The Space of Triangles, Vanishing Theorems, and Combinatorics

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Abstract

We consider compactifications of $(\mathbf{P}^n)^3 \setminus \bigcup \Delta_{ij}$, the space of triples of distinct points in projective space. One such space is a singular variety of configurations of points and lines; another is the smooth compactification of Fulton and MacPherson; and a third is the triangle space of Schubert and Semple.

We compute the sections of line bundles on these spaces, and show that they are equal as $GL(n)$ representations to the generalized Schur modules associated to "bad" generalized Young diagrams with three rows (Borel-Weil theorem). On the one hand, this yields Weyl-type character and dimension formulas for the Schur modules; on the other, a combinatorial picture of the space of sections. Cohomology vanishing theorems play a key role in our analysis.

Introduction

First, we present our main character in three different guises.

Combinatorial: the generalized Young diagram

$$D_3 = \begin{array}{ccccccc} \square & & & & \square & \square & \square \\ & \square & & \square & & \square & \square \\ & & \square & \square & \square & & \square \end{array}$$

whose columns correspond to the non-empty subsets of set of row-indices $\{1, 2, 3\}$. Also, we fix an integer $n \geq 3$.

Geometric: the configuration space $\mathcal{F}_{3,n}$, which is the variety of all 7-tuples $(p_1, p_2, p_3, l_1, l_2, l_3, P)$, where p_i are points, l_i lines, and P a plane, all in $\mathbf{P}^{n-1}$; subject to certain inclusions: the point p_i must lie on each of the lines l_j for $j \neq i$, and all must be contained in the plane P. This the *singular space of triangles in* $\mathbf{P}^{n-1}$, a projective variety of dimension $3n-3$. The seven entries correspond to the seven columns of the diagram, and each entry is a space whose projective dimension is one less than the number of boxes in the column. The inclusions of spaces correspond to horizontal inclusions of columns. (If we replaced D_3 by a Young diagram, the corresponding configuration space would be a flag variety.)

Algebraic: The $GL(n)$-representation $S_{D,n}$, a module defined by the Schur–Weyl construction applied to $D = D_3$. That is, let $\mathbf{C}^n$ be the defining representation of $GL(n)$, and consider the tensor power $(\mathbf{C}^n)^{\otimes D}$ with one factor for each square of the diagram. Then

$$S_{D_3,n} = (\mathbf{C}^n)^{\otimes D} \gamma_D,$$

where γ_D is a Young operation symmetrizing and anti-symmetrizing tensors according to the rows and columns of $D = D_3$. (If D were a Young diagram, $S_{D,n}$ would be a classical Schur module, an irreducible representation of $GL(n)$.)

These objects have been extensively explored. Combinatorists have examined the representations $S_{D,n}$ as part of the theory of generalized Schur modules and Young diagrams. (See [29], [33], [34], [35], [39].) The geometric theory of $\mathcal{F}_{3,n}$ and its desingularizations goes back to Schubert [37] and Semple [38], and has been illuminated recently by Fulton, MacPherson, and others ([10], [13], [20],[36], [7], [8]).

In the current paper, we explore the relations between the algebraic and geometric pictures. We prove a Borel-Weil theorem realizing $S_{D_3,n}$ (or the Schur module of any three-row diagram) as the sections of a line bundle over the triangle space, and we derive explicit character and dimension formulas generalizing those of Weyl for the irreducibles. Reversing the perspective, the Schur modules give a combinatorial construction of the space of sections of certain line bundles over the triangle spaces.

The key steps relating the geometry to the combinatorics require the vanishing of certain higher cohomology groups and explicit computations on spaces of sections. For these, we make a detailed examination of the defining equations, torus-fixed points, and birational maps of the triangle spaces, in order to apply the standard techniques for dealing with (almost) homogeneous varieties. Indeed, we intend this paper partly as a primer on Frobenius splitting, natural desingularizations, Lefschetz theorems, and rational singularities, as illustrated by our simple example.

Nevertheless, our character formulas for Schur modules are stated in purely elementary terms, and the interested reader can skip directly to sections 5.2 and 5.3.

In previous papers [22], [24], [23] we considered the same problems for diagrams D satisfying the "northwest" or "strongly separated" conditions, for which the geometry of $\mathcal{F}_D$ is particularly simple. D_3 is the smallest diagram which does not fall into these classes, and thus needs a different treatment. In the case of northwest or strongly separated diagrams, we may desingularize our varieties by the usual Bott–Samelson and Zelevinsky resolutions, but for the triangle space we must use more complicated special desingularizations which appear in the literature. We hope that our methods will shed light on defining non-singular spaces of tetrahedra and higher-dimensional simplices. Even for the space of triangles, some of the combinatorial results seem geometric in nature, but have no obvious geometric explanation. See especially section 5.4.

Contents.

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1 Definitions

A *diagram* is a finite subset of $\mathbf{N} \times \mathbf{N}$. Its elements $(i,j) \in D$ are called *squares*, and the square (i,j) is pictured in the i^{th} row and j^{th} column. We shall often think of D as a sequence $(C_1, C_2, \ldots, C_r)$ of columns $C_j \subset \mathbf{N}$.

Fix once and for all an integer $n \geq 3$. We denote the interval $[1,n] = \{1,2,\ldots,n\}$. We shall always write $G = GL(n,\mathbf{C})$, $B =$ the subgroup of upper triangular matrices, and $T =$ the subgroup of diagonal matrices. $\mathbf{C}^n$ is the defining representation of G.

We will assume our diagrams have at most n rows: $D \subset [1,n] \times \mathbf{N}$. Let Σ_D be the symmetric group permuting the squares of D, and for any diagram D, let

$$\mathrm{Col}(D) = \{\pi \in \Sigma_D \mid \pi(i,j) = (i',j) \ \exists i'\}$$

be the group permuting the squares of D within each column, and we define $\mathrm{Row}(D)$ similarly for rows. Define the idempotents α_D, β_D in the group algebra $\mathbf{C}[\Sigma_D]$ by

$$\alpha_D = \frac{1}{|\operatorname{Row} D|} \sum_{\pi \in \operatorname{Row} D} \pi, \qquad \beta_D = \frac{1}{|\operatorname{Col} D|} \sum_{\pi \in \operatorname{Col} D} \operatorname{sgn}(\pi)\pi,$$

where $\operatorname{sgn}(\pi)$ is the sign of the permutation.

Now, G acts diagonally on the left of the $|D|$-fold tensor product $(\mathbf{C}^n)^{\otimes D}$, and Σ_D acts on the right by permuting the tensor factors:

$$g(x_{t_1}, x_{t_2}, \ldots)\pi = (gx_{\pi t_1}, gx_{\pi t_2}, \ldots).$$

These actions commute. Define the *Schur module*

$$S_D \stackrel{\text{def}}{=} (\mathbf{C}^n)^{\otimes D} \alpha_D \beta_D \subset (\mathbf{C}^n)^{\otimes D},$$

a representation of G. Interchanging two columns (or two rows) of the diagram gives an isomorphic Schur module.

Now, let $e_1, \ldots, e_n$ be the standard basis of $\mathbf{C}^n$, and for $C \subset [1,n]$, define the coordinate subspace $E_C = \operatorname{Span}(e_i \mid i \in C) \in \operatorname{Gr}(|C|, \mathbf{C}^n)$. For a diagram $D = (C_1, \ldots, C_r)$, let

$$\operatorname{Gr}(D) = \operatorname{Gr}(|C_1|, \mathbf{C}^n) \times \cdots \times \operatorname{Gr}(|C_r|, \mathbf{C}^n) \ ,$$

and define the *configuration variety* as the closure in $\mathrm{Gr}(D)$ of the $GL(n)$-orbit of a configuration of coordinate subspaces:

$$\mathcal{F}_D = \text{closure } [G \cdot (E_{C_1}, \ldots, E_{C_r})] \subset \mathrm{Gr}(D) \ .$$

This is clearly an irreducible subvariety. We may also define the *inclusion variety* $\mathcal{I}_D \subset \mathrm{Gr}(D)$ by:

$$\mathcal{I}_D = \{(V_1, \ldots, V_r) \in \mathrm{Gr}(D) \mid C_i \subset C_j \Rightarrow V_i \subset V_j\}.$$

We clearly have $\mathcal{F}_D \subset \mathcal{I}_D$.

Consider the Plücker line bundle $\mathcal{O}(1) = \mathcal{O}(1, \ldots, 1)$ on the product of Grassmannians $\mathrm{Gr}(D)$. We may define a line bundle $\mathcal{L}_D$ on $\mathcal{F}_D$ and $\mathcal{I}_D$ as the restriction of $\mathcal{O}(1)$ to the subvarieties.

In case D is a Young diagram $\{(i,j) \mid 1 \le i \le \lambda_j\}$ for $\lambda = (\lambda_1 \ge \cdots \ge \lambda_n \ge 0)$, then $\mathcal{F}_D = \mathcal{I}_D$ is a flag variety, and $\mathcal{L}_D$ is the Borel-Weil line bundle whose sections are (the dual of) the irreducible Schur module $S_D = S_\lambda$.

In this paper, we will consider diagrams D with at most three rows: that is, all squares $(i,j) \in D$ have $i = 1$, 2, or 3. We may consider the diagram D_3 of the introduction as universal: it contains a column of each type, and any three-rowed diagram can be specified (up to the order of the columns) by the multiplicity $m_i \ge 0$, $i = 1, \ldots, 7$ of each column in D_3.

Note that the multiplicities m_i will not affect the varieties $\mathcal{F}_D$ or $\mathcal{I}_D$ (provided all $m_i > 0$). We will denote $\mathcal{F}_D = \mathcal{F}_{D_3} = \mathcal{F}_{3,n}$ and $\mathcal{I}_D = \mathcal{I}_{D_3} = \mathcal{I}_{3,n}$, both inside the product of Grassmannians:

$$\mathcal{F}_{3,n} \subset \mathcal{I}_{3,n} \subset (\mathbf{P}^{n-1})^3 \times \mathrm{Gr}(1, \mathbf{P}^{n-1})^3 \times \mathrm{Gr}(2, \mathbf{P}^{n-1}).$$

The variety $\mathcal{F}_{3,n}$ is the closure of the $GL(n)$-orbit of the coordinate 2-simplex in $\mathbf{P}^{n-1}$, and the inclusion variety is

$$\mathcal{I}_{3,n} = \{ \ (p_1, p_2, p_3, l_1, l_2, l_3, P) \ \mid \ p_i \subset l_j \ \ \forall\ i \neq j, \ \ l_j \subset P \ \ \forall\ j \ \}.$$

For a diagram D defined by integers m_i, $\mathcal{L}_D$ is the restriction of $\mathcal{O}(m_1, \ldots, m_7)$ on the product of Grassmannians. This is a (very) ample line bundle exactly when $m_i > 0$ for all i.

2 The Singular space

2.1 Borel-Weil theorem

Our first aim is to prove

Theorem 1 *We have*
(a) For any diagram D with at most three rows, we have $H^0(\mathcal{F}_{3,n}, \mathcal{L}_D) = S_D^*$ and $H^i(\mathcal{F}_{3,n}, \mathcal{L}_D) = 0$ for $i > 0$.
(b) $\mathcal{F}_{3,n} = \mathcal{I}_{3,n}$ is a normal, irreducible variety, and projectively normal with respect to $\mathcal{L}_D$.

The proof will occupy the following sections. We start with the elementary parts.

Claim: $\mathcal{I}_{3,n}$ is irreducible of dimension $3n-3$, and hence equal to its subvariety $\mathcal{F}_{3,n}$.

It will suffice to prove this for $\mathcal{I}_{3,3}$, since there is an obvious fiber bundle

$$\begin{array}{ccc} \mathcal{I}_{3,3} & \to & \mathcal{I}_{3,n} \\ & & \downarrow \\ & & \mathrm{Gr}(2, \mathbf{P}^{n-1}) \end{array}$$

given by mapping a triangle to the projective plane in which it lies. (The fiber is irreducible if and only if the whole bundle is, and $\dim \mathcal{I}_{3,n} = \dim \mathcal{I}_{3,3} + 3(n-3)$.)

Now, any triangle in $\mathcal{I}_{3,3}$ can be deformed under the $GL(3)$ action on $\mathbf{P}^2$ so as to approach a maximally degenerate configuration, in which all three vertices and edges are identical. Furthermore, $GL(3)$ acts transitively on this stratum of degenerate triangles, so it suffices to check the irreducibility in a neighborhood of a single degenerate triangle such as $(E_1, E_1, E_1, E_{12}, E_{12}, E_{12})$, for which the entries form a flag of coordinate subspaces of $\mathbf{P}^2$. Nearby, we can give affine coordinates so that a configuration in $(\mathbf{P}^2)^3 \times \mathrm{Gr}(1, \mathbf{P}^2)^3$ is represented by

$$\left[\begin{pmatrix} 1 \\ a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} 1 \\ a_2 \\ b_2 \end{pmatrix}, \begin{pmatrix} 1 \\ a_3 \\ b_3 \end{pmatrix}, \begin{pmatrix} c_1 \\ d_1 \\ 1 \end{pmatrix}, \begin{pmatrix} c_2 \\ d_2 \\ 1 \end{pmatrix}, \begin{pmatrix} c_3 \\ d_3 \\ 1 \end{pmatrix} \right]$$

subject to the six incidence conditions:

$$\begin{array}{lll} c_2 + a_1 d_2 + b_1 = 0 & c_1 + a_2 d_1 + b_2 = 0 & c_1 + a_3 d_1 + b_3 = 0 \\ c_3 + a_1 d_3 + b_1 = 0 & c_3 + a_2 d_3 + b_2 = 0 & c_2 + a_3 d_2 + b_3 = 0 \end{array}$$

By eliminating, we can reduce this to the seven variables a_1, a_2, a_3, d_1, d_2, d_3, c_1 subject to the single equation

$$(a_1 - a_2)(d_2 - d_3) + (a_3 - a_2)(d_1 - d_2) = 0.$$

A linear change of variables turns this into the product of an affine space and a quadric, an irreducible variety of dimension 6. This is what we wanted to show. Therefore $\mathcal{I}_{3,n}$ is irreducible and equal to $\mathcal{F}_{3,n}$.

Borel-Weil construction.

Next, we recall from [22] the elementary construction connecting the Schur module S_D of a three-row diagram with the triangle space. Let $V = \mathbf{C}^n$ and $U = V^*$ its dual space. By definition, S_D is the image of the composite map

$$S_D = \mathrm{Im}\left[V^{\otimes D}\alpha_D \stackrel{\mathrm{incl}}{\to} V^{\otimes D} \stackrel{\beta_D}{\to} V^{\otimes D}\beta_D\right].$$

Taking dual spaces, we have

$$S_D^* = \mathrm{Im}\left[U^{\otimes D}\beta_D \stackrel{\mathrm{incl}}{\to} U^{\otimes D} \stackrel{\alpha_D}{\to} U^{\otimes D}\alpha_D\right].$$

We translate this into geometric language as follows. Consider the product space $(\mathbf{P}^{n-1})^D$ as all $|D|$-tuples of points inscribed in the squares of D. Define

$$\phi : (\mathbf{P}^{n-1})^D \to \mathrm{Gr}(D)$$

to be the rational map taking a c-tuple of vectors in a column C to the space in $\mathrm{Gr}(c, \mathbf{C}^n)$ which they span. This is a rational map defined everywhere except a set of codimension ≥ 2, so it induces maps of locally free coherent sheaves as if it were regular. Also define the *row multidiagonal* $\Delta^D(\mathbf{P}^{n-1})$, the locus in $(\mathbf{P}^{n-1})^D$ where all points in the same row are equal: that is, for our $D = D_3$,

$$\Delta^D(\mathbf{P}^{n-1}) = \left\{ \begin{array}{cccccc} p_1 & & & & p_1 & p_1 & p_1 \\ & p_2 & & p_2 & & p_2 & p_2 \\ & & p_3 & p_3 & p_3 & & p_3 \end{array} \right\} \subset (\mathbf{P}^{n-1})^D \quad .$$

The composite image of these maps is precisely the configuration variety:

$$\mathcal{F}_D = \text{closure Im}\left[\Delta^D(\mathbf{P}^{n-1}) \stackrel{\text{incl}}{\to} (\mathbf{P}^{n-1})^D \stackrel{\phi}{\to} \text{Gr}(D)\right].$$

The above equation for the dual Schur module now translates easily into

$$\begin{aligned} S_D^* &= \text{Im}\left[\begin{array}{ccl} H^0(\text{Gr}(D), \mathcal{L}_D) & \stackrel{\phi^*}{\to} & H^0((\mathbf{P}^{n-1})^D, \mathcal{O}(1)) \\ & \stackrel{\text{rest}}{\to} & H^0(\Delta^D(\mathbf{P}^{n-1}), \text{rest}\,\mathcal{O}(1)) \end{array}\right] \\ &= \text{Im}\left[H^0(\text{Gr}(D), \mathcal{L}_D) \stackrel{\text{rest}}{\to} H^0(\mathcal{F}_D, \mathcal{L}_D)\right] , \end{aligned}$$

where rest $=$ incl*. This equation is true for an arbitrary diagram D. If $\mathcal{F}_D = \mathcal{F}_{3,n}$, the first equation in part (a) of the Theorem states that the above restriction map is onto, so that the dual Schur module is just equal to the sections of $\mathcal{L}_D$ over the triangle space $\mathcal{F}_D = \mathcal{F}_{3,n}$.

If D does not contain each column of D_3, then $\mathcal{F}_D \neq \mathcal{F}_{3,n}$, but there is a surjective map $\mathcal{F}_{3,n} \to \mathcal{F}_D$ given by forgetting the data associated to the columns which do not appear in D. We have the commutative diagram

$$\begin{array}{ccc} H^0(\text{Gr}(D_3), \mathcal{L}_D) & \stackrel{\text{rest}}{\to} & H^0(\mathcal{F}_{3,n}, \mathcal{L}_D) \\ \downarrow & & \downarrow \\ H^0(\text{Gr}(D), \mathcal{L}_D) & \stackrel{\text{rest}}{\to} & H^0(\mathcal{F}_D, \mathcal{L}_D) \end{array}$$

The vertical map between the sections over Grassmannians is clearly an isomorphism. Hence in this case, we must show the surjectivity of the top restriction map, *and* the bijectivity of the second vertical map.

Thus, the first equation of part (a) reduces in general to

Lemma 2 *The natural map $H^0(\text{Gr}(D), \mathcal{L}_D) \to H^0(\mathcal{F}_D, \mathcal{L}_D)$ is surjective, and the natural map $H^0(\mathcal{F}_{3,n}, \mathcal{L}_D) \to H^0(\mathcal{F}_D, \mathcal{L}_D)$ is bijective.*

To prove these facts and the rest of the Theorem, we will need more sophisticated techniques.

2.2 Frobenius splitting

The theory of Frobenius splittings invented by Mehta, Ramanan, and Ramanathan ([27], [30], [31], [18]) is a characteristic-p technique for proving surjectivity and vanishing results about coherent sheaves, even in characteristic zero. It is highly practical for dealing with homogeneous varieties because one can work over the integers in a characteristic-free way, and never consider the special features of characteristic-p geometry. In fact, the method reduces to classical questions about defining equations and canonical divisors of varieties. Most of the theorem of the last section will follow immediately from knowing that the pair $\mathcal{F}_{3,n} \subset \mathrm{Gr}(D_3)$ is "compatibly Frobenius split" in any characteristic.

Given two algebraic varieties $Y \subset X$ defined over an algebraically closed field F of characteristic $p > 0$, with Y a closed subvariety of X, we say that the pair $Y \subset X$ is *compatibly Frobenius split* if:
(i) the p^{th} power map $F : \mathcal{O}_X \to F_*\mathcal{O}_X$ has a splitting, i.e. an $\mathcal{O}_X$-module morphism $\phi : F_*\mathcal{O}_X \to \mathcal{O}_X$ such that ϕF is the identity; and
(ii) we have $\phi(F_*I) = I$, where I is the ideal sheaf of Y.

Because $\mathcal{L} \otimes F_*\mathcal{O}_X = F_*\mathcal{L}^p$ for any line bundle $\mathcal{L}$, a Frobenius splitting of Y allows one to embed the cohomology of an ample bundle on Y into the cohomology of its powers: $H^i(Y, \mathcal{L}) \subset H^i(Y, \mathcal{L}^{p^d})$ for all $d \geq 0$. Since the right-hand side becomes zero for large d by Serre vanishing, the $H^i(Y, \mathcal{L})$ itself must be zero ($i > 0$). If one can show this vanishing for reductions modulo p for all (or infinitely many) p, then semi-continuity implies $H^i(X(\mathbf{C}), \mathcal{L}) = 0$ as well. In fact, Mehta and Ramanathan prove the following

Proposition 3 *Let X be a projective variety, Y a closed subvariety, and $\mathcal{L}$ an ample line bundle on X. If $Y \subset X$ is compatibly split, then $H^i(Y, \mathcal{L}) = 0$ for all $i > 0$, and the restriction map $H^0(X, \mathcal{L}) \to H^0(Y, \mathcal{L})$ is surjective.*

Furthermore, if Y and X are defined and projective over $\mathbf{Z}$ (and hence over any field), and they are compatibly split over any field of positive characteristic, then the above vanishing and surjectivity statements also hold for all fields of characteristic zero.

Frobenius splitting is also sufficient to establish the normality of our varieties. The main theorem of Mehta and Srinivas [26] states that if Y is a Frobenius-split variety possessing a desingularization with connected fibers, then Y is normal. (Normality in all finite characteristics implies normality in characteristic 0).

These strong properties of split varieties will suffice to prove our Theorem, provided we construct a compatible splitting. This is rendered practical by a criterion that was made explicit in [21] in terms of a notion "residually normal crossing", which we now recall.

A divisor D defined by $f_0 = 0$ around a point P on a smooth affine variety X of dimension n has *residually normal crossing* at P if there exists a system of parameters $\{x_1, \cdots, x_n\}$ and functions $f_1, \cdots, f_{n-1} \in k[[x_1, \cdots, x_n]] = \widehat{\mathcal{O}_P}$ such that $f_i = x_{i+1} f_{i+1} \pmod{(x_1, \cdots, x_i)}$ for $i = 0, 1, \cdots, n-1$, where $f_n = 1$ (or a unit).

Now the criterion reads:

Proposition 4 *Let X be a smooth projective variety of dimension M over a field of characteristic $p > 0$, and let $Z_1, \ldots, Z_M$ be irreducible closed subvarieties of codimension 1. Suppose that there is a point $P \in X$ such that $Z_1 + \cdots + Z_M$ has residually normal crossing at P.*

Further suppose that there exists a global section s of the anti-canonical bundle K_X^{-1} such that $\operatorname{div} s = Z_1 + \cdots + Z_M$.

Then the section $\sigma = s^{p-1}$ gives a simultaneous compatible splitting of $Z_1, \ldots, Z_M$ in X. This is also a compatible splitting of any variety obtained from $Z_1, \ldots, Z_M$ by repeatedly taking intersections and irreducible components.

This works because there is bijection between sections s of K_X^{-1+p} and $\mathcal{O}_X$-module morphisms $\phi : F_* \mathcal{O}_X \to \mathcal{O}_X$, cf. appendix A3 of [18]. In the notation above, one shows that ϕ induces a map $F_* \widehat{\mathcal{O}_P}/(x_1, \ldots, x_i) \to \widehat{\mathcal{O}_P}/(x_1, \ldots, x_i)$ corresponding with the "residue" of s along $x_1 = \cdots = x_i = 0$ whose divisor is described by f_i.

Several other useful properties of Frobenius splittings can be found in [32].

We will apply our theory first in the case $n = 3$, and then indicate the modifications necessary for general n. Instead of directly splitting the pair $\mathcal{F}_{3,n} \subset \mathrm{Gr}(D_3)$, we will find it more convenient to use an intermediate subspace, which for $n = 3$ is just the triple product of flag varieties $(G/B)^3$. We embed $\mathcal{F}_{3,3} \subset (G/B)^3$ via the map $(p_1, p_2, p_3, l_1, l_2, l_3, P) \mapsto ((p_1, l_2), (p_2, l_3), (p_3, l_1))$.

Lemma 5 *The pair $\mathcal{F}_{3,3} \subset (G/B)^3$ is compatibly Frobenius split*

Proof. We describe the divisor giving our Frobenius splitting on $\mathcal{F}_{3,3}$. Given $(xB, yB, zB) \in (G/B)^3$, represented by matrices x, y, z, we consider the formula

$$
\begin{aligned}
s \quad = \quad & \begin{vmatrix} x_{11} & y_{11} & y_{12} \\ x_{21} & y_{21} & y_{22} \\ x_{31} & y_{31} & y_{32} \end{vmatrix} \cdot \begin{vmatrix} z_{11} & x_{11} & x_{12} \\ z_{21} & x_{21} & x_{22} \\ z_{31} & x_{31} & x_{32} \end{vmatrix} \cdot \begin{vmatrix} y_{11} & z_{11} & z_{12} \\ y_{21} & z_{21} & z_{22} \\ y_{31} & z_{31} & z_{32} \end{vmatrix} \cdot \\
& \left(\begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} \cdot \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} - \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \cdot \begin{vmatrix} y_{21} & y_{22} \\ y_{31} & y_{32} \end{vmatrix} \right) \cdot \\
& \begin{vmatrix} x_{11} & y_{11} \\ x_{31} & y_{31} \end{vmatrix} \cdot \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} \cdot z_{3,1}
\end{aligned}
$$

Each factor is a section of some line bundle over $(G/B)^3$. The first three factors describe the incidence relations $p_1 \in l_3$, $p_3 \in l_2$, $p_2 \in l_1$ respectively. Their product is a section of the pull-back to $(G/B)^3$ of the $\mathcal{O}(1,\ldots,1)$ bundle over $(\mathbf{P}^2)^3 \times \mathrm{Gr}(1,\mathbf{P}^2)^3$. (Just look which Plücker coordinates are used.) Similarly, the last four factors describe a section of that same bundle so that in total s is a section of the anti-canonical bundle of $(G/B)^3$, which is the pullback of $\mathcal{O}(2,\ldots,2)$.

Choosing local coordinates sensibly, one checks that $\operatorname{div} s$ has residually normal crossing at the totally degenerate triangle $(E_1, E_1, E_1, E_{12}, E_{12}, E_{12})$ and we thus have a splitting of $(G/B)^3$ compatible with the components of $\operatorname{div} s$ and with the intersection $\mathcal{F}_{3,3} = \mathcal{I}_{3,3}$ of the three components given by the first three factors of s.

One also computes that s vanishes to order five at the locus of totally degenerate triangles. By [21] this implies that the splitting extends to the blowup of $(G/B)^3$ along that locus, compatibly with the proper transform of $\mathcal{F}_{3,3}$.

2.3 Proof of Theorem 1

Proof for $n = 3$. If D contains each of the columns of D_3 at least once, then $\mathcal{L}_D$ is ample on $(G/B)^3$ and Proposition 3 implies that $H^i(\mathcal{F}_D, \mathcal{L}_D)$ vanishes for $i > 0$. Furthermore, $H^0((G/B)^3, \mathcal{L}_D) \to H^0(\mathcal{F}_D, \mathcal{L}_D)$ is surjective, and a well-known fact from representation theory [17, II 14.20] states that the restriction map $H^0((\mathbf{P}^2)^3 \times \mathrm{Gr}(1,\mathbf{P}^2)^3, \mathcal{L}) \to H^0((G/B)^3, \mathcal{L})$ is surjective for any effective $\mathcal{L}$. Thus $H^0(\mathrm{Gr}(D), \mathcal{L}_D) \to H^0(\mathcal{F}_D, \mathcal{L}_D)$ is surjective, and $H^0(\mathcal{F}_D, \mathcal{L}_D) \cong S_D^*$ by the previous section, proving part (a) of the Theorem in this case.

As for part (b), normality follows from the theorem of Mehta and Srinivas, provided $\mathcal{F}_{3,3}$ possesses a resolution of singularities with connected fibers. We will construct two such resolutions in later sections. The normality, together with the surjectivity of the restriction map above for any multiple of D, is essentially equivalent to projective normality by [15], Ch II, Ex 5.14(d), given the projective normality of the Plücker embedding.

If D does not contain all the columns of D_3, then we need a strengthening of Proposition 3. Indeed Ramanathan has proved [32, 1.12] that instead of requiring $\mathcal{L}$ to be ample it suffices to have a subdivisor E of $\operatorname{div} s$, not containing any component of Y, so that $\mathcal{L}^p \otimes \mathcal{O}(E)$ is ample. In our case such an E is provided by the last four factors of s, when $\mathcal{L} = \mathcal{L}_D$.

We also need to show that $H^0(\mathcal{F}_D, \mathcal{L}_D)$ may be identified with $H^0(\mathcal{F}_{3,3}, \mathcal{L}_D)$. For this one considers the projection map from $(\mathbf{P}^2)^3 \times \mathrm{Gr}(1, \mathbf{P}^2)^3$ to the subproduct corresponding to the columns that do occur in D (amongst the first six columns of D_3). What one needs to know is that $\pi_* \mathcal{O}_{\mathcal{F}_{3,3}} = \mathcal{O}_{\mathcal{F}_D}$, where $\pi : \mathcal{F}_{3,3} \to \mathcal{F}_D$ is the restriction of the projection map to $\mathcal{F}_{3,3} = \mathcal{I}_{3,3}$. Now the fibers of π are connected and [18, 6.1.6 and A.1.5] apply.

This finishes the proof for $n = 3$.

Case of general n.

For $n > 3$ we need to repeat the argument "fibered over $\mathrm{Gr}(2, \mathbf{P}^{n-1})$". An element of $\mathcal{F}_{3,n}$ may be represented by a tuple (M, x, y, z) where M is an $n \times 3$ matrix of rank three, whose columns span the plane P in $\mathbf{P}^{n-1}$, and x, y, z are three by three matrices as before. The flags $((p_1, l_2), (p_2, l_3), (p_3, l_1))$ are described by the first two columns of of Mx, My, Mz respectively. Let P_3 denote the stabilizer in G of the plane E_{123}. If X is any P_3-space, we denote by $G \times^{P_3} X$ the associated G-space fibered over $G/P_3 = \mathrm{Gr}(2, \mathbf{P}^{n-1})$, with fiber X over the point E_{123}.

If we replace in the formula for s each x by Mx, each y by My, each z by Mz, then we get a section of the relative anti-canonical bundle of the fibration

$$\begin{array}{ccc} (GL(3)/B(3))^3 & \to & G \times^{P_3} (GL(3)/B(3))^3 \\ & & \downarrow \\ & & \mathrm{Gr}(2, \mathbf{P}^{n-1}) \end{array}$$

Indeed it has the correct transformation properties under G and under P_3 and it restricts to our known section of the anti-canonical bundle of the fiber over E_{123}.

Thus to get a section of the anti-canonical bundle of the total space, one must still multiply with a section of the pull-back of the anti-canonical bundle of the base $\mathrm{Gr}(2, \mathbf{P}^{n-1})$. There is a choice here. Let us take one for which one can easily check that it has residually normal crossing at E_{123}, to wit the product of the n Plücker coordinates (subdeterminants of M) based on taking 3 consecutive rows of M, with the rows ordered cyclically.

The result of all this is a section of the anti-canonical bundle of the total space $G \times^{P_3} (GL(3)/B(3))^3$ that gives us a splitting with the same virtues as in the case $n = 3$. In particular, it is compatible with $\mathcal{F}_{3,n}$. (It suffices to check this in a neighborhood of the fiber of E_{123}.)

The proof of the Theorem now goes through exactly as before, except for one problem we still need to address: The analogue of the "well-known fact from representation theory" needs to be proved now. We need to show that the map $H^0(\mathrm{Gr}(D_3), \mathcal{L}) \to H^0(G \times^{P_3} (GL(3)/B(3))^3, \mathcal{L})$ is surjective for effective line bundles on $\mathrm{Gr}(D_3)$. This is indeed a fact in representation theory, for which we refer to the Appendix. Theorem 1 is proved.

2.4 Fixed points

In order to gain further information about the triangle space $\mathcal{F}_{3,n}$ and its line bundles, we will use the method of Lefschetz: that is, to study the fixed points of the torus of diagonal matrices $T \subset GL(n)$ acting on our space. The work of Atiyah, Bott, and others will then give us precise formulas for the cohomologies of coherent sheaves, expressed in terms of the combinatorial data of the T-fixed points and their tangent vectors. This technique applies only to smooth varieties, so in subsequent sections we will study desingularizations of the triangle space.

However, a desingularization map is an isomorphism on the smooth locus of the variety, so if a fixed point τ is a smooth point of $\mathcal{F}_{3,n}$, then the local tangent data will be the same for $\mathcal{F}_{3,n}$ and all desingularizations. We begin by examining these smooth points.

We adopt the combinatorial framework for dealing with general configuration varieties developed in [22], [24]. See also [14], Lect 16. A T-fixed point of the Grassmannian $\mathrm{Gr}(c, \mathbf{C}^n)$ is a c-dimensional subspace spanned by coordinate vectors $\{e_{i_1}, \ldots, e_{i_c}\}$, a subset of the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbf{C}^n$. That is, the fixed points are the spaces E_I for $I = \{i_1 \leq \cdots \leq i_c\} \subset [1, n]$.

Hence, we may index the fixed points of $\mathcal{F}_D$, for any diagram D, by *column tabloids* τ, which are maps $\tau : D \to [1,n]$, strictly increasing down each column, such that for every inclusion of columns $C \subset C'$, we have $\tau(C) \subset \tau(C')$. (More precisely, these are the fixed points of $\mathcal{I}_D$, but in our case this is identical to $\mathcal{F}_D$.) One may check for our $D = D_3$ that there are $11n(n-1)(n-2)$ such tabloids. As we shall see, all of them are smooth points of $\mathcal{F}_{3,n}$ except those corresponding to maximally degenerate triangles: namely, the singular fixed points are of the form

$$\tau_{ijk} = \begin{bmatrix} i & & & & i & i & i \\ & i & & i & & j & j \\ & & i & j & j & & k \end{bmatrix} ,$$

for i, j, k distinct integers in $[1, n]$.

For a space $V \in \mathrm{Gr}(c, \mathbf{C}^n)$, we may model the tangent space as $T_V \mathrm{Gr}(c, \mathbf{C}^n) \cong \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C}^n/V)$. (That is, the tangent bundle of the Grassmannian is isomorphic to Hom of the tautological subbundle into the tautological quotient bundle.) Furthermore, the incidence variety

$$\mathcal{I} = \{(U, V) \in \mathrm{Gr}(c, \mathbf{C}^n) \times \mathrm{Gr}(c', \mathbf{C}^n) \mid U \subset V\}$$

has tangent space

$$T_{(U,V)}\mathcal{I} = \{(\phi, \psi) \in \mathrm{Hom}_{\mathbf{C}}(U, \mathbf{C}^n/U) \times \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C}^n/V) \mid \psi|_U = \phi \bmod V\} .$$

From this, one can deduce as in [22]

Lemma 6 *The eigenvalues of T near a smooth fixed point τ of $\mathcal{F}_{3,n}$ are all of the form $\lambda(\mathrm{diag}(x_1, \dots, x_n)) = x_i^{-1} x_j$ for $i \neq j \in [1, n]$. The multiplicity $d_{ij}(\tau)$ of $x_i^{-1} x_j$ is the number of connected components of the following graph: vertices $= \{$columns C of $D \mid i \in \tau(C),\ j \notin \tau(C)\}$, edges $= \{(C, C') \mid C \subset C'$ or $C' \subset C\}$.*

For instance, consider the singular fixed point $\tau = \tau_{ijk}$ defined above. Suppose $l, m \neq i, j, k$. Then the multiplicities $d_{ab}(\tau)$ are given by the table:

$$\begin{array}{cc|ccccc} & & & & b & & \\ & d_{ab} & i & j & k & l & m \\ \hline & i & 0 & 3 & 1 & 1 & 1 \\ & j & 0 & 0 & 3 & 1 & 1 \\ a & k & 0 & 0 & 0 & 1 & 1 \\ & l & 0 & 0 & 0 & 0 & 0 \\ & m & 0 & 0 & 0 & 0 & 0 \end{array} .$$

Note that this makes a total of $3n-2$ eigenvectors, whereas $\mathcal{F}_{3,n}$ is $3n-3$ dimensional. The $3n-2$ vectors span the Zariski tangent space of $\mathcal{F}_{3,n} \subset \mathrm{Gr}(D_3)$ at the singular point. These eigenvectors also correspond to T-stable curves through τ: that is, $\Phi : \mathbf{C} \to \mathcal{F}_{3,n}$, with $\Phi(0) = \tau$ and $x \cdot \Phi(s) = \Phi(\lambda(x)s)$ for all $s \in \mathbf{C}$ and $x \in T$, and some eigenvalue character λ.

For all other tabloids, there are $3n-3$ eigenvectors, and the fixed points are smooth points of $\mathcal{F}_{3,n}$. Now, the singular locus of $\mathcal{F}_{3,n}$ is $GL(n)$-invariant, and every $GL(n)$-orbit which is not maximally degenerate approaches some T-fixed point which is not maximally degenerate, so we may conclude:

Lemma 7 *The singular locus of $\mathcal{F}_{3,n}$ consists of the maximally degenerate triangles (for which all three points and all three lines coincide).*

3 The Schubert–Semple space

Next we consider the smooth triangle space $\mathcal{F}_{3,n}^{\mathcal{SS}}$ first defined by Schubert [37], and given a modern construction by Semple [38]. See also Collino and Fulton [10]. The points of this space may be thought of intuitively as triangles with an extra piece of data: a circle passing through the three vertices, and tangent to the corresponding side if two vertices coincide. The circle is determined by the triangle in all cases except the maximally degenerate triangles, for which there is a $\mathbf{P}^1$ of compatible circles (including radius zero and infinity).

The rigorous definition is as follows. First, let $n = 3$. The space Q of conic curves in $\mathbf{P}^2$ can be identified as $Q = \mathbf{P}^*(\mathrm{Sym}^2\mathbf{C}^3) \cong \mathbf{P}^5$. A projective plane in this $\mathbf{P}^5$ is called a *net* of conics, and the space of nets is $\mathrm{Gr}(2, Q)$. Let $\mathcal{F}_{3,3}^{\circ}$ be the general triangles in $\mathbf{P}^2$. For any general triangle τ, the conics passing through its three vertices form a net N_τ, and we have an embedding

$$\begin{array}{rccc} \Phi : & \mathcal{F}_{3,3}^{\circ} & \to & \mathcal{F}_{3,3}^{\circ} \times \mathrm{Gr}(2,Q) \\ & \tau & \mapsto & (\tau, N_\tau) \end{array} .$$

The Schubert–Semple space is defined as the closure of the image:

$$\mathcal{F}_{3,3}^{\mathcal{SS}} = \text{closure } \mathrm{Im}(\Phi) \ \subset \ (\mathbf{P}^2)^3 \times \mathrm{Gr}(1, \mathbf{P}^2)^3 \times \mathrm{Gr}(2, Q) \ .$$

Now, for general n, we define $\mathcal{F}_{3,n}^{\mathcal{SS}}$ as a family of such spaces with the plane varying in $\mathbf{P}^{n-1}$:

$$\begin{array}{ccc} \mathcal{F}^{\mathcal{SS}}_{3,3} & \rightarrow & \mathcal{F}^{\mathcal{SS}}_{3,n} \\ & & \downarrow \\ & & \mathrm{Gr}(2,\mathbf{P}^{n-1}) \end{array}$$

More formally,

$$\mathcal{F}^{\mathcal{SS}}_{3,n} = GL(n) \times^{P_3} \mathcal{F}^{\mathcal{SS}}_{3,3} = \frac{GL(n) \times \mathcal{F}^{\mathcal{SS}}_{3,3}}{P_3} \ .$$

Here, $P_3 \subset GL(n)$ is again the parabolic subgroup such that $GL(n)/P_3 \cong \mathrm{Gr}(2,\mathbf{P}^{n-1})$, and P_3 acts on $GL(n) \times \mathcal{F}^{\mathcal{SS}}_{3,3}$ by $p \cdot (g,t) = (gp, b(p^{-1})t)$, $b : P_3 \rightarrow GL(3)$ being the obvious homomorphism. Semple shows that this is a smooth projective variety, and the obvious projection $\mathcal{F}^{\mathcal{SS}}_{3,n} \rightarrow \mathcal{F}_{3,n}$ is an isomorphism on the smooth locus of $\mathcal{F}_{3,n}$.

We wish to find the T-fixed points on $\mathcal{F}^{\mathcal{SS}}_{3,n}$, as well as their tangent eigenvectors. For a point in $\mathcal{F}^{\mathcal{SS}}_{3,n}$ whose image is a smooth point of $\mathcal{F}_{3,n}$, this follows immediately from the results of the previous section.

It remains to consider the fixed points of $\mathcal{F}^{\mathcal{SS}}_{3,n}$ lying above the singular fixed point

$$\tau_{ijk} = (E_i, E_i, E_i, E_{ij}, E_{ij}, E_{ij}, E_{ijk}) \in \mathcal{F}_{3,n} \ .$$

The degenerate triangle τ_{ijk} lies in the plane E_{ijk}, so that

$$\mathrm{Sym}^2(E_{ijk})^* = \langle u_i^2, u_j^2, u_k^2, u_iu_j, u_iu_k, u_ju_k \rangle \ ,$$

where $u_1, \ldots, u_n$ is the dual of the standard basis of $\mathbf{C}^n$. A diagonal matrix $x = \mathrm{diag}(x_1, \ldots, x_n) \in T$ acts on a monomial by the character $x \cdot u_iu_j = (x_ix_j)^{-1}u_iu_j$. According to [10], p. 79, the fiber above τ_{ijk} consists of all nets $N \in \mathrm{Gr}(2, \mathbf{P}^*(\mathrm{Sym}^2 E_{ijk}))$ such that

$$\langle u_k^2, u_ju_k \rangle \subset N \subset \langle u_k^2, u_ju_k, u_j^2, u_iu_k \rangle \ .$$

This is a copy of $\mathbf{P}^1 \subset \mathcal{F}^{\mathcal{SS}}_{3,n}$. More precisely, it is isomorphic as a T-space to $\mathbf{P}(\mathbf{C}_{x_j^{-2}} \oplus \mathbf{C}_{(x_ix_k)^{-1}})$, where $\mathbf{C}_\lambda$ is the one-dimensional representation of T with character λ. There are exactly two nets in this fiber which are fixed by T, namely

$$\eta_{ijk} = \langle u_k^2, u_ju_k, u_j^2 \rangle = [1:0]$$

$$\zeta_{ijk} = \langle u_k^2, u_j u_k, u_i u_k\rangle = [0:1] \ .$$

That is, each singular fixed point $\tau_{ijk} \in \mathcal{F}_{3,n}$ splits into two isolated fixed points $\eta_{ijk}, \zeta_{ijk} \in \mathcal{F}_{3,n}^{\mathcal{SS}}$.

Now we find the eigenvalues of T acting on the tangent spaces of these fixed points. First, consider the tangent vector at η_{ijk} pointing along the fiber above τ_{ijk}. The tangent space of this $\mathbf{P}^1$ at $\eta_{ijk} = [1:0]$ is

$$\mathrm{Hom}(\mathbf{C}_{x_j^{-2}}, \mathbf{C}_{(x_i x_k)^{-1}}) \cong \mathbf{C}_{x_j^2 (x_i x_k)^{-1}} \ ,$$

and the eigenvalue of our tangent vector is $x_j^2(x_i x_k)^{-1}$. Similarly, $(x_i x_k) x_j^{-2}$ is an eigenvalue of T acting on the tangent space at ζ_{ijk}.

The other eigenvalues of the smooth tangent spaces can all be found from examining the Zariski tangent space of the singular point τ_{ijk}. For example, consider the eigenvector defined by

$$\begin{aligned} \phi \in T_{\tau_{ijk}}\mathrm{Gr}(D_3) &= \mathrm{Hom}(E_i, \mathbf{C}^n/E_i)^3 \oplus \mathrm{Hom}(E_{ij}, \mathbf{C}^n/E_{ij})^3 \oplus \\ &\qquad \mathrm{Hom}(E_{ijk}, \mathbf{C}^n/E_{ijk})^3 \ , \\ \phi &= (\phi_{ij}, -\phi_{ij}, 0, 0, 0, 0, 0) \ , \end{aligned}$$

where $\phi_{ij}(e_l) = \delta_{il} e_j$. This eigenvector has eigenvalue $x_i^{-1} x_j$ and points along the T-stable curve $\Phi : \mathbf{C} \to \mathcal{F}_{3,n}$

$$\Phi(s) = (\Phi_{ij}(s), \Phi_{ij}(-s), E_i, E_{ij}, E_{ij}, E_{ij}, E_{ijk}) \ ,$$

where $\Phi_{ij} : \mathbf{C} \to \mathbf{P}^{n-1}$, $\Phi_{ij}(s) = E_i + sE_j$. Now, $\Phi(\mathbf{C} - 0)$ lies in the smooth locus of $\mathcal{F}_{3,n}$, so it lifts uniquely to a T-stable curve $\Phi^{\mathcal{SS}}$ in $\mathcal{F}_{3,n}^{\mathcal{SS}}$. Clearly $\Phi^{\mathcal{SS}}(0)$ is a fixed point above τ_{ijk}, and we may easily check that it is ζ_{ijk}. Differentiating $\Phi^{\mathcal{SS}}$ at $s = 0$, we obtain a tangent vector to ζ_{ijk} with eigenvalue $x_i^{-1} x_j$. We may argue similarly for the other Zariski tangent vectors which do not point along the singular locus of $\mathcal{F}_{3,n}$.

For a vector which *does* point along the singular locus, for instance

$$\phi = (\phi_{ij}, \phi_{ij}, \phi_{ij}, 0, 0, 0, 0) \ ,$$

the corresponding curve does not lift uniquely to a T-stable curve in $\mathcal{F}_{3,n}^{\mathcal{SS}}$. In fact, it lifts in exactly two ways, one leading to each point η_{ijk}, ζ_{ijk} above τ_{ijk}. Hence ϕ accounts for a tangent vector with eigenvalue $x_i^{-1} x_j$ at *each* of the lifted fixed points.

Summarizing, we get:

Lemma 8 *The eigenvalues at the fixed points in $\mathcal{F}_{3,n}^{\mathcal{SS}}$ above τ_{ijk} are as follows:*

$$\eta_{ijk} \qquad x_i^{-1}x_k^{-1}x_j^2,\ x_i^{-1}x_j,\ x_i^{-1}x_k,\ x_j^{-1}x_k \text{ (3 times)},\ x_i^{-1}x_l,\ x_j^{-1}x_l,\ x_k^{-1}x_l$$

$$\zeta_{ijk} \qquad x_ix_kx_j^{-2},\ x_i^{-1}x_j \text{ (3 times)},\ x_i^{-1}x_k,\ x_j^{-1}x_k,\ x_i^{-1}x_l,\ x_j^{-1}x_l,\ x_k^{-1}x_l\ ,$$

where l runs over $[1,n]\setminus\{i,j,k\}$.

4 The Fulton–MacPherson Space

We describe another desingularization of $\mathcal{F}_{3,n}$, a very special case of the construction of Fulton and MacPherson in [13]. It seems this space $\mathcal{F}_{3,n}^{\mathcal{FM}}$ is isomorphic to $\mathcal{F}_{3,n}^{\mathcal{SS}}$ as an abstract variety ([13], p. 189), but even if that is so, then it comes equipped with a different map $\mathcal{F}_{3,n}^{\mathcal{FM}} \to \mathcal{F}_{3,n}$ and a different $GL(n)$-action. In particular, the T-fixed points of $\mathcal{F}_{3,n}^{\mathcal{FM}}$ are not isolated, leading to a different type of local data for our fixed-point formulas below. Our analysis of the Fulton–MacPherson space will also show that $\mathcal{F}_{3,n}$ has rational singularities.

4.1 Strata

Again, we first consider the case $n = 3$, and we will have a fiber bundle $\mathcal{F}_{3,3}^{\mathcal{FM}} \to \mathcal{F}_{3,n}^{\mathcal{FM}} \to \mathrm{Gr}(2, \mathbf{P}^{n-1})$. It is shown in [13] how $\mathcal{F}_{3,3}^{\mathcal{FM}} = \mathbf{P}^2[3]$ can be constructed as a union of 8 strata, each consisting of certain configurations of points and tangent vectors in $\mathbf{P}^2$. For each stratum, there is a natural $GL(3)$ action and an equivariant map to $\mathcal{F}_{3,3}$.

- $D_\emptyset$, the configuration space $(\mathbf{P}^2)^3 \setminus \bigcup \Delta_{ij}$.
  An open set in $\mathcal{F}_{3,3}^{\mathcal{FM}}$.
  Triples $[p_1, p_2, p_3]$ of pairwise-distinct points.
  A triple maps to the triangle

$$(p_1, p_2, p_3, \mathbf{C}p_2 + \mathbf{C}p_3, \mathbf{C}p_1 + \mathbf{C}p_3, \mathbf{C}p_1 + \mathbf{C}p_2)\ ,$$

  where $\mathbf{C}p + \mathbf{C}p'$ means the projective line through the points.

- D_{12}, corresponding to the diagonal $\Delta_{12} \subset (\mathbf{P}^2)^3$.
Codimension 1 in $\mathcal{F}_{3,3}^{\mathcal{FM}}$.
Configurations $[p_{12}, p_3,\ \mathbf{C}^* v_3]$: distinct points p_{12} and p_3, and a non-zero tangent vector $v_3 \in T_{p_{12}}\mathbf{P}^{n-1}$, up to scaling of v_3.
Intuitively represents infinitesimally distinct points $(p_{12},\ p_{12}+v_3,\ p_3)$.
Maps to the triangle
$$(\, p_{12}, p_{12}, p_3,\ \mathbf{C}p_{12} + \mathbf{C}p_3,\ \mathbf{C}p_{12} + \mathbf{C}p_3,\ \mathbf{C}p_{12} + \mathbf{C}v_3 \,)\ .$$
Similarly $D_{23},\ D_{13}$.

- D_{123}, corresponding to the total diagonal Δ_{123} of $(\mathbf{P}^2)^3$.
Codimension 1 in $\mathcal{F}_{3,3}^{\mathcal{FM}}$.
Configurations $[p_{123},\ \mathbf{C}^*(v_1, v_2, v_3)]$: a point and three non-zero tangent vectors with $v_1 + v_2 + v_3 = 0$, up to simultaneous scaling.
Intuitively represents $(p_{123}, p_{123} + v_3, p_{123} - v_2)$.
Maps to the triangle
$$(\, p_{123}, p_{123}, p_{123},\ \mathbf{C}p_{123} + \mathbf{C}v_1,\ \mathbf{C}p_{123} + \mathbf{C}v_2,\ \mathbf{C}p_{123} + \mathbf{C}v_3 \,)\ .$$

- $D_{123,12}$.
Codimension 2 in $\mathcal{F}_{3,3}^{\mathcal{FM}}$.
Configurations $[p_{123}, \mathbf{C}^* v_{12}, \mathbf{C}^* v_3]$: a point and two non-zero tangent vectors up to scaling of each.
Intuitively represents $(\, p_{123}+v_{12},\ p_{123}+v_{12}+v_3,\ p_{123} \,)$, with v_3 infinitesimal compared to v_{12}, which is itself already infinitesimal.
Maps to the triangle
$$(\, p_{123}, p_{123}, p_{123},\ \mathbf{C}p_{123} + \mathbf{C}v_{12},\ \mathbf{C}p_{123} + \mathbf{C}v_{12},\ \mathbf{C}p_{123} + \mathbf{C}v_3 \,)\ .$$
Similarly $D_{123,23},\ D_{123,13}$.

The closures of the codimension 1 strata are smooth divisors: $\overline{D_{ij}} = D_{ij} \cup D_{123,ij}$, $\overline{D_{123}} = D_{123} \cup D_{123,12} \cup D_{123,23} \cup D_{123,13}$, and the intersections are transversal. The above description is enough to define coordinate charts for $\mathcal{F}_{3,3}^{\mathcal{FM}}$, gluing together the normal bundles of the strata appropriately.

Also, the maps described above piece together into a regular, equivariant desingularization $\pi : \mathcal{F}_{3,3}^{\mathcal{FM}} \to \mathcal{F}_{3,3}$. The map is an isomorphism outside the singular locus of $\mathcal{F}_{3,3}$, and the inverse image of a singular triangle

(p,p,p,l,l,l) is the set of configurations: $[p, \mathbf{C}^*(v_1,v_2,v_3)] \in D_{123}$ such that v_1, v_2, v_3 are non-zero and parallel to l, and $v_1+v_2+v_3=0$. (There are also three extra configurations in $D_{123,12}$, $D_{123,23}$, and $D_{123,13}$.) Thus, the fiber $\pi^{-1}(p,p,p,l,l,l)$ is a projective line, and it is easily seen that if (p,p,p,l,l,l) is fixed by the torus T, then each point of this fiber is also fixed.

4.2 Fixed lines

Now let us consider the general $\mathcal{F}_{3,n}^{\mathcal{FM}}$, for which everything we have said carries over. In particular, $\mathcal{F}_{3,n}^{\mathcal{FM}}$ has two classes of fixed points: the isolated ones, which map one-to-one to the non-singular fixed triangles in $\mathcal{F}_{3,n}$, and the fibers over the singular fixed triangles τ_{ijk},

$$\mathbf{P}^1_{ijk} = \pi^{-1}(\tau_{ijk}) \ .$$

The tangent data at the isolated points is identical to that in $\mathcal{F}_{3,n}$. For the $\mathbf{P}^1_{ijk}$, we will need to determine the types of their normal bundles as T-equivariant vector bundles over $\mathbf{P}^1$.

First, note that locally near $\mathbf{P}^1_{ijk}$, we have

$$\mathcal{F}_{3,n}^{\mathcal{FM}} \cong \mathcal{F}_{3,3}^{\mathcal{FM}}(E_{ijk}) \times T_{E_{ijk}}\mathrm{Gr}(2,\mathbf{P}^{n-1}).$$

That is, the normal bundles in the direction of the Grassmannian are trivial, and we reduce to $\mathcal{F}_{3,3}^{\mathcal{FM}}(E_{ijk})$, the Fulton–MacPherson space relative to the plane $\mathbf{P}^2 = \mathbf{P}(E_{ijk})$. Now we will use the alternative description ([13], p. 196) for $\mathcal{F}_{3,3}^{\mathcal{FM}}$ as a blowup of $(\mathbf{P}^2)^3$, first along the triple diagonal Δ_{123}, then along the proper transforms of the three partial diagonals Δ_{12}, Δ_{23}, Δ_{13}.

Let $\mathbf{C}_{ab}$ denote the one-dimensional T-space on which $\mathrm{diag}(x_1,\ldots,x_n)$ acts by the character $x_a^{-1}x_b$. We may take coordinates for a neighborhood $U \subset (\mathbf{P}^2)^3 = \mathbf{P}(E_{ijk})^3$ near the fixed point $\tau_i = (E_i,E_i,E_i)$ so that, as T-spaces, we have

$$\begin{array}{rcl} U & \cong & (\mathbf{C}_{ij}\times\mathbf{C}_{ik})^3 \\ \tau_i & \cong & (0,0,0,0,0,0) \\ \Delta_{12} & \cong & \{(0,0,a,b,c,d)\} \\ \Delta_{23} & \cong & \{(a,b,0,0,c,d)\} \\ \Delta_{13} & \cong & \{(a,b,a,b,c,d)\} \\ \Delta_{123} & \cong & \{(0,0,0,0,c,d)\} \ . \end{array}$$

In this blowup, there will be two T-fixed $\mathbf{P}^1$ above τ_i (namely, τ_{ijk} and τ_{ikj}), and we wish to determine their normal bundles.

Since all the centers of blowing up are products with the last factor $(\mathbf{C}_{ij} \times \mathbf{C}_{ik})$, the normal bundles will be trivial in these directions. Thus, we may reduce to

$$\begin{array}{rcl} U' & \cong & (\mathbf{C}_{ij} \times \mathbf{C}_{ik})^2 \\ \Delta'_{12} & \cong & \{(0,0,a,b)\} \\ \Delta'_{23} & \cong & \{(a,b,0,0)\} \\ \Delta'_{13} & \cong & \{(a,b,a,b)\} \\ \Delta'_{123} & \cong & \{(0,0,0,0)\} \ . \end{array}$$

Performing the first blowup along Δ'_{123}, we obtain:

$$\begin{array}{rcl} \mathrm{Bl}_{123}U' & \cong & \mathcal{O}(-1) \to \mathbf{P}\left((\mathbf{C}_{ij} \times \mathbf{C}_{ik})^2\right) \\ \tilde{\Delta}'_{12} & \cong & \mathcal{O}(-1) \to \{\,[0:0:a:b]\,\} \\ \tilde{\Delta}'_{23} & \cong & \mathcal{O}(-1) \to \{\,[a:b:0:0]\,\} \\ \tilde{\Delta}'_{13} & \cong & \mathcal{O}(-1) \to \{[a:b:a:b]\,\} \ . \end{array}$$

This has two T-fixed projective lines: let us focus on one of them, $\mathbf{P}(\mathbf{C}^2_{ij}) = \{[a:0:b:0]\}$. The normal bundle of a line $\mathbf{P}^1 \subset \mathbf{P}^2$ is $\mathcal{O}(1)$, so restricting to a neighborhood U'' of our fixed line gives

$$\begin{array}{rcl} U'' & \cong & \mathcal{O}_{ij}(-1) \oplus 2\mathcal{O}_{jk}(1) \to \mathbf{P}^1 \\ \Delta''_{12} & \cong & \{\,(v,0,w)\,\} \to [0:1] \\ \Delta''_{23} & \cong & \{\,(v,w,0)\,\} \to [1:0] \\ \Delta''_{13} & \cong & \{\,(v,w,w)\,\} \to [1:1] \ , \end{array}$$

where $\mathcal{O}_{ij}(m)$ indicates a line bundle over a T-fixed $\mathbf{P}^1$ with fibers of type $\mathbf{C}_{ij}$.

Now consider the next blowup, along Δ''_{12}. This is locally a product of $\mathcal{O}_{ij}(-1) \oplus \mathcal{O}_{jk}(1)$ and the locus

$$(\,0 \to [0:1]\,) \ \subset \ (\,\mathcal{O}_{jk}(1) \to \mathbf{P}^1\,) \ ,$$

so the blowup will not affect the first factors, and we may concentrate on the last. Thus, consider the total space of the line bundle $\mathcal{O}(m)$ over $\mathbf{P}^1$, and blow up at a point on the zero-section. It is easily seen in coordinates that the normal bundle of the proper transform of the zero-section is $\mathcal{O}(m-1)$.

Thus, the second, third, and fourth blowups will transform $\mathcal{O}_{ij}(-1) \oplus 2\mathcal{O}_{jk}(1)$ successively into

$$\mathcal{O}_{ij}(-1) \oplus \mathcal{O}_{jk} \oplus \mathcal{O}_{jk}(1), \quad \mathcal{O}_{ij}(-1) \oplus \mathcal{O}_{jk} \oplus \mathcal{O}_{jk}, \quad \mathcal{O}_{ij}(-1) \oplus \mathcal{O}_{jk}(-1) \oplus \mathcal{O}_{jk} \ .$$

Recalling the dimensions we dropped at the beginning, we obtain our final answer.

Lemma 9 *The normal bundle of the T-fixed component $\mathbf{P}^1_{ijk}$ in $\mathcal{F}^{\mathcal{FM}}_{3,n}$ is*

$$\mathcal{O}_{ij}(-1) \oplus \mathcal{O}_{jk}(-1) \oplus \mathcal{O}_{jk} \oplus \mathcal{O}_{ij} \oplus \mathcal{O}_{jk} \oplus \sum_{\substack{l \in [1,n] \\ l \neq i,j,k}} (\mathcal{O}_{il} \oplus \mathcal{O}_{jl} \oplus \mathcal{O}_{kl}) \ ,$$

where $\mathcal{O}_{ab}(m)$ is the line bundle with Chern class m and fibers of character $x_a^{-1} x_b$.

Recall that this describes not just the normal bundle, but an actual open neighborhood of the T-fixed component $\mathbf{P}^1_{ijk}$ in $\mathcal{F}^{\mathcal{FM}}_{3,n}$. It follows easily that the canonical bundle is trivial, as a line bundle, on that neighborhood. As $GL(3)$ acts transitively on the stratum of degenerate triangles, we get

Lemma 10 *The canonical bundle is trivial in a neighborhood of the fiber in $\mathcal{F}^{\mathcal{FM}}_{3,n}$ of any singular point.*

4.3 Rational singularities

In later sections, it will be convenient to know that our singular space $\mathcal{F}_{3,n}$ has rational singularities. Let us first recall Kempf's definition [19]. A birational proper map $\pi : Y \to X$ is called a *rational resolution* if Y is smooth, and
a. $\pi_* \mathcal{O}_Y = \mathcal{O}_X$ or, equivalently, X is normal,
b. $R^i \pi_* \mathcal{O}_Y = 0$ for $i > 0$,
c. $R^i \pi_* K_Y = 0$ for $i > 0$.
The last condition is automatic in characteristic 0. One says that X has *rational singularities* if there exists a rational resolution $\pi : Y \to X$. The usefulness of this notion lies in the

Lemma 11 *Let $\pi : Y \to X$ be a map satisfying conditions (a) and (b), and $\mathcal{L}$ a line bundle on X. Then $H^i(X, \mathcal{L}) = H^i(Y, \pi^* \mathcal{L})$ for all i.*

This follows from the projection formula and a degenerate case of the Leray spectral sequence [14, III, Ex. 8.1, 8.3]. We will use the Lemma below in the case of the triangle space and its desingularizations.

Now, we have seen that the singularity of $\mathcal{F}_{3,n}$ is that of the cone over a quadric in $\mathbf{P}^3$, and it is well known that this singularity is rational, but we shall prove it directly from the definition.

Proposition 12 *The map* $\pi : \mathcal{F}_{3,n}^{\mathcal{FM}} \to \mathcal{F}_{3,n}$ *is a rational resolution, and so is the map* $\mathcal{F}_{3,n}^{\mathcal{SS}} \to \mathcal{F}_{3,n}$.

Proof. The target $\mathcal{F}_{3,n}$ of π is normal by Theorem 1. For the second condition we use Grothendieck's theorem on formal functions. It tells us that we should try to show that $H^i(\pi^{-1}(P)_m, \mathcal{O}/\mathcal{I}^m)$ vanishes, where $\pi^{-1}(P)_m$ is the m-th order neighborhood of the fiber $\mathbf{P}^1$ over a point P of the singular locus and $\mathcal{I}$ is the ideal sheaf of this fiber. By dévissage we only need to show that $H^i(\pi^{-1}(P)_m, \mathcal{I}^{m-1}/\mathcal{I}^m) = H^i(\mathbf{P}^1, \mathcal{I}^{m-1}/\mathcal{I}^m)$ vanishes for $m > 0$. Now $\mathcal{I}^{m-1}/\mathcal{I}^m$ is just a power of the conormal bundle, so by the computation above (lemma 9), it is a sum of line bundles with nonnegative Chern class. The result follows.

From lemma 10 one sees that the $R^i\pi_*K_{\mathcal{F}_{3,n}^{\mathcal{FM}}}$ are locally the same as the $R^i\pi_*\mathcal{O}_{\mathcal{F}_{3,n}^{\mathcal{FM}}}$, so they vanish too. Alternatively, one checks that the Grauert-Riemenschneider vanishing theorem with Frobenius splitting [28] applies. For this, observe that our splitting of $\mathcal{F}_{3,n}$ gives one on the complement of the exceptional locus of π in $\mathcal{F}_{3,n}^{\mathcal{FM}}$. As this exceptional locus has codimension two the splitting extends and in fact our section s of the anti-canonical bundle extends. The divisor of the extended s contains the proper transform of the divisor of the factor $s_4 = ((-x_{2,2}x_{3,1} + x_{2,1}x_{3,2})(-y_{1,2}y_{2,1} + y_{1,1}y_{2,2}) - (-x_{1,2}x_{2,1} + x_{1,1}x_{2,2})(-y_{2,2}y_{3,1} + y_{2,1}y_{3,2}))$ of s, hence it contains the exceptional locus, as the equation of the divisor of s_4 only puts constraints on two lines in a configuration, no further restrictions on its points.

Before leaving the case of the $\mathcal{FM}$ resolution let us note that by lemma 10 there is a line bundle $\omega = \pi_*K_{\mathcal{F}_{3,n}^{\mathcal{FM}}}$ on $\mathcal{F}_{3,n}$ whose restriction to the smooth locus is the canonical bundle. Now let π denote the map $\mathcal{F}_{3,n}^{\mathcal{SS}} \to \mathcal{F}_{3,n}$ instead. The pull-back of ω to $\mathcal{F}_{3,n}^{\mathcal{SS}}$ agrees with the canonical bundle outside the exceptional locus, which has codimension two again. It follows that the pull-back is isomorphic with the canonical bundle, so the analogue of lemma

10 holds. Thus the vanishing of $R^i\pi_* K_{\mathcal{F}^{\mathcal{SS}}_{3,n}}$ is equivalent again to the vanishing of $R^i\pi_*\mathcal{O}_{\mathcal{F}^{\mathcal{SS}}_{3,n}}$. To apply the Grauert-Riemenschneider vanishing theorem we now use the factor $s_5 = (x_{3,1}y_{1,1} - x_{1,1}y_{3,1})$, whose divisor has a proper transform containing the exceptional locus.

Another reason that the proposition also holds for the $\mathcal{SS}$ resolution is that it locally looks the same as the $\mathcal{FM}$ resolution. If locally we see the singularity as a product of an affine space and a cone over a product of two projective lines, then it clearly has an automorphism that interchanges these two lines. One can pass between the $\mathcal{SS}$ resolution and the $\mathcal{FM}$ resolution by means of this local automorphism.

Corollary 13 *The desingularizations $\mathcal{F}^{\mathcal{SS}}_{3,n}$, $\mathcal{F}^{\mathcal{FM}}_{3,n}$, and $Bl_{sing\ locus}\mathcal{F}_{3,n}$ are all Frobenius split varieties in any characteristic.*

Finally, we remark that one can construct the blowup of $\mathcal{F}_{3,n}$ along its singular locus as the fibered product of $\mathcal{F}^{\mathcal{FM}}_{3,n}$ and $\mathcal{F}^{\mathcal{SS}}_{3,n}$ over $\mathcal{F}_{3,n}$.

5 Fixed-point formulas

We apply equivariant fixed-point theorems to the spaces of the preceding sections, putting together all the fixed-point data we have accumulated. This produces explicit formulas for the $GL(n)$-character and dimension of the Schur module S_D of any 3-row diagram D. The formulas are more complicated than those of [22] for northwest diagrams, but essentially similar.

We discuss the general fixed-point theorems in the first section, and in the following ones give a summary of the results in elementary language. We conclude by discussing the possibility of drawing geometric implications from the combinatorial formulas.

5.1 General theory

In what follows, X is a smooth projective variety of dimension M over $\mathbf{C}$, $L \to X$ an algebraic line bundle, and $T = (\mathbf{C}^*)^n$ a torus acting on X and L. *Throughout this section, we also assume the vanishing of the higher cohomology groups of L:*

$$H^i(X, L) = 0 \text{ for all } i > 0 .$$

The following formula is due to Atiyah and Bott [2].

Proposition 14 *Suppose the torus T acts X with isolated fixed points.*

Then the character of T acting on the space of global sections of L is given by:

$$\mathrm{tr}(x \mid H^0(X,L)) = \sum_{p \text{ fixed}} \frac{\mathrm{tr}(x \mid L|_p)}{\det(\mathrm{id} - x \mid T_p^* X)},$$

where p runs over the fixed points of T, $L|_p$ denotes the fiber of L above p, and $T_p^* X$ is the cotangent space.

We apply this to $X = \mathcal{F}_{3,n}^{\mathfrak{SS}}$ and $L = \pi^* \mathcal{L}_D$. By Lemma 11 and Theorem 1, we have $H^0(\mathcal{F}_{3,n}^{\mathfrak{SS}}, \pi^* \mathcal{L}_D) = H^0(\mathcal{F}_{3,n}, \mathcal{L}_D) = S_D^*$ and $H^i(\mathcal{F}_{3,n}^{\mathfrak{SS}}, \pi^* \mathcal{L}_D) = H^i(\mathcal{F}_{3,n}, \mathcal{L}_D) = 0$ for $i > 0$, so the above Proposition gives us a character formula for the Schur module in terms of the fixed-point data of Lemma 8. We write out the result in the next section.

For the $\mathcal{FM}$ space, we need a more general formula due to Atiyah, Bott, and Singer [3]. It requires the following characteristic classes for a vector bundle V over any smooth variety Y:

$$\mathcal{U}^\lambda(V) = \prod_i \frac{1 - \lambda^{-1} \exp(-r_i)}{1 - \lambda^{-1}}$$

$$\mathcal{T}(V) = \prod_i \frac{r_i}{1 - \exp(-r_i)} \ ,$$

where λ is a character, and r_i are the Chern roots of the bundle V. If $V = TY$, we denote $\mathcal{T}(V) = \mathcal{T}(Y)$.

Proposition 15 *Suppose $\mathcal{C}$ is the set of connected components of the fixed set X^T of T. For each component $c \in \mathcal{C}$, assume that each restriction $L|_c$ is a trivial bundle. Let $N(c) = \bigoplus_\lambda N_\lambda(c)$ denote the normal bundle with its T-eigenspace decomposition.*

Then the character of T acting on the space of global sections of L is given by:

$$\mathrm{tr}(x \mid H^0(X,L)) = \sum_{c \in \mathcal{C}} \left[\frac{tr(x \mid L|_c) \cdot \prod_\lambda \mathcal{U}^\lambda(N_\lambda(c))(x) \cdot \mathcal{T}(c)}{\det(\mathrm{id} - x \mid N^*(c))} \right] (\mathrm{Fund}\ c) \ ,$$

where the multiplication takes place in the cohomology ring of the component c, and Fund c *denotes the fundamental homology class.*

Applying this to $X = \mathcal{F}_{3,n}^{\mathcal{FM}}$, $L = \pi^*\mathcal{L}_D$, we again get a formula for the character of $S_D^* = H^0(\mathcal{F}_{3,n}^{\mathcal{FM}}, \pi^*\mathcal{L}_D)$, this time in terms of the data in Lemma 9.

The next result we shall use is based on the theorem of Hirzebruch-Riemann-Roch [3], combined with Bott's Residue Formula [6], [2], according to the method of Ellingsrud and Stromme [11].

Proposition 16 *Suppose the torus $T = \mathbf{C}^*$ is one-dimensional, and acts with isolated fixed points.*

Let $\mathbf{v} = 1$ in the Lie algebra $\mathbf{t} = \mathbf{C}$, and at each T-fixed point p, let $b(p) = \mathrm{tr}(\mathbf{v} \mid L|_p)$. Denote the $\mathbf{v}$-eigenspace decomposition of the tangent space by $T_pX = \oplus_{i=1}^{M}\mathbf{C}_{r_i(p)}$, where $r_i(p)$ are the integer eigenvalues. Also, define the polynomial

$$RR_M(b; r_1, \ldots, r_M) = \text{coeff at } U^M \text{ of } \left(\exp(bU)\prod_{i=1}^{M}\frac{r_iU}{1-\exp(-r_iU)}\right),$$

where the right-hand side is considered as a Taylor series in the formal variable U.

Then the dimension of the space of global sections of L is given by:

$$\dim H^0(X, L) = \sum_{p \text{ fixed}} \frac{RR_M(b(p); r_1(p), \ldots, r_M(p))}{\det(\mathbf{v} \mid T_pX)}.$$

We will consider $X = \mathcal{F}_{3,n}^{\mathcal{SS}}$ and take the T in the Proposition to be $\mathbf{C}^* \subset GL(n)$, $q \to \mathrm{diag}(q^{-1}, q^{-2}, \ldots, q^{-n})$. (This is the principal one-dimensional subtorus corresponding to the half-sum of positive roots.) Then the eigenvalue characters in Lemma 8 specialize to the subtorus, and give us the information required to compute the dimension of $S_D^* = H^0(\mathcal{F}_{3,n}^{\mathcal{SS}}, \pi^*\mathcal{L}_D)$. (We may check directly that the fixed points of the subtorus are identical to those of the large torus of all diagonal matrices.)

Let us also mention that for the smooth spaces $\mathcal{F}_{3,n}^{\mathcal{SS}}$ and $\mathcal{F}_{3,n}^{\mathcal{FM}}$, the theorem of Bialynicki-Birula [5] gives cell decompositions of these spaces using the fixed point data. Thus, one can compute their singular cohomology groups and Chow groups as is done in [10] and [13].

5.2 Character formulas

First, we recall the necessary combinatorial constructions. We specify a three-row diagram D of squares in the plane by assigning a multiplicity $m_C \geq$

0 to each column of the "universal three-row diagram"

$$D_3 \;=\; \begin{array}{ccccccc} m_1 & m_2 & m_3 & m_{1,2} & m_{2,3} & m_{1,3} & m_{1,2,3} \\ \square & & & \square & & \square & \square \\ & \square & & \square & \square & & \square \\ & & \square & & \square & \square & \square \end{array}$$

We define a *standard column tabloid* for D with respect to $GL(n)$, to be a filling (i.e. labeling) of the squares of D_3 by integers in $\{1, \ldots, n\}$, such that:
(i) the integers in each column are strictly increasing, and
(ii) if there is an inclusion $C \subset C'$ between two columns, then all the numbers in the filling of C also appear in the filling of C'. The tabloids describe the fixed points of the torus T acting on the configuration variety $\mathcal{F}_{3,n}$.

Given a tabloid τ for D, define its generating monomial

$$x^{\mathrm{wt}(\tau)} = \prod_{(i,j) \in D_3} x_{\tau(i,j)}^{m_j} \; .$$

That is, the power of x_i is the number of times i appears in the filling τ, counted with multiplicity.

Also, define integers $d_{ij}(\tau)$ to be the number of connected components of the following graph: the vertices are columns C of D_3 such that i appears in the filling of C, but j does not; the edges are (C, C') such that $C \subset C'$ or $C' \subset C$.

Now, our formula is a sum of terms corresponding to the column tabloids τ of D. For the smooth tabloids, the contribution $\mathcal{C}(\tau)$ is obtained by the same formula as in the northwest case discussed in previous works:

$$\mathcal{C}(\tau) = \frac{x^{\mathrm{wt}(\tau)}}{\prod_{i \neq j} (1 - x_i^{-1} x_j)^{d_{ij}(\tau)}}$$

However, for the tabloids where the configuration variety is singular, we substitute a special contribution which can be defined in two ways, corresponding to the two desingularizations $\mathcal{F}_{3,n}^{\mathcal{SS}}$ and $\mathcal{F}_{3,n}^{\mathcal{FM}}$. Surprisingly, these expressions reduce algebraically to another, simpler form which does not appear to be associated with any desingularization (c.f. section 5.4).

Theorem 17 *The character of the Schur module S_D for $GL(n)$ is*

$$\mathrm{char}_{S_D} = \sum_{\tau} \mathcal{C}(\tau) \ ,$$

where $\mathcal{C}(\tau)$ are given in the table below.

To get all tabloids from the types shown in the table, one should take all permutations of the first three and the second three columns. There are $11n(n-1)(n-2)$ tabloids altogether.

Set

$$\Delta_{ijk} = \prod_{\substack{l\in[1,n] \\ l\neq i,j,k}} \left(1 - \frac{x(l)}{x(i)}\right) \left(1 - \frac{x(l)}{x(j)}\right) \left(1 - \frac{x(l)}{x(k)}\right) \ .$$

Tabloid τ	Character contribution $\mathcal{C}(\tau)$	Desing
$\begin{array}{ccccccc} i & & & & i & i & i \\ & j & & j & & j & j \\ & & k & k & k & & k \end{array}$	$\dfrac{x^{\mathrm{wt}(\tau)}}{\left(1-\frac{x(i)}{x(j)}\right)\left(1-\frac{x(j)}{x(i)}\right)\left(1-\frac{x(i)}{x(k)}\right)\left(1-\frac{x(j)}{x(k)}\right)\left(1-\frac{x(k)}{x(i)}\right)\left(1-\frac{x(k)}{x(j)}\right)\Delta_{ijk}}$	smooth
$\begin{array}{ccccccc} i & & & & i & i & i \\ & i & & i & & j & j \\ & & j & j & j & & k \end{array}$	$\dfrac{x^{\mathrm{wt}(\tau)}}{\left(1-\frac{x(i)}{x(j)}\right)\left(1-\frac{x(j)}{x(i)}\right)\left(1-\frac{x(j)}{x(k)}\right)\left(1-\frac{x(k)}{x(i)}\right)^2\left(1-\frac{x(k)}{x(j)}\right)\Delta_{ijk}}$	smooth
$\begin{array}{ccccccc} i & & & & i & i & i \\ & i & & i & & k & j \\ & & j & j & j & & k \end{array}$	$\dfrac{x^{\mathrm{wt}(\tau)}}{\left(1-\frac{x(i)}{x(j)}\right)\left(1-\frac{x(j)}{x(i)}\right)^2\left(1-\frac{x(k)}{x(i)}\right)\left(1-\frac{x(k)}{x(j)}\right)^2\Delta_{ijk}}$	smooth
$\begin{array}{ccccccc} i & & & & i & i & i \\ & i & & i & & k & j \\ & & i & j & j & & k \end{array}$	$\dfrac{x^{\mathrm{wt}(\tau)}}{\left(1-\frac{x(j)}{x(i)}\right)^2\left(1-\frac{x(j)}{x(k)}\right)\left(1-\frac{x(k)}{x(i)}\right)\left(1-\frac{x(k)}{x(j)}\right)^2\Delta_{ijk}}$	smooth
$\begin{array}{ccccccc} i & & & & i & i & i \\ & i & & i & & j & j \\ & & i & j & j & & k \end{array}$	$\dfrac{x^{\mathrm{wt}(\tau)}\left(1-\left(1-\frac{x(i)}{x(j)}\right)^{-1}-\left(1-\frac{x(j)}{x(k)}\right)^{-1}\right)}{\left(1-\frac{x(j)}{x(i)}\right)^2\left(1-\frac{x(k)}{x(i)}\right)\left(1-\frac{x(k)}{x(j)}\right)^2\Delta_{ijk}}$	$\mathcal{FM}$
	$= \dfrac{x^{\mathrm{wt}(\tau)}}{\left(1-\frac{x(j)}{x(i)}\right)^3\left(1-\frac{x(k)}{x(i)}\right)\left(1-\frac{x(k)}{x(j)}\right)\left(1-\frac{x(i)\,x(k)}{x(j)^2}\right)\Delta_{ijk}} + \dfrac{x^{\mathrm{wt}(\tau)}}{\left(1-\frac{x(j)}{x(i)}\right)\left(1-\frac{x(k)}{x(i)}\right)\left(1-\frac{x(k)}{x(j)}\right)^3\left(1-\frac{x(j)^2}{x(i)\,x(k)}\right)\Delta_{ijk}}$	$\mathcal{SS}$
	$= \dfrac{x^{\mathrm{wt}(\tau)}}{\left(1-\frac{x(j)}{x(i)}\right)^3\left(1-\frac{x(k)}{x(j)}\right)^3\Delta_{ijk}}$	?

5.3 Dimension formula

We compute the dimension of the $GL(n)$-module S_D. Again, each tabloid gives a contribution, which is the value of a certain multivariable polynomial RR at a sequence of integers specific to the tabloid.

Let $M = 3n-3$, the dimension of the triangle space $\mathcal{F}_{3,n}$. Define an $(M+1)$-variable polynomial, homogeneous of degree M, by

$$\mathrm{RR}_M(b; r_1, \ldots, r_M) = \text{coeff at } U^M \text{ of } \left(\exp(bU) \prod_{i=1}^{M} \frac{r_i U}{1-\exp(-r_i U)}\right) ,$$

where the right side is understood as a Taylor series in U. For example, for $n=3$, $M=6$, $\mathrm{RR}_6(b; r_1, \ldots, r_6)$ is an irreducible 7-variable polynomial, homogeneous of degree 6, with 567 terms. However, since we will only evaluate this polynomial at $(M+1)$-tuples of small integers, this is within the range of computer calculations provided the column multiplicities m_j are not very large.

For the smooth tabloids, we again have a formula for the contributions in terms of the integers $d_{ij}(\tau)$. Namely, define a multiset of integers $r(\tau) = \{r_1, r_2, \ldots, r_M\}$ by inserting the entry $i-j$ with multiplicity $d_{ij}(\tau)$ for each ordered pair $i, j \in [1, n]$. That is, the total multiplicity of the integer k in $r(\tau)$ is

$$\sum_{\substack{i,j \in [1,n] \\ i-j=k}} d_{ij}(\tau) .$$

Let $b(\tau)$ be the sum of the entries of the tabloid τ, counting column multiplicities:

$$b(\tau) = \sum_{(i,j) \in D} m_j \, \tau(i,j) .$$

Then the contribution of τ to the dimension formula is

$$\begin{aligned} \mathcal{R}(\tau) &= \frac{\mathrm{RR}_M(b(\tau); r(\tau))}{\prod_{k=1}^{M} r(\tau)_i} \\ &= \frac{\mathrm{RR}_M(b(\tau); r(\tau))}{\prod_{i,j \in [1,n]} (i-j)} \end{aligned}$$

For the singular tabloids, we have only an expression corresponding to the $\mathcal{SS}$ desingularization, as well as a simplified expression with no geometric explanation, as before.

Theorem 18 *The dimension of the Schur module S_D for $GL(n)$ is*

$$\dim S_D = \sum_{\tau} \mathcal{R}(\tau) \ ,$$

where $\mathcal{R}(\tau)$ are given in the table below.

The entries in the table are terms of the form $\mathcal{R} = \mathrm{RR}_M(b; r)/\prod r$ for integers b and multisets r. Let $r'(i,j,k)$ be the standard multiset with entries $i-l$, $j-l$, $k-l$ for each integer $l \in [1,n]$, $l \neq i,j,k$. (For example, if $n=4$, we have $r'(1,2,3) = \{1-4,\ 2-4,\ 3-4\} = \{-3,-2,-1\}$.)

Tabloid τ	Dimension contribution $\mathcal{R}(\tau)$	Desing
$\begin{array}{ccccccc} i & & & & i & i & i \\ & j & & j & & j & j \\ & & k & k & k & & k \end{array}$	$\dfrac{\mathrm{RR}_M(b(\tau);\ j{-}i,\ i{-}j,\ k{-}i,\ k{-}j,\ i{-}k,\ j{-}k,\ r'(i,j,k))}{(j{-}i)(i{-}j)(k{-}i)(k{-}j)(i{-}k)(j{-}k)\ \prod r'(i,j,k)}$	smooth
$\begin{array}{ccccccc} i & & & & i & i & i \\ & i & & i & & j & j \\ & & j & j & j & & k \end{array}$	$\dfrac{\mathrm{RR}_M(b(\tau);\ j{-}i,\ i{-}j,\ k{-}j,\ i{-}k,\ i{-}k,\ j{-}k,\ r'(i,j,k))}{(j{-}i)(i{-}j)(k{-}j)(i{-}k)^2(j{-}k)\ \prod r'(i,j,k)}$	smooth
$\begin{array}{ccccccc} i & & & & i & i & i \\ & i & & i & & k & j \\ & & j & j & j & & k \end{array}$	$\dfrac{\mathrm{RR}_M(b(\tau);\ j{-}i,\ i{-}j,\ i{-}j,\ i{-}k,\ j{-}k,\ j{-}k,\ r'(i,j,k))}{(j{-}i)(i{-}j)^2(i{-}k)(j{-}k)^2\ \prod r'(i,j,k)}$	smooth
$\begin{array}{ccccccc} i & & & & i & i & i \\ & i & & i & & k & j \\ & & i & j & j & & k \end{array}$	$\dfrac{\mathrm{RR}_M(b(\tau);\ i{-}j,\ i{-}j,\ k{-}j,\ i{-}k,\ j{-}k,\ j{-}k,\ r'(i,j,k))}{(i{-}j)^2(k{-}j)(i{-}k)(j{-}k)^2\ \prod r'(i,j,k)}$	smooth
$\begin{array}{ccccccc} i & & & & i & i & i \\ & i & & i & & j & j \\ & & i & j & j & & k \end{array}$	$\dfrac{\mathrm{RR}_M(b(\tau);\ i{-}j,\ i{-}j,\ i{-}j,\ i{-}k,\ j{-}k,\ 2j{-}i{-}k,\ r'(i,j,k))}{(i{-}j)^3(i{-}k)(j{-}k)(2j{-}i{-}k)\ \prod r'(i,j,k)} + \dfrac{\mathrm{RR}_M(b(\tau);\ i{-}j,\ i{-}k,\ j{-}k,\ j{-}k,\ j{-}k,\ i{+}k{-}2j,\ r'(i,j,k))}{(i{-}j)(i{-}k)(j{-}k)^3(i{+}k{-}2j)\ \prod r'(i,j,k)}$	$\mathcal{SS}$
	$= \dfrac{\mathrm{RR}_M(b(\tau);\ i{-}j,\ i{-}j,\ i{-}j,\ j{-}k,\ j{-}k,\ j{-}k,\ r'(i,j,k))}{(i{-}j)^3(j{-}k)^3\ \prod r'(i,j,k)}$	?

5.4 Virtual desingularization

In the previous sections, we have drawn consequences about Schur modules from the geometry of the triangle space. However, one can imagine reversing this process.

For instance, consider the trivial line bundle over the triangle space, $m_1 = m_2 = \cdots = m_7 = 0$. Its space of sections is the trivial Schur module, with character 1. Hence, our character formulas for the $\mathcal{FM}$ and $\mathcal{SS}$ desingularizations each give ludicrously complicated expressions for 1. There is a non-trivial contribution $\mathcal{C}(\tau)$ for each T-fixed point τ of the smooth space: the numerators are reduced to 1, but the denominators still possess a factor for each eigenvalue of the tangent space at the fixed point. Because these eigenvalues determine a cell decomposition of the smooth space, one can read off from these "ludicrous formulas" a great deal of geometric information about the desingularizations of $\mathcal{F}_{3,n}$.

In fact, suppose we did not know of the existence of the $\mathcal{SS}$ desingularization. We could guess that there exists such a smooth space, with its cell decomposition, by finding a ludicrous formula for 1 with terms of Atiyah-Bott type.

This is not difficult: let us start by assuming there exists a $GL(n)$-equivariant desingularization of $\mathcal{F}_{3,3}$ with fibers of dimension 1 (the smallest possible), and with each singular fixed point lifting to only two fixed points in the smooth space. The singular fibers must then be isomorphic to $\mathbf{P}^1$, since this is the only curve possessing an appropriate torus action. Now, the contributions $\mathcal{C}(\tau)$ from the smooth tabloids of $\mathcal{F}_{3,n}$ are constrained to be the entries in the table. As for the singular tabloids, they must each correspond to two terms in the formula, one for each lifted fixed point. We know most of their eigenvalues from the T-stable curves in $\mathcal{F}_{3,n}$: each will lift to either two or one T-stable curves in the smooth space, depending on whether or not it lies in the singular locus. Since there are six eigenvectors at each fixed point of the smooth space, this leaves only one eigenvector to determine at each of the two lifted fixed points. Since the fiber is $\mathbf{P}^1$, these must be reciprocals of each other. Now, the six singular tabloids lie in a single GL(n) orbit, so the corresponding unknown eigenvalues are all images of each other. Therefore, there is only one variable left unknown, which we can solve for in the ludicrous equation: over τ_{123} the value must be $x_1 x_3 x_2^{-2}$, as in our formula.

Now, for the space of tetrahedra and higher-dimensional simplices (corresponding to general diagrams with four or more rows), there is no known explicit desingularization. One may hope to find evidence of one by arguments like the above, combined with induction on the dimension. That is, an appropriate ludicrous formula for 1 may be considered as a combinatorial or virtual desingularization.

Finally, let us point out one mystery in our results: the algebraic simplification of the character and dimension formulas, combining the complicated contributions given by our desingularizations into a single term of Atiyah-Bott type. In the above philosophy, this would mean that $\mathcal{F}_{3,n}$ is already "virtually smooth", with eigenvalues above τ_{ijk} equal to $x_i^{-1}x_j$ (three times), and $x_j^{-1}x_k$ (three times). Note that there can exist no actual G-equivariant desingularization of $\mathcal{F}_{3,n}$ for which the singular fixed points each lift uniquely, since this would use up only six of the seven eigenvalues given by the T-stable curves at τ_{ijk} (cf. Lemma 8). (Each of these curves must lift to at least one T-stable curve in the smooth space and lead to some lifted fixed point.)

6 Appendix: Restriction and induction

We now prove a result from the representation theory of reductive algebraic groups, needed in the proof of theorem 1. Let G be a connected reductive algebraic group, B a Borel subgroup, P a parabolic subgroup containing B. Let us call a weight λ *effective* if $\mathrm{ind}_B^G\lambda \neq 0$. (In [17] an effective weight is called dominant, in [18] it is called anti-dominant.) For the notions of 'induction', 'good filtration', 'excellent filtration', and the basic theorems concerning them we refer to [17], [25], [18].

Lemma 19 *Let λ be effective and let M be a P-module that has excellent filtration (as a B-module). Then $\mathrm{ind}_B^P(\lambda)\otimes M$ has an excellent filtration.*

Proof. As $\mathrm{ind}_B^P(\lambda)\otimes M = \mathrm{ind}_B^P(\lambda\otimes M)$ this follows from the main theorems on excellent filtrations as collected in [18].

Proposition 20 *For effective $\lambda_1, \ldots, \lambda_n$, the restriction map*

$$\mathrm{res} : \mathrm{ind}_B^G(\lambda_1)\otimes\cdots\otimes\mathrm{ind}_B^G(\lambda_n) \to \mathrm{ind}_B^G(\mathrm{ind}_B^P(\lambda_1)\otimes\cdots\otimes\mathrm{ind}_B^P(\lambda_n))$$

is surjective.

Proof. It suffices to show that for each i the kernel $\ker\phi_i$ of the surjective map ϕ_i:

$$\begin{array}{c}
\mathrm{ind}_B^G(\lambda_1)\otimes\cdots\otimes\mathrm{ind}_B^G(\lambda_{i-1})\otimes\mathrm{ind}_B^G(\lambda_i)\otimes\mathrm{ind}_B^P(\lambda_{i+1})\otimes\cdots\otimes\mathrm{ind}_B^P(\lambda_n)\\
\downarrow\\
\mathrm{ind}_B^G(\lambda_1)\otimes\cdots\otimes\mathrm{ind}_B^G(\lambda_{i-1})\otimes\mathrm{ind}_B^P(\lambda_i)\otimes\mathrm{ind}_B^P(\lambda_{i+1})\otimes\cdots\otimes\mathrm{ind}_B^P(\lambda_n)
\end{array}$$

is ind_B^G-acyclic. Indeed res may be viewed as $\mathrm{ind}_B^G(\phi_1)\circ\cdots\circ\mathrm{ind}_B^G(\phi_n)$. Now a module M is ind_B^G-acyclic if and only if $k[G]\otimes M$ is B-acyclic, so that the result follows from the lemma and the main theorems on excellent filtrations *etc.*

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