Singularly perturbed and non-local modulation equations for systems with interacting instability mechanisms

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Abstract

Two related systems of coupled modulation equations are studied and compared in this paper. The modulation equations are derived for a certain class of basic systems which are subject to two distinct, interacting, destabilizing mechanisms. We assume that, near criticality, the ratio of the widths of the unstable wavenumber-intervals of the two (weakly) unstable modes is small – as for instance can be the case in double-layer convection. Based on these assumptions we first derive a singularly perturbed modulation equation and then a modulation equation with a non-local term. The reduction of the singularly perturbed system to the non-local system can be interpreted as a limit in which the width of the smallest unstable interval vanishes. We study and compare the behaviour of the stationary solutions of both systems. It is found that spatially periodic stationary solutions of the non-local system exist under the same conditions as spatially periodic stationary solutions of the singularly perturbed system. Moreover, these solutions can be interpreted as representing the same quasi-periodic patterns in the underlying basic system. Thus, the 'Landau-reduction' to the non-local system has no significant influence on the stationary quasi-periodic patterns. However, a large variety of intricate heteroclinic and homoclinic connections is found for the singularly perturbed system. These orbits all correspond to so-called 'localised structures' in the underlying system: they connect simple periodic patterns at $x \to \pm\infty$. None of these patterns can be described by the non-local system. So, one may conclude that the reduction to the non-local system destroys a rich and important set of patterns.

1 Introduction

In the weakly nonlinear stability theory of the evolution of patterns one classically considers systems like

$$\psi_t = L_R\psi + N(\psi), \quad \psi(x,y,t): \mathbb{R}^n \times \Omega \times \mathbb{R}^+ \to \mathbb{R}^N, \tag{1.1}$$

where L_R (respectively N) is a linear (nonlinear) operator, R is a control, or bifurcation, parameter and Ω is a bounded domain $\subset \mathbb{R}^m$. We refer to the review paper by Eckhaus [9] for a survey and some of the numerous physical examples. This system is assumed to have a basic solution $\psi_0(y)$. The linearised stability of this solution is determined by setting

$$\psi = \psi_0 + f(y)e^{ikx+\mu t}$$

and solving, for any pair (k, R), an eigenvalue problem for $f(y)$ with eigenvalues $\mu = \mu(k, R)$. The neutral curve is defined as the set $\{\mathrm{Re}\mu_0(k, R) = 0\}$, where $\mu_0(k, R)$ is the critical eigenvalue (that is, the eigenvalue with the largest real part) for a given pair (k, R). The basic solution ψ_0 is linearly stable for $R = R_0$ if $\mathrm{Re}\mu_0(k, R_0) < 0$ for all k. A bifurcation occurs if one increases R such that $\{R = \text{const.}\}$ intersects the neutral curve at a minimum (k_c, R_c) of this curve: a small interval of 'linearly unstable waves' appears for $R > R_c$ (see Figure 1). To understand the behaviour of the solutions to (1.1) for R close to R_c, or $R - R_c = r\varepsilon^2$, $0 < \varepsilon \ll 1$, one first shows that the nonlinear evolution of solutions close ($= O(\varepsilon)$) to ψ_0 is governed by the (complex) amplitude $A(\xi, \tau)$ of the linearly 'most unstable mode' $f_c(y)e^{i(k_c x+\mu_c t)}$, where $f_c(y)$ is the critical eigenfunction at the eigenvalue $\mu(k_c, R_c) = i\mu_c$; ξ and τ are slow spatial and temporal variables. Then, one derives an equation for $A(\xi, \tau)$, the so-called Ginzburg-Landau equation:

$$A_\tau = r\mu_R A - \frac{1}{2}\mu_{kk} A_{\xi\xi} + cA|A|^2, \tag{1.2}$$

where $\mu_R = \frac{\partial\mu}{\partial R}(k_c, R_c)$, $\mu_{kk} = \frac{\partial^2\mu}{\partial k^2}(k_c, R_c)$ and $c \in \mathbf{C}$ is the so-called Landau constant. Note that $\mathrm{Re}\mu_R > 0$ and $\mathrm{Re}\mu_{kk} < 0$ since the neutral curve can be approximated near the minimum (k_c, R_c) by the parabola

$$R = R_c - \frac{1}{2}\frac{\mathrm{Re}\mu_{kk}}{\mathrm{Re}\mu_R}(k - k_c)^2. \tag{1.3}$$

Details of this procedure are, for instance, given in [9]. Recently, a number of papers on the mathematical validity of the Ginzburg-Landau approximation have appeared, we refer to [2] for a survey and relevant references. The Landau-equation associated to this nonlinear stability problem can be obtained from (1.2) by setting $A(\xi, \tau) = A(\tau)$. This 'Landau-reduction' can be interpreted by saying that one neglects the width of the band of unstable waves centered around $k = k_c$ for $R = R_c + r\varepsilon^2$. Historically, the Landau equation was derived a decade earlier than the Ginzburg-Landau equation (see [30]).

In this paper we consider a class of physical problems which have two distinct, interacting, instability mechanisms at near-critical conditions. This means, in the above setting, that the neutral curve $\{\mathrm{Re}\mu_0(k, R) = 0\}$ has two local minima, (k_1, R_1) and (k_2, R_2), such that $|R_1 - R_2|$ is small (see Figure 1). So, if one chooses R close to criticality in this case, one expects two independent, interacting, 'linearly most unstable waves', $f_1(y)e^{i(k_1 x+\mu_1 t)}$ near (k_1, R_1) with complex amplitude $A(\xi, \tau)$ and $f_2(y)e^{i(k_2 x+\mu_2 t)}$ near (k_2, R_2) with amplitude $B(\xi, \tau)$. The nonlinear behaviour of patterns near criticality is then described by a coupled system of Ginzburg-Landau equations.

When a neutral curve has more than one local minimum one does not expect that those minima occur for (approximately) the same value of the bifurcation parameter R. However, the relative position of the minima can very often be changed as a second parameter S is varied. Thus, by changing this second parameter S the neutral curve transforms from a curve with an absolute minimum in (k_1, R_1) to a curve with an absolute minimum (k_2, R_2) (or vice versa). If (k_1, R_1) is the absolute minimum then we are in the above described classical case and the evolution of patterns near criticality is governed by (1.2); if (k_2, R_2) is the absolute minimum then the situation is again classical and is governed by a Ginzburg-Landau equation for $B(\xi, \tau)$. The two unstable modes interact in the transition region. This situation occurs in many applications and the above described coupled system of equations has been derived by many authors. We mention here some physical examples: double-layer convection [25], [24], [18]; crystal-growing experiments (where the convective and morphological modes can interact) [13], [26], [21]; gasless combustion [19]; sand ripple formation [31]. The coupled system of modulation equations has, for instance, been derived in [19], [17], [22].

In section 2 we will give a short sketch of the derivation of the coupled system in the case of (non-resonantly) interacting instability mechanisms. The model problem considered in this paper is assumed to have a reflection symmetry in the one-dimensional unbounded variable x. Therefore, all coefficients in the coupled system of modulation equations will be real:

$$\left\{ \begin{array}{l} A_\tau = rA + A_{\xi\xi} + A(t_1|A|^2 + c_1|B|^2) \\ B_\tau = sB + DB_{\xi\xi} + B(t_2|B|^2 + c_2|A|^2) \end{array} \right. \tag{1.4}$$

where r and s measure the distance between R and $R_{1,2}$ (see section 2 for more details). By rescaling, we have simplified the coefficients of the linear terms. Due to the reflection symmetry – which for instance occurs naturally in convection experiments – the ξ-variable is not moving (it is independent of t). This is a consequence of the fact that all eigenvalues $\mu(k, R)$ are real. If this is not the case, then the A and B amplitudes will be traveling with the group speed of the linearly unstable waves. This speed is in general not the same for the A- and B-modes, so the interaction of the A and B patterns cannot be described by (1.4). In this case one has to apply some kind of averaging formalism in order to derive a so-called mean field Ginzburg-Landau equation, see for instance [17], [19] and [23] for a validity result.

Here, we focus on the significance of the diffusion parameter D (> 0) in (1.4): D measures the relative widths of the bands of unstable modes just above the minima (k_1, R_1) and (k_2, R_2). More precise: as in (1.2), the diffusion coefficients are determined by $\frac{\partial^2 \mu}{\partial k^2}(k_{1,2}, R_{1,2})$, which measure the curvature of the neutral curve at the minimum (see (1.3)). We have rescaled the diffusion coefficient in the A-equation to 1: D measures the ratio of $\frac{\partial^2 \mu}{\partial k^2}(k_1, R_1)$ and $\frac{\partial^2 \mu}{\partial k^2}(k_2, R_2)$. Thus, $D \gg 1$ means that the neutral curve near (k_2, R_2) is much 'sharper', or narrower, than near (k_1, R_1) (see Figure 1 and section 2 for more details). This occurs for instance naturally in experiments on double-layer convection (where the depth of the layers differs significantly) and in experiments on crystal-growth (see [13], [25] and [22], [18] for a discussion). If this is the case one can introduce a second small parameter $0 < \delta \ll 1$ by setting $D = \frac{1}{\delta^2}$ and write down a *singularly perturbed* system:

$$\left\{ \begin{array}{l} A_\tau = rA + A_{\xi\xi} + A(t_1|A|^2 + c_1|B|^2) \\ B_\tau = sB + \frac{1}{\delta^2}B_{\xi\xi} + B(t_2|B|^2 + c_2|A|^2) \end{array} \right. \tag{1.5}$$

There is another, equivalent, way of interpreting this singular term: both instability mechanisms are associated to a natural spatial scale at which the patterns evolve. In this paper we consider the case in which the magnitudes of these scales differ significantly. Due to the rescaling we can say that the natural scale associated to A is ξ, while it is $\delta\xi$ for B:

$B(\xi, \tau)$ only varies very slowly on the ξ-scale. Returning to the above interpretation this means that the width of the (k_2, R_2)-parabola (see (1.3)) is $O(\delta)$ compared to the width of the (k_1, R_1)-parabola (Figure 1). In this situation it is natural to apply the above described 'Landau-reduction' for B: $B(\xi, \tau) = B(\tau)$. In section 2 we show that (1.5) then reduces to the following *non-local* system:

$$\begin{cases} A_\tau = rA + A_{\xi\xi} + A(t_1|A|^2 + c_1|B|^2) \\ B_\tau = sB + B(t_2|B|^2 + c_2 \lim_{M\to\infty} \frac{1}{2M} \int_{-M}^{M} |A|^2 d\xi) \end{cases} \tag{1.6}$$

We shall also show that this reduction is only valid when $A(\xi, \tau)$ (and $B(\tau)$) satisfy an extra solvability condition

$$c_2 B \int_{-\infty}^{\infty} \left(|A|^2 - \lim_{M\to\infty} \frac{1}{2M} \int_{-M}^{M} |A|^2 d\hat{\xi} \right) e^{iK\xi} d\xi = O(\varepsilon^2) \text{ for } K = O(\delta). \tag{1.7}$$

This condition cannot be satisfied by all solutions of (1.6) (see section 3.1, remark 3.1). The idea of a Landau-reduction has also been applied by Metzener and Proctor ([22]) in their analysis of the evolution of patterns at 'disparate scales'. Note that our approach is not exactly the same as in [22]: there k_2, instead of D, has been taken as a small parameter. In section 2 we relate our approach to the one in [22]. Some fundamental properties of a modulation equation with a non-local term, such as the existence, uniqueness, regularity of solutions, the dimension of attractors etc., have been studied in [7].

The main goal of this paper is to understand the impact of this Landau-reduction for B on the patterns described by (1.5). We focus on the analysis of the stationary solutions of (1.5) and (1.6). First we search for spatially periodic solutions. For both systems, the analysis is based on the fact that the (stationary) equation for A is integrable when B is fixed at a constant value (see [4] for references to the stationary problem of the (uncoupled) 'real' Ginzburg-Landau equation). Thus, the stationary problem associated to (1.5) is a (singularly) perturbed integrable system; periodic orbits in the fast field can be found by constructing a Poincaré map. We find that both systems have a similar set of periodic solutions which exists under the same conditions on the parameters and, most importantly, which describe – up to $\mathcal{O}(\delta)$ corrections – the same family of quasi-periodic patterns in the basic system. The most important difference between the quasi-periodic patterns described by (1.5) and (1.6) is that $|B|$ is periodic with an $O(\delta)$ amplitude around a certain value b in (1.5), while the corresponding solution described by (1.6) has $|B| \equiv b$. Thus, the above Landau-reduction for the B-mode has no significant influence here.

By introducing polar coordinates for A and B it is possible to write the stationary singularly perturbed problem as a 4-dimensional ODE with two fast directions, x and y corresponding to A, and two slow directions, z and w corresponding to B:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + x(x^2 - c_1 z^2) \\ \dot{z} = \delta w \\ \dot{w} = \delta(-sz + z(z^2 - c_2 x^2)) \end{cases} \tag{1.8}$$

where we have scaled (r, t_1, t_2) in (1.5) to $(1, -1, -1)$ (thus we chose $t_{1,2} < 0$, as occurs most frequently in applications); the 'dot' refers to differentiation with respect to 'time' t, where

t is now a rescaled version of ξ. The reduction from the expected 8-dimensional system to a 4-dimensional system is due to the phase invariance in the equations for A and B and to the fact that there are two integrals, Ω_1 and Ω_2, in the full system. These integrals are uncoupled, in the sense that they are identical to the integrals of the uncoupled equations for A and B (see section 3 and [4]). We simplify the analysis by choosing $\Omega_1 = \Omega_2 = 0$. Both, 'most stable' (see [20]) Stokes-wave solutions, ($A =$ const., $B \equiv 0$) and ($B =$ const., $A \equiv 0$), satisfy $\Omega_1 = \Omega_2 = 0$ and are thus described by (1.8). This 4-dimensional system can be analysed (for instance) by the geometric theory for singularly perturbed systems, originally developed by Fenichel [11], see also the contribution of Jones to [1]. Due to the results of Fenichel we establish the existence of two so-called slow, invariant, manifolds Γ_l and Γ_r. We find a very rich structure of heteroclinic and homoclinic orbits which 'jump up and down' between Γ_l and Γ_r. More precise: there are 4 critical points on the slow manifolds: $P_l, Q_l \in \Gamma_l$ and $P_r, Q_r \in \Gamma_r$. For any $N > 0$ there are $\mathcal{N}(N)$ different 'N-jump' heteroclinic or homoclinic orbits which connect two of the above four points and which consist of $N+1$ slow parts near Γ_l or Γ_r and N jumps through the fast field. The number $\mathcal{N}(N)$ can be explicitly calculated: $\mathcal{N}(N) = 4\times$ the $(N+2)$-th Fibonacci number (see Theorems 1 and 2 in section 4.2). These results are obtained by carefully tracking the 3-dimensional stable and unstable manifolds of Γ_l, Γ_r and the 2-dimensional stable and unstable manifolds of $P_{l,r}, Q_{l,r}$ through the fast field and near the slow manifolds. The Hamiltonian structure of (1.8), see section 4, is a key ingredient of the proof of our results. Based on the methods developed in [6] we are also able to show the existence of homoclinic orbits which do not jump immediately from Γ_l to Γ_r (or vice versa), but remain in the fast field for a 'longer time' (see Theorem 3).

These orbits all correspond to so-called 'localised structures' in the underlying system: they connect simple, spatially periodic patterns at $x \to \pm\infty$. This type of patterns can be stable in the uncoupled Ginzburg-Landau equation (see for instance [28] for a survey). However, none of these patterns can be described by the non-local reduction (1.6). There are two reasons for this. The first reason is that the most important ingredient of the construction of the heteroclinic and homoclinic orbits is the existence of the slow manifolds Γ_l and Γ_r. These manifolds can (of course) not exist in the Landau reduction since B, and thus z and w in (1.8), cannot evolve slowly. However, there exist a small number of heteroclinic orbits in the stationary problem associated to (1.6) which do have a counterpart in (1.8): the orbits only remain near $\Gamma_{l,r}$ for an $O(\delta)$-distance. These 'localised patterns' cannot be described by the Landau-reduction due to a second, independent reason: they do not satisfy the extra solvability condition (1.7), see Remark 3.1.

Thus we conclude that the reduction to a Landau approach for B destroys a rich and important set of patterns.

We end this introduction by a short sketch of the structure of this paper. In section 2 we derive equations (1.4), (1.5) and (1.6), with extra condition (1.7). We also pay some attention to the problem studied by Metzener and Proctor [22] and relate it to our approach. Section 3 is devoted to the derivation of stationary problems associated to (1.5) and (1.6) and the analysis of the non-local problem. The singularly perturbed problem is studied in section 4: first we show the existence of (fast) periodic solutions using a Poincaré map, then we employ the ideas of geometric singular perturbation theory. We end the paper by a short discussion.

2 The derivation of the equations

We consider the following model problem

$$\frac{\partial\psi}{\partial t} = L_{R,S,T}(\psi) + N(\psi) \text{ where } \psi(x,t) : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}, \tag{2.1}$$

which is a simplification of (1.1) since we restrict ourselves to a one-dimensional problem without a bounded y-variable. Furthermore we assume, as in the introduction, that there is a reflection symmetry $x \to -x$ in (2.1) and that the basic solution $\psi_0 \equiv 0$. Here, the linear operator depends on three bifurcation-parameters R, S and T. The 'eigenvalue' $\mu(k,R)$ as defined in the introduction is in this case equal to the symbol of the linear operator $L_{R,S,T}$:

$$L_{R,S,T}(e^{ikx}) = \mu(k,R;S,T)e^{ikx}. \tag{2.2}$$

We consider this very simple model in order to simplify the derivation of the modulation equation as much as possible. Introducing transversal y-dimensions will merely increase the technical difficulties. The validity of the Ginzburg-Landau equation (1.2) for systems like (2.1) has been proved in [12].

As in the introduction, we define the neutral curve $\{\mathrm{Re}\mu(k,R) = 0\}$. Here, we will study the case that this curve has two minima: (k_1,R_1) and (k_2,R_2), with $k_2 < k_1$. The neutral curve near (k_1,R_1) can be scaled such that

$$R = 1 + (k-1)^2 + h.o.t. \tag{2.3}$$

locally, thus $(k_1,R_1) = (1,1)$ (compare with the general expression (1.3)). The two conditional parameters S and T can now be interpreted. By changing T, the relative position of R_2 with respect to $R_1 = 1$ can be adapted. The relative width of the critical curve, or the band of unstable waves, at $k_1 = 1$ and at k_2 is changed by S. The neutral curve near (k_2,R_2) can be written as

$$R = k_2 + D(k-k_2)^2 + h.o.t., \tag{2.4}$$

thus $D = D(S)$ measures the relative widths of the $(1,1)$- and (k_2,R_2)-parabolas. For simplicity one might say that T controls the position of R_2 and S controls D.

The object of nonlinear (stability) theory is to describe the nonlinear evolution of the perturbation for R close to the critical value R_c. If $O(R-R_1) \neq O(R-R_2)$ one derives a single uncoupled Ginzburg-Landau equation (1.2) in the weakly non-linear stability analysis, either near $(1,1)$ if $1 < R_2$ or near (k_2,R_2) if $1 > R_2$. Coupling occurs if we assume that

$$R - 1 = r\varepsilon^2, R - R_2 = s\varepsilon^2, 0 < \varepsilon \ll 1. \tag{2.5}$$

This can be seen as follows: one models the perturbation of the basic solution as slow modulations of the critical waves, e^{ix} and e^{ik_2x}, and their complex conjugates:

$$\psi(x,t) = \varepsilon A(\xi,\tau)e^{ix} + \varepsilon B(\xi,\tau)e^{ik_2x} + c.c. + O(\varepsilon^2),$$

where A and B are unknown 'amplitudes' of the slow space and time variables $\xi = \varepsilon x$ and $\tau = \varepsilon^2 t$. The non-linear terms in (2.1) will generate harmonics of these simple linear waves.

Thus, the $\varepsilon^2, \varepsilon^3$-terms are constructed from a product of the two most unstable waves, e^{ix} and e^{ik_2x}

$$\psi(x,t) = \begin{array}{rll} \varepsilon^2\phi_{02}+ & \cdots & \\ e^{ix}[\varepsilon A + \varepsilon^2\phi_{12}+ & \varepsilon^3\phi_{13}+\cdots & \\ e^{ik_2x}[\varepsilon B + \varepsilon^2\psi_{12}+ & \varepsilon^3\psi_{13}+\cdots & \\ e^{2ix}[\varepsilon^2\phi_{22}+ & \cdots & +c.c. \\ e^{2ik_2x}[\varepsilon^2\psi_{22}+ & \cdots & \\ e^{ix(1+k_2)}[\varepsilon^2\Phi_{12}+ & \cdots & \\ e^{ix(1-k_2)}[\varepsilon^2\Psi_{12}+ & \cdots & \end{array} \tag{2.6}$$

Here the $A, B, \phi_{ij}, \psi_{ij}, \Phi_{ij}$ and Ψ_{ij} are functions of ξ and τ for every $i, j \in \mathbb{N}$. All scalings are classical, see for instance [9]. The validity of this expansion is proven in [10] for the case that there is one minimum.

This expansion is valid as long as there are no resonances between $k_1 = 1$ and k_2. It is clear that for $k_2 \neq \frac{1}{2}$ all above interaction terms are different. For $k_2 = \frac{1}{2}$ some of these terms coincide. As a consequence one has to choose other temporal and spatial scales and there will appear quadratic terms in the governing evolution equations (see for instance [24] and [31] for physical examples). This is called resonance; we will not discuss this in more detail in this paper. Resonance also occurs for $k_2 = \frac{1}{3}$ on the $O(\varepsilon^3)$-level. Since the dominant terms of the modulation equations are determined at the $O(\varepsilon^3)$-level there are no other resonant values of k_2.

The idea behind the derivation of the modulation equation is simple: one substitutes the expansion for ψ into (2.1), one expands and then gathers terms of the form $\varepsilon^a e^{ix(b_1+b_2k_2)}$, $a, b_1, b_2 \in \mathbb{N}$. The equations at the $a = 2$-level can be solved: the functions in expansion (2.6) can all be expressed in terms of A and B. The solvability conditions for ϕ_{13} and ψ_{13} at the levels $a = 3,\ b_1 = 1,\ b_2 = 0$ and $a = 3,\ b_1 = 0,\ b_2 = 1$ yield, after some trivial rescalings, the coupled system (1.4) for A and B given in the introduction .

In this paper we study the situation in which the local parabola near (k_2, R_2), (2.4), is very narrow with respect to (2.3) (see the introduction and Figure 1), thus we assume that

$$D = \frac{1}{\delta^2} \text{ with } 0 < \delta \ll 1. \tag{2.7}$$

This automatically yields the *singularly perturbed* system (1.5) given in the introduction. The appearance of the singular term $\frac{1}{\delta^2}\frac{\partial^2 B}{\partial \xi^2}$ can also be understood directly from the derivational point of view: the width of the (k_2, R_2)-parabola at $R = R_2 + O(\varepsilon^2)$ is $O(\varepsilon\delta)$, so the natural spatial scale associated to the B-mode is $\xi_2 = \varepsilon\delta x = \delta\xi$. Thus, B evolves on a slow spatial scale, compared to A. Therefore, it is natural to assume a 'Landau-ansatz' for B: $B = B(\tau)$, that is, B is independent of ξ, as has for instance been done in [22]. This Ansatz means that we approximate the (k_2, R_2)-parabola by a line. Repeating the above derivation process we see that this Landau-reduction has no influence on the equation for A. However, the equation for ψ_{13} (see expansion (2.6)) now reads

$$L_{R,S,T}(\psi_{13}e^{ik_2x}) = [\alpha_1\frac{\partial B}{\partial \tau} - (\alpha_2 B + \alpha_3 B|B|^2 + \alpha_4 B|A|^2)]e^{ik_2x} \tag{2.8}$$

where the $\alpha_1, \ldots, \alpha_4$ are the non-scaled counterparts of the constants in (1.4). This equation can be written as

$$L_{R,S,T}(\psi_{13}e^{ik_2x})e^{-ik_2x} = f(\xi,\tau) + g(\tau), \tag{2.9}$$

where $f(\xi,\tau) = -\alpha_4 B|A|^2$ is the only term which depends on ξ. We define the averages

$$\begin{aligned} \overline{\psi_{13}}(\tau) &= \lim_{M\to\infty} \frac{1}{2M}\int_{-M}^{M} \psi_{13}(\xi,\tau)d\xi \\ \overline{f}(\tau) &= \lim_{M\to\infty} \frac{1}{2M}\int_{-M}^{M} f(\xi,\tau)d\xi \end{aligned}$$

and separate f and ψ_{13} into a part which only depends on τ and a part which still depends on both ξ and τ:

$$\begin{aligned} \psi_{13}(\xi,\tau) &= \overline{\psi_{13}}(\tau) + \Psi(\xi,\tau) \\ f(\xi,\tau) &= \overline{f}(\tau) + F(\xi,\tau). \end{aligned}$$

Note that although we of course assume that ψ_{13} and f are bounded on $\mathbb{R}$, these averages do not automatically exist for all ψ_{13} and f. However, we shall see that $\overline{\psi_{13}}$ and $\overline{f}$ exist for the functions studied here (see Remark 3.1 in section 3.1). We substitute these expressions into (2.9) and obtain

$$L_{R,S,T}(\overline{\psi_{13}}e^{ik_2x})e^{-ik_2x} + L_{R,S,T}(\Psi e^{ik_2x})e^{-ik_2x} = (\overline{f} + g) + F(\xi,\tau). \tag{2.10}$$

Because $\overline{\psi_{13}}$ only depends on τ the first term of this expression can be calculated as

$$L_{R,S,T}(\overline{\psi_{13}}e^{ik_2x})e^{-ik_2x} = \mu(k_2,R_2)\overline{\psi_{13}} = 0.$$

Thus we get

$$L_{R,S,T}(\Psi e^{ik_2x})e^{-ik_2x} = (\overline{f} + g) + F. \tag{2.11}$$

Taking the above defined average on both sides leads to the following solvability equation

$$\alpha_1 \frac{\partial B}{\partial \tau} = \alpha_2 B + \alpha_3 B|B|^2 + \alpha_4 B \lim_{M\to\infty} \frac{1}{2M}\int_{-M}^{M} |A|^2 d\xi.$$

This equation follows from (2.11), however, it is not a sufficient condition to solve (2.11): the equation for Ψ is still left. Writing Ψ and F as (formal) Fourier integrals, we have

$$\int_{-\infty}^{\infty} \left(\mu(k_2+\varepsilon K, R)\hat{\Psi}(K) - \hat{F}(K)\right) e^{iK\xi} dK = 0 \tag{2.12}$$

in the sense of distributions. Now we note that $\mu(k_2+\varepsilon K, R) = O(\varepsilon^2)$ if $|K| = O(\delta)$ (since the local neutral (k_2, R_2)-parabola is only of $O(\delta)$ width). Thus we see that (2.12) cannot be solved for a bounded $O(1)$ solution Ψ if $\hat{F}(K) \neq O(\varepsilon^2)$ for $|K| = O(\delta)$. This yields a *second solvability condition* on $F = f - \overline{f} = \alpha_4 B|A|^2 - \overline{f}$:

$$f_2 B \int_{-\infty}^{\infty} \left(|A|^2 - \lim_{M\to\infty} \frac{1}{2M}\int_{-M}^{M} |A|^2 d\hat{\xi}\right) e^{iK\xi} d\xi = 0 + h.o.t. \tag{2.13}$$

for $|K| \leq O(\delta)$. Observe that (2.11) can now be solved. After rescaling, the above analysis leads to the *non-local* system (1.6) given in the introduction, where again $t_1,\ t_2,\ r,\ s,\ c_1,\ c_2$

have exactly the same values as in (1.5); extra condition (2.13) coincides with (1.7). In [3] a proof has been given of the asymptotic validity of a Ginzburg-Landau equation with an extra non-local term combined with some additional conditions for a certain version of the Poiseuille flow problem. Note that intuitively the relation between the singularly perturbed system and the non-local system is quite simple: δ has become so small that one is forced to assume that B can not be a function of ξ (at highest order). This has no influence on the equation for A_τ in (1.5). However, the $B_{\xi\xi}$ has to disappear in the B_τ-equation, and one has to eliminate the ξ-dependence of the $|A|^2$-term.

Remark 2.1 The above derivation of the non-local system (1.6) is not completely rigorous. In order to improve this one should work with the Fourier transform $\hat{\psi}$ of ψ, the solution of (2.1), and interpret it as a distribution, see for instance [12] and [3].

Remark 2.2 In their analysis of pattern evolution with disparate scales Metzener and Proctor [22] do not use the relative width of the local parabolas as small parameter, but the second critical wavenumber k_2: $k_2 = \sigma \ll 1 = k_1$. If this is the case, it is easy to show that the width of the (k_2, R_2)-parabola must also be small, so the above derivation covers this case. Note that we assume -as in [22]- that the (k_2, R_2)-parabola yields a classical Ginzburg-Landau equation if we omit the A-mode (see the introduction); in other words: we assume that $\mu(0, R_1) > 0$ and $O(1)$ (see [18] for a short discussion). In this remark we sketch the extra complications encountered by taking σ small. If one derives the coupled system in this case, one finds if $\sigma \gg \varepsilon$:

$$\left\{ \begin{array}{rl} \frac{\partial A}{\partial \tau} = & rA + \frac{\partial^2 A}{\partial \xi^2} + A(t_1|A|^2 + \frac{\tilde{c}_1}{\sigma^2}|B|^2) \\ \frac{\partial B}{\partial \tau} = & sB + \frac{d}{\sigma^2}\frac{\partial^2 B}{\partial \xi^2} + B(t_2|B|^2 + \frac{\tilde{c}_2}{\sigma^2}|A|^2). \end{array} \right. \tag{2.14}$$

Thus, the coupling terms $A|B|^2$ and $B|A|^2$ must also be large, $O(\frac{1}{\sigma^2})$. This is due to the fact that the terms $e^{i(1\pm\sigma)x}$ in the expansion of ψ (see (2.6)) are now close to the critical wave e^{ix}. Thus, solving the equations for Φ_{12}, Ψ_{12}, (2.6), yields terms like $\frac{AB, A\overline{B}}{\mu(1\pm\sigma,1)} = O(\frac{1}{\sigma^2})$. This case is much harder to study than the above case. Therefore, we focus in this paper on the assumption $k_2 = O(1), D = \frac{1}{\delta^2}, \delta \ll 1$.

Moreover, one encounters many other complications in deriving and studying the appropriate equations as the relative magnitudes of ε and σ change. For instance, for $\varepsilon = O(\sigma)$ the system becomes (at leading order)

$$\left\{ \begin{array}{rl} \frac{\partial A}{\partial \tau} & = rA + \frac{\partial^2 A}{\partial \xi^2} + cA(Be^{iL\xi} + \overline{B}e^{-iL\xi}) \\ \frac{\partial B}{\partial \tau} & = sB + a_1 \lim_{M\to\infty} \frac{1}{2M}\int_{-M}^{M} |A|^2 e^{-iL\xi} d\xi, \end{array} \right. \tag{2.15}$$

where $k_2 = \sigma = \varepsilon L$. In the derivation of these last equations the magnitude of the perturbation had to be taken of order ε^2 instead of ε. Solutions of the above system can be found explicitly and it can be checked that these solutions are unstable. From this it follows that although one first has to take the perturbation of magnitude $O(\varepsilon^2)$, they will grow to a magnitude of $O(\varepsilon)$. This leads to a rather complicated system. The above example (2.15) is just added to indicate the complications caused by decreasing σ further. Deriving and analysing the full set of equations for every possible combination of the magnitude of ε relative to that of σ is a task we will not pursue in this paper. We refer to [29] in which such a complete non-linear stability analysis has been performed for a, in a sense, simpler case: a weakly, periodically driven, system. There, five essentially different types of modulation equations have been derived.

In [22] a different system has been proposed to describe the weakly non-linear evolution of patterns at 'disparate scales'. This system is in some sense a combination of (1.6) and (2.15):

$$\left\{\begin{array}{lcl} \frac{\partial A}{\partial \tau} & = & \mu A - |A|^2 A + \frac{\partial^2 A}{\partial X^2} + A(Be^{iX} + B^* e^{-iX}) \\ \frac{\partial B}{\partial \tau} & = & \nu B - c|B|^2 B - s \lim_{M\to\infty} \frac{1}{2M} \int_{-M}^{M} |A|^2 e^{-iX} \, dX. \end{array}\right. \tag{2.16}$$

In order to give a foundation to this system one has to assume relations between the parameters of (1.6) and (2.15) and ε (see [18]): that is the only way to have quadratic and cubic terms of the same magnitude. Since these parameters are in principle completely independent of ε this assumption will be violated in general. Therefore, one will not find this system if one pursues the above sketched task of deriving all relevant 'generic' equations.

3 Stationary solutions

From now on we focus on the analysis of non-local system (1.6) and singularly perturbed system (1.5). We will study and compare the solutions of these systems. System (1.6) can be considered as a limit of (1.5), thus we expect that some classes of solutions represent similar patterns in the underlying basic system. The most simple solutions are the stationary solutions, which we will study here. The stationary solutions are certainly of a physical relevance, for instance in the convection context where stable stationary patterns exist (see section 5). The stationary problem associated to the singularly perturbed system reads

$$\left\{\begin{array}{lcl} \frac{\partial^2 A}{\partial \xi^2} & = & -rA - A(t_1|A|^2 + c_1|B|^2) \\ \frac{\partial^2 B}{\partial \xi^2} & = & \delta^2(-sB - B(t_2|B|^2 + c_2|A|^2)) \end{array}\right. \tag{3.1}$$

To reduce the amount of unknown variables we introduce polar coordinates. Let

$$A = \rho_1 e^{i\theta_1}, \; B = \rho_2 e^{i\theta_2}, \tag{3.2}$$

where ρ_1, ρ_2, θ_1 and θ_2 depend on ξ. We insert these expressions into the above system, separate the real and complex parts and find

$$\left\{\begin{array}{rl} r\rho_1 + \frac{\partial^2 \rho_1}{\partial \xi^2} - \rho_1(\frac{\partial \theta_1}{\partial \xi})^2 + \rho_1(t_1\rho_1^2 + c_1\rho_2^2) & = 0 \\ s\rho_2 + \frac{1}{\delta^2}\frac{\partial^2 \rho_2}{\partial \xi^2} - \frac{1}{\delta^2}\rho_2(\frac{\partial \theta_2}{\partial \xi})^2 + \rho_2(t_2\rho_2^2 + c_2\rho_1^2) & = 0 \\ 2\frac{\partial \rho_i}{\partial \xi}\frac{\partial \theta_i}{\partial \xi} + \rho_i \frac{\partial^2 \theta_i}{\partial \xi^2} & = 0, \end{array}\right. \tag{3.3}$$

for $i = 1, 2$. Because

$$\frac{1}{\rho_i}\frac{\partial}{\partial \xi}(\rho_i^2 \frac{\partial \theta_i}{\partial \xi}) = 2\frac{\partial \rho_i}{\partial \xi}\frac{\partial \theta_i}{\partial \xi} + \rho_i \frac{\partial^2 \theta_i}{\partial \xi^2}$$

we obtain that

$$\frac{\partial}{\partial \xi}(\rho_i^2 \frac{\partial \theta_i}{\partial \xi}) = 0$$

and therefore introduce the integrals Ω_i:

$$\rho_i^2 \frac{\partial \theta_i}{\partial \xi} = \Omega_i.$$

Since $\frac{\partial B}{\partial \xi} = O(\delta)$ we know that $\frac{\partial \theta_2}{\partial \xi}$ has to be of $O(\delta)$, which implies that Ω_2 is $O(\delta)$. We introduce $\tilde{\Omega}_2$, where $\Omega_2 = \delta\tilde{\Omega}_2$ and $\tilde{\Omega}_2 = O(1)$. Substituting this into (3.3) leads to

$$\left\{ \begin{array}{ll} \frac{\partial^2 \rho_1}{\partial \xi^2} & = -r\rho_1 - \rho_1(t_1\rho_1^2 + c_1\rho_2^2) + \frac{\Omega_1^2}{\rho_1^3} \\ \frac{\partial^2 \rho_2}{\partial \xi^2} & = \delta^2(-s\rho_2 - \rho_2(t_2\rho_2^2 + c_2\rho_1^2) + \frac{\tilde{\Omega}_2^2}{\rho_2^3}). \end{array} \right. \tag{3.4}$$

Equivalently, we determine the ODE associated to the stationary solutions of (1.6):

$$\left\{ \begin{array}{lll} \frac{\partial^2 A}{\partial \xi^2} & = & -rA - A(t_1|A|^2 + c_1|B|^2) \\ 0 & = & B(s + t_2|B|^2 + c_2 \lim_{M\to\infty} \frac{1}{2M}\int_{-M}^{M} |A|^2 d\xi). \end{array} \right. \tag{3.5}$$

Thus, there are possible values for B

1. $B = 0$
2. $|B|^2 = \dfrac{-s}{t_2} + \dfrac{-c_2}{t_2}C(A)$ where $C(A) = \lim\limits_{M\to\infty} \dfrac{1}{2M}\displaystyle\int_{-M}^{M} |A|^2 d\xi.$

The first case leads to the stationary uncoupled Ginzburg-Landau equation for A. The second case leads to the following equation for A

$$\frac{\partial^2 A}{\partial \xi^2} = (-r + \frac{c_1}{t_2}(s + c_2C(A))A - t_1A|A|^2.$$

As above, we introduce polar coordinates for A, $A = \rho_1 e^{i\theta_1}$:

$$\frac{\partial^2 \rho_1}{\partial \xi^2} = (-r + \frac{c_1}{t_2}(s + c_2C(\rho_1)))\rho_1 - t_1\rho_1^3 + \frac{\Omega_1^2}{\rho_1^3}, \tag{3.6}$$

where, as above, $\Omega_1 = \rho_1^2\frac{\partial \theta_1}{\partial \xi}$ is an integral. To simplify the calculations, we set in *both* systems (3.4) and (3.6) $r = 1,\ t_1 = -1,\ t_2 = -1$. Note that this can be obtained by straightforward rescalings with additional assumptions on the signs of $r,\ t_1,\ t_2$. These assumptions are so that the equations are directly related to the single Ginzburg-Landau equation mostly studied in the literature. Furthermore we assume that $\Omega_1 = \tilde{\Omega}_2 = 0$. Taking $\Omega_1 = \tilde{\Omega}_2 = 0$ simplifies the analysis, however for $\Omega_1 \neq 0$ and $\tilde{\Omega}_2 \neq 0$ the analysis is in essence the same. Because $\Omega_i = \rho_i^2\frac{\partial \theta_i}{\partial \xi}$ for $i = 1, 2$, $\Omega_1 = \tilde{\Omega}_2 = 0$ implies that $\frac{\partial \theta_1}{\partial \xi} = \frac{\partial \theta_2}{\partial \xi} = 0$. This yields that θ_1 and θ_2 do not depend on ξ thus θ_1 and θ_2 are constants. Thus due to the phase shift invariance in (1.5) and (1.6), one can say that we restrict ourselves to studying real solutions of these systems. Moreover, we note that the Stokes waves, (A=constant, $B = 0$) and ($A = 0$, B=constant), are on the $\Omega_1 = \tilde{\Omega}_2 = 0$ level sets. We refer to [4] for a detailed discussion of the relation between $\Omega = 0$ and $\Omega \neq 0$ in the single real Ginzburg-Landau case.

3.1 Stationary solutions of the non-local equations

We introduce $x = \rho_1$ in (3.6) with $\Omega_1 = 0, r = 1,\ t_1 = -1$ and $t_2 = -1$:

$$\ddot{x} = -(1 + c_1(s + c_2C(x)))x + x^3 \text{ with } C(x) = \lim_{M\to\infty} \frac{1}{2M}\int_{-M}^{M} x^2 dt, \tag{3.7}$$

where the dot means differentiating with respect to 'time' $t = \xi$. We are only interested in bounded solutions of these equations since A, where $|A| = \rho_1 = x$, must remain bounded.

Moreover $C(x)$ is only defined for bounded x. First we set $C(x) = \mathcal{C}$, where $\mathcal{C}$ is a fixed constant. System (3.7) then becomes

$$\ddot{x} = -ax + x^3 \text{ where } a = 1 + c_1(s + c_2\mathcal{C}). \tag{3.8}$$

We will describe the phase portrait of this equation in some detail, because later on we will come across this equation again. For $a < 0$, (3.8) has only one critical point: $(0,0)$, which is a saddle point. This yields that there are no bounded solutions, except for the trivial critical point. For $a > 0$, the system has three critical points: $(0,0)$ and $(\pm\sqrt{a}, 0)$. In this case $(0,0)$ is a centre point and $(\pm\sqrt{a}, 0)$ are saddle points. There exists a heteroclinic connection between the two saddle points. Inside this heteroclinic solution there are bounded periodic solutions and outside all orbits are unbounded. This means that bounded solutions of (3.8) will always lie inside the heteroclinic cycle formed by the two connections. These solutions are periodic (see Figure 2). System (3.8) is integrable with integral or energy k:

$$k = \frac{1}{2}y^2 + \frac{1}{2}ax^2 - \frac{1}{4}x^4, \text{ where } y = \dot{x}. \tag{3.9}$$

It is possible to determine $C(x)$ explicitly for a periodic solution of (3.8), with period T_0. Note that

$$C(x) = \lim_{M\to\infty} \frac{2M}{2MT_0} \int_0^{T_0} x^2 dt = \frac{1}{T_0} \int_0^{T_0} x^2 dt \tag{3.10}$$

for a periodic solution x. Hence

$$C(x) = \frac{\int_0^{T_0} x^2 dt}{\int_0^{T_0} dt} = \frac{\int_{-x_1}^{x_1} \frac{x^2}{\sqrt{G(x,k)}} dx}{\int_{-x_1}^{x_1} \frac{1}{\sqrt{G(x,k)}} dx} \tag{3.11}$$

by changing variables, where $G(x,k) = 2k - ax^2 + \frac{1}{2}x^4$. Here $-\sqrt{a} < -x_1 < 0 < x_1 < \sqrt{a}$ are the intersection points of the solution x, on the energy level set k, with $y = 0$. Introduce $X = x^2$ and $X_1 = x_1^2$ then

$$C(x) = \frac{T_1(k)}{T_0(k)}, \text{ where } T_i(k) = \oint \frac{X^i dX}{\sqrt{2kX - aX^2 + \frac{1}{2}X^3}}. \tag{3.12}$$

This is a contour integral in the complex plane around the interval $[0, X_1]$ on the real axis. We define

$$\chi(k) = \frac{T_1(k)}{T_0(k)}. \tag{3.13}$$

Because x is a periodic solution which lies inside the heteroclinic cycle, the k-value of x lies between 0 and $\frac{1}{4}a^2$. Below we will show that χ is a monotonic function of k, so we can conclude that $0 < \chi < a$ (since $\chi(0) = 0$ and $\lim_{k\uparrow\frac{1}{4}a^2} \chi(k) = a$). Although this result is a special case of a more general result proved in [4], we will sketch the derivation of the monotonicity result: $\chi(k)$ is an important quantity which will also appear in subsequent sections. Note that

$$\frac{\partial\chi}{\partial k} = \frac{\partial}{\partial k}\frac{T_1}{T_0} = \frac{T_0\frac{\partial T_1}{\partial k} - T_1\frac{\partial T_0}{\partial k}}{T_0^2}.$$

We define $G_1(k,X) = \sqrt{2kX - aX^2 + \frac{1}{2}X^3}$ and define

$$J_i(k) = \oint \frac{X^i dX}{(G_1(k,X))^3},$$

which yields that

$$\frac{\partial \chi}{\partial k} = \frac{-T_0 J_2 + T_1 J_1}{T_0^2}. \tag{3.14}$$

Rewriting $T_i(k)$ into these $J_i(k)$'s leads to the following relation for every $i \geq 0$

$$T_i = 2kJ_{i+1} - aJ_{i+2} + \frac{1}{2}J_{i+3}. \tag{3.15}$$

Since

$$\oint \frac{d}{dX} \frac{X^i dX}{(G_1(k,X))} = 0$$

for every $i \geq 1$, we also find:

$$k(2i-1)J_i + (1-i)aJ_{i+1} + (\frac{1}{2}i - \frac{3}{4})J_{i+2} = 0. \tag{3.16}$$

Setting in (3.15) $i = 0, 1$ and in (3.16) $i = 1, 2$ leads to a system of four equations from which J_1 and J_2 can be solved. Substitution of these expressions into (3.14) gives

$$\frac{\partial \chi}{\partial k} = \frac{1}{4k(a^2 - 4k)}(a\chi^2 - 8k\chi + 4ka).$$

We define $f(k) = 4k(a^2 - 4k)$ and $P(\chi) = a\chi^2 - 8k\chi + 4ka$, and see that $f(k) > 0$ for $0 < k < \frac{1}{4}a^2$, which is exactly the interval we are studying. There are no solutions of $P(\chi) = 0$ and so, because $a > 0$, $P(\chi) > 0$ for every χ. Combining this finally results in $\frac{\partial \chi}{\partial k} > 0$. This yields that $0 < \chi(k) < a$, where $\chi(k) = 0$ corresponds to the centre point $(0,0)$ and $\chi(k) = a$ to the heteroclinic orbit. Since $C(x) = \chi(k)$ one observes that bounded solutions can only exist if $0 < C(x) < a$. We can now explicitly solve (3.7): for a given value of $a = 1 + c_1(s + c_2\mathcal{C})$ in (3.8) we see that the bounded orbits have values of $\mathcal{C}$ ranging from 0 to a, thus if

$$C(x) \in (0, 1 + c_1(s + c_2C(x))), \tag{3.17}$$

we see that one bounded orbit of system (3.8) is selected as solution of system (3.7). Again $C(x) = 0$ corresponds to the selection of the centre point $(0,0)$ and $C(x) = 1 + c_1(s + c_2C(x))$ to the selection of a heteroclinic orbit. Recall that $|B|^2 = s + c_2C(x)$, which relates the value of $|B|$ to every $C(x)$. Thus, the non local system (1.6) only has bounded stationary solutions $(A(\xi), B)$ with 'average' $C(A)$ if $0 < C(A) < 1 + c_1(s + c_2C(A))$ and $|B|^2 = s + c_2C(A) > 0$.

Remark 3.1 The two boundaries of the interval given by (3.17) determine, in a sense, the bifurcations at which the periodic solutions of (3.7) (dis)appear. As already noted above the periodic solution shrinks into a critical point at $C(x) = 0$. The other boundary, at $C(x) = \frac{1+c_1 s}{1-c_1 c_2}$ determines a global bifurcation at which the periodic orbit merges with a heteroclinic cycle of (3.7). From the derivation of the system we also obtained an extra condition (1.7) which the solutions have to satisfy. As can be easily seen, the critical points of the system do satisfy the condition. This is also true for the periodic solutions as long as their period is not too large. This can be seen as follows. Note that (1.7) determines the Fourier transform of the (periodic) function $|A|^2 - C(A)$. This is of course a discrete spectrum with 'peaks' at $K = \frac{2\pi n}{T_0}, n = \pm 1, \pm 2, \ldots$ where T_0 is the period of $|A|^2$. Observe that there is no peak at $K = 0$ due to the subtraction '$-C(A)$'. Since T_0 becomes unbounded as $k \uparrow \frac{1}{4}a^2$ (see [4]), we see that the $n = \pm 1$ peaks approach the $K = O(\delta)$ region if k approaches $\frac{1}{4}a^2$. In other words: the periodic orbits satisfy the extra condition (1.7) as long as $T_0 \neq O(\frac{1}{\delta})$. The period orbits with a very long period do not satisfy (1.7). The same is true for the heteroclinic orbits: they do not satisfy (1.7) (this can be checked by using the explicit expression (4.15) given in section 4.2). Thus, we conclude that the only stationary solutions described by the 'Landau-reduction' are the solutions with $|A|$=constant or $|A|$ is periodic with $O(1)$ period. All other solutions of solvability condition (1.6) do not satisfy solvability condition (1.7). Furthermore, we note that the 'average' described in section 2 is defined for all solutions of the stationary problem associated to (1.6).

4 The singularly perturbed system

In this section we will study the bounded solutions of singularly perturbed system (3.4) with $\Omega_1 = \tilde{\Omega}_2 = 0$ (and $r = 1, t_1 = t_2 = -1$). We introduce $x = \rho_1$ and $y = \frac{\partial \rho_1}{\partial \xi} = \dot{x}$ where the dot means differentiation with respect to 'time' $t = \xi$; equivalently we write $z = \rho_2$ and $\delta w = \dot{z}$:

$$\begin{cases} \dot{x} = & y \\ \dot{y} = & -x + x(x^2 - c_1 z^2) \\ \dot{z} = & \delta w \\ \dot{w} = & \delta(-sz + z(z^2 - c_2 x^2)). \end{cases} \tag{4.1}$$

So, x and y can be considered as the fast($=O(1)$) moving coordinates and z and w as the slow($=O(\delta)$) coordinates. The system contains a lot of useful symmetries: (4.1) is equivariant under

$$\begin{array}{lll} \{x \to -x, y \to -y\}, & \{x \to -x, t \to -t, w \to -w\}, & \{x \to -x, t \to -t, z \to -z\} \\ \{z \to -z, w \to -w\}, & \{y \to -y, t \to -t, w \to -w\}, & \{y \to -y, t \to -t, z \to -z\}. \end{array} \tag{4.2}$$

System (4.1) can be considered, in some sense, as a Hamiltonian system. We introduce the arbitrary rescalings $x = \alpha\tilde{x}, y = \alpha\tilde{y}, z = \beta\tilde{z}$ and $w = \beta\tilde{w}$, for $\alpha, \beta > 0$. This gives

$$\begin{cases} \dot{\tilde{x}} & = \tilde{y} \\ \dot{\tilde{y}} & = -\tilde{x} + \tilde{x}(\alpha^2\tilde{x}^2 - c_1\beta^2\tilde{z}^2) \\ \dot{\tilde{z}} & = \delta\tilde{w} \\ \dot{\tilde{w}} & = \delta(-s\tilde{z} + \tilde{z}(\beta^2\tilde{z}^2 - c_2\alpha^2\tilde{x}^2)). \end{cases} \tag{4.3}$$

The Hamiltonian H which could belong to this system must be of the form

$$H = \frac{1}{2}(\tilde{x}^2 + \tilde{y}^2 + \delta s\tilde{z}^2 + \delta\tilde{w}^2) - \frac{1}{4}\alpha^2\tilde{x}^4 - \frac{1}{4}\beta^2\tilde{z}^4 + m\tilde{x}^2\tilde{z}^2,$$

where m still has to be determined. This implies that we must impose that $c_1\beta^2 = \delta c_2\alpha^2$. Which yields that

$$\frac{\beta^2}{\alpha^2} = \frac{\delta c_2}{c_1}, \text{ when } sign(c_1) = sign(c_2).$$

Thus, it is possible to rescale (4.1) so that it becomes a Hamiltonian system. This rescaling has to satisfy $\frac{\beta^2}{\alpha^2} = O(\delta)$; which means that $O(1)$ solutions in the Hamiltonian system corresponds to solutions of (4.1) of which either the pair (x, y) or the pair (z, w) (or both) is not $O(1)$. However, amplitude A in (1.5) corresponds to (x, y) and B to (z, w) and both A and B must be $O(1)$, due to the structure of the derivation process. Therefore we will not write (4.1) as a Hamiltonian system, but we will use that the energy is conserved. Note that the expression for the energy contains only even powers of x, y, z and w. The above is also true in the case that $sign(c_1) = -sign(c_2)$.

Before we study the perturbed system we are interested in the dynamics of the unperturbed system. Setting $\delta = 0$ leads to

$$\left\{ \begin{array}{rcl} \dot{x} & = & y \\ \dot{y} & = & -x + x(x^2 - c_1 z_0^2). \end{array} \right. \tag{4.4}$$

Here $z = z_0$ and $w = w_0$ where z_0 and w_0 are constants of motion, because setting $\delta = 0$ implies that $\dot{z} = 0$ and $\dot{w} = 0$. The above system can also be written as

$$\ddot{x} = -(1 + c_1 z_0^2)x + x^3. \tag{4.5}$$

This is the same as equation (3.8) with $a = 1 + c_1 z_0^2$, which we already studied in (3.1). Therefore the same results as for (3.8) can be obtained.

Next we consider the critical points of the perturbed system. Here the critical points are given with their characterisation, where the first part is the characterisation in the fast directions and the second in the slow directions (that is, the first pair of eigenvalues is $O(1)$, the second pair $O(\delta)$).

$(0,0,0,0)$	for	$s > 0$	centre/centre
	for	$s < 0$	centre/saddle
$(0,0,\pm\sqrt{s},0)$	for	$sc_1 + 1 < 0$	saddle/saddle
	for	$sc_1 + 1 > 0$	centre/saddle
$(\pm 1, 0, 0, 0)$	for	$c_2 + s < 0$	saddle/saddle
	for	$c_2 + s > 0$	saddle/centre
$(\pm\sqrt{\frac{c_1 s+1}{1-c_1 c_2}}, 0, \pm\sqrt{\frac{s+c_2}{1-c_1 c_2}}, 0)$	for	$\frac{2(c_2+s)(1+c_1 c_2)}{1-c_1 c_2} < 0$	saddle/centre
	for	$\frac{2(c_2+s)(1+c_1 c_2)}{1-c_1 c_2} > 0$	saddle/saddle.

These critical points only exist when the expressions under the square root are positive. So the second two critical points only exist for $s > 0$ and the last four critical points only exist for $\frac{c_1 s+1}{1-c_1 c_2} > 0$ and $\frac{s+c_2}{1-c_1 c_2} > 0$. These critical points give rise to solutions of the original system. The first critical point gives the trivial solution, the second gives the Stokes wave $A = 0$ and $B \pm \sqrt{s} e^{i\theta_2}$ where θ_2 is a constant; the third gives another Stokes wave $A = \pm e^{i\theta_1}$ and $B = 0$ with θ_1 a constant. The last four critical points correspond to the mixed patterns $A \equiv \pm\sqrt{\frac{c_1 s+1}{1-c_1 c_2}} e^{i\theta_1}$ and $B \equiv \pm\sqrt{\frac{s+c_2}{1-c_1 c_2}} e^{i\theta_2}$ with θ_1 and θ_2 constants. The heteroclinic and homoclinic orbits of section 4.2 will have their 'begin and endpoint' at one of these four points.

4.1 Periodic solutions

The singularly perturbed system possesses two time scales, (x, y) are the fast variables and (z, w) are the slow variables. In the non-local system there is no slow behaviour. When studying behaviour which is dominated by the fast field in the singularly perturbed system, we have to take $\dot{z}$ and $\dot{w}$ almost 0, which implies that B is independent of ξ. Recall that this is exactly the condition which is imposed on B when deriving the non-local system. Therefore we expect to observe approximately the same behaviour for solutions which remain in the fast field of the singularly perturbed system as for the solutions of the non-local system. In this (sub)section we will study the solutions of (4.1) which are dominated by the fast field. We will focus on the periodic solutions. We expect to find approximately the same existence conditions (and solutions) as for the non-local system. In the next section we will study solutions to (4.1) which consist of fast and slow parts. These solutions can not exist in the non-local system.

We will construct solutions to (4.1) by using the fact that the unperturbed limit $\delta = 0$ is fully integrable:

$$\left\{\begin{array}{rcl} \dot{x} & = & y \\ \dot{y} & = & -x + x(x^2 - c_1 z^2) \\ \dot{z} & = & 0 \\ \dot{w} & = & 0 \end{array}\right. \tag{4.6}$$

with the three integrals

$$k = \frac{1}{2}(y^2 + x^2 - \frac{1}{2}x^4 + c_1 x^2 z^2), \; k_1 = z, \; k_2 = w. \tag{4.7}$$

The behaviour of fast periodic solutions of (4.1) is dominated by the unperturbed system, since the solutions to (4.1) remain $O(\delta)$ close to solutions of (4.6) for $O(1)$ time. Thus, a fast periodic solution to (4.1) will be $O(\delta)$ close to a periodic solution of (4.6): we can study the existence of periodic solutions of the singularly perturbed system by constructing and approximating a Poincaré map which measures the changes in the k, z and w-values for a solution of the perturbed system. This Poincaré mapping is defined as follows

$$\begin{array}{rcl} P(k, k_1, k_2) & = & (k + \Delta K(k, k_1, k_2), k_1 + \Delta K_1(k, k_1, k_2), k_2 + \Delta K_2(k, k_1, k_2)) \\ & = & (k + \Delta K(k, z, w), z + \Delta K_1(k, z, w), w + \Delta K_2(k, z, w)) \end{array} \tag{4.8}$$

The quantities $\Delta K(k_0, z_0, w_0)$, $\Delta K_1(k_0, z_0, w_0)$ and $\Delta K_2(k_0, z_0, w_0)$ measure the accumulated change in the slow variables k, z and w from a solution of the perturbed system. Due to reasons which will become clear later on we define the Poincaré map in a somewhat nonstandard way. The Poincaré map is defined by a solution which consists of two parts which are joined together. The first part starts on the cross section $\{x = 0, y > 0\}$, travels forwards in time and ends by intersecting the cross section $\{x = 0, y < 0\}$. The time it takes this solution to intersect with the cross section $\{x = 0, y < 0\}$ is denoted by T_δ. The other part travels backwards in time and ends by intersecting $\{x = 0, y < 0\}$.(see Figure 3) The time it takes this solution to intersect with $\{x = 0, y < 0\}$ is denoted by $T_{-\delta}$. Here (k_0, z_0, w_0) is the initial value of these solutions, hence k_0 is so that $x_0 = 0$ and $y_0 > 0$. The accumulated change of the integral k over this orbit is given by

$$\Delta K(k_0, z_0, w_0) = \int_{-T_{-\delta}}^{T_\delta} \dot{k}(x_\delta, y_\delta, z_\delta, w_\delta) dt,$$

where $(x_\delta(t), y_\delta(t), z_\delta(t), w_\delta(t))$ is the above constructed solution of the perturbed system. The quantities ΔK_1 and ΔK_2 can be expressed in the same way. Substituting the expression for k, (4.7), gives

$$\Delta K(k_0, z_0, w_0) = \delta \int_{-T_{-\delta}}^{T_\delta} c_1 x_\delta^2 w_\delta z_\delta dt.$$

The solution $(x_\delta(t), y_\delta(t), z_\delta(t), w_\delta(t))$ can be approximated with $O(\delta)$-error by the solution $(x_0(t), y_0(t), z_0, w_0)$ of the unperturbed system with energy $k = k_0$ which starts on the cross section $\{x = 0, y < 0\}$ with the same initial data as $(x_\delta(t), y_\delta(t), z_\delta(t), w_\delta(t))$; $T_{\pm\delta}$ can be approximated by $\pm\frac{1}{2}T_0$, the period of the solution $(x_0(t), y_0(t), z_0, w_0)$. Therefore

$$\Delta K(k_0, z_0, w_0) = \delta \int_{-\frac{1}{2}T_0}^{\frac{1}{2}T_0} c_1 x_0^2 w_0 z_0 dt + O(\delta^2).$$

We define $G(k, z, x) = 2k - x^2 + \frac{1}{2}x^4 - c_1 x^2 z^2$, note that this is nearly the same function as defined in section 3.1. Transforming the coordinates leads to

$$\Delta K(k_0, z_0, w_0) = 2\delta c_1 w_0 z_0 \int_{-x_1}^{x_1} \frac{x^2 dx}{\sqrt{G(k_0, z_0, x)}} + O(\delta^2).$$

Here $-\sqrt{1 + c_1 z_0^2} < -x_1 < 0 < x_1 < \sqrt{1 + c_1 z_0^2}$ are the intersection points with $y = 0$ of the solution with $k = k_0$. We set $X = x^2$ and $X_1 = x_1^2$, then

$$\Delta K(k_0, z_0, w_0) = \delta c_1 w_0 z_0 T_1(k_0, z_0) + O(\delta^2), \tag{4.9}$$

where

$$T_i(k, z) = \oint \frac{X^i dX}{\sqrt{2kX - X^2 + \frac{1}{2}X^3 - c_1 X^2 z^2}}.$$

This contour integral around the interval $[0, X_1]$ is again very similar to the one defined in section 3.1. In the same way we obtain

$$\begin{aligned} \Delta K_1(k_0, z_0, w_0) &= \delta w_0 T_0(k_0, z_0) + O(\delta^2), \\ \Delta K_2(k_0, z_0, w_0) &= -\delta z_0((s - z_0^2)T_0(k_0, z_0) + c_2 T_1(k_0, z_0)) + O(\delta^2). \end{aligned} \tag{4.10}$$

A solution is periodic if $\Delta K = \Delta K_1 = \Delta K_2 = 0$. Since $T_0(k, z)$ and $T_1(k, z)$ are strictly positive this yields

$$\begin{aligned} c_1 w_0 z_0 + O(\delta) &= 0 \\ w_0 + O(\delta) &= 0 \\ z_0((s - z_0^2)T_0(k_0, z_0) + c_2 T_1(k_0, z_0)) + O(\delta) &= 0. \end{aligned} \tag{4.11}$$

This seems to lead to two possibilities

1. $z_0 = O(\delta)$ and $w_0 = O(\delta)$
2. $w_0 = O(\delta)$ and $(s - z_0^2)T_0(k_0, z_0) + c_2 T_1(k_0, z_0) = O(\delta)$.

However, (4.12) is a singular system in the limit $\delta \to 0$. Thus we can not solve this system for $\delta = 0$ by applying the implicit function theorem and concluding that there is a solution to the perturbed system $O(\delta)$ near the $\delta = 0$ solution. On the contrary: one must expect that the solution of the full problem will be much more complicated. Nevertheless, we can show that the above two 'singular' solutions are correct.

Approximating the solution of the perturbed system by a solution of the unperturbed system throws away to much of the dynamics of the system, thus we have to look at the expressions without approximating the solution. The exact expressions for $\Delta K, \Delta K_1$ and ΔK_2 are

$$\begin{aligned}
\Delta K(k_0, z_0, w_0) &= \delta \int_{-T_{-\delta}}^{T_\delta} c_1 x_\delta^2 w_\delta z_\delta dt \\
\Delta K_1(k_0, z_0, w_0) &= \delta \int_{-T_{-\delta}}^{T_\delta} w_\delta dt \\
\Delta K_2(k_0, z_0, w_0) &= \delta \int_{-T_{-\delta}}^{T_\delta} (-s z_\delta + z_\delta^3 - c_2 x_\delta^2 z_\delta) dt.
\end{aligned}$$

We note that $w_\delta(t) \equiv 0$ and $z_\delta(t) \equiv 0$ if $z_0 = w_0 = 0$. Thus $\Delta K(k_0, 0, 0) = \Delta K_1(k_0, 0, 0) = \Delta K_2(k_0, 0, 0) = 0$. Therefore periodic solutions exist. This can also be seen by noting that if $z = w = 0$, the dynamics of (4.1) are described by

$$\ddot{x} = -x + x^3.$$

This leads to periodic solutions in the $(z, w) = (0, 0)$-plane. These solutions are also solutions of the unperturbed system.

We now consider the second possible solution to (4.12). On the cross section we have $x_0 = x(0) = 0$. Let's again consider $w_0 = w(0) = 0$, thus $\dot{z}(0) = 0$. It can be shown for a solution $(x_\delta, y_\delta, z_\delta, w_\delta)$ of (4.1) with these initial conditions that $x_\delta(t)$ (respectively $z_\delta(t)$) is an odd (resp. even) function of t. This can be done by inductively checking that $x(0) = 0$ and $\dot{z}(0) = 0$ in (4.1) yields that $x^{(2n)}(0) = 0, z^{(2n+1)}(0) = 0$ for every n. From the fact that x is odd it follows that $T_{-\delta} = T_\delta$. Because z is even, $w = \dot{z}$ is odd. Thus

$$\Delta K(k_0, z_0, w_0) = \delta \int_{-\frac{1}{2}T_\delta}^{\frac{1}{2}T_\delta} c_1 x_\delta^2 w_\delta z_\delta dt \equiv 0$$

since $x_\delta^2 w_\delta z_\delta$ is an odd function of t. Analogously, since w_δ is odd

$$\Delta K_1(k_0, z_0, w_0) \equiv 0.$$

We still have to use the solution of the unperturbed system to approximate ΔK_2, see (4.10). Thus, a periodic solution with initial data $x_0 = z_0 = 0$ must satisfy

$$z_0 = 0 \text{ or } (z_0^2 - s)T_0(k_0, z_0) - c_2 T_1(k_0, z_0) = 0.$$

This is equivalent to $\chi(k_0, z_0) = \frac{z_0^2 - s}{c_2}$ where χ is in essence the function defined in (3.13). In section (3.1) we showed that $\chi \in [0, a]$, where a in (3.8) corresponds to $1 + c_1 z^2$ in (4.5), and that χ is a monotonically increasing function of k from 0 to $a = 1 + c_1 z^2$. Since s, c_1 and c_2 are known from the equations, an interval for z_0 can be determined so that $\chi(k_0, z_0) = \frac{z_0^2 - s}{c_2}$ holds:

$$0 \leq \frac{z_0^2 - s}{c_2} \leq 1 + c_1 z_0^2. \tag{4.12}$$

As in the case for existence of solutions of the non-local system, there are, for different values of the coefficients, different intervals for z_0 where periodic solutions exist.

Since we expected similar conditions for the existence of periodic solutions for the singularly perturbed system as for solutions of the non-local system we are now going to compare the conditions for this existence. Recall that in the non-local system bounded solutions exist for $B = 0$ and for $|B|^2 = s + c_2C(x)$ where $C(x) \in (0, 1 + c_1(s + c_2C(x)))$. In the singularly perturbed system there exist periodic solutions for $z = 0$ and for z with $0 \leq \frac{z^2-s}{c_2} \leq 1 + c_1z^2$. In section 3 we set $B = \rho_2 e^{i\theta_2}$, later(section 4) we introduced z by $\rho_2 = z$. This implies that $B = 0$ is the same as $z = 0$. Thus the first possibilities for existence of the periodic solutions coincide, which is of course not surprising since the non-local system (1.6) and the singularly perturbed system(1.5) are exactly the same for the subcase $B = 0$. The other two cases are somewhat more difficult to compare but finally these appear to be essentially the same. Since $|B| = z$ in the singularly perturbed case and $|B|^2 = s + c_2C(x)$ in the non-local case, both existence conditions (3.17) and (4.12) can be compared by substituting $z^2 = s + c_2C(x)$ into (4.12) to obtain

$$0 \leq \frac{c_2C(x) + s - s}{c_2} \leq 1 + c_1(s + c_2C(x)).$$

This is exactly the condition (3.17) for existence of bounded periodic solutions for the non-local system. This yields that bounded solutions of the non-local system and fast periodic solutions of the singularly perturbed system exist exactly under the same conditions.

Moreover, there is a direct relation between these two families of periodic solutions. The construction of the periodic solutions in the non-local case of section 3.1 shows that these solutions are exactly the same as the uniquely defined periodic solutions of the unperturbed system of the singularly perturbed case which survive the perturbation. In other words: for any periodic solution of the non-local system there is a fast periodic solution of the singularly perturbed system which is $O(\delta)$ close. The difference can be seen most clearly by comparing B in both cases: B=constant in the non-local case while $|B|$ varies periodically with an $O(\delta)$ amplitude around that same constant in the singularly perturbed case.

In the next (sub)section we will focus on heteroclinic and homoclinic orbits. From the analysis in the fast field we might expect that there exists a heteroclinic cycle for the extreme value $\frac{z_0^2-s}{c_2} = 1 + c_1z_0^2$, similar to the heteroclinic orbits found for the non-local system in section 3.1. However, we have to be careful here: using the Poincaré-map (4.8) we can only hope to connect the 1-dimensional strong unstable/stable manifolds of the critical points $(\pm\sqrt{\frac{c_1s+1}{1-c_1c_2}}, 0, \pm\sqrt{\frac{s+c_2}{1-c_1c_2}}, 0)$. By a simple geometric counting argument we see that one should not expect that such connections exist for any parameter combination (c_1, c_2). In the next section we shall see that these connections only exist for $c_2 = 0$. For $c_2 \neq 0$ there also are heteroclinic orbits, this is necessary since the periodic orbit has to disappear into some kind of heteroclinic/homoclinic structure, but these orbits will consist of fast and slow parts.

4.2 Heteroclinic and homoclinic orbits

In section 3.1 we studied the dynamics of the stationary solutions of the non-local system. In section 4.1 we showed that these solutions can in essence also be found in the singularly perturbed system by studying solutions which do not have slow parts. In this section we focus on solutions of (4.1) which do consist of distinct slow and fast parts. A solution evolves slowly if it is close to a so-called slow manifold of the system. A slow manifold is an invariant manifold on which the flow is $O(\delta)$-slow. The existence of these manifolds follows from the theory originally developed by Fenichel (see [11] and references there, or [1]). There it is shown that a manifold of critical points of the unperturbed limit $\delta \to 0$ with a normally hyperbolic structure persists under the perturbation as a slow manifold. These slow manifolds play an important role in the organisation of the total flow induced by the singularly perturbed system. In this section we will construct various families of heteroclinic and homoclinic orbits which are (exponentially) close to these slow manifolds except for a number of fast 'jumps' through the fast field. These heteroclinic and homoclinic solutions are especially important as solutions of the full PDE (4.1) since they correspond to so called 'localised structures' such as fronts or pulses. These localised structures are again very important for understanding the dynamics of the solutions of the PDE. We refer to [28] and the references given there for an extensive discussion of the existence and stability of these solutions in the single Ginzburg-Landau equation.

Before we apply the theory of Fenichel we note that we can find another explicit slow manifold just by setting $A = 0$ in (1.5) or equivalently $x = y = 0$ in (4.1). This is also an invariant manifold on which the flow is slow, however, its existence can not be deduced from the general theory. By (4.4) we find that the eigenvalues of the critical point $(0, 0)$ are given by $\lambda_\pm = \pm i\sqrt{1 + c_1 z^2}$. Thus, the manifold $x = y = 0$ can never be globally normally hyperbolic (it can be normally hyperbolic for certain values of z if $c_1 < 0$). However, the slow manifold exists since we have an explicit expression: $\{x = y = 0\}$. The flow on this manifold is given by

$$\ddot{z} = \delta^2(-sz + z^3).$$

So we note that there are two slow heteroclinic orbits between the fixed points $(0, 0, \pm\sqrt{s}, 0)$ if $s > 0$. There is another trivial but non-slow invariant manifold which will play a role in the forthcoming analysis: $B = 0$, or $z = w = 0$ in (4.1). We already encountered this manifold in the previous section.

Setting $\delta = 0$ in the singularly perturbed system leads to two globally normally hyperbolic manifolds of critical points defined by $x^2 = 1 + c_1 z^2$ and $y = 0$, see (4.4). Note that the eigenvalues of the critical points $(\pm\sqrt{1 + c_1 z^2}, 0)$ in (4.4) are given by $\lambda_\pm = \pm\sqrt{2(1 + c_1 z^2)}$, thus, the invariant manifolds are globally normally hyperbolic, but only for $c_1 > 0$ they are unbounded. Thus, by the work of Fenichel[11] we know that for $\delta \neq 0$, sufficiently small, there are nearby($=O(\delta)$) invariant slow manifolds on which the flow is $O(\delta)$. We denote these slow manifolds by Γ_l, for $x < 0$, and by Γ_r, for $x > 0$. The highest order approximation of the flow on the slow manifolds is given by

$$\left\{ \begin{array}{rcl} \dot{z} & = & \delta w \\ \dot{w} & = & \delta(-(s + c_2)z + (1 - c_1 c_2)z^3), \end{array} \right. \tag{4.13}$$

since $x^2 = 1 + c_1 z^2 + O(\delta)$ (see also [1]). There are a few possibilities for the structure of the phase space on Γ_l and Γ_r. The system has the following critical points: $(0, 0)$ and $(\pm\sqrt{\frac{s+c_2}{1-c_1 c_2}}, 0)$ for $\frac{s+c_2}{1-c_1 c_2} > 0$. The critical point $(0, 0)$ is for $s + c_2 > 0$ a centre point and for $s + c_2 < 0$ a saddle point while for $s + c_2 > 0$ the $(\pm\sqrt{\frac{s+c_2}{1-c_1 c_2}}, 0)$ are saddle points and for $s + c_2 < 0$ the points are centre points. So one is tempted to conclude that the possible phase space are as in Figure 4. However (4.13) only gives an approximation of the flow up to $O(\delta)$.

On the other hand, we know that the system has a Hamiltonian structure, thus there are only a limited number of possibilities for the exact, integrable, behaviour of the flow on the slow manifold. By the symmetry $z \to -z, w \to -w$ we conclude that the topological structure of the flow on the slow manifold must be exactly as in Figure 4. We focus in this section on the case $s + c_2 > 0$, this choice is not at all essential, we will summarize the analogous results for $s + c_2 < 0$ at the end of this section in Remark 4.1. We denote the saddle point on Γ_l (respectively Γ_r) with $z > 0$ by P_l (respectively P_r) and with $z < 0$ by Q_l (respectively Q_r):

$$P_{l,r}, Q_{l,r} = (\pm\sqrt{\frac{c_1 s + 1}{1 - c_1 c_2}}, 0, \pm\sqrt{\frac{s + c_2}{1 - c_1 c_2}}, 0). \tag{4.14}$$

As a direct consequence of the structure of the flows on $\Gamma_{l,r}$(Figure 4), we conclude that there exist two distinct slow heteroclinic connections between P_l and Q_l and between P_r and Q_r (for $s + c_2 > 0$).

The slow manifolds Γ_l and Γ_r possess stable and unstable manifolds, $W^s(\Gamma_l)$ and $W^u(\Gamma_l)$ resp. $W^s(\Gamma_r)$ and $W^u(\Gamma_r)$ (see again [11] and[1]). These manifolds consist of points (x_0, y_0, z_0, w_0) such that the orbits $\gamma_0(t)$ through these points satisfy

$$\lim_{t \to \pm\infty} dist(\Gamma_{l,r} - \gamma_0(t)) = 0$$

where the + (resp. -) corresponds to the stable (resp. unstable) manifold. Note that $W^{u,s}(\Gamma_{l,r})$ merge with the manifold of stable and unstable manifolds of the (degenerate) critical points $(\pm\sqrt{1 + c_1 z^2}, 0, z, 0)$, $z \in \mathbb{R}$, of the unperturbed limit (4.4). In this paper we restrict our attention to those parts of $W^{u,s}(\Gamma_{l,r})$ which merge with the heteroclinic cycles which exist in the unperturbed limit (see Figure 5). In other words: we do not pay attention to those parts of $W^{u,s}(\Gamma_{l,r})$ which are unbounded in the limit $\delta \downarrow 0$. Thus $W^u(\Gamma_l) = W^s(\Gamma_r)$ and $W^u(\Gamma_r) = W^s(\Gamma_l)$ in the limit $\delta \downarrow 0$. These identities fail to hold as soon as $\delta \neq 0$. However, since all $W^{u,s}(\Gamma_{l,r})$ are three-dimensional (and the space is four dimensional) we expect to find two-dimensional intersections $W^u(\Gamma_l) \cap W^s(\Gamma_r)$ and $W^u(\Gamma_r) \cap W^s(\Gamma_l)$.

By using the Melnikov method for slowly varying systems the separation and thus the intersection of these stable and unstable manifolds can be calculated. See for example [27], [32]. The method is derived for the case that the unperturbed limit has homoclinic orbits, but the extension to the heteroclinic case is straightforward. Assuming that $\delta \neq 0$, sufficiently small, the distance between $W^u(\Gamma_l)$ and $W^s(\Gamma_r)$ is calculated at the cross section $\{x = 0, y > 0\}$. We define y^u_δ and y^s_δ as the intersection points of orbits on $W^u(\Gamma_l)$ resp. $W^s(\Gamma_r)$ with $\{z = z_0, w = w_0\}$ on $\{x = 0\}$. The solutions $\gamma^u_\delta(t) = (x^u_\delta(t), y^u_\delta(t), z^u_\delta(t), w^u_\delta(t))$ in $W^u(\Gamma_l)$ and $\gamma^s_\delta(t) = (x^s_\delta(t), y^s_\delta(t), z^s_\delta(t), w^s_\delta(t))$ in $W^s(\Gamma_r)$ of (4.1) are determined by the initial condition $\gamma^{u,s}_\delta(0) = (0, y^{u,s}_\delta, z_0, w_0)$; $\gamma_0(t) = (x_0(t), y_0(t), z_0, w_0)$ is the heteroclinic solution of the unperturbed system with $\gamma_0(0) = (0, \frac{1}{2}\sqrt{2}a, z_0, w_0)$ where $a = 1 + c_1 z_0^2$. There is an explicit expression for this solution:

$$(x_0(t), y_0(t)) = (\frac{2}{\sqrt{2}} b \tanh(bt), \sqrt{2} b^2 (1 - \tanh^2(bt))), \tag{4.15}$$

where $b = \frac{1}{2}\sqrt{2(1+c_1 z_0^2)}$. As usual in Melnikov theory, we define the following time-dependent distance function

$$\Delta(t, z_0, w_0) = \begin{pmatrix} \frac{\partial}{\partial\delta}(x_\delta^u(t) - x_\delta^s(t)) \\ \frac{\partial}{\partial\delta}(y_\delta^u(t) - y_\delta^s(t)) \end{pmatrix} \wedge \begin{pmatrix} y_0(t) \\ -x_0(t) + x_0^3(t) - c_1 x_0(t) z_0^2 \end{pmatrix},$$

where the wedge product represents the scalar cross product in the plane. The distance between $W^u(\Gamma_l)$ and $W^s(\Gamma_r)$ in the $\{z = z_0, w = w_0\}$-plane is given by $\Delta(0, z_0, w_0)$. Similar to the derivation of the Melnikov function for periodically driven systems, one shows, see [27], that for (4.1)

$$\Delta(0, z_0, w_0) \stackrel{\text{def}}{=} \Delta(z_0, w_0) = \int_{-\infty}^{\infty} \begin{pmatrix} 0 \\ -2c_1 x_0(t) z_0 \frac{\partial z}{\partial\delta}(t) \end{pmatrix} \wedge \begin{pmatrix} y_0(t) \\ -x_0(t) + x_0^3(t) - c_1 x_0(t) z_0^2 \end{pmatrix} dt,$$

where $\frac{\partial z}{\partial\delta}(t)$ is a solution of $\frac{d}{dt}\frac{\partial z}{\partial\delta} = w_0$ with $\frac{\partial z}{\partial\delta}(0) = 0$, thus $\frac{\partial z}{\partial\delta} = w_0 t$. The separation is then given by

$$\Delta(z_0, w_0) = -2c_1 w_0 z_0 \int_{-\infty}^{\infty} t x_0(t) y_0(t) dt, \tag{4.16}$$

where $(x_0(t), y_0(t))$ is the heteroclinic solution of the unperturbed system. Substituting this expression into (4.16) results in

$$\Delta(z_0, w_0) = -4c_1 w_0 z_0 b^3 \int_{-\infty}^{\infty} t \tanh(bt)(1 - \tanh^2(bt)) dt. \tag{4.17}$$

Since

$$\int_{-\infty}^{\infty} t \tanh(bt)(1 - \tanh^2(bt)) dt = \frac{1}{b^2},$$

we obtain (in leading order)

$$\Delta(w_0, z_0) = -2c_1 w_0 z_0 \sqrt{2(1 + c_1 z^2)}. \tag{4.18}$$

This yields that $W^u(\Gamma_l) \cap W^s(\Gamma_r) \cap \{x = 0, y > 0\}$ is either $O(\delta)$ close to the $\{w_0 = 0\}$-plane or to the $\{z_0 = 0\}$-plane. See Figure 6 for a three dimensional sketch of this four dimensional intersection. Using the symmetries (4.2) one derives a similar expression for $W^s(\Gamma_l) \cap W^u(\Gamma_r) \cap x = 0, y < 0$.

So far, we have found a number of trivial heteroclinic connections, such as those found in the invariant planes $\{x = y = 0\}$ and $\{z = w = 0\}$. Note that the former pair between the points $(-1, 0, 0, 0) \in \Gamma_l$ and $(1, 0, 0, 0) \in \Gamma_r$ corresponds to the zeros of (4.18) at $z_0 = w_0 = 0$ for $W^u(\Gamma_l) \cap W^s(\Gamma_r) \cap \{x = 0\}$ and $W^s(\Gamma_l) \cap W^u(\Gamma_r) \cap \{x = 0\}$. Two additional pairs of heteroclinic orbits are found in Γ_l and Γ_r. These orbits connect P_l to Q_l, respectively P_r to Q_r; we denote these orbits as $\gamma^{(0)}_{P_l Q_l}(t), \gamma^{(0)}_{Q_l P_l}(t) \in \Gamma_l$, where $\lim_{t\to\infty} \gamma^{(0)}_{P_l Q_l}(t)$ (resp. $\gamma^{(0)}_{Q_l P_l}(t)$) $= Q_l$ (resp. P_l), and analogously $\gamma^{(0)}_{P_r Q_r}(t), \gamma^{(0)}_{Q_r P_r}(t) \in \Gamma_r$. The flow on these orbits is, of course, everywhere $O(\delta)$.

Next, we will construct heteroclinic orbits between the points $P_{l,r}, Q_{l,r}$ which consist of distinct slow and fast parts. We will do that by analysing the intersections of the stable and unstable manifolds of these points with the $\{x = 0\}$-plane. These manifolds are subsets of

$W^{s,u}(\Gamma_l, \Gamma_r)$. The 'toffee'-like structure in Γ_l formed by the unstable manifolds of P_l and Q_l will be mapped by the flow on $W^u(P_l) \cup W^u(Q_l)$ to a topologically similar structure which is contained in $W^u(\Gamma_l) \cap \{x = 0\}$ (moreover, the z- and w-coordinates are only $O(\delta)$ modified by the fast field). Thus, by (4.18), there will be four intersection points with $W^s(\Gamma_r) \cap \{x = 0\}$ (two near $z_0 = 0$ and two near $w_0 = 0$). Note that all four intersections are transversal. These intersection points correspond to orbits which are biasymptotic to Γ_l and Γ_r. By construction, they satisfy $\lim_{t\to-\infty} = P_l$ or Q_l. These orbits are all on the same energy level set as P_l and Q_l. Thus, by the Hamiltonian structure of the flow, they can only be asymptotic to orbits on Γ_r with that same energy: the stable and unstable manifolds of P_r and Q_r. This indicates that the four orbits are heteroclinic connections between P_l, Q_l and P_r, Q_r. A similar argument yields four connections which travel from Γ_r to Γ_l. However, we have to use the symmetries (4.2) of the system to get a more precise result:

Theorem 1 *For any s, c_1 and c_2 which satisfy*

$$1 + c_1 s > 0, 1 - c_1 c_2 > 0 \textit{ and } s + c_2 > 0 \tag{4.19}$$

eight heteroclinic orbits of the following type exist (in (4.1)):

$$\begin{array}{ll} \gamma^{(1)}_{LR}(t) \textit{ with} & \lim_{t\to-\infty} \gamma^{(1)}_{LR}(t) = L, \lim_{t\to\infty} \gamma^{(1)}_{LR}(t) = R \\ & \qquad \textit{and } L = P_l, Q_l; R = P_r, Q_r. \\ \gamma^{(1)}_{RL}(t) \textit{ with} & \lim_{t\to-\infty} \gamma^{(1)}_{RL}(t) = R, \lim_{t\to\infty} \gamma^{(1)}_{RL}(t) = L \\ & \qquad \textit{and } L = P_l, Q_l; R = P_r, Q_r. \end{array}$$

All eight orbits consist of three parts: two slow parts near either Γ_l or Γ_r and one 'jump' through the fast field.

Note that condition (4.19) just ascertains the existence of the critical points $P_{l,r}$, $Q_{l,r}$ in the case $s + c_2 > 0$. Schematic sketches of all eight orbits are given in Figures 7, 8, 9.

Proof: Let $\gamma_p(t) = (x_p(t), y_p(t), z_p(t), w_p(t))$ be a solution of (4.1) on $W^u(P_l)$ which is exponentially close to Γ_l between the points P_l and $p = (p_x, p_y, p_z, p_w) \in W^u(P_l) \cap W^s(Q_l) \cap \Gamma_l$ (thus, γ_p leaves Γ_l $O(\delta)$ near p). Note that $W^u(P_l) \cap W^s(Q_l) \cap \Gamma_l$ is the above defined 'trivial' heteroclinic orbit $\gamma^{(0)}_{P_l Q_l}$. We denote by $\gamma_p(0)$ the (first) intersection of γ_p with the $\{x = 0\}$-plane: $\gamma_p(0) = (0, y_p(0), z_p(0), w_p(0))$. Using the symmetries (4.2) we define the orbits $\gamma^z_p(t) = (-x_p(-t), y_p(-t), -z_p(-t), w_p(-t))$ and $\gamma^w_p(t) = (-x_p(-t), y_p(-t), z_p(-t), -w_p(-t))$. Note that $\lim_{t\to\infty} \gamma^z_p(t) = Q_r$, $\lim_{t\to\infty} \gamma^w_p(t) = P_r$ and that $\gamma^z_p(0) = (0, y_p(0), -z_p(0), w_p(0))$, $\gamma^w_p(0) = (0, y_p(0), z_p(0), -w_p(0))$. Thus, the symmetries of (4.1) yield a heteroclinic solution between P_l and Q_r (respectively P_r) if we can choose p such that $z_p(0) = 0$ (resp. $w_p(0) = 0$), since then γ^z_p (resp. γ^w_p) coincides with γ_p at $t = 0$.

The fast field between p and $(0, y_p(0), z_p(0), w_p(0))$ only has an $O(\delta)$ influence on the slow (z, w)-coordinates of γ_p: $z_p(0)$ is $O(\delta)$ close to the z-coordinate p_z of $p \in \Gamma_l$. Since p_z can be varied between the z-coordinates of P_l and Q_l (4.14) we see that there must be a p^*_z such that $\gamma_{p^*} = \gamma^z_{p^*}$ is a heteroclinic solution between P_l and Q_r of the type described by the Theorem. By the symmetries $\{x \to -x, y \to -y\}$, $\{z \to -z, w \to -w\}$ and $\{t \to -t, y \to -y, w \to -w\}$ three distinct, symmetric counterparts of γ_{p^*} can be constructed. Thus, we have proved the existence of the solutions $\gamma^{(1)}_{P_lQ_r}$, $\gamma^{(1)}_{P_rQ_l}$, $\gamma^{(1)}_{Q_lP_r}$ and $\gamma^{(1)}_{Q_rP_l}$ described in the Theorem (see Figure 7).

One has to be more careful in constructing the other four orbits. First we have to construct a connection between P_l and P_r by determining a p such that $\gamma_p(t)$ and $\gamma_p^w(t)$ can be identified. Thus, we have to find a $p \in W^u(P_l) \cap W^s(Q_l) \cap \Gamma_l$ such that $w_p(0) = 0$. However, the w-coordinate of p, w_p, does not change sign. Since the w-coordinates of P_l and Q_l are 0 and $w_p(0)$ is $O(\delta)$ close to w_p, we can only expect to find a heteroclinic connection between P_l and P_r if we choose p $O(\delta)$ close to P_l or Q_l. It is possible to compute $w_p(0)$ up to $O(\delta^2)$ accuracy for these values of p by the Poincaré map P (4.8). We set $k = 0$ and $(z, w) =$ the coordinates of P_l or Q_l (4.14) in (4.1), where we have to change the interval of integration $(-T_\delta, T_\delta)$ into $(-\infty, 0)$. This way P measures the accumulated change in k, z, w on the one-dimensional (purely) strong unstable manifolds of P_l and Q_l between Γ_l and $\{x = 0\}$. Note that all three integrals converge and that ΔK and ΔK_1 are $0 + O(\delta^2)$; the computation of ΔK_2 yields:

$$w_p(0) = \pm\sqrt{2}\delta c_2 \sqrt{\frac{1 + c_1 s}{1 - c_1 c_2}} \sqrt{\frac{s + c_2}{1 + s c_1}}, \tag{4.20}$$

where the $+$ (resp. $-$) sign corresponds to $p = P_l$ (resp.Q_l). Note that c_2 is the only parameter which has influence on the sign of $w_p(0)$. However, by (4.20), we observe that for all $c_2 \neq 0$ $w_p(0)$ has to change sign (at least) once if p is varied from P_l to Q_l. Moreover, if $c_2 = 0$ we see that $(\dot{z}, \dot{w})$ decouple from the $(\dot{x}, \dot{y})$ in (4.1): there are exact (integrable) connections between the one-dimensional strong stable/unstable manifolds of P_l and P_r if $z \equiv$ the z- coordinate of P_l in (4.14). Thus there exists a $p \in W^u(P_l) \cap W^s(Q_l) \cap \Gamma_l$ such that $w_p(0) = 0$, and for this p: $\gamma_p = \gamma_p^w$. We conclude that there exists a heteroclinic orbit between P_l and P_r for all c_2 such that (4.19) holds. As in the previous case, we can construct from this orbit, $\gamma^{(1)}_{P_lP_r}(t)$ three symmetrical counterparts, $\gamma^{(1)}_{P_rP_l}(t)$, $\gamma^{(1)}_{Q_lQ_r}(t)$ and $\gamma^{(1)}_{Q_rQ_l}(t)$ by applying the symmetries $\{x \to -x, y \to -y\}$, $\{z \to -z, w \to -w\}$ and $\{t \to -t, y \to -y, z \to -z\}$. This concludes the proof of the Theorem (see Figures 8, 9). $\square$

At this point we can compare the behaviour of the 'localised structures' in the non-local and in the singularly perturbed system. We ignore, just for the moment, the fact that we derived the extra condition (1.7). Remember that the heteroclinic orbits found in section (3.1) do not satisfy the extra condition (1.7). First, we note that the solutions $\gamma^{(1)}_{P_lQ_r}$, $\gamma^{(1)}_{P_rQ_l}$, $\gamma^{(1)}_{Q_lP_r}$ and $\gamma^{(1)}_{Q_rP_l}$, which intersect $\{x = 0\}$ at $z = 0$, cannot have a counterpart in the non-local system, simply because the w-coordinates of these solutions are $O(1)$ during the fast 'jump' while all solutions of the non-local system must correspond to $w = \dot{z} = 0$. The other four solutions have a w-coordinate of $O(\delta)$ during the jump. Moreover, the jumps take place $O(\delta)$ near P_l and P_r (or Q_l and Q_r) and it is easy to check that the jumps are $O(\delta)$ close to the two pairs of non-local, integrable, heteroclinic orbits found in section (3.1). Thus, it is natural to conclude that $\gamma^{(1)}_{P_lP_r}(t)$, $\gamma^{(1)}_{P_rP_l}(t)$, $\gamma^{(1)}_{Q_lQ_r}(t)$ and $\gamma^{(1)}_{Q_rQ_l}(t)$ are the counterparts of the non-local heteroclinic orbits.

However, this conclusion can only be justified if the coupling coefficient, c_2, in the B-equation, is positive. This follows from (4.20): the jump of the connection $\gamma_p(t)$ takes place $O(\delta)$ near P_l if $c_2 > 0$. Thus all four heteroclinic orbits are as in Figure 8 : they are $O(\delta)$ close to the purely fast connections of the unperturbed or the non-local problem. The parts of the orbits near Γ_l and Γ_r are only of an $O(\delta)$ length. If $c_2 < 0$ the connection $\gamma^{(1)}_{P_l P_r}(t)$ makes its jump $O(\delta)$ near the unperturbed, fast connections between Q_l and Q_r. The solution follows $W^u(P_l) \cap W^s(Q_l) \cap \Gamma_l$ from P_l almost ($O(\delta)$) up to Q_l. The same happens in/near Γ_r (see Figure 9). Analogously, the unperturbed fast connections between P_l and P_r correspond to heteroclinic orbits from Q_l to Q_r of the perturbed system. Moreover, the w-coordinate of these solutions become $O(1)$ during their (long) stays near Γ_l and Γ_r. Note that this significant distinction between the cases $c_2 > 0$, $c_2 < 0$ also has its impact on the periodic orbits found in section 3.1: these solutions will have to merge with the heteroclinic cycles $\{\gamma^{(1)}_{P_l P_r}(t), \gamma^{(1)}_{P_r P_l}(t)\}$ and $\{\gamma^{(1)}_{Q_l Q_r}(t), \gamma^{(1)}_{Q_r Q_l}(t)\}$ as z_0 approaches $\pm\sqrt{(c_2+s)/(1-c_1c_2)}$, the z-coordinates of $P_{l,r}, Q_{l,r}$ (see (4.14)). For $c_2 < 0$ the periodic orbits with $z_0 > 0$ (resp. $z_0 < 0$) will 'grow' large, slow, parts (exponentially) close to Γ_l and Γ_r which follow the cycle $\{\gamma^{(1)}_{Q_l Q_r}(t), \gamma^{(1)}_{Q_r Q_l}(t)\}$ (resp. $\{\gamma^{(1)}_{P_l P_r}(t), \gamma^{(1)}_{P_r P_l}(t)\}$) as z_0 approaches $\sqrt{(c_2+s)/(1-c_1c_2)}$ (see Figure 9). In a sense one can interpret this as a four-dimensional 'canard-like' behaviour (see [8]), since this $O(1)$ change in the periodic orbits takes place for an exponentially small change in z_0. Thus, a significant part of the structure of the solutions disappears in the transition from the non-local system to the singularly perturbed system if $c_2 < 0$.

Below we will study the possible existence of 'multi-jump' heteroclinic orbits, that is, heteroclinic solutions connecting two of the critical points $P_{l,r}, Q_{l,r}$ by various jumps through the fast field alternated with slow parts near Γ_l, Γ_r. These solutions can have no counterpart in the non-local limit. We will find that they can only exist for $c_2 < 0$. First we focus on orbits which only make jumps from Γ_l to Γ_r, or vice versa, without following the periodic flow of the fast field for more than half a circuit. Later, we will construct orbits from Γ_l (or Γ_r) to itself which make one complete circuit through the fast field. Before we formulate the Theorem we give a construction of one of the most simple multi-=2-jump heteroclinic orbits in the case $c_2 < 0$ and show that this construction cannot work if $c_2 > 0$.

We consider the part of the two-dimensional unstable manifold $W^u(P_l)$ on which the orbits approach P_l (as $t \to -\infty$) 'from the right' tangential to the trivial heteroclinic orbit $\gamma^{(0)}_{P_l Q_l}$. Thus, as in the definition of $W^u(\Gamma_l)$, we only consider those parts of $W^{s,u}(P_{l,r}Q_{l,r})$ which merge with the family of heteroclinic connections in the limit $\delta \downarrow 0$. For simplicity we also denote this subset of $W^u(\Gamma_l)$ by $W^u(P_l)$. In the sequel we will use similar restrictions on the 'full' manifolds $W^{u,s}(P_{l,r}, Q_{l,r})$, also without adapting the notation. By the above Theorem we know that $W^u(P_l)$ intersects $W^s(P_r)$: this is the orbit $\gamma^{(1)}_{P_l P_r}(t)$ which has, if $c_2 < 0$ (resp. $c_2 > 0$), (z,w)-coordinates $O(\delta)$ close to those of Q_l (resp. P_l) during its jump through the fast field.

First we consider the case $c_2 < 0$. Let $\mathcal{L}_1 \subset W^u(P_l) \cap \{x = 0\}$ be a (one-dimensional) neighbourhood of $\gamma^{(1)}_{P_l P_r} \cap \{x = 0\}$; $\mathcal{L}_1$ intersects $W^s(P_r)$ transversally (by (4.18)). Define for $q \in \mathcal{L}_1$ the orbit through q by $\gamma_q(t) \in W^u(P_l)$. Thus when $q_0 = \mathcal{L}_1 \cap W^s(P_r)$, $\gamma_{q_0} = \gamma^{(1)}_{P_l P_r}$. The orbit γ_q will follow γ_{q_0} along Γ_r for an $O(1)$ distance, if q is exponentially close to q_0. Such an orbit γ_q will leave the neighbourhood of Γ_r exponentially close to $W^u(Q_r)$ since $\gamma_{q_0} \in W^s(P_r)$ and $W^u(Q_r) \cap \Gamma_r = W^s(P_r) \cap \Gamma_r$ (see Figure 9). We take $\mathcal{L}_1$ of exponentially short length; $\mathcal{L}_1$ is divided into two distinct parts by $W^s(\Gamma_r) \cap \{x = 0\}$ with $q_0 = \gamma^{(1)}_{P_l P_r} \cap \{x = 0\}$ as separatrix. Therefore, the two-dimensional manifold $\mathcal{F}(\mathcal{L}_1)$ of orbits γ_q through $\mathcal{L}_1$ is separated into two parts, an 'inner' and an 'outer' part, by the three-dimensional stable manifold $W^s(\Gamma_r)$ of Γ_r before it approaches Γ_r. Orbits $\gamma_q(t)$ on the outer part of $\mathcal{F}(\mathcal{L}_1)$ will again leave the neighbourhood of Γ_r in the direction opposite to Γ_l (their x-coordinates increase): they cannot return to either Γ_l or Γ_r and become unbounded. Orbits on the inner part of $\mathcal{F}(\mathcal{L}_1)$ will follow $W^u(\Gamma_r)$ – where we use the restricted definition (see above)– and return to the $\{x = 0\}$-hyperplane. The flow near Γ_r twists $\mathcal{F}(\mathcal{L}_1)$ such that the inner part leaves the neighbourhood of Γ_r as a 'sheet' exponentially close to $W^u(Q_r)$. We refer to [14] and especially [15] (since this paper applies to system (1.5) for a general treatment of the deformation of manifolds near slow manifolds (see also [1]). Thus, the intersection $\mathcal{F}(\mathcal{L}_1) \cap \{x = 0\}$, after the (first) passage through a neighbourhood of Γ_r, consists of a curve exponentially close to the curve $W^u(Q_r) \cap \{x = 0\} \subset W^u(\Gamma_r) \cap \{x = 0\}$. By Theorem 1 we know that $W^u(Q_r) \cap \{x = 0\}$ intersects the two-dimensional manifold $W^s(\Gamma_l) \cap \{x = 0\}$ twice transversally (the orbit $\gamma^{(1)}_{Q_r P_l}$ near $\{z = 0\}$ (Figure 7 and the orbit $\gamma^{(1)}_{Q_r Q_l}$ near $\{w = 0\}$ (Figure 9)). Now we note that the orbit $\gamma_{q_0} = \gamma^{(1)}_{P_l P_r}$ already 'touches down' on Γ_r (or better: approaches Γ_r exponentially close) $O(\delta)$ close to Q_r. Thus, γ_{q_0} is exponentially close to $\gamma^{(1)}_{Q_r P_l}$ and $\gamma^{(1)}_{Q_r Q_l}$ for certain parts of $O(1)$ length of these orbits (near Γ_r). As a consequence, we know that the curve $\mathcal{F}(\mathcal{L}_1) \cap \{x = 0\}$ must be extended along the entire length of $W^u(Q_r) \cap \{x = 0\}$ (except for an $O(\delta)$ part), and that it thus has to intersect $W^s(\Gamma_l) \cap \{x = 0\}$ twice, exponentially close to $\gamma^{(1)}_{Q_r P_l}$ and $\gamma^{(1)}_{Q_r Q_l}$. This yields that there are two orbits on $\mathcal{F}(\mathcal{L}_1)$ which are asymptotic to Γ_l. We again apply the argument that system (4.1) has a Hamiltonian structure, so that the 'energy' H is conserved on orbits: $\mathcal{F}(\mathcal{L}_1)$ can only intersect $W^s(\Gamma_l)$ along $W^s(P_l)$ or $W^s(Q_l)$.

So we conclude that there exist two 2-jump orbits, which consist of a five parts: a slow part near Γ_l, a fast jump, a slow part near Γ_r, a second jump and a third slow part near Γ_l: the heteroclinic orbit $\gamma^{(2,1)}_{P_l Q_l}$ and the homoclinic orbit $\gamma^{(2,1)}_{P_l P_l}$. By the symmetries (4.2) we can create a family of 8 distinct 2-jump homoclinic orbits and 4 2-jump heteroclinic orbits. Note that there thus exist, for instance, two different homoclinic 2-jump orbits to P_l: $\gamma^{(2,1)}_{P_l P_l}$ and $\gamma^{(2,2)}_{P_l P_l}$, related to each other by the symmetry $\{y \to -y, t \to -t, w \to -w\}$ (see below).

Before we extend the above argument to 3,4,...-jump heteroclinic and homoclinic orbits we consider the case $c_2 > 0$. The above construction is impossible in this case. The construction is based on the orbit $\gamma^{(1)}_{P_l P_r}$. This orbit exists also for $c_2 > 0$, but now, as we already noted above, this orbit only has parts of $O(\delta)$ length near the slow manifolds (Figure 8). Thus, the intersection of $\mathcal{F}(\mathcal{L}_1)$ with $\{x = 0\}$ (after passing Γ_r) is also only of $O(\delta)$ length and cannot intersect $W^s(\Gamma_l) \cap \{x = 0\}$: $\mathcal{F}(\mathcal{L}_1) \cap W^s(\Gamma_l) = \emptyset$. Of course one could try to construct 2-jump orbits based on one of the 1-jump orbits which jump near $\{z = 0\}$ after following a trivial heteroclinic orbit on Γ_l for half its length (Figure 7). Let's for instance consider $\gamma^{(1)}_{P_l Q_r}$ (this is no restriction, due to the symmetries (4.2)). It is only possible to construct a 2-jump orbit if $\gamma^{(1)}_{P_l Q_r}$ has parts exponentially close to one of the two 1-jump connections which depart from P_r (since $W^u(P_r) \cap \Gamma_r = W^s(Q_r) \cap \Gamma_r$). It is clear that $\gamma^{(1)}_{P_r Q_l}$ is the only possible candidate (see Figure 7). More precisely: a 2-jump combining $\gamma^{(1)}_{P_l Q_r}$ and $\gamma^{(1)}_{P_r Q_l}$ is possible if the 'touch down' point of $\gamma^{(1)}_{P_l Q_r}$ on Γ_r has a z-coordinate which is larger than the z-coordinate of the 'take off' point of $\gamma^{(1)}_{P_rQ_l}$. Note that these two orbits are related to each other by the symmetry $\{x \to -x, y \to -y\}$. Since the w-coordinate of both orbits is (strictly) negative during the jump through the fast field we find by (4.1) that the z-coordinate of both orbits decreases monotonically. Thus, the touch down point is 'below' the take off point: there cannot be a 2-jump orbit if $c_2 > 0$. However, in Theorem 3 we shall show, using a different argument, that there exists a solution connecting P_l and Q_l with two slow parts near Γ_l and no slow parts near Γ_r: it makes a complete circuit through the fast field and does not touch down on Γ_r.

A priori one would assume that the 1-jump orbits which jump near the $\{z = 0\}$-plane can be used to construct other, new, 2-jump orbits in the case $c_2 < 0$. By the above arguments it is possible to construct a 2-jump orbit which is exponentially close to $\gamma^{(1)}_{P_lQ_r}$ until it takes off from Γ_r to follow the 1-jump orbit $\gamma^{(1)}_{P_rP_l}$. However, the thus constructed 2-jump homoclinic solution to P_l is the symmetric counterpart $\gamma^{(2,2)}_{P_lP_l}$ of the above constructed orbit $\gamma^{(2,1)}_{P_lP_l}$ under the symmetry $\{y \to -y, t \to -t, w \to -w\}$.

We can now formulate the Theorem on N-jump orbits:

Theorem 2 *Assume that $c_2 < 0$ and s and c_1 are such that (4.19) holds. Then, for any $N \geq 2$, there are $\mathcal{N}(N)$ distinct N-jump heteroclinic or homoclinic orbits $\gamma^{(N,k)}_{S,T}(t)$ between the critical points $S, T \in \{P_l, P_r, Q_l, Q_r\}$; k denotes the fact that there can be more than one orbit between S and T. The number $\mathcal{N}(N)$ satisfies the recurrence relation*

$$\mathcal{N}(N) = \mathcal{N}(N-1) + \mathcal{N}(N-2) \text{ with } \mathcal{N}(1) = 8, \mathcal{N}(2) = 12. \tag{4.21}$$

These orbits consist of $N+1$ slow passages near $\Gamma_{l,r}$ alternated by N jumps through the fast field and are all exponentially close to the 'skeleton' spanned by the 1-jump solutions constructed in Theorem 1.

Proof: The 3-jump orbits are based on the 2-jump orbits, just as the 2-jump orbits are based on the 1-jump orbits. We consider, for instance, the heteroclinic orbit $\gamma^{(2,1)}_{P_lQ_l}(t)$. Let $\mathcal{L}_2$ be an exponentially small, one-dimensional, neighbourhood of $\gamma^{(2,1)}_{P_lQ_l}(t) \cap \{x = 0\}$ in $\mathcal{F}(\mathcal{L}_1) \cap \{x = 0\}$, after the first passage of $\mathcal{F}(\mathcal{L}_1)$ of Γ_r (where $\mathcal{F}(\mathcal{L}_1)$ is defined above); $\mathcal{L}_2$ will play a role similar to $\mathcal{L}_1$ in the above construction of the 2-jump orbits. We define $\mathcal{F}(\mathcal{L}_2) \subset \mathcal{F}(\mathcal{L}_1)$ as the manifold of orbits through $\mathcal{L}_2$; $\mathcal{F}(\mathcal{L}_2)$ is separated into two parts by $W^s(\Gamma_l)$ with $\gamma^{(2,1)}_{P_lQ_l}(t)$ as separatrix. Thus, $\mathcal{F}(\mathcal{L}_2)$ is splitted and twisted by the slow flow near Γ_l: it becomes a sheet exponentially close to $W^u(P_l)$ when it again leaves the neighbourhood of Γ_l. The intersection $\mathcal{F}(\mathcal{L}_2) \cap \{x = 0\}$ after the passage of Γ_l consists of a curve exponentially close to the first intersection of $W^u(P_l)$ with $\{x = 0\}$ and of the same length as $W^u(P_l) \cap \{x = 0\}$ (up to $O(\delta)$-terms). Therefore, $\mathcal{F}(\mathcal{L}_2) \cap \{x = 0\}$ intersects $W^s(\Gamma_r) \cap \{x = 0\}$ two times transversally: an intersection exponentially close to $\gamma^{(1)}_{P_lQ_r}(t) \cap \{x = 0\}$ and another exponentially close to $\gamma^{(1)}_{P_lP_r}(t) \cap \{x = 0\}$. By the Hamiltonian character of the flow we know that these intersections must correspond to the 3-jump orbits $\gamma^{(3,1)}_{P_lQ_r}(t)$ and $\gamma^{(3,1)}_{P_lP_r}(t)$.

It is clear that this construction can be repeated for all N: based on $\gamma^{(3,1)}_{P_lP_r}(t)$ we define $\mathcal{L}_3 \subset \mathcal{F}(\mathcal{L}_2) \cap \{x = 0\}$, exponentially close to the third intersection of $\gamma^{(3,1)}_{P_lP_r}(t)$ with $\{x = 0\}$.

The manifold $\mathcal{F}(\mathcal{L}_3)$ gets twisted and separated near Γ_r so that it intersects $W^s(\Gamma_l) \cap \{x = 0\}$, after its passage of Γ_r, two times: the 4-jump orbits $\gamma^{(4,1)}_{P_l P_l}(t)$ and $\gamma^{(4,1)}_{P_l Q_l}(t)$, $i = 1, 2$.

Note that the number $\mathcal{N}(N)$ of N-jump orbits increases quite rapidly with N. Let's construct $\mathcal{N}(2)$ from $\mathcal{N}(1) = 8$ (Theorem 1). If $\gamma^{(1)}$ jumps through the fast field near $\{w = 0\}$ then we have shown above that one can construct two 2-jump orbits based on this one: one which makes its second jump near $\{z = 0\}$ and one which makes its second jump again near $\{w = 0\}$. If $\gamma^{(1)}$ jumps near $\{z = 0\}$, then there only exists one 2-jump orbit based on this $\gamma^{(1)}$, which makes its second jump near $\{w = 0\}$. Thus, the four 1-jump orbits near $\{w = 0\}$ lead to 8 2-jump orbits, 4 of them make their last jump near $\{w = 0\}$, the other 4 make their last jump near $\{z = 0\}$. The 4 1-jump orbits near $\{z = 0\}$ lead to 4 2-jump orbits with a second jump near $\{w = 0\}$. Thus, as we already found by the symmetries (4.2), $\mathcal{N}(2) = 12$, since all these orbits are distinct, by construction.

This method of counting can be used for every transition from N to $N + 1$. Define $\mathcal{W}(N)$, respectively $\mathcal{Z}(N)$, the number of N-jump orbits of $\mathcal{W}$, resp. $\mathcal{Z}$, type (by definition) which make their final jump through the fast field near $\{w = 0\}$, resp. $\{z = 0\}$. By the above construction, every $\mathcal{W}$-orbit yields one $\mathcal{W}$-type orbit and one $\mathcal{Z}$-type orbit; a $\mathcal{Z}$-orbit yields a $\mathcal{W}$-orbit; thus,

$$\left\{ \begin{array}{rcl} \mathcal{W}(N+1) & = & \mathcal{W}(N) + \mathcal{Z}(N) \\ \mathcal{Z}(N+1) & = & \mathcal{W}(N). \end{array} \right.$$

Since $\mathcal{N}(N) = \mathcal{W}(N) + \mathcal{Z}(N)$ we recover (4.21). Note that $\mathcal{N}(N) = 4p_{N+2}$, where p_N is the N-th Fibonacci number. Thus $\mathcal{N}(N) \approx \frac{1}{2}(1 + \sqrt{5})\mathcal{N}(N-1)$ for large N. □

The set of intersections of all N-jump orbits with $\{x = 0\}$ is a Cantor-like set of exponentionally small dimension. There are of course more points in this set than the N-jump heteroclinic/homoclinic orbits. One can, for instance, construct many types of different periodic orbits between Γ_l and Γ_r, which consist of alternating slow and fast parts. Thus, these periodic orbits differ significantly from those found in section 4.1. None of them can have a counterpart which can be described by the non-local system. Note that these periodic orbits correspond to limit points of the above described Cantor-like set in the $\{x = 0\}$-hyperplane. One of the simplest periodic orbit of this type consists of four parts: a slow part exponentially close to Γ_l and to $\gamma^{(1)}_{P_l P_r} \cap \gamma^{(1)}_{Q_r Q_l}$, a fast part near the fast jump of $\gamma^{(1)}_{P_l P_r}$, a slow part near Γ_r and $\gamma^{(1)}_{P_l P_r} \cap \gamma^{(1)}_{Q_r Q_l}$ and the second fast jump near $\gamma^{(1)}_{Q_r Q_l}$ (see Figure 9).

So far we studied orbits which only make jumps directly from Γ_l to Γ_r, or vice versa. Now we want to construct orbits which make one complete circuit through the fast field. Here we focus, for simplicity, on constructing a heteroclinic orbit from P_l to Q_l which has two slow parts both near Γ_l alternated by one 'double' jump which makes a complete circuit through the fast field. This orbit only can be constructed for $c_1 < 0$, where the sign of c_2 is arbitrary. For $c_1 > 0$ the above orbit generally does not exist. Using the symmetries (4.2) in system (4.1), one can obtain from this orbit other heteroclinic orbits which have two slow parts near Γ_l or Γ_r connected by a fast 'double' jump. The idea of the proof of the following Theorem is based on the methods developed in [6].

Theorem 3 *Assume that $c_1 < 0$ and s and c_2 are such that (4.19) holds. Then, there exist four heteroclinic orbits $\gamma^{(d)}_{P_lQ_l}(t), \gamma^{(d)}_{Q_lP_l}(t), \gamma^{(d)}_{P_rQ_r}(t)$ and $\gamma^{(d)}_{Q_rP_r}(t)$. These orbits consist of two slow parts which are both near Γ_l (or Γ_r), alternated by a complete circuit (or a 'double' jump) through the fast field.*

Proof: We only construct $\gamma^{(d)}_{P_lQ_l}(t)$, the other three orbits can be found by applying the symmetries (4.2). Let $\mathcal{L}_4 \subset W^u(P_l) \cap \{x = 0\}$ be a (one-dimensional) neighbourhood of $\gamma^{(1)}_{P_lQ_r} \cap \{x = 0\}$, $\mathcal{L}_4$ intersects $W^s(Q_r)$ transversally. Define for $p \in \mathcal{L}_4$ the orbit through p by $\gamma_p(t) \in W^u(P_l)$. Thus for $p_0 = \mathcal{L}_4 \cap W^s(Q_r)$, $\gamma_{p_0}(t) = \gamma^{(1)}_{P_lQ_r}(t)$. If we take $p_1 \in \mathcal{L}_4$ exponentially close to p_0, the orbit will follow γ_{p_0} along Γ_r for an $O(1)$ distance. Such an orbit γ_{p_1} will leave the neighbourhood of Γ_r exponentially close to $W^u(Q_r)$ and will still be exponentially close to $W^u(Q_r)$ at its next intersection with the hyperplane $\{x = 0\}$; we denote this intersection point by q_1. At this intersection, it will be 'outside' $W^s(\Gamma_l)$. Here, an orbit is said to be outside $W^s(\Gamma_l)$ when, after the passage near Γ_l, it leaves the neighbourhood of Γ_l in the direction opposite to Γ_r (its x-coordinate decreases): it cannot return to Γ_l or Γ_r and becomes unbounded. On the other hand an orbit is inside $W^s(\Gamma_l)$ when it does return to the $\{x = 0\}$-hyperplane. In other words: an 'inside' orbit leaves the neighbourhood of Γ_l near the structure of heteroclinic connections between Γ_l and Γ_r which exist in the limit $\delta \downarrow 0$. The fact that an orbit is 'outside' or 'inside' is determined by (4.18) and thus by the sign of c_1. Now we take $p_2 \in \mathcal{L}_4$ at an $O(1)$ distance from p_0, where the z-coordinate of p_2, p_2^z is larger than the z-coordinate of p_0. Here we also make sure that p_2^z is not at $O(\delta)$ distance from the z-coordinate of P_l. This assures that the next intersection of γ_{p_2} with $\{x = 0\}$ is inside $W^s(\Gamma_l)$, this intersection point is denoted by q_2. Note that $\gamma_{p_2}(t)$ only approaches Γ_r $O(\sqrt{\delta})$-close. We denote the two-dimensional manifold of orbits γ_p through $\mathcal{L}_4$ by $\mathcal{F}(\mathcal{L}_4)$. From the above it follows that the next intersection of $\mathcal{F}(\mathcal{L}_4)$ with the hyperplane $\{x = 0\}$ contains a curve connecting q_1 and q_2. Since q_1 is outside $W^s(\Gamma_l)$ and q_2 is inside $W^s(\Gamma_l)$, there exists a $p^* \in \mathcal{L}_4$ so that the orbit through p^* intersects $W^s(\Gamma_l)$. Due to the Hamiltonian structure of the flow we know that p^* must be on $W^s(Q_l)$. Thus we constructed a heteroclinic cycle $\gamma_{p^*} = \gamma^{(d)}_{P_lQ_l}$ with two slow parts near Γ_l and one fast complete circuit. From the fact that the point $\gamma^{(1)}_{P_lP_r} \cap \{x = 0\}$ is not in the interval $[p_1, p_2] \subset \mathcal{L}_4$ we see that, for $c_2 < 0$, the above orbit is not the one we constructed in Theorem 2, $\gamma^{(2)}_{P_lQ_l}$: $\gamma^{(d)}_{P_lQ_l}$ does not come closer to Γ_r than $O(\sqrt{\delta})$. □

We can show by analogous analysis that such an orbit as constructed above is generally not found for $c_1 < 0$. Again let $p_0 = \gamma_{P_lQ_r} \cap \{x = 0\}$. Then the orbit through a point on $W^u(P_l) \cap \{x = 0\}$ which is exponentially close to p_0 is at its next intersection with the $\{x = 0\}$-hyperplane exponentially close to $W^u(Q_r)$ and is inside $W^s(\Gamma_l)$. However, the orbit through a point which is at $O(1)$ distance from p_0 (and with a z-coordinate which now has to be chosen smaller than p_0^z) is at its next intersection with $\{x = 0\}$ also inside $W^s(\Gamma_l)$. Thus, the line between these points does, in general, not intersect $W^s(\Gamma_l)$ (compare to the 'inside' and 'outside' cases defined in [6]). This implies that such an orbit as constructed in the above Theorem does generally not exist for $c_1 > 0$.

Remark 4.1 In all the above theorems we assumed that $s + c_2 > 0$. However, for $s + c_2 < 0$ similar statements hold. Recall that in this case the integrable flow on $\Gamma_{l,r}$ has a 'figure 8' structure: there are two homoclinic orbits to the points $(\pm 1, 0, 0, 0)$ on $\Gamma_{l,r}$ (Figure 4) .

The points $P_{l,r}, Q_{l,r}$ have become center points (on $\Gamma_{l,r}$). Analogous to Theorem 1 one can prove that there exist eight heteroclinic orbits between the critical points $(\pm 1, 0, 0, 0) \in \Gamma_{l,r}$ which consist of two slow parts near Γ_l and Γ_r, alternated by one fast jump. There also exist N-jump homoclinic orbits, independent of the signs of the coefficients. However, the number of N-jumps is not the same as before. There are 2 2-jump orbits and, for every $N \geq 3$, there exists only one N-jump orbit. Thus choosing $s + c_2 < 0$ reduces the number of heteroclinic and homoclinic orbits drastically, although the general behaviour remains the same.

5 Discussion

In this paper we derived and studied two different types of modulation equations which describe the same physical phenomena. Pattern formation in a reflection symmetric system which is subject to two interacting destabilizing mechanisms is described by two non-linearly coupled Ginzburg-Landau equations (1.4). If the natural spatial scales associated to those mechanisms differ significantly (see Figure 1) one can either describe the behaviour near criticality by a singularly perturbed modulation equation (1.5), or one can apply a so-called Landau-reduction and derive a non-local modulation equation (1.6), see also Metzener and Proctor [22] for the application of this idea. As a necessary consequence of the derivation process we showed that there is an extra, again non-local, solvability condition in the non-local case (1.7).

Our main goal has been to compare the set of solutions described by the singularly perturbed equation to that of the non-local system. We restricted ourselves to the stationary solutions. Note that it is natural to expect stationary patterns in systems with a reflection symmetry, such as convection experiments. We therefore, for instance, refer to the theoretical and analytical study of double-layer convection by Rasenat et al. [25]: under certain conditions these experiments can be described by the equations studied in this paper (see for instance the neutral curve in Figure 6 in [25]); the experiments performed for this paper exhibit stationary patterns (although the patterns can certainly be non-stationary). Of course it could be expected that the singularly perturbed equation has a richer set of solutions than the non-local reduction. However, in section 3 we have shown that the non-local system, combined with extra condition (1.7), cannot describe any other patterns than purely (spatially) periodic, or quasi-periodic with only two independent frequencies. These patterns are also described by the singularly perturbed system, but, this system also governs a very complicated set of 'localised' patterns, corresponding to heteroclinic and homoclinic solutions. These types of patterns are important in the dynamics of the uncoupled Ginzburg-Landau equation (see for instance [28]). The large families of 'multi-jump' and 'complete circuit' orbits found in section 4 only make up a small part of the entire set of possible solutions which have an alternating slow-fast structure. The heteroclinic and homoclinic orbits found in section 4 correspond to only a very small subset of the Cantor-like sets formed by the intersections $W^{u,s}(P_{l,r}, Q_{l,r}) \cap \{x = 0\}$, which we only briefly discussed in that section. Moreover, we did not pay any attention to connections between the slow manifolds Γ_l and Γ_r which are not on the 'energy'-level of the critical points $P_{l,r}, Q_{l,r}$. By the methods developed in this paper it is also possible to show the existence of orbits which connect, through the fast field, corresponding periodic orbits on Γ_l and Γ_r. Furthermore, the essence of the analysis also works for other values of the Ω_i-integrals (see section 3) than $\Omega_1 = \Omega_2 = 0$.

This is another aspect of the paper: we have shown that the singularly perturbed Hamiltonian system (1.8) has a very intricately structured phase-space. Moreover, we have been able to unravel much of the structure of this phase-space using in essence topological, or geometrical, methods. These methods are based on the ideas for instance described in [11], [1] and [6].

Thus, the geometrical methods have enabled us to show that the reduction of the singularly perturbed system to the non-local system destroys a very large set of 'localised' patterns.

Finally we make just one short remark about the stability of patterns as described by the modulation equations (1.5) and (1.6). We did not pay any attention to that aspect in this paper. There is much literature on this. We refer to Matkowsky and Volpert [20] where the stability of purely periodic patterns to systems like (1.4), and thus (1.5), has been studied. The same ideas can be used to study corresponding solutions to (1.6). We have not done this in this paper because the analysis is rather straightforward, while the results depend in a complicated manner on the values of the coefficients in the equations. The stability of the quasi-periodic and 'localised' patterns is a much more complicated issue. Only recently the instability of stationary quasi-periodic patterns to the uncoupled real Ginzburg-Landau equation has been proved in [5]. Note that the quasi-periodic solutions found in this paper correspond directly to the quasi-periodic solutions studied in [5]. There are many stability/instability results on 'localised' patterns in an uncoupled Ginzburg-Landau equation. These results only exist for patterns which are much less complicated than most of the ones constructed in this paper. Here, we only refer to [28] and the recent paper [16], in which the approach is also geometrical, and the references given there.

Acknowledgements The authors thank Peter Bollerman for his useful remarks on the derivation of the modulation equations in section 2.

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Figure captions

Figure 1: A critical curve with two local minima, one at (k_1, R_1) and the other at (k_2, R_2). Here, R is fixed at a value above R_1 and R_2: there are two intervals of 'unstable waves'. The small parameter δ corresponds to the ratio of the widths of these intervals.

Figure 2: The phase portrait for the equation $\ddot{x} = -ax + x^3$ where $a > 0$.

Figure 3: The construction of the Poincaré map. The thin lines represent solutions to the unperturbed problem (4.6) in the plane $\{z = z_0, w = w_0\}$. The thick line is the projection of a solution of (4.1) on the plane $\{z = z_0, w = w_0\}$.

Figure 4: The flow on the slow manifolds for $s + c_2 > 0$ (a) and for $s + c_2 < 0$ (b).

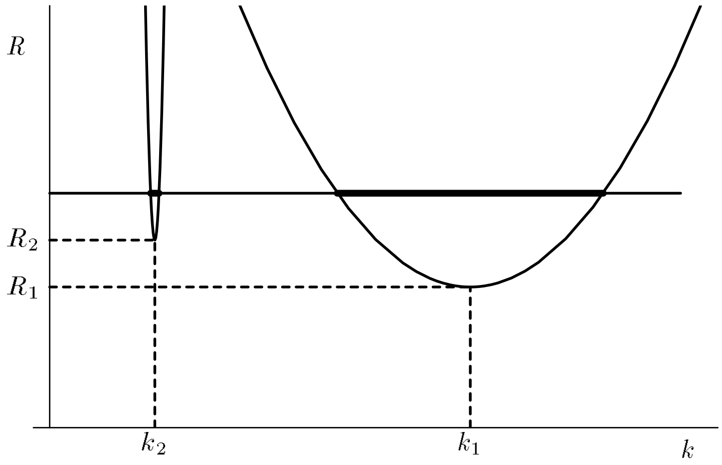
Figure 5: A three dimensional sketch of the four dimensional phase space of the unperturbed system. The two-dimensional slow manifolds Γ_l and Γ_r are represented by curves. The positions of the critical points of the perturbed system are also indicated.

Figure 6: The intersection of the two-dimensional manifolds $W^u(\Gamma_l) \cap \{x = 0, y > 0\}$ and $W^s(\Gamma_r) \cap \{x = 0, y > 0\}$ for $c_1 < 0$ and $c_2 < 0$. Parts of the one-dimensional curves $W^u(P_l, Q_l) \cap \{x = 0, y > 0\} \subset W^u(\Gamma_l) \cap \{x = 0, y > 0\}$ and $W^s(P_r, Q_r) \cap \{x = 0, y > 0\} \subset W^s(\Gamma_r) \cap \{x = 0, y > 0\}$ are also shown.

Figure 7: A schematic sketch of the four 1-jump orbits $\gamma^{(1)}_{P_lQ_r}$, $\gamma^{(1)}_{P_rQ_l}$, $\gamma^{(1)}_{Q_lP_r}$ and $\gamma^{(1)}_{Q_rP_l}$. The slow parts are exponentially close to the heteroclinic cycles on $\Gamma_{l,r}$ (the thin lines). The fast parts 'jump' through the full four-dimensional phase-space, $O(\delta)$ close to the $\{z = 0\}$-hyperplane.

Figure 8: A schematic sketch of the four 1-jump orbits $\gamma^{(1)}_{P_lP_r}(t), \gamma^{(1)}_{P_rP_l}(t), \gamma^{(1)}_{Q_lQ_r}(t)$ and $\gamma^{(1)}_{Q_rQ_l}(t)$ for $c_2 > 0$ which 'jump' $O(\delta)$ close to the $\{w = 0\}$-hyperplane.

Figure 9: A schematic sketch of the four orbits $\gamma^{(1)}_{P_lP_r}(t), \gamma^{(1)}_{P_rP_l}(t), \gamma^{(1)}_{Q_lQ_r}(t)$ and $\gamma^{(1)}_{Q_rQ_l}(t)$ for $c_2 < 0$ which 'jump' $O(\delta)$ close to the $\{w = 0\}$-hyperplane.



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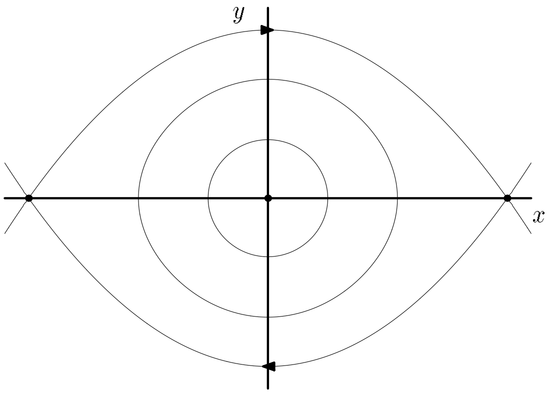

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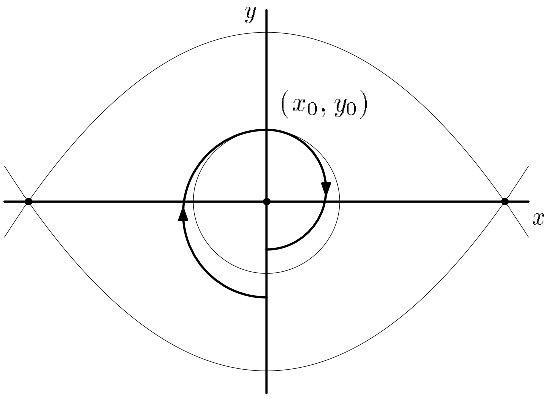

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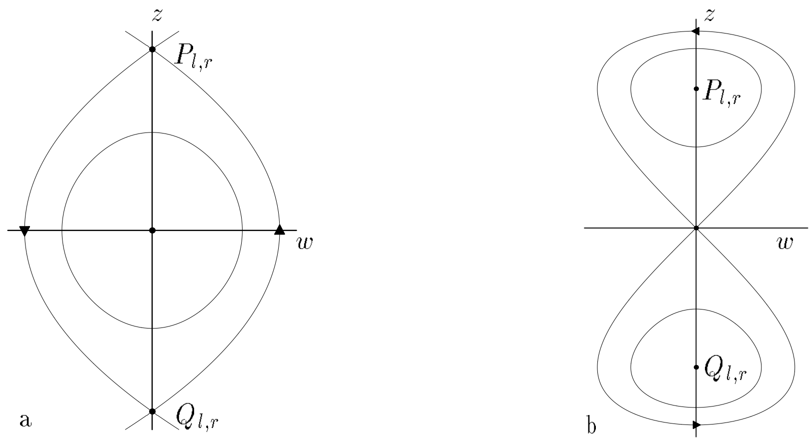

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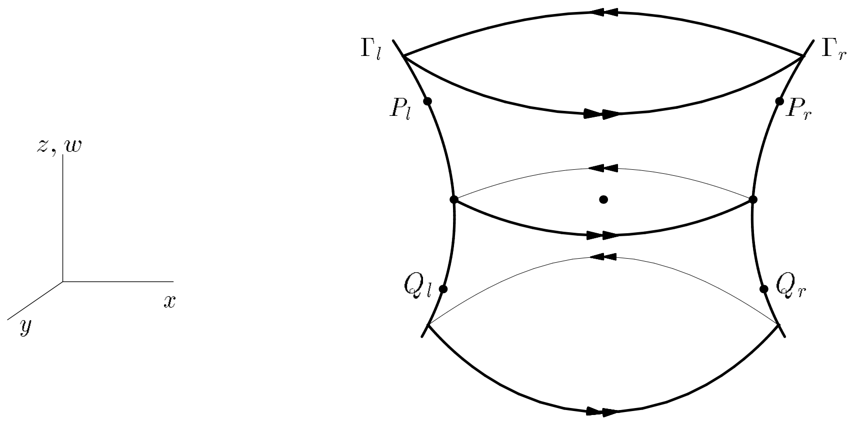

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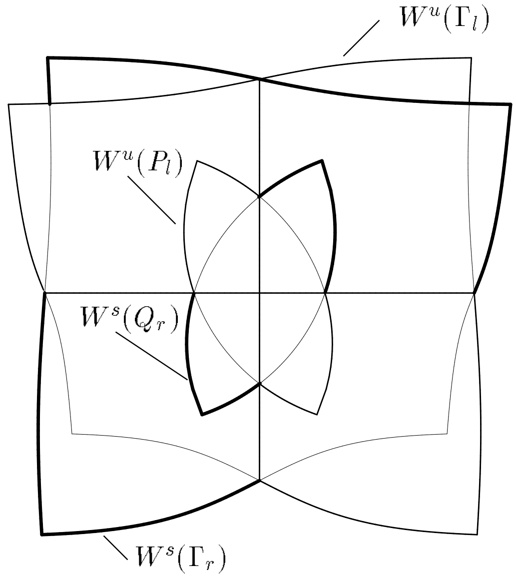

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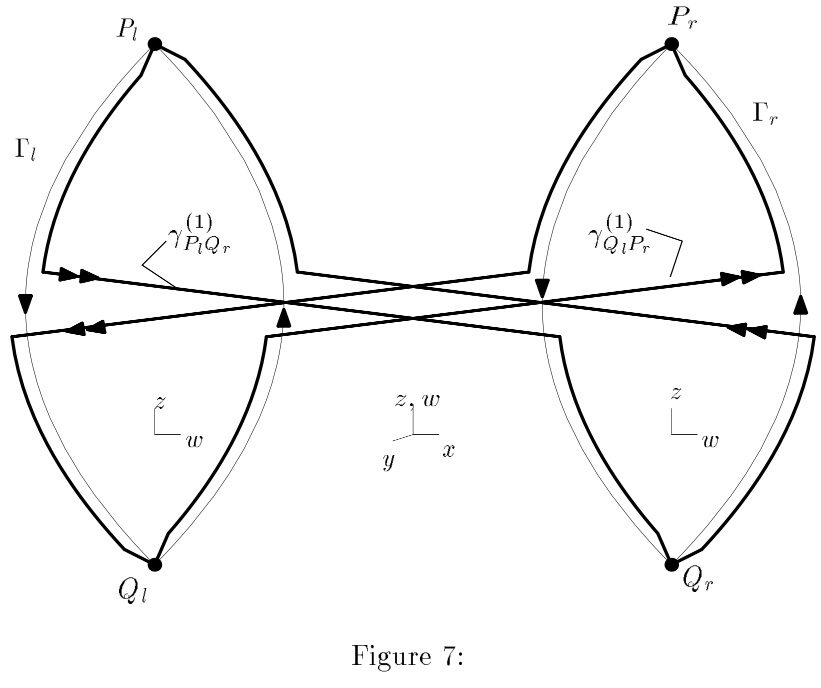

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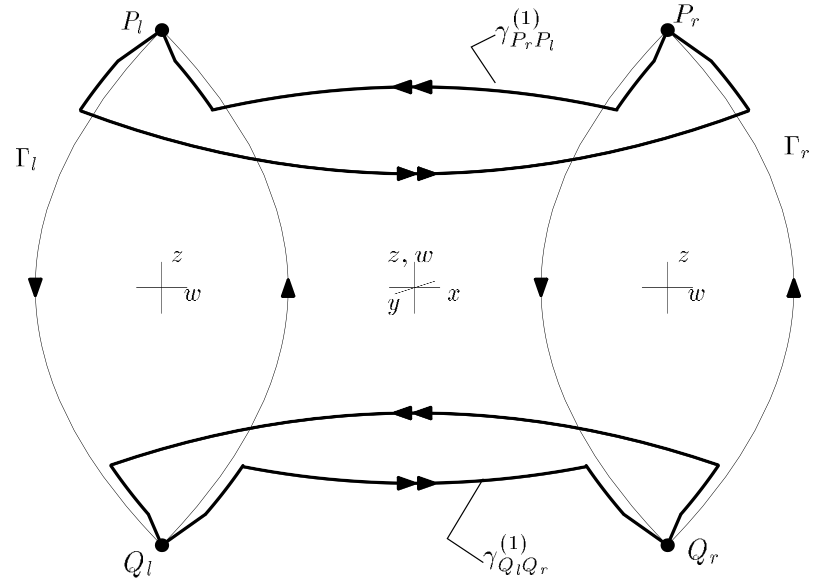

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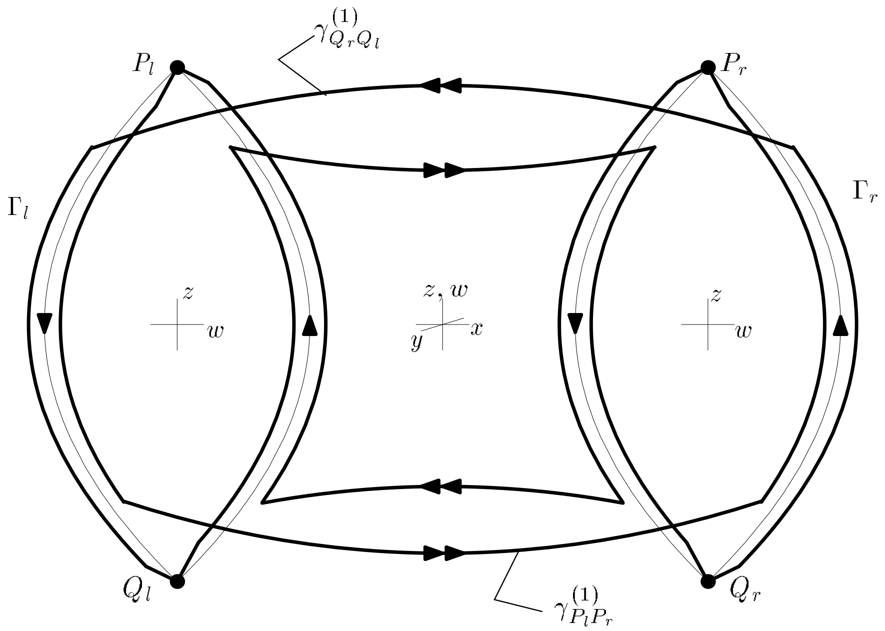

Figure 9: