the tangent subbundle N_g for all $g \in G$. This structure will be denoted by $(\cdot,\cdot)_N$. It is not difficult to check that $(\cdot,\cdot)_{N_e}$ is $\mathrm{Ad}(H)$–invariant and thus the corresponding Riemannian structure $(\cdot,\cdot)_N$ is H–right invariant.

Now we are ready to compose two structures $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_N$ in order to construct one making π a Riemannian submersion. Define on T_eG for $X, Y \in T_eG$

$$\langle X, Y\rangle := (X|_{K_e}, Y|_{K_e})_0 + (X|_{N_e}, Y|_{N_e})_{N_e}, \tag{4}$$

where $X|_{K_e}, Y|_{K_e}$ and $X|_{N_e}, Y|_{N_e}$ are projections of X, Y on K_e and N_e respectively. This inner product is clearly $\mathrm{Ad}(H)$–invariant and can be expanded by applying dl_g G–left and H–right invariantly on all T_gG. From formulas (3) and (4) it follows immediately that all $d_g\pi$ are partial isometries (isometries from N_g to $T_{\pi(g)}M$),

$$\langle \bar{X}, \bar{Y}\rangle = (\bar{X}, \bar{Y})_{N_e} =< X, Y >_{M_p},$$

$\bar{X}, \bar{Y} \in N_e \subset T_eG$ being the horizontal lifts of $X, Y \in T_pM$. This means that π is a Riemannian submersion.

3 Actions of the projection π

In this section we will fix the algebraic notation and establish some auxiliary results concerning convolutions on symmetric spaces. We adopt Schwartz's notation $\mathcal{E}(M)$ for the space $\mathcal{C}^\infty(M)$ of smooth functions on M and $\mathcal{D}(M)$ for smooth compactly supported functions.

The group G acts on $\mathcal{E}(M)$ by left regular action τ_g, $\tau_g\phi(x) = \phi(l_{g^{-1}}x) = \phi(g^{-1}x)$, $\phi \in \mathcal{E}(M)$. The restriction of τ_g on $\mathcal{D}(M)$ gives a representation of G in $\mathcal{D}(M)$, the space of test functions on M. Let $T \in \mathcal{D}'(M)$ be a distribution on M. Then distribution τ_gT is defined by

$$< \tau_gT, \phi >=< T, \tau_{g^{-1}}\phi >,$$

where $< \cdot, \cdot >$ here stands for the duality relation between spaces $\mathcal{D}'(M)$ and $\mathcal{D}(M)$. The restriction of τ_g to $\mathcal{E}'(M)$ gives the left regular representation of G in $\mathcal{E}'(M)$, the space of compactly supported distributions on M.

In the same way one defines the left regular representation of G in the space $\mathcal{D}'(G)$ of distributions on G, starting with $\phi \in \mathcal{E}(G)$ as above. The right regular representation R of G in $\mathcal{D}'(G)$ is defined by $R_g\phi(k) = \phi(r_{g^{-1}}k) = \phi(kg^{-1})$, $g, k \in G$, $\phi \in \mathcal{D}(G)$ and $< R_gT, \phi >=< T, R_{g^{-1}}\phi >$ for $T \in \mathcal{D}'(G)$.

Let $T \in \mathcal{D}'(M)$ be *H–invariant*: $\tau_hT = T$ for all $h \in H$. The space of all such distributions will be denoted by $\mathcal{D}'(M)^H$. We want to define an action of M on $\mathcal{D}'(M)^H$. Note, that M has no group structure in general, when H is not a normal subgroup of G. Let $x = l_g(p) = gp \in M$. Define τ_x by $\tau_xT \stackrel{\mathrm{def}}{=} \tau_gT$. One readily checks that this definition is independent of the choice of a representative of the equivalence class $x = \pi(g)$.

Continuous inclusions $\mathcal{D}(M), \mathcal{E}(M), \mathcal{E}'(M) \hookrightarrow \mathcal{D}'(M)$ induce action τ_x of M on spaces $\mathcal{D}(M)^H, \mathcal{E}(M)^H, \mathcal{E}'(M)^H$, which will be also denoted by τ_x.

An invariant measure on M has the following property, which, in fact, can be taken as its definition:

$$\int_M f(x)dx = \int_G (f \circ \pi)(g)dg, \tag{5}$$

where dg is Haar measure on G. It turns out that any other invariant measure on M is proportional to dx, the Riemannian measure in particular, cf.[2, Thm.1.9,p.90 and Remark,p.93]. Note also, that we can make no difference between left and right Haar measures, G being semisimple, hence unimodular.

One of our next goals is to define convolution on M and fix all the relative notation. The convolution of two distributions $T, S \in \mathcal{D}'(G)$, one of which is compactly supported, can be defined by

$$< T * S, \phi >=< T \oplus S, \Phi >,$$

where $\phi \in \mathcal{D}(G)$, $\Phi(x,y) = \phi(xy)$ and $T \oplus S$ stands for the tensor product of T and S.

However, the following, more practical definition, will be also useful. Let $\phi, \psi \in \mathcal{C}(G)$, one of them compactly supported. Then by the convolution of ϕ and ψ we mean function

$$(\phi * \psi)(g) = \int_G R_k\phi(g)\psi(k)dk = \int_G \phi(k)\tau_k\psi(g)dk, \tag{6}$$

where dk is Haar measure on G. For $\phi \in \mathcal{D}(G)$ define $\check{\phi} \in \mathcal{D}(G)$ by $\check{\phi}(g) = \phi(g^{-1})$. Then, for $T \in \mathcal{E}'(G)$ the mapping $\phi \to \phi * T$, defined by

$$< \phi * T, \psi >=< T, \check{\phi} * \psi >, \ \psi \in \mathcal{D}(G) \tag{7}$$

is an endomorphism of $\mathcal{D}(G)$. Now, for $S \in \mathcal{D}'(G)$ and $T \in \mathcal{E}'(G)$ we define

$$< S * T, \phi >=< S, \phi * \check{T} >,$$

where distribution $\check{T}$ is defined by $< \check{T}, \phi >=< T, \check{\phi} >$.

We will need a "flat" mapping described in the following simple Lemma, a proof of which is left to the reader.

Lemma 1 *Let $\phi \in \mathcal{C}_c(G)$ be a continuous compactly supported function on G. Then for $x = \pi(g)$ the function $\phi^\flat(x)$ is correctly defined by*

$$\phi^\flat(x) = \int_H \phi(gh)dh, \tag{8}$$

where dh is the normalized Haar measure on H. The function $\phi^\flat$ is continuous and compactly supported on M: $\phi^\flat \in \mathcal{C}_c(M)$. Moreover, mapping $\phi \to \phi^\flat$ is linear surjective from $\mathcal{C}_c(G)$ to $\mathcal{C}_c(M)$.

Corollary 1 *The mapping $\phi \to \phi^\flat$, restricted to $\mathcal{D}(G)$ is a linear surjective mapping from $\mathcal{D}(G)$ to $\mathcal{D}(M)$. The transpose of this restriction is then correctly defined by*

$$< T^\sharp, \phi >=< T, \phi^\flat >, \tag{9}$$

where $T \in \mathcal{D}'(M)$ and the mapping $T \to T^\sharp$ maps $\mathcal{D}'(M) \to \mathcal{D}'(G)$ injectively.

The second assertion of the Corollary follows from the surjectivity of $\phi \to \phi^\flat$.

Lemma 2 *Let $S \in \mathcal{D}'(G)$. Then $R_h S = S$ for all $h \in H$ if and only if there exists $T \in \mathcal{D}'(M)$, such that $S = T^\sharp$.*

For the proof one shows first that the assertion holds for continuous functions and then an application of Banach–Steinhaus theorem together with standard density argument yields the result for distributions. The details are left to the reader.

Now we are ready to define convolution of distributions on M. Let $T, S \in \mathcal{D}'(M)$, one compactly supported. Then, $T^\sharp, S^\sharp \in \mathcal{D}'(G)$, $T^\sharp$ or $S^\sharp$ compactly supported. Because of Lemma 2, $S^\sharp$ is right H–invariant and, therefore, $T^\sharp * S^\sharp$ too. By Lemma 2 there exists $U \in \mathcal{D}'(M)$, such that $T^\sharp * S^\sharp = U^\sharp$. We will call U to be the convolution of T and S. Reformulating this definition, one has

$$(T * S)^\sharp = T^\sharp * S^\sharp. \tag{10}$$

The following simple fact will be useful in the sequel.

Lemma 3 *Let $T \in \mathcal{D}'(M)$. Then $T * \delta_p = T$, where δ_p stands for the probability density on M, concentrated at p (delta function at p).*

For the proof we notice that $\delta_H = \delta_p^\sharp$ is Haar measure on H and one readily checks that it follows from the definitions that for every $S \in \mathcal{D}'(G)$, $S * \delta_H = S$.

From (5) one can naturally pass to $\int_M |f(x)|^q dx = \int_G |(f \circ \pi)(g)|^q dg$ for $1 \le q < \infty$. This means, that $f \in L^q(M)$ if and only if $f \circ \pi \in L^q(G)$. Combination of this with Corollary 1 yields that $f^\sharp = f \circ \pi$ and $f \in L^q_{loc}(M)$ if and only if $f^\sharp \in L^q_{loc}(G)$. For $q = \infty$ obviously $f \in L^\infty(M)$ if and only if $f^\sharp \in L^\infty(G)$.

One has the following property, which in fact holds for general locally compact groups (cf.[4]).

Lemma 4 *One can generalize a notion of the convolution for functions in L^p spaces and the following holds. Let $\phi \in L^p(G)$ and $\psi \in L^q(G)$, where $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $1 \le p, q, r \le \infty$. Then the convolution $\phi * \psi \in L^r(G)$ and $||\phi * \psi||_r \le ||\phi||_p ||\psi||_q$. If $p = \infty$, $q = 1$, then the convolution $\phi * \psi$ is a bounded continuous function.*

The following simple relation will be also needed.

Lemma 5 *Let Y be a horizontal lift of a vector field X on M and let $T \in \mathcal{D}'(M)$ be a distribution on M. Then $Y(T^\sharp) = (XT)^\sharp$.*

If $f \in \mathcal{E}(M)$ is a smooth function, then $f^\sharp = f \circ \pi$, and

$$Y(f^\sharp) = Y(f \circ \pi) = (d\pi(Y)f) \circ \pi = (Xf) \circ \pi = (Xf)^\sharp.$$

One can use density or duality argument to establish the result for arbitrary distributions $T \in \mathcal{D}'(M)$.

4 Invariant fundamental solutions

In this section we will construct invariant fundamental solutions of the Laplacian and investigate some of their properties. Note firstly, that the Laplace–Beltrami operator commutes with the group action τ_g, which is an isometry due to the invariance of the Riemannian structure on M (cf.[2, Ch.II,§2,Prop.2.4]).

Lemma 6 *Let $K_0 \in \mathcal{D}'(M)$ be such that $\Delta K_0 = \delta_p$. Then there also exists an H–invariant fundamental solution, namely $K \in \mathcal{D}'(M)$ such that $\Delta K = \delta_p$ and $\tau_h K = K$ for all $h \in H$.*

Proof Let $K_g = \tau_g K_0$. For $h \in H$, K_h is a fundamental solution: $\Delta\tau_h K_0 = \tau_h \Delta K_0 = \tau_h \delta_p = \delta_p$, so $\{\tau_h K_0\}_{h\in H}$ forms a family of fundamental solutions of Δ in p. Define $K = \int_H \tau_h K_0 dh$, where dh is the normalized Haar measure for the compact topological group H. Then $\Delta K = \int_H \Delta\tau_h K_0 dh = \int_H \delta_p dh = \delta_p$. In particular this means that $K \not\equiv 0$. Clearly K is H–invariant, dh being Haar measure.

Lemma 7 *There exist $\alpha \in \mathcal{D}(M)$ such that $\alpha \geq 0$, $\int_M \alpha dx = 1$ and $\tau_h\alpha = \alpha$ for all $h \in H$. Moreover, α can be chosen such that $\alpha(x) = 1$ for all x in some neighborhood of p.*

Proof Take $\alpha_0 \in \mathcal{D}(M)$, nonnegative, such that $\int_M \alpha_0 dx = 1$. Define $\alpha = \int_H \tau_h\alpha_0 dh$. Let us verify now that α satisfies the desired conditions. Differentiation under integral sign implies smoothness of α. Let us show that α is compactly supported. Now, for $x \in A = \{x \in M : \alpha(x) \neq 0\}$, it follows that $\tau_h\alpha_0(x) \neq 0$ for some $h \in H$ and this means that $l_{h^{-1}}x \in \mathrm{supp}\alpha_0$, or $x \in l_h\mathrm{supp}\alpha_0 \subset l_H\mathrm{supp}\alpha_0$. The latter set is compact being an image of the compact set $H \times \mathrm{supp}\alpha_0$ under continuous mapping $(h, x) \to l_h x$. This implies $\mathrm{supp}\alpha = \mathrm{cl}A \subset l_H\mathrm{supp}\alpha_0$ is compact. Therefore, $\alpha \in \mathcal{D}(M)$. Clearly $\alpha \geq 0$ and it is H–invariant. Application of Fubini's theorem yields

$$\int_M \alpha dx = \int_H (\int_M \tau_h\alpha_0 dx) dh = 1.$$

In particular it implies $\alpha \not\equiv 0$.

Now we will show that there exist α satisfying the properties above and a neighborhood of p, where $\alpha \equiv 1$. Suppose $\alpha_0(x) = 1$ for $x \in B(p, \epsilon)$, the ball centered at p with radius ϵ. In view of the invariance of the Riemannian metric on M it follows that τ_h leaves $B(p, \epsilon)$ invariant. Therefore, $\tau_h\alpha_0(x) = 1$ for all $x \in B(p, \epsilon)$ and $h \in H$. The definition of α implies $\alpha(x) = 1$. This completes the proof.

Corollary 2 *If follows from the proof that if we start with α_0 with* $\mathrm{supp}\alpha_0 \subset B(p, \epsilon)$, *where $B(p, \epsilon) = \{x \in M : d(x, p) < \epsilon\}$, then* $\mathrm{supp}\alpha \subset B(p, \epsilon)$.

We will need the following observation.

Lemma 8 *Let $K \in \mathcal{D}'(M)^H$ be as in Lemma 6 and let $\alpha \in \mathcal{D}(M)^H$ be as in Lemma 7. Then the following holds:*

$$\delta_p = \Delta(\alpha K) + \beta, \tag{11}$$

where $\beta \in \mathcal{D}(M)^H$ is smooth, compactly supported and H–invariant.

Proof Take $\beta = \delta_p - \Delta(\alpha K)$. We will show that such β satisfies the conditions in the Lemma. Function αK is smooth outside p. In fact, α is smooth everywhere and K is smooth outside p as a fundamental solution of an elliptic operator, δ_p being smooth outside p. In a small neighborhood of p, $\alpha K = K$, α being 1 in U_p of Lemma 7 and, therefore, $\beta = 0$. This proves that $\beta \in \mathcal{E}(M)$. It is also compactly supported being a difference of two compactly supported distributions. Clearly β is H–invariant. In fact, it is sufficient to show that $\Delta(\alpha K)$ is H–invariant. Let $h \in H$. Then $\tau_h \Delta(\alpha K) = \Delta \tau_h(\alpha K) = \Delta(\tau_h \alpha \tau_h K) = \Delta(\alpha K)$. This proves the Lemma.

5 Proof

In this section we will give a proof of Theorem 1. Let u be as in Theorem 1: $u, \Delta u \in L^p(M)$. We will obtain a convolutive representation of u and then lift its derivative with respect to a (not necessarily invariant) vector field X to the group. Then, using the constructed in Section 1 Riemannian structure on G we will descent the problem back to M and reduce it to the integrability properties of a fundamental solution K. The latter can be expressed in terms of Neumann function on T_pM. We will establish also some auxiliary analiticity facts for Bessel functions.

A composition of Lemma 3 and (11) of Lemma 8 yields

$$u = u * \delta_p = u * \Delta(\alpha K) + u * \beta. \tag{12}$$

For a representation τ of G in $\mathcal{D}'(M)$, $\tau : G \to GL(\mathcal{D}'(M))$, we have its differential $d\tau$ mapping $d\tau : \mathfrak{g} \to \mathrm{End}(\mathcal{D}'(M))$, where $\mathfrak{g}$ denotes Lie algebra of G. By the universal property it uniquely extends to the universal enveloping algebra U. This extension we will also denote by $d\tau : U \to \mathrm{End}(\mathcal{D}'(M))$. Recall, that $U = \mathbf{D}(G)$ is isomorphic to the algebra of left invariant differential operators on G and is generated by Lie algebra $\mathfrak{g}$. The Laplace-Beltrami operator is the image of the Casimir operator ω on G: $\Delta = d\tau(\omega)$ and ω belongs to the center of U.

In view of (12) and definition (10) the pullback $u^\sharp$ of u is:

$$u^\sharp = (u * \Delta(\alpha K))^\sharp + (u * \beta)^\sharp = u^\sharp * \omega(\alpha K)^\sharp + u^\sharp * \beta^\sharp.$$

Now ω, being bi-invariant, commutes with left and right actions of G and, therefore, with left and right convolutions. It follows that

$$u^\sharp = \omega u^\sharp * (\alpha K)^\sharp + u^\sharp * \beta^\sharp = (\Delta u)^\sharp * (\alpha K)^\sharp + u^\sharp * \beta^\sharp. \tag{13}$$

Let $X \in \Gamma^\infty(TM)$ be a smooth bounded vector field on M. For the simplicity we assume that X is bounded by one: $||X_x|| \leq 1$. Let Y denote the horizontal lift of X. Recall, that this means

1. $d_g\pi(Y_g) = X_x$, where $x = \pi(g)$.
2. $Y_g \in N_g = (\mathrm{Ker}\ d_g\pi)^\perp$.

In Section 1 the Riemannian structure on G was chosen such that π is a Riemannian submersion. This means that $||Y_g|| = ||X_x||$, where $||\cdot||$ are norms corresponding to the Riemannian structures on G and M respectively. In particular, $||Y_g|| \leq 1$.

We will show now that the vector field Y is smooth. Let $X_1, \ldots, X_n$ be the derivatives along the coordinate paths in a chart V_0 of $(x_1, \ldots, x_n)$ at p, $n = \mathrm{dim} M$. They constitute an orthonormal basis of the tangent space T_pM if the coordinate paths are orthogonal. Otherwise we could take $X_1, \ldots, X_n$ orthonormal and $x_1, \ldots, x_n$ the corresponding Riemannian coordinates at p. Let $(g_1, \ldots, g_N)$, $N = \mathrm{dim} G$, be a chart U_0 of e in G, $\pi(U_0) \subset V_0$. Let Z_i be horizontal lifts of X_i, $(Z_i)_g \in N_g, g \in U_0$. Then $(Z_i)_g, i = 1, \ldots, n$, constitute a basis for N_g. In fact, if $\sum_{i=1}^n a_i Z_i = 0$, then $\sum_{i=1}^n a_i X_i = d\pi(\sum_{i=1}^n a_i Z_i) = 0$ and, therefore, all $a_i = 0$. At the same time the dimension of N_g equals n, $d\pi$ being an isomorphism between N_g and T_xM, $x = \pi(g)$.

The projection π is a submersion, this implies that the rang of the Jacobian matrix is equal to n. Without any loss of generality we will assume that the corresponding coordinates on G are $(g_1, \ldots, g_n)$: $\det\{\frac{\partial x_i}{\partial g_j}|_{g=e}\}_{1\leq i,j\leq n} \neq 0$. It follows from the implicit function theorem that there exist neighborhoods U of e and V of p, such that the system $x_i = x_i(g_1, \ldots, g_N)$, $i = 1, \ldots, n$ of coordinate expressions for $\pi : U_0 \to V_0$, is uniquely and smoothly solvable with respect to $g_1, \ldots, g_n$. Neighborhoods U and V can be taken contained in the charts U_0 of e and V_0 of p respectively. This implies that for a coordinate system on G at e we can take $z_1 = x_1, \ldots, z_n = x_n, z_{n+1} = g_{n+1}, \ldots, z_N = g_N$. Then in the local coordinates π becomes a projection to the first n coordinates and $d\pi = Id \oplus d\pi|_{K_g} : N_g \oplus K_g \to T_xM$ for g in U, $x = \pi(g) \in V$. Let $f \in \mathcal{C}^\infty(U)$. The identity of $d\pi$ on N_g implies $Z_i f(z_1, \ldots, z_N) = \frac{\partial f}{\partial z_i}(z_1, \ldots, z_N)$ is smooth. Therefore, Z_i are smooth vector fields in U, $i = 1, \ldots, n$. Moreover, $d\pi$ being the identity on N_g, implies that $Y(z_1, \ldots, z_n)$ has the same decomposition with respect to $Z_i(z_1, \ldots, z_n)$ as $X(x_1, \ldots, x_n)$ with respect to $X_i(x_1, \ldots, x_n)$. X being smooth implies smoothness of Y as a smooth combination of smooth vector fields $Z_1, \ldots, Z_n$.

Let $Y_1, \ldots, Y_N$ be an orthonormal basis of the Lie algebra $\mathfrak{g}$, such that $(Y_1)_g, \ldots, (Y_n)_g \in N_g$ for all $g \in G$ (we can take vectors $(Y_1)_e, \ldots, (Y_n)_e$ as an orthonormal basis of N_e and then expand them invariantly on the whole G, using that $d_e l_g = (d_g\pi)^{-1} \circ d_p l_{\pi(g)} \circ d_e\pi$ maps N_e isomorphically onto N_g). Vector field Y can be decomposed with respect to the basis $Y_1, \ldots, Y_n$ at every point $g \in G$:

$$Y_g = \sum_{i=1}^n a_i(g) Y_{i,g} \in N_g \subset T_gG, \tag{14}$$

where $Y_{i,g} = (Y_i)_g = d_e l_g (Y_i)_e$ are values at g of the left invariant vector fields Y_i. Note, that such decomposition is pointwise because Y need not be left invariant in general, we use that $Y_g \in N_g$ and the fact that $Y_{1,g}, \ldots, Y_{n,g}$ constitute a basis for a linear space N_g. It also has a global character and functions $a_1, \ldots, a_n$ are smooth due to the smoothness of Y and $Y_1, \ldots, Y_n$.

The norm of Y_g at $T_g G$ is $||Y_g||^2 = \sum_{i=1}^n |a_i(g)|^2$. In particular, $|a_i(g)| \le 1$ for all $g \in G$. Now we differentiate $u^\sharp$ with respect to the basis vector fields Y_i and (13) together with left invariance of Y_i yield:

$$Y_i u^\sharp = (\Delta u)^\sharp * Y_i(\alpha K)^\sharp + u^\sharp * Y_i \beta^\sharp. \tag{15}$$

Obviously $Y_i \beta^\sharp \in \mathcal{D}(G) \subset L^1(G)$. We want to make use of Lemma 4 and our next goal will be to establish the integrability of $Y_i(\alpha K)$.

Lemma 9 *For $i = 1, \ldots, n$ the derivative of $(\alpha K)^\sharp$ with respect to the basis vector field Y_i is integrable:*

$$Y_i(\alpha K)^\sharp \in L^1(G).$$

Proof We will use the Corollary from Lemma 7 stating that α can be chosen with arbitrarily small support. Balls $B(p, \epsilon)$ form a fundamental system of neighborhoods on M at p, it follows that if we take α supported in $B(p, \epsilon) \subset V$, the coordinate notations above apply.

The vector field Y_i locally defines a first order differential operator annihilating constants, this means that in terms of Z_j it allows the decomposition $Y_i = \sum_{j=1}^n c_{ij}(g) Z_j$ with smooth functions $c_{ij} \in \mathcal{C}^\infty(U)$. They are bounded and this implies that it is sufficient to prove the integrability of the derivatives $Z_j(\alpha K)^\sharp$ for all j.

By Lemma 5 we have $Z_j(\alpha K)^\sharp = (X_j(\alpha K))^\sharp$ and then integrability of $Z_j(\alpha K)^\sharp$ is equivalent to the integrability of $(X_j(\alpha K))^\sharp$ on G, and, therefore, the integrability of $X_j(\alpha K)$ on M. Note, that in our coordinate notation the distributional support $\mathrm{supp} X_j(\alpha K) \subset V$. It is sufficient to show the integrability only in an arbitrarily small neighborhood of p, the distribution αK being a smooth function outside the origin p.

Let Exp denote the exponential mapping for M, $Exp : T_p M \to M$. This Exp is a diffeomorphism ([3, p.567]), and in a neighborhood of zero in $T_p M$ the Jacobians of Exp and Exp^{-1} are bounded functions, implying that a function (distribution) $Q \in L^1_{loc}(M)$ if and only if $Q \circ Exp \in L^1_{loc}(T_p M)$.

We will use a special form of a fundamental solution of the Laplace–Beltrami operator. Such fundamental solution is not unique, thus we have some freedom with which K_0 in Lemma 6 to start. Distribution K_0, which we intend to use, will be of the form of distribution Q in [3, Theorem 5.4, p.582]. Differentiation under the integral sign yields that it is sufficient for us to prove local integrability of such $K_0 = \alpha K_0$ in some small neighborhood of p.

Let K_0 be as in [3, Theorem 5.4, p.582]. Then $K_0 \in L^1_{loc}(M)$ and $K_0 \circ Exp$ can be estimated by a constant times $Q_0(X) = |X|^{-n/2+1} N_{n/2-1}(|\rho||X|)$, where $n = \dim T_p M$ and N_ν is the Neumann function,

$$N_\nu(r) = \frac{\cos\nu\pi J_\nu(r) - J_{-\nu}(r)}{\sin\nu\pi}$$

for noninteger ν. For $\nu = p$ an integer, $N_p(r)$ is defined by the limit procedure:

$$N_p(r) = \lim_{\nu\to p} N_\nu(r). \tag{16}$$

In our case $\nu = \frac{n}{2} - 1$. According to the discussion above, it is sufficient to prove local integrability of $\frac{\partial Q_0(X)}{\partial x_j}$, where x_j is the j-th component of X.

We will need the following Lemma.

Lemma 10 *Let J_ν be the ν–th Bessel function of the first type:*

$$J_\nu(r) = \left(\frac{r}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-1)^k(\frac{r}{2})^{2k}}{k!\Gamma(\nu+k+1)}. \tag{17}$$

Let $\nu \in \mathbf{R}$ satisfy $\nu > \frac{1}{2}$. Then function $r^{-\nu}J_\nu(r)$ allows an extension to the entire analytic function, say $\phi(z)$, satisfying

$$|\phi(z)| \le Ce^{4\pi|Im\ z|},\ z \in \mathbf{C},$$

for some constant $C > 0$. In particular, a local integrability of $r^{-\nu}J_\nu(r)$ follows.

Proof We will use the following formula (cf.[6, IX,1;70])

$$J_\nu(r) = \frac{1}{\Gamma(\nu+1/2)\sqrt{\pi}}\left(\frac{r}{2}\right)^\nu \int_{-1}^1 (1-t^2)^{\nu-1/2}e^{-itr}dt.$$

Now, denoting $f(t) = (1-t^2)^{\nu-1/2}$ for $-1 \le t \le 1$ and zero otherwise, we get $r^{-\nu}J_\nu(r) = c^{st}\mathcal{F}(f)(r)$, where $\mathcal{F} \in \mathcal{L}(L^2(\mathbf{R}))$ stands for the Fourier transform in L^2. It follows now, that $r^{-\nu}J_\nu(r)$ is a Fourier image of a (continuous) compactly supported L^2 function, therefore, by Paley-Wiener theorem, it allows the desired extension, satisfying the estimate of our Lemma. The Lemma is proved.

First we will treat the case of odd n. We have $Q_0(r) = \pm|X|^{-n/2+1}J_{-n/2+1}(|X|)$. Function $\frac{\partial Q_0(X)}{\partial x_j}$ as a function of $r = |X|$ has the only singularity at 0, thus a local integrability automatically holds outside zero.

Derivative of $Q_0(X)$ with respect to x_j is

$$\frac{\partial Q_0(X)}{\partial x_j} = C_1 \overbrace{|X|^{-n/2}\frac{x_j}{|X|}J_{-n/2+1}(|X|)}^{I} + C_2 \overbrace{|X|^{-n/2+1}\frac{\partial J_{-n/2+1}(|X|)}{\partial x_j}}^{II},$$

where x_j is the j-th component of X. Now, $\frac{|x_j|}{|X|} \le 1$ and $|I| \le c^{st}\frac{Q_0(X)}{|X|}$. In a small neighborhood of zero $Q_0(X)|X|^{n-2}$ is a bounded function. In particular, this means for small $|X|$, $X \ne 0$, $\frac{Q_0(X)}{|X|} \le C|X|^{-(n-1)}$, implying the local integrability of I.

The derivative of Bessel function J_ν can be represented as

$$J'_\nu(r) = J_{\nu-1}(r) - \frac{\nu}{r}J_\nu(r) \tag{18}$$

Thus,

$$|II| \le C_3|X|^{-n/2+1}[J_{-n/2}(|X|) - (-n/2+1)|X|^{-1}J_{-n/2+1}(|X|)]\frac{x_j}{|X|}.$$

The second summand is $O(|X|^{-n/2}J_{-n/2+1}(|X|))$ and can be treated in the same way as I. The first summand is $O(J_{-n/2}(|X|)|X|^{-n/2+1})$ and from the definition of the Bessel functions (17) it follows that this is $|X|^{-n+1}O(1)$, which has locally integrable singularity at zero. This implies integrability of II, finishing the proof for odd n.

Assume now n to be even, $n = 2, 4, \ldots$. The Neumann function (16) can be written as (cf.[6, IX,1;39]):

$$\pi N_p(r) = \overbrace{2\left(\log\frac{r}{2}+\gamma\right)J_p(r)}^{I} - \overbrace{\left(\frac{r}{2}\right)^{-p}\sum_{k=0}^{p-1}\left(\frac{r}{2}\right)^{2k}\frac{1}{k!}(p-k-1)!}^{II} -$$

$$\overbrace{\left(\frac{r}{2}\right)^{p}\frac{1}{p!}\sum_{l=1}^{p}\frac{1}{l}}^{III} - \overbrace{\left(\frac{r}{2}\right)^{p}\sum_{k=1}^{\infty}\frac{(-1)^k(\frac{r}{2})^{2k}}{k!(p+k)!}\left(\sum_{l=1}^{k}\frac{2}{l}+\sum_{l=k+1}^{p+k}\frac{1}{l}\right)}^{IV} \tag{19}$$

with some constant γ ($\gamma = -\Gamma'(1)$). It is sufficient to show the integrability at zero of the derivatives of the representation (19), because of the boundedness of the derivatives $\frac{\partial r}{\partial x_j}, r = |X|$ as in the case of odd n. We will treat $(I), (II), (III), (IV)$ consequently. The derivative of (I) is

$$(I)' = \frac{2}{r}J_p(r) + (2\gamma + 2\log\frac{r}{2})J'_p(r)$$

and in view of (18) we have

$$(I)' = (2-2\gamma p)\frac{J_p(r)}{r} + 2\gamma J_{p-1}(r) + 2\log\frac{r}{2}J_{p-1}(r) - \frac{2p}{r}\log\frac{r}{2}J_p(r). \tag{20}$$

Now, $\frac{J_p(r)}{r} = (r^{-p}J_p(r))r^{p-1}$ and the first term is continuous for $p > 0$ by Lemma 10, meanwhile the second term is integrable in $\mathbf{R}^n, n \ge 2$. For the rest of (20), J_{p-1} is locally integrable, $\log\frac{r}{2}J_{p-1}(r) = (r\log\frac{r}{2})(r^{-(p-1)}J_{p-1}(r))r^{p-2} = O(r^{p-2})$ and $\frac{1}{r}\log\frac{r}{2}J_p(r) = (r\log\frac{r}{2})(r^{-p}J_p(r))r^{p-2} = O(r^{p-2})$ with r^{p-2} locally integrable for $p \ge 1$ in $\mathbf{R}^n, n \ge 4$. It remains the case of $p = 0$ or, equivalently, $n = 2$. The last two terms of the expression (20) become $2\log\frac{r}{2}J_{-1}(r) = -2\log\frac{r}{2}J_1(r) = -2(\frac{J_1(r)}{r})(r\log\frac{r}{2})$ and the last term is bounded at zero by Lemma 10. Thus, we have proved $(I)' \in L^1_{loc}$.

Now we turn to (II). This is a rational function of r, its derivative at zero is $O(r^{-(p+1)})$. The latter is integrable at zero if $p+1 < n$ which is obviously satisfied for all n, $p = \frac{n}{2}-1$.

The expression (III) is a polynomial, its derivative is bounded, hence locally integrable. The last term in (19), (IV), is r^p times an analytic function of r, it can be differentiated termwise and is continuous in some neighborhood of zero. Thus, Lemma is proved now for all n.

Now we continue our estimates of $Y_i u^\sharp$. We will use Lemma 4 together with formula (15) to prove the estimate of Theorem 1. Let $A_i = ||Y_i(\alpha K)^\sharp||_1$ and $B_i = ||Y_i \beta^\sharp||_1$. These constants are finite in view of Lemma 9. Application of Lemma 4 to (15) yields

$$||Y_i u^\sharp||_p \leq A_i||(\Delta u)^\sharp||_p + B_i||u^\sharp||_p.$$

As it was shown before, $||(\Delta u)^\sharp||_p = ||\Delta u||_p$ and $||u^\sharp||_p = ||u||_p$, where norms at the left hand side of the equalities are the norms in $L^p(G)$ and norms at the right hand side are the norms in $L^p(M)$. Now, decomposition (14) implies $Yu^\sharp = \sum_{i=1}^n a_i(g)Y_i u^\sharp$ and $||Yu^\sharp||_p \leq \sum_{i=1}^n ||Y_i u^\sharp||_p$ in view of $|a_i| \leq 1$. This implies

$$||Yu^\sharp||_p \leq A||\Delta u||_p + B||u||_p$$

with $A = \sum_{i=1}^n A_i$ and $B = \sum_{i=1}^n B_i$. Application of Lemma 5 to u yields $||Yu^\sharp||_p = ||Xu||_p$. Finally, we get

$$||Xu||_p \leq A||\Delta u||_p + B||u||_p. \tag{21}$$

Thus, we established the estimate (1) of Theorem 1. We will argue now the continuity of Xu for $p = \infty$. In view of Lemma 4, equality (15), Lemma 9 and our assumptions on u, we get that $Y_i u^\sharp$ are continuous functions. Formula (14) shows that $Yu^\sharp = \sum_{i=1}^n a_i(g)Y_i u^\sharp$ with smooth functions a_i, implying the continuity of $Yu^\sharp$. Now, Lemma 5 implies $(Xu)^\sharp = Yu^\sharp$ is continuous. Recall finally, that M is equipped with quotient topology, i.e. strongest topology, for which π is a continuous function. This implies the continuity of Xu on M.

References

[1] J. Jost, H. Karcher, 'Geometrische Methoden zur Gewinnung von a-priori-Schranken für harmonische Abbildungen', *Manuscripta Math.* **40** (1982), 27-77.

[2] S. Helgason, *Groups and Geometric Analysis*, Pure and Applied Math. 113, Academic Press, 1984.

[3] S. Helgason, 'Fundamental solutions of invariant differential operators on symmetric space', *Amer.J.Math.* **86** (1964), 565-601.

[4] E. Hewitt, K. A. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, 1963.

[5] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1966.

[6] L. Schwartz, *Méthodes Mathématiques pour les Sciences Physiques*, Hermann, Paris, 1965.

[7] E. Stein, *Harmonic Analysis*, Priceton Univ. Press, Princeton, 1993.

M.Ruzhansky
Mathematical Institute
University of Utrecht
P.O.Box 80.010
3508 TA Utrecht
The Netherlands
e-mail: M.Ruzhansky@math.ruu.nl