Quotients of tame fundamental groups of affine curves

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1 Introduction

This paper contains some results on the tame fundamental group of affine curves in positive characteristic. Let D be a curve over an algebraically closed field k of characteristic $p > 0$. Choose closed points $\lambda_1, \ldots, \lambda_r$ and put $U = D - \{\lambda_1, \ldots, \lambda_r\}$. We will suppose that $r \geq 1$. About the structure of $\pi_1^t(U)$ not much is known. The problem studied in this paper is to calculate some of the finite quotients of $\pi_1^t(U)$. In particular we look at the quotients which are extensions of $(\mathbb{Z}/p)^n$ by cyclic groups of order prime to p. We will give a necessary condition for these quotients to exist. In some cases we show that this condition is actually sufficient. In studying these quotients one is led to studying the p-rank of tame covers of D.

For the full fundamental group of an affine curve the finite quotients are known: a finite group G is a quotient of $\pi_1(U)$ if and only if $G/p(G)$ can be generated by $2g + r - 1$ elements. Here $p(G)$ is the subgroup of G generated by its p-Sylow groups. (See [4] and [11]). However for the finite quotients of $\pi_1^t(U)$ there is not even a conjecture. The two cases are essentially different. Whereas the set of finite quotients of $\pi_1(U)$ depends only on the number $2g + r$ this is not the case for $\pi_1^t(U)$ (cf. Section 6). Since $\pi_1^t(U)$ is finitely generated it is determined by its finite quotients (see [1, Prop. 15.4]). This is not true for $\pi_1(U)$ (see [5, Prop. 1.6]). For the fundamental group of projective curves see [16].

We now turn to a more technical discussion of the results. In this paper we will study the number of quotients of $\pi_1^t(U)$ of the form $(\mathbb{Z}/p)^n \rtimes H$, where H is an abelian group of prime-to-p order. We will expres the existence of these quotients in terms of certain invariants of D and the branchpoints λ_i (see Section 7). These invariants, called generalized Hasse–Witt invariants, are defined as follows (see Section 3). Consider an ℓ-cyclic cover $C \to D$ with ℓ prime to p. Write $H^1(C, \mathcal{O}_C) = \oplus_{i=0}^{\ell-1} L_i$, where the L_i are the eigenspaces of $H^1(C, \mathcal{O}_C)$ with respect to the action of $\mathrm{Gal}(C, D)$. Then some power f of the Frobenius morphism F sends L_i into itself. Define γ_i as the dimension of the largest subspace of L_i on which F^f is a bijection. The γ_i depend on the monodromy of the cover and the choice of the branchpoints. The γ_i can be viewed as portions of the p-rank of C; we have p-rank $(C) = \sum_{i \in I} \gamma_i f_i$, where the sum is taken over the cosets of $<p>$ in $\mathbb{Z}/\ell - \{0\}$ and f_i is the length of the ith coset. Hence if one can calculate the γ_i one can calculate the p-rank of the ℓ-cyclic cover $C \to D$. This is the second main topic of this paper.

We will formulate a conjecture for the invariants γ_i in the case that $D = \mathbb{P}^1$ and the branchpoints are chosen sufficiently general (see 3.13). This conjecture says roughly the following. Consider C_λ defined by $y^\ell = \prod_{i=1}^r (x-\lambda_i)^{a_i}$ and let $\pi : C_\lambda \to \mathbb{P}^1$ be the ℓ-cyclic cover defined by $(x, y) \mapsto x$. Fix ℓ and the a_i and let the λ_i vary. We will see in Section 3 that there exists an explicit upperbound for the p-rank of C (or more precisely for the γ_i). In general this upperbound is strictly smaller than the genus of C. Hence there exist $a_1, \ldots, a_r$ such that C_λ is not ordinary for all choices of the λ_i. The conjecture states that there exist $\lambda_1, \ldots, \lambda_r$ for which this upperbound is reached. We will prove the conjecture eg if the number of branchpoints is at most four, and if $p \equiv \pm 1 \pmod{\ell}$. We also show that if we prove the conjecture for $\mathbb{P}^1$ then we get a similar statement for generic curves.

The structure of the paper is as follows. In Section 3 we define the generalized Hasse–Witt invariants γ_i. We prove some easy arithmetical properties of the γ_i and formulate the conjecture mentioned above. In Section 4 we prove the conjecture in the case that $\dim L_i = 1$ and $p + 2 \geq$ the number of branchpoints. In Section 5 we work out the theory for covers of semistable curves. We use deformations of (admissible) covers of semistable curves to covers of nonsingular curves to establish the conjecture in some more cases. In Section 6 we study covers of generic curves. In Section 7 we show that the number of $(\mathbb{Z}/p)^n \rtimes \mathbb{Z}/\ell$-quotients of $\pi_1^t(U)$ can be expressed in terms of suitable γ_i. We also show that the analogous question for $(\mathbb{Z}/p)^n \rtimes H$ with H abelian of prime-to-p order, can be reduced to the case of H a cyclic group.

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2 Notations

Let k be an algebraically closed field of characteristic $p > 0$. By ℓ we denote an integer prime to p. All curves are supposed to be nonsingular projective irreducible and defined over k, unless explicitely stated otherwise. Eg in Section 3 we consider semistable curves, these are connected projective curves with only ordinary double points as singularities. By $\sigma(C)$ we denote the p-rank of a curve C. A cover is a map wich is finite flat and generically etale. A cover is called G-Galois if G is a finite group of automorphisms of C and $D \simeq C/G$. A cover is ℓ-cyclic if it is Galois with Galois group $\mathbb{Z}/\ell$. If A is a complete discrete valuation ring and $S = \mathrm{Spec}\, A$ and $\mathcal{D} \to S$ is a scheme over S then we denote by $\mathcal{D}_s$ the special fibre and by $\mathcal{D}_\eta = \mathcal{D} \otimes \overline{Q(A)}$ the generic fibre.

3 Generalized Hasse–Witt invariants

Nakajima ([8]) has defined generalized Hasse–Witt invariants of a curve D. These invariants are related to the p-ranks of the etale cyclic covers of D of prime-to-p order. Nakajima shows that the generalized Hasse–Witt invariants give information about the fundamental group $\pi_1(D)$ of a projective curve D. Furthermore he shows that there is a dense open subset U_ℓ of the moduli space of curves of genus g such that for all $D \in U_\ell$ all the invariants corresponding to the etale ℓ-cyclic covers of D are maximal. Nakajima calls these curves ℓ-ordinary.

The definition of Nakajima can be extended to include ramified cyclic covers. This will be done in this section. We will see that the results of Nakajima extend to this case (cf. Lemma 5.6 and Section 7). However it turns out that the generalized Hasse–Witt invariants corresponding to ramified covers are much easier to calculate than Nakajima's invariants. Sections 4 and 5 will be devoted to this. In Section 5 we use deformations of covers of semistable curves to covers of nonsingular curves to study generalized Hasse–Witt invariants, therefore we also define the invariants for ℓ-cyclic covers of semistable curves.

3.1 In the rest of this section we fix the following notations. Let $\pi : C \to D$ be an ℓ-cyclic cover of semistable curves, defined over an algebraically closed field of characteristic p. Starting from 3.5 we will suppose C, D furthermore to be nonsingular. We assume $(\ell, p) = 1$. Let ϕ be a generator of $G := \mathrm{Gal}(C/D)$. Fix a primitive ℓth root of unity $\zeta \in k$. Write $H^1(C, \mathcal{O}_C) = \oplus_{i=0}^{\ell-1} L_i$, where L_i is the eigenspace of ϕ with eigenvalue ζ^i. Note $L_0 = H^1(C, \mathcal{O}_C)^G \simeq H^1(D, \mathcal{O}_D)$. Frobenius acts on the L_i by sending L_i into $L_{\overline{pi}}$, with $0 \leq \overline{pi} < \ell$ such that $\overline{pi} \equiv pi \pmod{\ell}$, we will omit the bar. Because if $\xi \in L_i$ then $\phi^\star F\xi = F\phi^\star\xi = \zeta^{pi} F\xi$. Put $\ell_i = \ell/(\ell, i)$, and f_i equal to the order of p in $(\mathbb{Z}/\ell_i)^\star$, then $F^{f_i} : L_i \to L_i$.

Definition 3.2 Let $1 \leq i \leq \ell - 1$. The *generalized Hasse–Witt invariants* $\gamma_i(\pi) = \gamma_i$ of π are defined as the dimension of the largest subspace of L_i on which F^{f_i} is a bijection.

We could have defined γ_0 as the dimension of largest eigenspace of L_0 on which F is a bijection. This γ_0 is equal to the p-rank of D, and is not a new invariant.

Lemma 3.3 *Notations as above*

(i) $\gamma_i = \gamma_{pi}$,

(ii) $\sigma(C) = \sigma(D) + \sum_{i \in I} f_i \cdot \gamma_i$,
where I is a set of representatives of the cosets of $<p>$ in $\mathbb{Z}/\ell - \{0\}$.

Proof. The first statement is immediate. For the last statement, choose a basis of $H^1(C, \mathcal{O}_C)$ consisting of eigenvectors of ϕ. Let A be the Hasse–Witt matrix of C with respect to this basis, and let A_i be the submatrix of A giving the action of Frobenius on $\oplus_{j=1}^{f_i} L_{p^j i}$. Then the p-rank of C is equal to the rank of the matrix

$$B := A \cdot A^{(p)} \cdots A^{(p^{f-1})},$$

where for a matrix $M = (m_{i,j})$ we denote $M^{(p^n)} := (m_{i,j}^{p^n})$ (see [15, p. 39]). This is the matrix of F^f on $H^1(C, \mathcal{O}_C)$. Analogously γ_i is equal to the rank of the matrix

$$B_i := A_i \cdot A_i^{(p)} \cdots A_i^{(p^{f-1})}.$$

This is the matrix of $F^f : L_i \to L_i$. Put $B_0 = A_0 \cdots A_0^{(p^{f-1})}$, where A_0 is the Hasse–Witt matrix of D. The statement now follows since B is a block matrix with the B_i for $i \in I \cup \{0\}$ as blocks on the diagonal. $\square$

3.4 i) Suppose $\ell = n \cdot m$ with $n, m \neq 1$. Then $H^1(C, \mathcal{O}_C)^{<\phi^m>} = \oplus_{i=0}^{n-1} L_{mi}$, and the γ_{mi} are the generalized Hasse–Witt invariants of the n-cyclic cover $C/\phi^m \to D$.
ii) Nakajima's generalized Hasse–Witt invariants are the γ_i of all the etale cyclic covers of prime-to-p order.

3.5 In the rest of the section $\pi : C \to D$ will be an ℓ-cyclic cover of nonsingular irreducible projective curves, with branchpoints $\lambda_1, \ldots, \lambda_r$. We are going to analyse how the generalized Hasse–Witt invariants depend on the monodromy. For this we use the following notation.

Notation 3.6 Let u_i be a uniformizing parameter at some point y_i of $\pi^{-1}(\lambda_i)$. Let $a_i \in \mathbb{Z}$, with $0 \leq a_i < \ell$. For each i put $n_i = \gcd(a_i, \ell)$. This n_i is equal to the number of points in the fibre above λ_i. We say that π is of *type* $(\ell; a_1, \ldots, a_r)$ if for all $i \in \{1, \ldots, r\}$ we have $(\phi^{n_i})^\star u_i = \zeta^{b_i} u_i \pmod{(u_i^2)}$, where $b_i \equiv (a_i/n_i)^{-1} \pmod{\ell/n_i}$.

Note that the type is independent of the choice of y_i and u_i. Replacing ϕ or ζ changes $(l; a_1, \ldots, a_r)$ into $(l; na_1, \ldots, na_r)$ for some $n \in (\mathbb{Z}/l)^\star$, so a type is uniquely determined up to multiplication by an element of $(\mathbb{Z}/\ell)^\star$. The definition of type can also be defined a bit different. First define it in the totally ramified case as above. If π is not totally ramified above λ_i, i.e. $n_i \neq 1$, then define the condition in terms of the generator ϕ^{n_i} of the inertiagroup of a points in the fibre above λ_i. Of course this yields the same definition.

We can interprete the type of an ℓ-cyclic cover $\pi : C \to D$ in terms of the map $f : \pi_1^t(D - \{\lambda_1, \ldots, \lambda_r\}) \to \mathbb{Z}/\ell$ corresponding to the cover. Choose generating elements $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \delta_1, \ldots, \delta_r$ of $\pi_1^t(D - \{\lambda_i\})$ such that $\prod[\alpha_i, \beta_i] \prod \delta_i = 1$ and such that for each continuous $\pi_1^t(D - \{\lambda_i\}) \to G$, with G finite, the images of the δ_i generate an inertiagroup above λ_i ([3, XIII.2.12]). The cover π is of type $(\ell; f(\delta_1), \ldots, f(\delta_r))$. If $(\ell; a_1, \ldots a_r)$ is a type, then $\sum a_i \equiv 0 \pmod{\ell}$. Conversely if $\sum a_i \equiv 0 \pmod{\ell}$ then there exists a (not necessarily connected) cover of type $(\ell; a_1, \ldots a_r)$.

Example 3.7 Suppose $g(D) = 0$ and choose $\lambda_1, \ldots, \lambda_r \in D$. If $\sum a_i \equiv 0 \pmod{\ell}$ then there is a unique cover $C \to \mathbb{P}^1$, with C nonsingular of type $(\ell; a_1, \ldots a_r)$, unbranched outside $\lambda_1, \ldots \lambda_r \in \mathbb{P}^1_k$. This curve is the nonsingular curve associated to the equation

$$y^\ell = (x - \lambda_1)^{a_1}(x - \lambda_2)^{a_2} \cdots (x - \lambda_r)^{a_r}.$$

The curve C is connected if $\gcd(\ell, a_1, \ldots, a_r) = 1$.

Lemma 3.8 *Let $\pi : C \to D$ be of type $(\ell; a_1, \ldots a_r)$. For $i \in (\mathbb{Z}/\ell) - \{0\}$, we have*

$$\dim_k L_i = \sum_{j=1}^{r} \langle \frac{i a_j}{\ell} \rangle - 1 + g(D),$$

where $\langle \cdot \rangle$ denotes the fractional part.

Notation 3.9 For $\mathbf{a} = (\ell; a_1, \dots a_r)$ we will denote $\|i\| = \|i\|_{\mathbf{a}} = \sum_{j=1}^{r} \langle ia_j/\ell \rangle - 1$. This $\|i\|_{\mathbf{a}}$ is equal to the dimension of the ith eigenspace L_i of $H^1(C, \mathcal{O}_C)$, where $C \to \mathbb{P}^1$ is a cover of type $\mathbf{a}$.

Proof. Write $\pi_* \mathcal{O}_C = \oplus_{i=0}^{\ell-1} \mathcal{L}_i$ with $\mathcal{L}_i$ the eigenspace of ϕ with eigenvalue ζ^i. Consider

$$\begin{array}{ccc} \mathcal{L}_i^{\otimes \ell} & \to & \mathcal{L}_0 \simeq \mathcal{O}_D, \\ s \otimes \cdots \otimes s & \mapsto & s \cdot \cdots \cdot s \end{array}$$

(multiplication in $\pi_* \mathcal{O}_C$). Then $\pi_*(u_j^{\langle ia_j/\ell \rangle \ell})$ generates $\mathcal{L}_i$ locally around λ_j. So $\mathcal{L}_i^{\otimes \ell} \simeq \mathcal{O}_D(-\sum_{j=1}^{r} \langle ia_j/\ell \rangle \ell \lambda_i)$. Therefore

$$\deg \mathcal{L}_i = \frac{1}{\ell} \sum_{j=1}^{r} \langle ia_j/\ell \rangle \ell.$$

Note that $H^1(D, \mathcal{L}_i) \simeq L_i$. So

$$\dim_k L_i = \dim_k H^1(D, \mathcal{L}_i) = (\sum_{j=1}^{r} \langle \frac{ia_j}{\ell} \rangle) + g(D) - 1.$$

by the Riemann–Roch Theorem. $\square$

If $C \to D$ is an etale ℓ-cyclic cover then $\dim_k L_i = g(D) - 1$ for $i \neq 0$. Since necessarily $\gamma_i = \gamma_{p^j i} \leq \dim_k L_{ip^j}$ for all j we obtain

Corollary 3.10 *Let $\pi : C \to D$ be of type $(\ell; a_1, \dots a_r)$. Then*

$$\begin{array}{l} \gamma_i(\pi) \leq \min_j \|ip^j\| + g(D) \text{ and} \\ \sigma(C) \leq (\ell - 1) g(D) + \sigma(D) + \sum_{i \in I} f_i \min_j(\|ip^j\|). \end{array}$$

Lemma 3.11 *Fix a type $(\ell; a_1, \dots a_r)$. Let s be the number of a_i unequal to $0 \pmod{\ell/(i, \ell)}$. Then*

$$\begin{array}{ll} \|i\| + \|-i\| = s - 2 & \text{if } \ell/(i,\ell) \neq 2, \\ \|i\| = \frac{s}{2} - 1 & \text{if } \ell/(i,\ell) = 2. \end{array}$$

In particular $\|i\| \leq s - 2$.

Proof. Using the remark in 3.4 we may asume $(i, \ell) = 1$. After multiplying all a_j with i, we may assume $i = 1$. Assume further $0 < a_j < \ell$, so $s = r$. If $\ell \neq 2$ then

$$\|1\| + \|-1\| = \sum_{j=1}^{r} \frac{a_j}{\ell} - 1 + \sum_{j=1}^{r} \langle \frac{-a_i}{\ell} \rangle - 1 = \sum_{j=1}^{r} (\frac{a_j}{\ell} + \frac{\ell - a_j}{\ell}) - 2 = r - 2.$$

If $\ell = 2$, then $a_i = 1$ for all i and $\|1\| = r/\ell - 1 = r/2 - 1$. $\square$

Lemma 3.8 shows that the dimension of the eigenspaces of $H^1(C, \mathcal{O}_C)$ does not depend on the ramification points, but only on the type of the cover. Therefore if we fix the type the bound for the generalized Hasse–Witt invariants given in Corollary 3.10 is the same for all choices of the branchpoints.

Example 3.12 Let us consider the bounds of Corollary 3.10 in an example. Let D be a curve of genus zero, and let C_λ be the nonsingular curve associated to the equation

$$y^5 = x(x-1)(x-\lambda)$$

in characteristic $p \neq 5$. Then $\pi : C_\lambda \to D, (x, y) \mapsto x$ is an ℓ-cyclic cover of type $(5; 1, 1, 1, 2)$ and for the dimensions of the eigenspaces of $H^1(C_\lambda, \mathcal{O}_{C_\lambda})$ we find: $\|1\| = 0, \|2\| = 1, \|3\| = 1, \|4\| = 2$. Suppose p is equivalent to 2 or 3 mod ℓ, then p generates $(\mathbb{Z}/\ell)^\star$, and the minimal dimension in the coset is zero. Therefore Corollary 3.10 shows that $\sigma(C_\lambda) = 0$ for all λ. If $p \equiv -1 \pmod{\ell}$ then there are two cosets: $\{1, 4\}$ and $\{2, 3\}$. The bound of Corollary 3.10 becomes $\sigma(C_\lambda) \leq 2$. For $p \equiv 1 \pmod{\ell}$ the bound becomes $\sigma(C_\lambda) \leq 4 = g(C_\lambda)$. In particular we see that there exist p and a type $\mathbf{a}$ such that for all choices of the branchpoints the curve C_λ is not ordinary. For explicit values of p it is easy to check that for almost all λ the p-rank of C_λ is maximal by calculating the coefficients of the Hasse–Witt matrix and using the proof of Lemma 3.3. For special values of λ the γ_i drop, eg if $p \equiv 1 \pmod{\ell}$ then this happens for the λ which are zero's of the determinant of the Hasse–Witt matrix of C_λ. For explicit calculation see the next section. This seems to be the general pattern.

Conjecture 3.13 *Let $\mathbf{a} = (\ell; a_1, \ldots a_r)$ be a type. There exist $\lambda_1, \ldots \lambda_r \in \mathbb{P}^1$ such that each cover of $\mathbb{P}^1$ of type $\mathbf{a}$ unbranched outside the λ_i satisfies*

$$\sigma(C) = \sum_{i \in I} f_i \cdot \min_j(\|ip^j\|), \tag{1}$$

where I is a set of representatives of the cosets of p in $\mathbb{Z}/\ell - \{0\}$, and f_i is the number of elements of the coset containing i.

4 Cyclic covers of a curve of genus zero

Let $\pi : C \to \mathbb{P}^1$ be an ℓ-cyclic cover. The generalized Hasse–Witt invariant γ_i is equal to the rank of the matrix B_i corresponding to the action of some power of Frobenius F^f on the ith eigenspace L_i of $H^1(C, \mathcal{O}_C)$. The invariants γ_i are less or equal to $\dim L_{p^j i}$ for all j and $\gamma_i = \dim L_i$ iff $\det B_i \neq 0$. In this section we will show that if $\dim L_{p^j i} \geq 1$ for all j and $\dim L_i = 1$ and $p + 2 \geq$ to the number of branchpoints then $\det B_i$ is a polynomial in the branchpoints which is not identical to zero. This implies that in this case γ_1 is maximal, i.e. $\gamma_1 = 1$.

4.1 We will use the following notations. Let $\pi : C \to \mathbb{P}^1$ be an ℓ-cyclic cover branched at r points, which we will denote by $\lambda_1, \ldots, \lambda_r$. We let $\mathbf{a} = (\ell; a_1, \ldots a_r)$ be the type of the cover π as defined in 3.6. By $\|i\|$ we will denote the dimension of the ith eigenspace L_i of $H^1(C, \mathcal{O}_C)$ with respect to a fixed primitive ℓth root of unity and a fixed generator of $\mathrm{Gal}(C/\mathbb{P}^1)$ as in 3.9. The integer f is the order of p in $(\mathbb{Z}/\ell)^\star$. Put $m = (p^f - 1)/\ell$.

The curve C is defined by

$$y^\ell = (x - \lambda_1)^{a_1}(x - \lambda_2)^{a_2} \cdots (x - \lambda_r)^{a_r},$$

with $0 < a_i < \ell$ and $a_1 + \cdots + a_r \equiv 0 \pmod{\ell}$. Note that we suppose that $C \to \mathbb{P}^1$ is unbranched outside ∞, this is no restriction and will make the formulae simpler. We are now going to calculate the coefficients of the matrix of F^f on L_i. Put

$$\begin{aligned} v_i &= y^i (x - \lambda_1)^{-[i\frac{a_1}{\ell}]} \cdots (x - \lambda_r)^{-[i\frac{a_r}{\ell}]}, \\ \xi_{i,j} &= x^{-j} v_i. \end{aligned}$$

Then $\{\xi_{i,j} | 0 < i < \ell, 0 < j \le \|i\|\}$ forms a basis of $H^1(C, \mathcal{O}_C)$ (Čech cohomology). Let f be the order of p in $(\mathbb{Z}/p)^\star$ and put $m = (p^f - 1)/\ell$. To calculate the $(i,j),(i,j')$-th coefficient of the matrix corresponding to the action of F^f on $H^1(C, \mathcal{O}_C)$ we have to find the coefficient of $\xi_{i,j'}$ in $(\xi_{i,j})^{p^f}$. We have

$$(\xi_{i,j})^{p^f} = x^{-jp^f} y^{ip^f} (x - \lambda_1)^{-p^f[i\frac{a_1}{\ell}]} \cdots (x - \lambda_r)^{-p^f[i\frac{a_r}{\ell}]}.$$

Note $-p^f[ia/\ell] = -[p^f ia/\ell] + [p^f \langle ia/\ell \rangle]$. So

$$(x^{-j} v_i)^{p^f} = ((x)^{-p^f j} (x - \lambda_1)^{[p^f \langle \frac{ia_1}{\ell} \rangle]} \cdots (x - \lambda_r)^{[p^f \langle \frac{ia_r}{\ell} \rangle]}) v_i.$$

Hence the coefficient of $x^{-j'} v_i$ is

$$(-1)^L \sum_{n_1 + \cdots + n_r = L} \binom{m\ell \langle \frac{ia_1}{\ell} \rangle}{n_1} \cdots \binom{m\ell \langle \frac{ia_r}{\ell} \rangle}{n_r} \lambda_1^{n_1} \cdots \lambda_r^{n_r}, \tag{2}$$

where $L = (\|i\| + 1 - j)\ell m + j' - j$. All other coefficients are equal to zero.

To calculate the $(i,j),(i',j')$-th coefficient of the Hasse–Witt matrix of C we have to find the coefficient of $\xi_{i',j'}$ in $(\xi_{i,j})^p$. This coefficient is zero if $i' \not\equiv pi \pmod{\ell}$. If $i' \equiv pi \pmod{\ell}$ then the $(i,j),(i',j')$-th coefficient is equal to

$$(-1)^L \sum_{\substack{n_1 + \cdots + n_r = L \\ 0 \le n_j \le [p\langle \frac{ia_j}{\ell} \rangle]}} \binom{[p\langle \frac{ia_1}{\ell} \rangle]}{n_1} \cdots \binom{[p\langle \frac{ia_r}{\ell} \rangle]}{n_r} \lambda_1^{n_1} \cdots \lambda_r^{n_r}, \tag{3}$$

with $L = j' - jp + [p\langle ia_1/\ell \rangle] + \cdots + [p\langle ia_r/\ell \rangle]$. As follows from a computation similar to the above one.

Lemma 4.2 *Suppose $\|ip^n\| \ge 1$ for all n and $p \ge r-2$. Then the $(i,j),(i,j')$-th coefficient of F^f on $H^1(C, \mathcal{O})$ is a polynomial in $\lambda_1, \ldots, \lambda_r$ which is not identical to zero.*

Proof. We assume $i = 1$, and $\|p^n\| \geq \|1\| \geq 1$ for all n. This is no restriction. We write

$$ma_i = \sum_{k=0}^{f-1} \alpha_i(k) p^k \text{ and } n_i = \sum_{k=0}^{f-1} n_i(k) p^k,$$

with $0 \leq \alpha_i(k), n_i(k) \leq p-1$. The binomial coefficient

$$\binom{ma_i}{n_i}$$

is unequal to zero mod p if and only if $n_i(k) \leq \alpha_i(k)$ for all k. Note $L = (\|1\| + 1 - j)\ell m + j' - j = (\|1\| - j)p^f + \sum_{i=1}^{f-1}(p-1)p^f + p - 1 - \|1\| + j'$. We will show that we can choose $n_i(k)$ with

$$\begin{array}{l} 0 \leq n_i(k) \leq \alpha_i(k) \text{ all } i \text{ and } k, \\ \sum_{i=1}^r n_i(k) = \left\{ \begin{array}{ccr} (\|1\| - j)p + p - 1 & \text{if} & k = f-1 \\ p-1 & \text{if} & 1 \leq k \leq f-2 \\ p - 1 - \|1\| + j' & \text{if} & k = 0. \end{array} \right. \end{array} \tag{4}$$

For $n_i(k)$ satisfying these conditions we know that

$$\prod_{i=1}^r \binom{ma_i}{n_i}$$

is unequal to zero mod p. This then will imply that the $(1, j), (1, j')$-th coefficient of the matrix of F^f is not identical to zero. To show that we can choose $n_i(k)$ satisfying (4) note

$$\|1\| + 1 = \frac{m(a_1 + \cdots + a_r)}{\ell m} \leq \frac{(\sum_{i=1}^r \alpha_i(f-1)p^{f-1})}{p^f - 1} + r\frac{p^{f-1} - 1}{p^f - 1}.$$

So

$$\sum_{i=1}^r \alpha_i(f-1) \geq (\|1\| + 1)\frac{(p^f - 1)}{p^{f-1}} - r\frac{(p^{f-1} - 1)}{p^{f-1}} = (\|1\| + 1)p - r + \frac{r - \|1\| - 1}{p^{f-1}}.$$

From Lemma 3.11 follows that $r > \|1\| + 1$, so

$$\sum_{i=1}^r \alpha_i(f-1) \geq (\|1\| + 1)p - r + 1.$$

To get this estimation also for $\sum_{i=1}^r \alpha_i(k)$ we use
Claim: $m\langle \frac{pa_i}{\ell} \rangle \ell = \sum_{k=1}^{f-1} \alpha_i(k-1)p^k + \alpha_i(f-1)$.
Proof of the claim. Note $mpa_i = \sum_{k=1}^{f-1} \alpha_i(k-1)p^k + \alpha_i(f-1)(\ell m + 1)$. Write $pa_i = \bar{a} + \ell n$ with $0 \leq \bar{a} < \ell$. So $\bar{a} = \langle pa_i/\ell \rangle \ell$ and $n = [pa_i/\ell]$. From $ma_i < \ell m = p^f - 1$ follows that $[pa_i/\ell] = [pa_i m/\ell m] = \alpha_i(f-1)$. So

$$m\langle \frac{pa_i}{\ell} \rangle \ell = mpa_i - m\ell n = \sum_{k=1}^{f-1} \alpha_i(k-1)p^k + \alpha_i(f-1).$$

This proves the claim.

We can now repeat the argument we gave for $k = f-1$ and get an estimation

$$\sum_{i=1}^{r} \alpha_i(k) \geq (\|p^{f-1-k}\| + 1)p - r + 1.$$

We have to prove that under the assumptions we made we have

$$\sum_{i=1}^{r} \alpha_i(k) \geq \begin{cases} (\|1\| - j)p + p - 1 & \text{if} \quad k = f-1 \\ p-1 & \text{if} \quad 1 \leq k \leq f-2 \\ p - 1 - \|1\| + j' & \text{if} \quad k = 0. \end{cases}$$

These equations are satisfied if

$$(\|p^{f-1-k}\| + 1)p - r + 1 \geq \begin{cases} (\|1\| - j)p + p - 1 & \text{if} \quad k = f-1 \\ p-1 & \text{if} \quad 1 \leq k \leq f-2 \\ p - 1 - \|1\| + j' & \text{if} \quad k = 0. \end{cases}$$

We assumed that $p \geq r-2$, so $(\|p^{f-1-k}\|+1)p-r+1 \geq p-1+(\|p^{f-1-k}\|p-r+2) \geq p-1$. And for $k = f-1$ we have $(\|1\|+1)p-r+1 \geq (\|1\|-j+1)p-1+jp-r+2 \geq (\|1\|-j+1)p-1$.

The only thing left to check is that a term $\lambda_1^{k_1} \cdots \lambda_r^{k_r}$ occuring in the polynomial does not occur twice, i.e. $k_1, \ldots, k_r$ determine the $n_i(j)$ occuring in the summation. For this write $k_i = \sum_{j=0}^{f-1} k_i(j)p^j$, with $0 \leq k_i(j) \leq p-1$. Then necessarily $k_i(j) = n_i(j)$ since $n_i(j) \leq \alpha_i(j) \leq p-1$. This proves the lemma. □

Remark 4.3 i) The bounds in the previous lemma are not optimal. The methods of the proof of the lemma do not determine automatically all non vanishing monomials of the coefficients of the matrix giving the action of F^f on $H^1(C, \mathcal{O}_C)$.

ii) If $\|p^n\| = 1$ for all n then we do not need the restrictions on p in the previous lemma. In this case it suffices to prove that certain coefficients of the Hasse–Witt matrix are not identical to zero. Consider the $(i,j),(i',j')$th coefficient of the Hasse–Witt matrix. If $i' \not\equiv pi$ then this coefficient is zero. The rest of the coefficients are polynomials in the λ_i which are not identical zero. This is easy to see since $[p\langle ka_i/\ell\rangle] < p$, so if $0 \leq n_i \leq [p\langle ka_i/\ell\rangle]$ then the binomial coefficient

$$\binom{[p\langle ka_i/\ell\rangle]}{n_i}$$

does not vanish mod p. Now proceed as in the proof of the lemma.

Corollary 4.4 *Fix a type* $\mathbf{a} = (\ell; a_1, \ldots a_r)$. *Suppose* $p \geq r-2$. *Suppose further that* $\|i\| = 1$ *and* $\|ip^j\| \geq 1$ *for all* j. *Then* $\gamma_i = 1$ *for almost all choices of* $\lambda_1, \ldots, \lambda_r$.

Proof. As before we may suppose $i = 1$. Recall that γ_1 is equal to the dimension as k-vectorspace of the maximal subspace of L_1 on which F^f is a bijection. Because $\dim_k L_1 = \|1\| = 1$, we have that $\gamma_1 = 1$ if and only if the branchpoints $\lambda_1, \ldots, \lambda_r$ are not zero's of the $(1,1)$-th coefficient of the matrix of F^f. This happens for almost all choices of $\lambda_1, \ldots, \lambda_r$ by the previous lemma. □

If $\|ip^j\| = 0$ for some j, then the matrix of Frobenius on $\oplus L_{ip^j}$ is nilpotent, hence $\gamma_i = 0$. Therefore we need the condition $\|ip^j\| \geq 1$ for all j. The condition $p \geq r-2$ is needed in the prove of Lemma 4.2, but does not seem to be essential. In explicit examples for small p and r one finds that although the monomials considered in the proof of Lemma 4.2 might be zero still the relevant coefficient itself does not vanish.

Corollary 4.5 *Conjecture 3.13 is true for $r = 3$.*

Proof. This follows immediately from the above corollary, since Lemma 3.11 implies that the dimension of the eigenspaces is at most one in this case. □

This result can also be deduced from the results of [17] by using the formula for the zetafunction of C, where $C \to \mathbb{P}^1$ is an ℓ-cyclic cover unbrached outside $0, 1, \infty$ ([2, Section 1]). Actually in this way we can calculate not only the p-rank of C, but also the isogeny type of the p-divisible group of the Jacobian of C.

5 Deformation

In this section we will calculate some γ_i by deforming admissible covers of two $\mathbb{P}^1$'s meeting in one point to covers of $\mathbb{P}^1$ and using an induction argument. We will show that 3.13 holds for covers branched at at most four points by combining the result of the previous section with a deformation argument (see 5.7). For $p \equiv 1 \pmod{\ell}$ the conjecture states that if $\lambda_1, \ldots, \lambda_r \in \mathbb{P}^1$ are choosen sufficiently general then each cover of $\mathbb{P}^1$ unbranched outside the λ_i is ordinary. In this case the proof is rather straight forward and only uses that the p-rank goes up under deformation and the case $r = 3$. For $p \equiv -1 \pmod{\ell}$ we have to be a little more carefull. We have to show that the γ_i go up under deformation and start the induction at $r = 4$ (see 5.8). In general this approach does not seem to work.

The following notation is fixed in this section. Suppose $k = \bar{k}$ and characteristic of k is $p > 0$. Let $A := k[[t]]$ have maximal ideal m_A, residue field k and field of fractions K. Put $S = \mathrm{Spec}(A)$.

Definition 5.1 Let $\pi : C \to D$ be a tame Galois cover of semistable curves over k. Then π is called *admissible* if for each singular point τ of C the following condition is satisfied. Let ϕ be a generator of the inertia group of τ. Let u_i for $i = 1, 2$ be a uniformizing parameter at τ of the branches of C meeting at τ. We require $\phi^\star u_1 = \zeta u_1 \pmod{(u_1^2)}$ and $\phi^\star u_2 = \zeta^{-1} u_2 \pmod{(u_2^2)}$, for some root of unity $\zeta \in k$.

Proposition 5.2 *(i) Let $\pi_0 : C_0 \to D_0$ be an admissible ℓ-cyclic cover, with D_0 a semistable curve over k and $(\ell, p) = 1$. Let $\mathcal{D}$ be a curve over S with $\mathcal{D}_s \simeq D_0$ and $\mathcal{D}_\eta$ nonsingular. Choose sections $x_1, \ldots, x_r : S \to \mathcal{D}$, which induce distinct points $x_i(s)$ in the smooth locus of $\mathcal{D}_s$. Suppose that the branchlocus of π_0 is contained in $\{x_1(s) \ldots, x_r(s)\} \cup \mathrm{Sing}(\mathcal{D}_s)$. Then there exists a deformation $\pi : \mathcal{C} \to \mathcal{D}$ of $\pi_0 : C_0 \to D_0$, unbranched outside $\{x_1, \ldots x_r\} \cup \mathrm{Sing}(\mathcal{D}_s)$.*

(ii) Let $\mathcal{D} \to S$ be a curve with $\mathcal{D}_s$ semistable, and $\mathcal{D}_\eta$ nonsingular. Let $\mathcal{C}_\eta \to \mathcal{D}_\eta$ be an ℓ-cyclic cover, unbranched outside points $x_1, \ldots, x_r$. Suppose that the x_i specialize to distinct points in the smooth locus of $\mathcal{D}_s$. Let $\mathcal{C}$ be the normalization of $\mathcal{D}$ in the function field of $\mathcal{C}_\eta$. Suppose $\mathcal{C}_s$ is reduced. Then $\mathcal{C}_s \to \mathcal{D}_s$ is an admissible cover, unbranched outside $\{x_1(s), \ldots, x_r(s)\} \cup \mathrm{Sing}(\mathcal{D}_s)$.

Proof. (ad i) The case needed here follows from [13, Remarque 3].
(ad ii) This is a special case of [14, Chap. 2, Th. 1.1.2]. □

For a semistable curve C over k let $J(C)$ be its generalized Jacobian, $f : \tilde{C} \to C$ its normalization and Γ_C the graph associated to C. Then the p-rank $\sigma(C)$ of C is defined as $\mathrm{Hom}(\mu_p, J(C))$. This number is equal to the largest subspace of $H^1(C, \mathcal{O}_C)$ on which Frobenius F is a bijection. We have $\sigma(C) = \sigma(\tilde{C}) + \dim_{\mathbb{Z}} H^1(\Gamma_C, \mathbb{Z})$ (see [12, p. 1279]). Also $p_a(C) = p_a(\tilde{C}) + \dim_{\mathbb{Z}} H^1(\Gamma_C, \mathbb{Z})$ and $p_a(\tilde{C}) = \sum g(C_i)$, where the sum is taken over all components of C. Therefore C is ordinary if and only if all its components are ordinary.

There exists an analogon for the generalized Hasse–Witt invariants of the formula

$$\sigma(C) = \sum_{C_i} \sigma(C_i) + \text{ the number of loops,} \tag{5}$$

where the sum is taken over all components of C. The exact sequence

$$0 \to \mathcal{O}_C \to f_\star \mathcal{O}_{\tilde{C}} \to Q \to 0,$$

with $Q = f_\star \mathcal{O}_{\tilde{C}} / \mathcal{O}_C$, gives rise to a long exact sequence

$$0 \to H^0(C, \mathcal{O}_C) \to H^0(C, \mathcal{O}_{\tilde{C}}) \to H^0(C, Q) \to H^1(C, \mathcal{O}_C) \to H^1(C, \mathcal{O}_{\tilde{C}}) \to 0. \tag{6}$$

Let n be the number of components of C and m be the number of singular points. Then the long exact sequence becomes

$$0 \to k \to k^n \to k^m \to H^1(C, \mathcal{O}_C) \to H^1(C, \mathcal{O}_{\tilde{C}}) \to 0. \tag{7}$$

The Formula 5 follows from this. To get a formula for the generalized Hasse–Witt invariants one has to consider the action of ϕ on the whole sequence. Since the formula becomes rather messy we will only work it out in the only special case we will need. The idea is that the ith generalized Hasse–Witt invariant is equal to the sum of the ith generalized Hasse–Witt invariants of the components plus a contribution of the loops.

5.3 Suppose D is a semistable curve of genus zero consisting of two nonsingular components D_1, D_2 meeting in one point τ where they have an ordinary double point. Let $\pi : C \to D$ be an ℓ-cyclic cover, with C semistable, and suppose that all singular points of C map to τ. This condition is for example fulfilled if π is admissible. Write $C_i = \pi^{-1}(D_i)$. Choose a generator ϕ of $\mathrm{Gal}(C/D)$. Let m be the number of singular points of C. The singular points of C are cyclically permuted by ϕ. Hence the eigenvalues of ϕ on k^m in Equation 7 are the m-th roots of unity, all with multiplicity one.

Let n_i be the number of irreducible components of C_i. Note that n_1, n_2 are divisors of m and $\gcd(n_1, n_2) = 1$ since C is connected. The irreducible components of C_1 (resp. C_2) are cyclically permuted by ϕ. Hence the term k^n in Equation 7 can be written as $k^{n_1} \oplus k^{n_2}$, and the eigenvalues of ϕ on k^{n_1} (resp. k^{n_2}) are the n_1th (resp. the n_2th) roots of unity. Equation 6 implies that

$$H^1(C, \mathcal{O}_C) \simeq \prod_i H^1(X_i, \mathcal{O}_{X_i}) \prod_i H^1(Y_i, \mathcal{O}_{Y_i}) \prod L,$$

where the first product is taken over all components of C_1, the second over all components of C_2 and L is the image of k^m in $H^1(C, \mathcal{O}_C)$. Hence the eigenvalues of ϕ on L are exactely the mth roots of unity which are neither n_1th nor n_2th roots of unity.

If $\mathcal{C}/S$ is a curve with semistable fibres then $\sigma(\mathcal{C} \otimes \bar{K}) \geq \sigma(\mathcal{C}_s)$ ([12, p. 1281]). The analogous statement for generalized Hasse-Witt invariant if also true.

Proposition 5.4 *Let $\mathcal{D} \to S$ be a curve with $\mathcal{D}_s$ semistable, and $\mathcal{D}_\eta$ nonsingular. Let $\pi : \mathcal{C} \to \mathcal{D}$ be an ℓ-cyclic cover with $\pi \otimes k : \mathcal{C}_s \to \mathcal{D}_s$ admissible. Then for each i*

$$\gamma_i(\pi \otimes \bar{K}) \geq \gamma_i(\pi \otimes k).$$

Proof. Let C be a semistable curve over k with an automorphism ϕ of order ℓ. The automorphism ϕ acts on $H^1(C, \mathcal{O}_C)^F$, since if $F\xi = \xi$ then $\phi\xi = \phi F \xi = F\phi\xi$, so $\phi\xi \in H^1(C, \mathcal{O}_C)^F$. There exist isomorphisms

$$H^1(C, \mathcal{O}_C)^F \simeq H^1_{et}(C, \mathbb{Z}/p) \simeq (\mathbb{Z}/p)^{\sigma(C)}$$

(see [7, III.4.12]). This defines an action of ϕ on $(\mathbb{Z}/p)^{\sigma(C)}$. Now the ith generalized Hasse–Witt invariant is equal to the multiplicity of the eigenvalue ζ^i of the action of ϕ on $(\mathbb{Z}/p)^\sigma$. Note that the primitive ℓth root of unity ζ is contained in a finite extension $\mathbb{F}_{p^f}$ of $\mathbb{F}_p$. If $p \not\equiv 1 \pmod{\ell}$ then the action of ϕ on $(\mathbb{Z}/p)^\sigma$ is not diagonalizable over $\mathbb{F}_p$.

Let $\mathcal{C} \to \mathcal{D}$ be as in the proposition, and let ϕ be a generator of $\mathrm{Gal}(\mathcal{C}, \mathcal{D})$. There exist canonical maps

$$H^1(\mathcal{C}_s, \mathbb{Z}/p) \xrightarrow{\sim} H^1(\mathcal{C}, \mathbb{Z}/p) \hookrightarrow H^1(\mathcal{C} \otimes \bar{K}, \mathbb{Z}/p)$$

([12, p. 1281]). Since ϕ acts on the whole sequence this implies

$$\gamma_i(\pi \otimes k) \leq \gamma_i(\pi \otimes \bar{K}).$$

□

Proposition 5.5 *Let $\ell | p - 1$. There exists a dense open subset $U \subset (\mathbb{P}^1_k)^r$, such that if $(\lambda_1, \ldots, \lambda_r) \in U$, then any ℓ-cyclic cover $\pi : C \to \mathbb{P}^1_k$, unbranched outside $\lambda_1, \ldots, \lambda_r$, is ordinary.*

Proof. We use induction on r. For $r = 2$ there is nothing to prove, since each cover of $\mathbb{P}^1$ branched at 2 points has genus zero. For $r = 3$ we proved the conjecture in 4.1.

Suppose the statement is true for all $s < r$. Construct a curve $\mathcal{D} \to S$, with special fibre consisting of two curves D_1 and D_2 of genus 0 meeting in one point τ, and generic fibre a nonsingular curve of genus 0. Choose sections $\lambda_1, \ldots, \lambda_r : S \to \mathcal{D}$, such that λ_1, λ_2 meet the smooth part of D_1 and $\lambda_3, \ldots, \lambda_r$ meet the smooth part of D_2. We can choose the λ_i such that for all $m|\ell$, the m-cyclic covers of D_2, unbranched outside $\lambda_3, \ldots, \lambda_r, \tau$ are ordinary by induction.

Let $\pi_0 : C_0 \to \mathcal{D}_s$ be an admissible ℓ-cyclic cover, unbranched outside $\lambda_1, \ldots \lambda_r, \tau$. Let X be a component of C_0. Let m be the order of the inertia group of X. Then $X/(\phi^{\ell/m})$ is isomorphic to D_i for i equal to 1 or 2, so $X \to D_i$ is an m-cyclic cover of D_i, and X is ordinary by assumption. Since all components of C_0 are ordinary, C_0 itself is ordinary. By Proposition 5.2 we can deform π_0 to an ℓ-cyclic cover $\pi : \mathcal{C} \to \mathcal{D}$. We have

$$g(\mathcal{C} \otimes \bar{K}) \geq \sigma(\mathcal{C} \otimes \bar{K}) \geq \sigma(\mathcal{C}_s) = p_a(\mathcal{C}_s).$$

Because $\mathcal{C} \to S$ is flat, $p_a(\mathcal{C}_s) = g(\mathcal{C} \otimes \bar{K})$, so $\mathcal{C} \otimes \bar{K}$ is ordinary.

We have to check that all ℓ-cyclic covers of $\mathcal{C}_\eta$ unbranched outside $x_1, \ldots, x_r$ are obtained in this way. Let $\pi : \mathcal{C}_\eta \to \mathcal{D}$ be such a cover. By construction the x_i specialize to distinct points in the smooth locus of $\mathcal{D}_s$. Let $\mathcal{C}$ be the normalization of $\mathcal{D}$ in the function field of $\mathcal{C}_\eta$. After a finite extension of A the special fibre $\mathcal{D}_s$ is reduced. Therefore we can apply Proposition 5.2. We conclude that all ℓ-cyclic covers of $\mathcal{D}_\eta$ unbranched outside the x_i, are obtained as deformations of admissible covers of $\mathcal{D}_s$, unbranched outside $\{x_1(s), \ldots, x_r(s), \tau\}$. Hence all these covers of $\mathcal{D}_\eta$ are ordinary.

For each type $\mathbf{a} = (\ell; a_1, \ldots a_r)$ define $U_{\mathbf{a}} \subset (\mathbb{P}^1_k)^r$ as the set of $(\lambda_1, \ldots, \lambda_r) \in (\mathbb{P}^1_k)^r$ such that all ℓ-cyclic covers of $\mathbb{P}^1$ of type $(\ell; a_1, \ldots a_r)$ branched at the λ_i are ordinary. The next lemma shows that $U_{\mathbf{a}}$ is open. The subset U defined in the statement of the proposition is equal to $\cap_{\mathbf{a}} U_{\mathbf{a}}$, hence is also open since for fixed ℓ and r the number of types is finite. The set U is nonempty, since we have constructed a $\bar{K}$-valued point in it. $\square$

Lemma 5.6 *Let* $\mathbf{a}$ *be a type and* D *any curve. Denote by* Δ *the generalized diagonal of* $(D)^r$. *Define* $U_{\mathbf{a}} \subset (D)^r - \Delta$ *as the set of* $(\lambda_1, \ldots, \lambda_r) \in (D)^r - \Delta$ *such that for all* ℓ*-cyclic covers of* D *unbranched outside the* λ_i *of type* $\mathbf{a}$ *Equation 1 holds. Then* $U_{\mathbf{a}}$ *is open.*

Proof. For each $(\lambda_1, \ldots, \lambda_r) \in D^r \Delta$ we define $C_{\lambda_1, \ldots, \lambda_r} \to D$ as the smallest cover which has all covers of D of type $\mathbf{a}$ branched at the λ_i as subcovers. If we choose generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \delta_1, \ldots, \delta_r$ of $\pi_1^t(D - \{\lambda_1, \ldots, \lambda_r\})$ such that for each finite quotient the images of the δ_i generate an inertia group above λ_i then this cover corresponds to the maximal quotient

$$\pi_1^t(D - \{\lambda_1, \ldots, \lambda_r\}) \twoheadrightarrow (\mathbb{Z}/\ell)^n$$

which sends δ_i to $a_i, \ldots, a_i$. Eg if ℓ is prime the integer $n = \ell^{2g}$, where g is the genus of D. For suitable ℓ-cyclic subcovers of $C_{\lambda_1, \ldots, \lambda_r} \to D$ we have

$$J(C_{\lambda_1, \ldots, \lambda_r}) \sim \prod_{j=1}^{n} J(C_j/D) \times J(D).$$

This implies $\sigma(C_{\lambda_1,\ldots,\lambda_r}) \leq \sigma(D) + n((\ell-1)g(D) + \sum_{i\in I} f_i \min_j \|p^j i\|) =: s$. We get a family $f : \mathcal{A} \to D^r - \Delta$, where $\mathcal{A}_{\lambda_1,\ldots,\lambda_r} = J(C_{\lambda_1,\ldots,\lambda_r})$. Let $W_{\mathbf{a}} = D^r - \Delta - U_{\mathbf{a}} = \{(\lambda_1,\ldots,\lambda_r) | \sigma(C_{\lambda_1,\ldots,\lambda_r}) \leq s-1\}$. From [9, Cor. 1.5] follows that $W_{\mathbf{a}}$ is closed. This proves the lemma. □

Proposition 5.7 *Conjecture 3.13 is true for $r=4$.*

Proof. We suppose $i=1$ and $\gcd(a_1,\ldots,a_4,\ell)=1$. From Lemma 3.11 we know that $\|k\| \leq r-2 \leq 2$. Therefore we can distinguish three cases;

(i) $\|p^j\| = 0$ for some j,

(ii) $\|1\| = 1$ and $\|p^j\| \geq 1$ for all j,

(iii) $\|p^j\| = 2$ for all j.

In case (i) the action of Frobenius on $\oplus_j L_{p^j}$ is nilpotent, so $\gamma_1 = 0$. Case (ii) follows from Corollary 4.4. Suppose we are in case (iii).

Let $A = k[[t]]$. Construct a curve $\mathcal{D} \to \mathrm{Spec}(A) =: S$ with $\mathcal{D}_\eta$ a complete nonsingular irreducible curve of genus zero and $\mathcal{D}_s$ consisting of two (nonsingular projective) connected components D_1, D_2 of genus zero meeting in one point τ, where they have a normal crossing. We choose sections $s_1,\ldots,s_4$ of $S \to \mathcal{C}$, such that s_1, s_2 (resp. s_3, s_4) meet the smooth locus of D_1 (resp. D_2). Construct ℓ-cyclic covers $C_1 \to D_1$, branched at s_1, s_2, τ of type (a_1, a_2, a_3+a_4) and $C_2 \to D_2$, branched at τ, s_3, s_4 of type $(-(a_3+a_4),, a_3, a_4)$. This yields an admissible cover $\pi_0 : C_0 \to \mathcal{D}_s$. Let $\pi : \mathcal{C} \to \mathcal{D}$ be as in Proposition 5.2. Note C_1 and C_2 need not be connected, but C_0 is since we assumed that $\gcd(a_1,\ldots,a_4,\ell)=1$. The assumption $\|p^j\|_{\mathcal{C}_\eta} = 2$ implies $\|-p^j\|_{\mathcal{C}_\eta} = 0$ by Lemma 3.11. Hence $\|-p^j\|_{C_j} = 0$ for all j. Let $d = \gcd(a_3+a_4,\ell)$. Then d is the number of points of C_0 above τ. As in 5.3

$$H^1(C, \mathcal{O}_C) \simeq H^1(C_1, \mathcal{O}_{C_1}) \oplus H^1(C_2, \mathcal{O}_{C_2}) \oplus L,$$

where L is the part coming from the loops. Since the points above τ are permuted by $\phi^{\ell/d}$ the eigenvalues of ϕ occuring in L are dth roots of unity. Let $n_1 = \gcd(a_1,a_2,\ell)$ be the number of components of C_1 and $n_2 = \gcd(a_3,a_4,\ell)$ be the number of components of C_2. In 5.3 we proved that the eigenvalues of ϕ on L are exactely the dth roots of unity which are neither n_1th roots of unity nor n_2th roots of unity. This implies that if ζ^{p^j} is an eigenvalue of ϕ on L, then so is ζ^{-p^j}. Since we already saw that $\|-p^j\|_{C_i} = 0$ for all j, this implies that for each j the eigenspace of ϕ on $H^1(C,\mathcal{O})$ is contained in $H^1(C_1,\mathcal{O}_{C_1}) \oplus H^1(C_2,\mathcal{O}_{C_2})$. Hence also

$$\gamma_j(C_0) = \gamma_j(C_1) + \gamma_j(C_2).$$

Since for cover of $\mathbb{P}^1$ branched at 3 points we have $\|k\| \leq 1$ we must have $\|p^j\|_{C_i} = 1$ for all j and $i = 1,2$. We have already proved the result for $r=3$, so $\gamma_1(C_1) = \gamma_1(C_2) = 1$. Hence

$$2 = \|1\|_{\mathcal{C}_\eta} \geq \gamma_1(\mathcal{C}_\eta) \geq \gamma_1(C_1) + \gamma_1(C_2) = 2,$$

therefore $\gamma_1(\mathcal{C}_\eta) = 2$. The proposition now follows from Lemma 5.6. □

If we consider covers branched at at least five points then the covers with $2 \leq \|k\| < r-2$ and $\|p^j k\| \geq \|k\|$ for all j, cannot be handled by the methods used in the proof of the previous proposition.

Proposition 5.8 *Conjecture 3.13 is true for* $p \equiv -1 \pmod{\ell}$.

Proof. Let $\pi : D \to \mathbb{P}^1$ be a cover of type $\mathbf{a} = (\ell; a_1, \ldots a_r)$ with $0 < a_i < \ell \neq 2$. Suppose that $p \equiv -1 \pmod{\ell}$. Assume $i = 1$. The coset of $<p>$ in $(\mathbb{Z}/\ell)^*$ containing 1 consists of two elements 1 and -1. From Lemma 3.11 we know that $\|1\| + \|-1\| = r-2$. Put $k = \|1\|$. We may suppose $0 \leq k \leq [(r-2)/2]$. If $k = 0$ the submatrix of the Hasse–Witt matrix of C corresponding to $1, -1$ is nilpotent and $\gamma_1 = 0$, so suppose that $k > 0$.

We will prove the proposition by induction. For $r = 3, 4$ the statement follows from Proposition 5.7. Suppose the statement holds for $r' < r$.

Case 1. Suppose $k < (r-2)/2$. For r odd this is always true. Choose $i \neq j$ with $a_i + a_j < \ell$. We can do this since if $a_{2n-1} + a_{2n} \geq \ell$ for each n, then $a_1 + \cdots + a_r \geq [(r-1)/2]\ell$. But $a_1 + \cdots + a_r = k\ell < (r-2)\ell/2 \leq [(r-1)/2]\ell$, so we get a contradiction. Put $\mathbf{a}_1 = (a_i, a_j, -a_i - a_j)$ and $\mathbf{a}_2 = (a_i + a_j, a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots)$. For a cover $f : X \to \mathbb{P}^1$ of type $\mathbf{a}_1$ we find

$$\|1\|_{\mathbf{a}_1} = \frac{a_i + a_j + \ell - a_i - a_j}{\ell} - 1 = 0 \text{ and } \|-1\|_{\mathbf{a}_1} = 1.$$

For a cover $g : Y \to \mathbb{P}^1$ of type $\mathbf{a}_2$ we find

$$\|1\|_{\mathbf{a}_2} = \frac{a_1 + \cdots + a_r}{\ell} - 1 = k \text{ and } \|-1\|_{\mathbf{a}_2} = r - 3 - k.$$

Note $k \leq r - k - 3$. By induction $\gamma_1(f) = 0$ and $\gamma_1(g) = k$. So Proposition 5.4 implies $\gamma_1(\pi) \geq \gamma_1(g) = k$. Hence $\gamma_1(\pi) = k$.

Case 2. Suppose $r = 2s$ and $s > 2$ and $k = s - 1$. Then $a_1 + \ldots + a_r = s\ell$.

Claim: there exist distinct i, j, k, such that $\ell < a_i + a_j + a_k < 2\ell$. Put $x_i = a_i/\ell - 1/2$. Note $-1/2 < x_i < 1/2$ and $x_1 + \ldots + x_r = 0$. We will see that there exists distinct $i_1, \ldots, i_k \in \{1, \ldots, r\}$ such that $-1/2 < \sum_{j=1}^{k} x_{i_j} < 1/2$. The claim follows from this by taking $k = 3$.

We use induction. For $k = 1$ it is trivial. Let $k > 1$. Choose $I = \{i_1, \ldots i_{k-1}\}$ such that $-1/2 < S_{k-1} := \sum_{i \in I} x_i < 1/2$. If $S_{k-1} \geq 0$ then $(\sum x_i) - S_{k-1} < 0$, so there is an $j \notin I$ with $x_j < 0$. We can add j to I to fulfull the condition for k. Analogous for $S_{k-1} < 0$. This proves the claim.

Choose i, j, k as in the claim. Put $\mathbf{a}_1 = (a_i, a_j, a_k, -a_i - a_j - a_k)$, and $\mathbf{a}_2 = (a_i + a_j + a_k$, rest of the $a_n)$. Then $\|1\|_{\mathbf{a}_1} = \|-1\|_{\mathbf{a}_1} = 1$ and $\|1\|_{\mathbf{a}_2} = \|-1\|_{\mathbf{a}_2} = s - 1$. As in the previous case the proposition follows by induction. □

Using only the deformation statement will not lead to a proof of Conecture 3.13. For example let C be as in Example 3.12 and suppose $p \equiv -1 \pmod{\ell}$. Then $\sigma(C) = 2$ for almost all λ. We could not have proved this using deformations since all ℓ-cyclic covers of

$\mathbb{P}^1$ branched at three points have p-rank zero. Also it seems in the general case not possible to use a more complicated deformation argument as we did for $p \equiv -1 \pmod{\ell}$. The case $p \equiv -1 \pmod{\ell}$ is very special since in this case the dimensions of the eigenspaces in one coset are related by $\|i\| + \|-i\| = r - 2$.

6 Cyclic covers of generic curves

In this section we no longer restrict to covers of $\mathbb{P}^1$, but study cyclic covers of arbitrary curves, in particular generic curves. Recall that for an ℓ-cyclic cover $\pi : C \to D$ we proved that

$$\gamma_i(\pi) \leq \min_j \|ip^j\| + g(D),$$

where $\|k\| + g(D)\|$ is the dimension of the kth eigenspace of $H^1(D, \mathcal{O}_D)$. We can ask (as in the genus zero case) whether we get equality for all covers of a fixed type $\mathbf{a}$ for certain choices of the branchpoints. We will prove this if D is the generic curve of genus g under the assumption that Conjecture 3.13 is true for $\mathbf{a}$.

Lemma 6.1 *Let D be a curve of genus g. Let $\mathbf{a} = (\ell; a_1, \dots a_r)$ be a type for which Conjecture 3.13 holds. Assume that there exists an etale ℓ-cyclic cover $D' \to D$ with $\sigma(D') - \sigma(D) = g(D') - g(D)$. Then there exists a cover $\pi : C \to D$ of type $\mathbf{a}$ with*

$$\begin{array}{l} \gamma_i(\pi) = \min_j \|ip^j\| + g \text{ if } i \neq 0, \\ \sigma(C) = \sum_{i \in I} f_i \min_j \|ip^j\| + (\ell - 1)g + \sigma(D), \end{array} \tag{8}$$

where I is a set of representatives of the cosets of $<p>$ in $\mathbb{Z}/\ell - \{0\}$ and f_i is the length of the coset corresponding to i.

If for a cover $C \to D$ Equation 8 holds we will say that C has *maximal p-rank.*

Proof. Let $S = \operatorname{Spec} k[[t]]$ and $K = Q(k[[t]])$. Construct a curve $\mathcal{D}/S$ with generic fibre $\mathcal{D} \otimes_k \bar{K} \simeq D \otimes_k \bar{K}$ and special fibre $\mathcal{D}_s$ consisting of two components, one isomorphic to D and one component D_1 nonsingular of genus zero, such that the two components meet in one point τ where they have an ordinary double point. Choose distinct sections $x_1, \dots, x_r : S \to \mathcal{D}$ which meet the special fibre in $D_1 - \{\tau\}$. Choose these sections in such a way that the ℓ-cyclic cover $\pi_1 : C_1 \to D_1$ of type $\mathbf{a}$ branched at $x_1(s), \dots, x_r(s)$ has maximal p-rank, that is we assume that $\gamma_i(\pi_1) = \min_j \|ip^j\|$ for $i \neq 0$. We can do this by assumption. Note that π_1 is unbranched at τ. Let $\pi_2 : D' \to D$ be as in the statement of the lemma. Identify the points in $\pi_1^{-1}(\tau)$ and $\pi_2^{-1}(\tau)$ suitably to obtain an admissible cover $\pi_0 : C_0 \to \mathcal{D}_s$. Since π_0 is unbranched at τ Example 5.3 implies $\gamma_i(\pi_0) = \gamma_i(\pi_1) + \gamma_i(\pi_2) + 1$ for $i \neq 0$, with $\gamma_i(\pi_1) = \min_j \|ip^j\|$ and $\gamma_i(\pi_2) = g(D) - 1$. The last equation follows from the assumption that $J(D')/J(D)$ is ordinary, this implies that $\gamma_i(\pi_2)$ is equal to the dimension of the ith eigenspace of $H^1(C_1, \mathcal{O}_{C_1})$. Hence $\gamma_i(\pi_0) = \min_j \|ip^j\| + g(D)$. Proposition 5.2 implies that

$$\gamma_i(\pi_\eta) \geq \gamma_i(\pi_0) = \min_j \|ip^j\| + g(D)$$

for $i \neq 0$ and the first statement follows from Corollary 3.10. The second statement follows from Lemma 3.3. □

The condition that there exists an etale ℓ-cyclic cover $D' \to D$ with $\sigma(D') - \sigma(D) = g(D') - g(D)$ is eg fulfilled if D is elliptic or if $g \geq 2$ and $\ell + 1 \geq (p-1)g$, see [10, Théorème 4.3.1].

Proposition 6.2 *Let D be the generic curve of genus $g \geq 2$ or an ordinary elliptic curve. Let $\mathbf{a}$ be a type for which Conjecture 3.13 holds. Then there exists a dense open subset $U_{\mathbf{a}} \subset D^r$ such that for $(x_1, \ldots, x_r) \in U_{\mathbf{a}}$ each cover of D of type $\mathbf{a}$ branched at the x_i has maximal p-rank.*

Proof. Let D be the generic curve of genus $g \geq 2$ or an ordinary elliptic curve and let k be an algebraically closed field of characteristic p over which D is defined. Let $\mathbf{a}$ be a type. Put $S = \operatorname{Spec} k[[t]]$ and let $\mathcal{D} \to S$ be as in the proof of the previous lemma, i.e. the generic fibre of $\mathcal{D}$ is a trivial deformation of D, and the special fibre of $\mathcal{D}$ is a copy of D with a component D_1 of genus zero attached at one point τ. We choose sections $x_1, \ldots, x_r : S \to \mathcal{D}$ which meet the special fibre in the nonsingular part of the genus zero component. As in the proof of the previous proposition we suppose that all covers of D_1 of type $\mathbf{a}$ branched at the x_i have maximal p-rank. We want to apply the Proposition 5.2 and 5.4 to get information about the ℓ-cyclic covers of $\mathcal{D}_\eta$. We therefore need to calculate the p-ranks of the admissible covers of $\mathcal{D}_s$ of type $\mathbf{a}$.

Let $\pi_0 : C_0 \to \mathcal{D}_s$ be an admissible cover of $\mathcal{D}_s$ of type $\mathbf{a}$ branched at $x_1, \ldots, x_r$. Then $\pi_0|_D : C \to D$ is a (not necessarily connected) etale cover of D. It is well known that all etale ℓ-cyclic covers of a generic curve D of genus $g \geq 2$ are ordinary, this is easily seen eg by considering a deformation of D to a chain of ordinary elliptic curves (cf. [8, Section 5.II]). If D is an ordinary elliptic curve then we also have that the etale covers of D are ordinary. Hence all components of C are ordinary and also C itself. The cover $\pi_0|_{D_1} : C_1 \to D_1$ is a (not necessarily connected) cover of D_1 of type $\mathbf{a}$. By assumption on the branchpoints this cover has maximal p-rank.

Let $\pi : \mathcal{C} \to \mathcal{D}$ be a deformation of π_0 as in Proposition 5.2. As in the proof of the previous lemma we get that $\mathcal{C}_\eta$ has maximal p-rank, i.e. Equation 8 holds. By the second part of Proposition 5.2 we obtain all covers of $\mathcal{D}_\eta$ of type $\mathbf{a}$ branched at the x_i in this way. hence all these covers satsfy Equation 8. The statement that the same holds for almost all choices of the branchlocus follows from Lemma 5.6. □

If E is a supersingular elliptic curve then the condition $\sigma(E') - \sigma(E) = g(E') - g(E)$ is fulfilled for every etale cover $E' \to E$. Still the argument we used for generic curves does not imply immediately that for generic choice of the branchpoints all covers of E of a fixed type have maximal p-rank. The reason for this if that if $\pi_0 : C_0 \to \mathcal{D}_s$ is an admissible ℓ-cyclic cover as in the proof of the proposition and $\pi_0|_E$ is not connected then π_0 does not satisfy Equation 8. One can show that in this case the conclusion of the proposition still holds by using the connectedness of a suitable Hurwitz space. I hope to come back to this later.

7 Quotients of tame fundamental groups

In this section we no longer restrict to cyclic covers. For any H-Galois cover $\pi : C \to D$ and any character χ of H we will define an invariant $\gamma(\chi)$. We will see that if H is an abelian group of prime-to-p order we can express the $\gamma(\chi)$'s in terms of suitable generalized Hasse–Witt invariants. The results of the previous sections can be generalized to this case. Furthermore we will give a criterium for a group $(\mathbb{Z}/p)^n \rtimes H$, with H abelian and of prime-to-p order, to be a quotient of a fixed tame fundamental group in terms of the generalized Hasse-Witt invariants. This implies for instance that we can calculate all quotients of this form of $\pi_1^t(\mathbb{P}^1 - \{0, 1\infty\})$, since in this case we can calculate all generalized Hasse–Witt invariants.

Let $\pi : C \to D$ be an H-Galois cover, with H any finite group. For any character χ of H define

$$L_\chi = \{\xi \in H^1(C, \mathcal{O}_C) | h \cdot \xi = \chi(h)\xi \text{ for all } h \in H\}.$$

Note that Frobenius sends L_χ into L_{χ^p}. Define $f(\chi)$ as the smallest integer f such that $F^f : L_\chi \to L_\chi$. Define $\gamma(\chi)$ as the largest subspace of L_χ on which $F^{f(\chi)}$ is a bijection. The invariants $\gamma(\chi)$ have the following properties.

Lemma 7.1 *(i) If $\chi = \chi_1 + \chi_2$ then $\gamma(\chi) = \gamma(\chi_1) + \gamma(\chi_2)$.*

(ii) If ξ is the character of the regular representation then $\gamma(\chi) = \sigma(C)$.

Proof. Immediate from the definition, cf. [6] for more properties. □

In the rest of the section we will restrict to the case where H is an abelian group whose order m we suppose prime to the characteristic. Suppose χ is an irreducible character of H over k. Then χ is an homomorphism $\chi : H \to \mu_m(k)$, and its image is cyclic say of order ℓ. We choose a primitive mth root of unity $\zeta_m \in k$. For each $m' | m$ put $\zeta_{m'} = \zeta_m^{m/m'}$. Put $H_\chi = \ker(\chi)$ and $Y = Y_\chi = C/H_\chi$. Note that the eigenspaces L_{χ^j} can be identified with the eigenspaces of $H^1(Y, \mathcal{O}_Y)$ with respect to the action of $\mathrm{Gal}(Y, D)$. Let $\phi = \phi_\chi$ be a generator of the (cyclic) group $\mathrm{Gal}(Y, D)$ such that

$$H^1(C, \mathcal{O}_C)^{\chi^j} = \{\xi \in H^1(Y, \mathcal{O}_Y) | \phi^\star \xi = \zeta_\ell^j \xi\}.$$

Lemma 7.2 *Using the above notation we get*

$$\gamma(\chi) = \gamma_1(Y \to D).$$

Hence we can calculate the new invariant $\gamma(\chi)$ in terms of the cyclic cover $Y \to D$. To make this more explicit we need to describe the type of the cover $Y \to D$ in terms of the cover $C \to D$.

Definition 7.3 Let $x \in D$ be a branchpoint of an H-Galois cover $\pi : C \to D$. Let $z \in C$ be an element of $\pi^{-1}(x)$ and let u be a local parameter at z. We define the *canonical generator of inertia at x* as $\sigma_x \in H$ such that $\sigma_x u = \zeta_{n_x} u \pmod{u^2}$, where n_x is the order of the inertia group at z and ζ_{n_x} the before hand chosen n_xth root of unity.

Let $x_1, \dots x_r$ be the branchpoints of $\pi : C \to D$. Choose $z_i \in \pi^{-1}(x_i)$ and let y_i be the image of z_i in Y. Let σ_i be the canonical generator of inertia at z_i. Note that its image $\bar{\sigma}_i \in H/H_\chi$ is a canonical generator of inertia at y_i. Define $a_i \in \mathbb{Z}$ by $\bar{\sigma}_i = \phi^{a_i}$ and $0 \leq a_i < \ell$.

Lemma 7.4 *The cover $Y \to D$ is of type $(\ell; a_1, \dots a_r)$.*

Proof. Let m_i be the order of the inertia group at y_i and u_i a local parameter at y_i. Then $\gcd(a_i, \ell) = \ell/m_i =: n_i$, we have that ϕ^{n_i} generates the inertia group at y_i. Hence n_i divides a_i, since by definition ϕ^{a_i} also generates the inertia group at y_i. We have $\phi^{n_i} u_i = \zeta_{m_i}^{b_i} u_i \pmod{u_i^2}$ for some integer b_i. Hence $\zeta_{m_i} u_i = \phi^{a_i} u_i = (\phi^{n_i})^{a-i/n-i} u_i = \zeta_{m_i}^{a_i b_i/n_i} u_i$. This implies that $\phi^{n_i} u_i = \zeta_\ell^{b_i} u_i \pmod{u_i^2}$ with $b_i \equiv (a_i/n_i)^{-1} \pmod{m_i}$ as we had to prove. □

Let $\chi_1, \dots \chi_s$ be the k-irreducible characters of H (up to isomorphism). Frobenius acts on the χ_i be sending χ_i to χ_i^p. Let I be a set of the representatives of the orbits of this action. Let $(\ell_i; a_{i,1}, \dots, a_{i,r})$ be the type of $Y_{\chi_i} \to D$. As in Corollary 3.10 we get an inequality

$$\sigma(C) = \sum_{\chi_i \in I} f(\chi_i)\gamma(\chi_i) \leq (\ell - 1)g(D) + \sigma(D) + \sum_{\chi_i \in I} f(\chi_i) \min_j \left(\sum_{k=1}^{r} \langle \frac{j a_{i,k}}{\ell_i} \rangle - 1 \right). \quad (9)$$

Proposition 7.5 *Let H a finite abelian group of prime-to-p order and let χ be a k-irreducible character of H, we denote the order of its image by ℓ. Let $\pi : C \to D$ be an H-Galois cover with branchpoints $x_1, \dots, x_r$. Suppose for each ℓ-cyclic cover $X \to D$ unbranched outside the x_i we have (in the notation of Corollary 3.10)*

$$\sigma(X) = (\ell - 1)g(D) + \sigma(D) + \sum_{i \in I} f_i \min_j (\|ip^j\|).$$

Then

$$\gamma(\chi) = \min_j \dim H^1(C, \mathcal{O}_C)^{\chi^{p^j}} = \min_j \left(\sum_{i=1}^{r} \langle \frac{j a_i}{\ell} \rangle - 1 \right),$$

where $(\ell; a_1, \dots a_r)$ is the type of the cover $Y_\chi \to D$. In particular the inequality in Equation 9 is an equality.

The type of the cover $Y_\chi \to D$ can be expressed in terms of χ and π by Lemma 7.4. This proposition says that if all cyclic covers of D branched at $x_1, \dots, x_r$ have maximal p-rank then the same holds for all abelian covers of D with the same branchlocus.

Proof. Immediate. □

Next we are going to relate the generalized Hasse–Witt invariants to certain quotients of tame fundamental groups. Let G be a finite group satisfying

$$0 \to (\mathbb{Z}/p)^n \to G \to H \to 0, \tag{10}$$

where H is an abelian group of prime-to-p order. This sequence necessarily splits and $G = (\mathbb{Z}/p)^n \rtimes H$. For a group G satisfying Equation 10 we denote by χ_G the character associated to the action of H on $(\mathbb{Z}/p)^n$. We will study the tame G-Galois covers $X \to D$ which factor through a given H-Galois cover $\pi : C \to D$.

Proposition 7.6 *Suppose G satisfies Equation 10. Then*

$$\{G - \textit{Galois covers } X \to D \textit{ which factor through } \pi\} \xrightarrow{1:1}$$

$$\{V \subset \mathrm{Hom}(\pi_1(C), \mathbb{Z}/p) \textit{ with } \dim_{\mathbb{F}_p} V = n \textit{ on which } H \textit{ acts via } \chi_G^{-1}\}.$$

Proof. Given a G-Galois cover $X \to D$ which factors through π we can define $V = \mathrm{Hom}(\mathrm{Gal}(X, D), \mathbb{Z}/p)$. We can regard V as a subpace of $\mathrm{Hom}(\pi_1(C), \mathbb{Z}/p)$ via the exact sequence

$$0 \to \pi_1(X) \to \pi_1(C) \to \mathrm{Gal}(X, C) \to 0.$$

Then H acts on V via χ_G^{-1}.

Conversely, suppose given $V \subset \mathrm{Hom}(\pi_1(C), \mathbb{Z}/p)$ with $\dim_{\mathbb{F}_p} V = n$. Then V corresponds to an etale $(\mathbb{Z}/p)^n$-Galois cover $X \to C$. The cover $X \to D$ is Galois iff V is H-invariant. The space V with the action of H can be identified with $\mathrm{Hom}(\mathrm{Gal}(X, C), \mathbb{Z}/p)$, this implies the statement about the characters. □

The space $\mathrm{Hom}(\pi_1(C), \mathbb{Z}/p)$ is isomorphic to $H^1(C, \mathcal{O}_C)^F$ (see [15, Prop. 12]), this isomorphism commutes with the action of H. Write

$$\chi_G \simeq \eta_1^{n_1} + \cdots + \eta_s^{n_s},$$

where the η_i are distict $\mathbb{F}_p$-irreducible characters. For each i we write

$$\eta_i \simeq \chi_i + \chi_i^p + \cdots + \chi_i^{p^{f-1}}$$

for some k-irreducible character χ_i. This is clearly possible. Note that $\gamma(\eta_i) = f \cdot \gamma(\chi_i)$ since $\gamma(\chi) = \gamma(\chi^p) = \cdots = \gamma(\chi^{p^{f-1}})$.

Proposition 7.7 *There exists a tame G-Galois cover $X \to D$ factoring through π iff*

$$n_i \leq \gamma(\chi_i^{-1}) \textit{ for each } i.$$

Up to isomorphism the number of these covers is

$$\prod_{i=1}^{s} \frac{(q_i^{\gamma(\chi_i^{-1})} - 1)(q_i^{\gamma(\chi_i^{-1})} - q_i) \cdots (q_i^{\gamma(\chi_i^{-1})} - q_i^{n_i - 1})}{(q_i^{n_i} - 1) \cdots (q_i^{n_i} - q_i^{n_i - 1})},$$

where $q_i = p^{f(\chi_i)}$.

Proof. Suppose $X \to D$ is a G-Galois cover. The group G is given as an $\mathbb{F}_p$-vectorspace $V = (\mathbb{Z}/p)^n$ with an action of H on it. Write $V = \oplus V_j$, where the V_j are $\mathbb{F}_p$-irreducible modules with character η_j. Note that the η_j need not be distinct. Now $G_j = V/V_j$ can be considered as a subgroup of $(\mathbb{Z}/p)^n$. This subgroup is stabelized by H, hence a normal subgroup, since $(\mathbb{Z}/p)^n$ is abelian. Define $X_j = X/G_j$. We obtain

$$
\begin{array}{ccccc}
 & & X & & \\
 & \swarrow & & \searrow & \\
X_1 & & \cdots & & X_s \\
 & \searrow & & \swarrow & \\
 & & C & & \\
 & & \downarrow & & \\
 & & D & & .
\end{array}
$$

Furthermore $J(X) \sim J(C) \times \prod_j J(X_j)/J(C)$. The curves X_j correspond to subspaces $V_j \subset \mathrm{Hom}(\pi_1(C), \mathbb{Z}/p)$ as in the previous proposition. Let χ_{i_j} be an k-irreducible character such that $\eta_j = \chi_{i_j} + \ldots + \chi_{i_j}^{p^{f-1}}$. Here the χ_k are as before a set of distinct $\mathbb{F}_{p^f}$-irreducible characters. The group H acts on V_j via $\chi_{i_j}^{-1}$ and the intersection $V_j \cap V_{j'} = \emptyset (j \neq j')$, since the maximal common subcover of $X_j \to D$ and $X_{j'} \to D$ is $C \to D$. This shows that in this case $n_i \leq \gamma(\chi_i^{-1})$. Conversely suppose that $n_i \leq \gamma(\chi_i^{-1})$ then choose independent H-stable $V_j \subset \mathrm{Hom}(\pi_1(C), \mathbb{Z}/p)$, where the action of H on exactely n_i of the V_j is via the character χ_i^{-1}. This is possible by assumption. Let $X_j \to D$ be the cover corresponding to V_j. Define X as the fibre product of the X_j over C, then $X \to D$ is G-Galois. This proves the first part of the proposition. The number of G_i-Galois covers factoring through π is equal to the number of $(\mathbb{Z}/p)^{f(\chi_i)} \times \cdots (\mathbb{Z}/p)^{f(\chi_i)}$ (n_i times) subspaces of $(\mathbb{Z}/p)^{f(\chi_i)\gamma(\chi_i^{-1})}$ for each i. $\square$

We have now eg calculated the number of $(\mathbb{Z}/p)^n \rtimes H$ covers of $\mathbb{P}^1$ branched at $0, 1, \infty$, where H is abelian and has an order which is prime to p. In the previous proposition we gave a criterium for these covers to exists in terms of $\gamma(\chi)$'s. In the Lemma's 7.2 and 7.4 we interpreted the invariants $\gamma(\chi)$ in terms of suitable generalized Hasse–Witt invariants. In 5.7 we showed that these generalized Hasse–Witt invariants are given by the explicit expression from 3.13. For covers of $\mathbb{P}^1$ branched at four points we can do the same if the branchpoints are choosen sufficientely general.

If H is a nonabelian group of prime-to-p order then there exist absolutely irreducible characters χ whose image is not cyclic. Hence in this case we can not reduce to the cyclic case in the same way as we did for abelian groups. This suggest that $\gamma(\chi)$ is a new invariant, independent of the generalized Hasse–Witt invariants of the cyclic covers.

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