Ergodic Theorems for Subadditive Superstationary Families of Random Sets with Values in Banach Spaces

Grzegorz Krupa

Mathematical Institute, University of Utrecht, Utrecht, the Netherlands

Abstract

Pointwise and mean ergodic theorems under different assumptions for subadditive superstationary families of random sets whose values are weakly (or strongly) compact convex subsets of a separable Banach space are presented. The results generalize the results of [14], where random sets in $\mathbb{R}^d$ are considered. Techniques used here are inspired by [3].

AMS Classification: 28B20, 26E25, 28D99, 47A35.
Keywords and phrases: multivalued ergodic theorems, measurable multifunctions, random sets, subadditive superstationary processes, set convergence.

1 Introduction

A generalization of Birkhoff's pointwise ergodic theorem for superstationary families of random variables was given by Krengel [13]. In [12] Kingman proved Birkhoff's ergodic theorem for stationary subadditive processes. Abid [1] generalized previous results and showed ergodic theorem for subadditive superstationary families of $\mathbb{R}$-valued random variables. Using Abid's result Schürger [14] proved pointwise and mean ergodic theorems for subadditive superstationary families of convex compact random sets in $\mathbb{R}^d$. Here results of [14] are generalized. Subadditive superstationary families $(F_{s,t})$ of multivalued functions with values being weakly (respectively strongly) compact convex subsets of a separable Banach space are considered. Certain compactness conditions are imposed upon $(F_{s,t})$. Namely, it is assumed that $\frac{1}{t}F_{0,t}(\omega)$ are a.e. contained in some, dependent on ω, ball-compact set for all $t \in \mathbb{N}$, or, in the case when the Banach space and its dual both have the Radon-Nikodym property, $\operatorname{cl}\operatorname{co} \bigcup_{t=1}^{\infty} \frac{1}{t} \int_A F_{0,t}$ is supposed to be w-compact for all subsets A of an underlying σ-algebra. It is noteworthy that in the finite-dimensional case there is no distinction between weak and strong topology and that the conditions mentioned above are automatically satisfied. Later, under some additional conditions, convergence of subadditive superstationary families of subsets of a Banach space to a constant limit is proved.

The main idea used in the proofs is to scalarize elements of subadditive superstationary families using support functions, then use Abid's one-dimensional results and prove the existence of multivalued infinite-dimensional limits using techniques used in [3] and [10].

2 Preliminaries

Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let X be a separable Banach space with the norm $\|\cdot\|$. X^* will denote the dual of X and $\langle \cdot, \cdot \rangle$ will stand for the usual duality. The strong and weak topology on X will be denoted by s and w respectively. Let $\mathcal{P}_{wk}(X)$ (respectively $\mathcal{P}_{sk}(X)$) denote a family of w-compact (s-compact) subsets of X. $\mathcal{P}_{wkc}(X)$ and $\mathcal{P}_{skc}(X)$ will be used for the families of w-compact convex and s-compact convex subsets of X. A subset of X will be called *w-ball-compact (s-ball-compact)* if its intersection with any closed ball with the center at the origin is w-compact (s-compact). Denote the family of w-ball-compact (resp. s-ball-compact) sets by

$\mathcal{R}_w$ (resp. $\mathcal{R}_s$). A *multifunction* is any mapping $F:\Omega \to \mathcal{P}_{wkc}(X)$. A multifunction F is said to be *(Effrös) measurable* if the *preimage* $F^- U := \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\}$ belongs to $\mathcal{A}$ for any s-open set $U \subset X$. Measurable multivalued functions will be considered and the adjective will be often omitted. Also the term *random set (r.s.)* will be used to denote measurable multivalued functions. The *support function* and the *radius* of the set $C \in \mathcal{P}_{wkc}(X)$ will be defined in the following way:

$$s(x^*, C) := \sup_{x \in C} \langle x, x^* \rangle, \qquad \|C\| := \sup_{x \in C} \|x\|.$$

A sequence $(C_n) \subset \mathcal{P}_{wkc}(X)$ *converges scalarly* to a $C \in \mathcal{P}_{wkc}(X)$ (notation: $C_n \rightharpoonup C$) if

$$\lim s(x^*, C_n) = s(x^*, C) \text{ for all } x^* \in X^*.$$

The *sequential weak Kuratowski limes superior* (w-Ls C_n) of a sequence $(C_n) \subset X$ is the set of all w-limits of subsequences (x_{n_j}), such that $x_{n_j} \in C_{n_j}$ for all n_j.

Denote by D a countable subset of the unit ball B^* in X^* dense with respect to the Mackey topology τ. Let H be the set spanned by all rational linear combinations of vectors in D. For details see [5, III.32]. Notice also that, by [5, III.34] for any $C \in \mathcal{P}_{wkc}(X)$

$$C = \bigcap_{x^* \in H} \{x \in X : \langle x, x^* \rangle \leq s(x^*, C)\}. \tag{1}$$

Denote $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$. Let $\Delta := \{(s,t) \in (\mathbb{N}_0)^2 : s < t\}$. Families $(F_{s,t})_{(s,t)\in\Delta}$ in $\mathcal{P}_{wkc}(X)$ (or $\mathcal{P}_{skc}(X)$) will be considered. They will be denoted simply $(F_{s,t})$. Concepts of subadditivity and superstationarity are defined in the following way:

Definition 2.1 *A family $(C_{s,t})_{(s,t)\in\Delta} \subset \mathcal{P}_{wkc}(X)$ (or $\mathcal{P}_{skc}(X)$) is called* subadditive *if*

$$C_{s,t} \subset C_{s,u} + C_{u,t} \text{ for all } s < u < t.$$

Let $\mathcal{D}$ be a space of families $F = (F_{s,t})$ in $\mathcal{P}_{wkc}(X)^\Delta$ (or $\mathcal{P}_{skc}(X)^\Delta$) :

$$F = \begin{bmatrix} F_{0,1} & F_{1,2} & F_{2,3} & \dots \\ F_{0,2} & F_{1,3} & F_{2,4} & \dots \\ \vdots & \vdots & \vdots & \vdots \\ F_{0,n} & F_{1,n+1} & F_{2,n+2} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

endowed with the product topology. Let $(i,j) \in \mathbb{N}^2$ and $\pi_{i,j}:\mathcal{D} \to \mathcal{P}_{wkc}(X)$ (or $\mathcal{P}_{skc}(X)$) be a projection that gives the (i,j)-th coordinate of $F \in \mathcal{D}$, so $\pi_{i,j}(F) = F_{j-1,i+j-1}$.

Define the shift $T : \mathcal{D} \to \mathcal{D}$, such that $\pi_{i,j}(T(F)) = F_{j,i+j}$ for all $i,j \in \mathbb{N}$. Let $\mathcal{M}(\mathcal{D})$ be a family of probability measures defined on $\mathcal{B}(\mathcal{D})$, the Borel σ-algebra of subsets of $\mathcal{D}$.

A probability measure $P_1 \in \mathcal{M}(\mathcal{D})$ is *stochastically smaller* than $P_2 \in \mathcal{M}(\mathcal{D})$ (notation: $P_1 \prec P_2$) if

$$\int_{\mathcal{D}} f dP_1 \leq \int_{\mathcal{D}} f dP_2$$

for all bounded measurable functions $f:\mathcal{D} \to \mathbb{R}$ which are increasing i.e. $C \lesssim D$ implies that $f(C) \leq f(D)$, where $C \lesssim D$ denotes that $C_{i,j} \subseteq D_{i,j}$ for all $(i,j) \in \Delta$.

Definition 2.2 *Let $F = (F_{s,t})$ be a family of $\mathcal{P}_{wkc}(X)$-valued random sets defined on $(\Omega, \mathcal{A}, P)$. Let Q_i denote the probability distribution of $T^i F$ i.e. $Q_i(A) = P(T^i F \in A)$, $A \in \mathcal{B}(\mathcal{D})$. The family F is called* superstationary *if $Q_1 \prec Q_0$.*

Note that also $Q_{i+1} \prec Q_i$ for all $i \in \mathbb{N}$ if F is a superstationary process.

For example given a sequence (G_n) of i.i.d. random sets $F_{s,t}(\cdot) := \sum_{i=s}^{t-1} G_i(\cdot)$ defines a superstationary process.

3 Results

Theorems 3.1, 3.2, 3.3 give different conditions under which theorem 4.1 of [14] can be generalized for r.s. with values in a Banach space.

We begin with pointwise ergodic theorems.

Theorem 3.1 *Let $F = (F_{s,t})$ be a subadditive superstationary family in $\mathcal{P}_{wkc}(X)$ satisfying the following assumptions:*

i) $(F_{s,s+1})$ is $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$-bounded i.e. there exists a constant $\tilde{k}$ such that $\int_\Omega \|F_{s,s+1}\| \le \tilde{k}$ for all $s \in \mathbb{N}_0$,

ii) for almost all $\omega \in \Omega$, $\bigcup_{t=1}^{\infty} \frac{1}{t} F_{0,t}(\omega)$ is contained in some, dependent on ω, element of $\mathcal{R}_w$.

Then there exists $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ such that

a) $\frac{1}{t} F_{0,t} \rightarrow F_\infty$,

b) $F_\infty(\omega) \subset \mathrm{cl}\,\mathrm{co}\, w\text{-Ls}\, \frac{1}{t} F_{0,t}(\omega)$ a.e. in Ω.

Note that if the space X is reflexive then balls in X are w-compact thus condition $ii)$ of theorem 3.1 is trivially satisfied. Namely X itself is w-ball compact.

The next theorem is a version of the previous one for $\mathcal{P}_{skc}(X)$-valued functions.

Theorem 3.2 *Under the assumptions of theorem 3.1, where $\mathcal{P}_{wkc}(X)$ and $\mathcal{R}_w$ were replaced by $\mathcal{P}_{skc}(X)$ and $\mathcal{R}_s$ respectively, the following holds: there exists an $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{skc}(X)}$ such that*

a) $\lim_t \rho_H(\frac{1}{t} F_{0,t}, F_\infty) = 0$, where ρ_H denotes the Hausdorff distance,

b) $F_\infty(\omega) \subset \mathrm{cl}\,\mathrm{co}\, s\text{-Ls}\, \frac{1}{t} F_{0,t}(\omega)$ a.e. in Ω.

Another generalization of theorem 4.1 in [14] is possible. The assumptions in the next theorem are inspired by [3]. In this result both X and X^* are required to have the Radon-Nikodym property (RNP). Recall that X has the RNP with respect to $(\Omega, \mathcal{A}, P)$ if any P-absolutely continuous measure Q with bounded variation has a density $f \in \mathcal{L}^1_X$ with respect to P, that is $Q(A) = \int_A f dP$ for all $A \in \mathcal{A}$.

Theorem 3.3 *Suppose that the Banach space X and its dual X^* (with the dual norm) have both the RNP. Let $F = (F_{s,t})$ be a subadditive superstationary family in $\mathcal{P}_{wkc}(X)$ satisfying the following assumptions:*

i) $(F_{s,s+1})$ is $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$-bounded i.e. there exists a constant $\tilde{k}$ such that $\int_\Omega \|F_{s,s+1}\| \le \tilde{k}$, for all $s \in \mathbb{N}_0$,

ii) the set $\mathrm{cl}\,\mathrm{co} \bigcup_{t=1}^{\infty} \frac{1}{t} \int_A F_{0,t}$ is w-compact for all $A \in \mathcal{A}$.

Then there exists $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ such that

a) $\frac{1}{t} F_{0,t} \rightarrow F_\infty$,

b) $F_\infty(\omega) \subset \bigcap_m \mathrm{cl}\,\mathrm{co} \bigcup_{t>m} \frac{1}{t} F_{0,t}(\omega)$ a.e. in Ω.

Observe that here condition $ii)$ is of more global nature than in theorem 3.1. Note also that for a separable Banach space X, X^* has the RNP if and only if X^* is separable for the dual norm (see [8, Stegall's theorem p.195]).

The following theorems give conditions under which limits that occur in theorems 3.1, 3.2, 3.3 are constant a.e. These conditions appeared in [9], [14].

Theorem 3.4 *Let $F = (F_{s,t})$ be a family of random sets satisfying the assumptions of theorem 3.1. Suppose also* *i)* $\int_\Omega s(x^*, F_{0,t}) < \infty$, *for all* $t \in \mathbb{N}$, $x^* \in H$,

ii) $\lim_t \frac{1}{t^2} \int_\Omega (s(x^*, F_{0,t}))^2 = \alpha_{x^*}$ *exists and is finite for all* $x^* \in H$,

iii) There is a constant $0 \leq \delta_{x^*} < 1$ *depending on* $x^* \in H$ *such that* $Var(s(x^*, F_{0,2t})) + \left(\int_\Omega s(x^*, F_{0,2t})\right)^2 \leq 2(1+\delta_{x^*})\, Var(s(x^*, F_{0,t})) + 4\left(\int_\Omega s(x^*, F_{0,t})\right)^2$ *for all* $x^* \in H$, *where Var denotes variance.*

Then there exists a set $C \in \mathcal{P}_{wkc}(X)$ *such that*

a) $\frac{1}{t} F_{0,t} \to C$,

b) $\lim_t \frac{1}{t} s(x^*, F_{0,t}) = s(x^*, C)$ *in* $\mathcal{L}^2_{\mathbb{R}}$, *for all* $x^* \in X^*$.

We have an immediate analogue of theorem 3.4 for r.s. with values in $\mathcal{P}_{skc}(X)$.

Theorem 3.5 *Let* $F = (F_{s,t})$ *be a family of random sets satisfying the assumptions of theorem 3.2 and i), ii), iii) of theorem 3.4. Then there exists a set* $C \in \mathcal{P}_{skc}(X)$ *such that*

a) $\lim_t \rho_H \left(\frac{1}{t} F_{0,t}, C\right) = 0$

b) $\lim_t \frac{1}{t} s(x^*, F_{0,t}) = s(x^*, C)$ *in* $\mathcal{L}^2_{\mathbb{R}}$, *for all* $x^* \in X^*$.

Theorem 3.6 *Suppose both* X *and* X^* *have the RNP. Let* $F = (F_{s,t})$ *be a family of random sets satisfying the assumptions of theorem 3.3 and i), ii), iii) of theorem 3.4. Then there exists a set* $C \in \mathcal{P}_{wkc}(X)$ *such that a), b) of theorem 3.4 hold.*

Remark 3.1 No analogue of theorem 3.6 exists for $\mathcal{P}_{skc}(X)$-valued r.s. The Radon-Nikodym theorem for multimeasures, which is used in the proof of theorem 3.6, provides only the existence of a derivative multifunction with w-compact values.

Let us now present mean ergodic theorems.

Proposition 3.1 *Let* (F_n) *be a sequence of* $\mathcal{P}_{wkc}(X)$*-valued random sets, such that for almost all* $\omega \in \Omega$, $\bigcup_{n \in \mathbb{N}} F_n(\omega)$ *is a subset of some w-ball-compact set dependent on* ω. *Suppose that for some* $m \in \mathbb{N}$, $\lim_n s(x^*, F_n)$ *exists a.e. in* $\mathcal{L}^m_{\mathbb{R}}$ *for all* $x^* \in H$ *and that* $(\|F_n\|)$ *converges in* $\mathcal{L}^m_{\mathbb{R}}$. *Then there exists an* $F_\infty \in \mathcal{L}^m_{\mathcal{P}_{wkc}(X)}$ *such that*

$$\int_\Omega |s(x^*, F_n) - s(x^*, F_\infty)|^m \to 0.$$

Theorem 3.7 *Let* $F = (F_{s,t})$ *be a family of random sets in* $\mathcal{P}_{wkc}(X)$ *satisfying conditions of theorem 3.1. Then there exists a random set* $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ *such that*

$$\lim_t \int_\Omega \left| s(x^*, \frac{1}{t} F_{0,t}) - s(x^*, F_\infty) \right| = 0$$

Remark 3.2 Suppose that both X and X^* have the RNP. It is possible to derive analogues of theorem 3.7 (and also proposition 3.1, which is used in the proof of theorem 3.7). In that case the condition that $\bigcup_{n \in \mathbb{N}} F_n(\omega)$ is a.e. contained in a w-ball-compact set, which depends on ω, is replaced by the following one: $\mathrm{cl\,co} \int_A F_n$ is w-compact for all $A \in \mathcal{A}$.

Much more interesting results can be obtained for r.s. whose values are in $\mathcal{P}_{skc}(X)$. Let us show first a $\mathcal{P}_{skc}(X)$-valued version of proposition 3.1.

Proposition 3.2 *Let (F_n) be a sequence of random sets in $\mathcal{P}_{skc}(X)$ such that for almost all $\omega \in \Omega$, $\bigcup_{n\in\mathbb{N}} F_n(\omega)$ is a subset of some, dependent on ω, s-ball-compact set. Suppose that for some $m \in \mathbb{N}$, $\lim s(x^*, F_n)$ exists in $\mathcal{L}^m_{\mathbb{R}}$ a.e. for all $x^* \in H$ and that $(\|F_n(\omega)\|)$ converges in $\mathcal{L}^m_{\mathbb{R}}$. Then there exists $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{skc}(X)}$ such that*

$$\int_\Omega \rho^m_H(F_n, F_\infty) \to 0.$$

Now we have an analogue of theorem 3.7.

Theorem 3.8 *Let $F = (F_{s,t})$ be a family of random sets in $\mathcal{P}_{skc}(X)$ satisfying the assumptions of theorem 3.2. Then there exists an $\mathcal{L}^1_{\mathcal{P}_{skc}(X)}$-bounded random set F_∞ satisfying the following condition*

$$\lim_{t\to\infty} \int_\Omega \rho_H(\frac{1}{t}F_{0,t}, F_\infty) = 0$$

Under additional assumptions (namely, those that appear in theorem 3.4) it is possible to derive mean-square convergence result.

Theorem 3.9 *Let $F = (F_{s,t})$ be a family of random sets in $\mathcal{P}_{skc}(X)$ satisfying the assumptions of theorem 3.2 and i), ii), iii) of theorem 3.4. Then there exists a constant set $C \in \mathcal{P}_{skc}(X)$ such that*

$$\lim_{t\to\infty} \int_\Omega \rho^2_H(\frac{1}{t}F_{0,t}, C) = 0$$

Remark 3.3 Artstein and Hansen showed in [2] that strong law of large numbers for convex-valued random sets can be extended to non-convex case by applying smart observation that given a sequence (K_i) of s-compact sets in a Banach space X and an s-compact set K_0 such that $\rho_H(\frac{1}{n}\sum_{i=1}^n \operatorname{co} K_i, K_0) \to 0$ as $n \to \infty$, then also $\rho_H(\frac{1}{n}\sum_{i=1}^n K_i, K_0) \to 0$ as $n \to \infty$. In the case when compact subsets of $\mathbb{R}^d$ are considered, well known result of Shapley-Folkmann-Starr can be applied. However this line of argument does not apply in case of subadditive, superstationary processes. A counterexample (already in $\mathbb{R}^d$) is given in [14].

4 Applications

Recall that if the space X is reflexive then closed balls in X are w-compact. Keeping in mind that remark it is easy to see that the condition which appears in the results of the section 3, namely, that $\bigcup_{t=1}^\infty F_{0,t}(\omega)$ is a subset of some element in $\mathcal{R}_w$, is automatically satisfied in reflexive spaces. (X itself is then w-ball-compact.) Thus Schürger's results for random sets in $\mathbb{R}^d$ with compact, convex values follow from the results presented in section 3. Notice also that scalar convergence topology is equivalent to the topology generated by Hausdorff distance in that case.

Let $\mathcal{P}_{kc}(\mathbb{R}^d)$ denote a family of all convex compact subsets of $\mathbb{R}^d$. We have an easy corollary from any of theorems 3.1, 3.2, 3.3.

Corollary 4.1 (Theorem 4.1, [14]) *Let $F := (F_{s,t})$ be a subadditive superstationary family of $\mathcal{P}_{kc}(\mathbb{R}^d)$-valued random sets defined on a common probability space $(\Omega, \mathcal{A}, P)$. Assume that there exists a constant $\tilde{K} > 0$ such that $\int_\Omega \|F_{s,s+1}\| \le \tilde{K}$ for $s \in \mathbb{N}_0$. Then $\lim_t \frac{1}{t}F_{0,t}$ exists a.e. in $(\mathcal{P}_{kc}(\mathbb{R}^d), \rho_H)$.*

Theorem 3.4 (or 3.5, 3.6) implies:

Corollary 4.2 (Theorem 4.16, [14]) *Let $F := (F_{s,t})$ be a subadditive superstationary family of $\mathcal{P}_{kc}(\mathbb{R}^d)$-valued random sets satisfying the conditions of corollary 4.1 and conditions i), ii), iii) of theorem 3.4. Then there exists a set $C \in \mathcal{P}_{kc}(\mathbb{R}^d)$ such that $\lim_t \frac{1}{t}F_{0,t} = C$ a.e., $\varphi_{x^*} = s^2(x^*, C)$ and for all x^*, $\lim_t \frac{1}{t}s(x^*, F_{0,t}) = s(x^*, C)$ in $\mathcal{L}^2_{\mathbb{R}}$.*

Theorem 3.8 and 3.9 yield respectively:

Corollary 4.3 (Theorem 4.32, [14]) *Let $F := (F_{s,t})$ be a subadditive superstationary family of $\mathcal{P}_{kc}(\mathbb{R}^d)$-valued random sets satisfying the assumptions of theorem 4.1. Then there exists a $\mathcal{P}_{kc}(\mathbb{R}^d)$-valued random set G having the following properties: $\int_\Omega \|G\| dP < \infty$ and $\lim_t \int_\Omega \rho_H(\frac{1}{t}F_{0,t}, G)dP = 0$.*

Corollary 4.4 (Theorem 4.35, [14]) *Let $F := (F_{s,t})$ be a subadditive superstationary family of $\mathcal{P}_{kc}(\mathbb{R}^d)$-valued random sets satisfying the assumptions of theorem 4.1. Then there exists a $G \in \mathcal{P}_{kc}(\mathbb{R}^d)$ such that $\lim_t \int_\Omega \rho_H^2(\frac{1}{t}F_{0,t}, G)dP = 0$.*

5 Proofs of the results

The following result, due to Abid ([1]) plays a key role in the proofs of the results of this paper. Let us recall, after [1], the definition of a real-valued subadditive superstationary process. We consider a family $x = (x_{s,t}) \subset \mathbb{R}^\Delta$ where $x_{s,t}$ are real-valued random variables. $\mathbb{R}^\Delta$ is equipped with the product topology. The shift T is defined analogously as in the case of $\mathcal{P}_{wkc}(X)$-valued families.

Definition 5.1 *We say that a real-valued process $x := (x_{s,t})$ is* subadditive *and* superstationary *if*

i) $x_{s,t} \le x_{s,u} + x_{u,t}$, for any triple $s, u, t \in \mathbb{N}_0$ such that $s < u < t$,

ii) the distribution of x is stochastically smaller than the distribution of Tx,

iii) $\int_\Omega x_{0,t} < +\infty$ for any $t \in \mathbb{N}$,

iv) there exists an $M > 0$ such that $\inf_{s\ge 0} \int_\Omega x_{s,s+t} \ge -Mt$, for any $t \in \mathbb{N}$.

Lemma 5.1 ([1]) *For any subadditive superstationary family $x := (x_{s,t})$, $\lim_t \frac{1}{t} x_{0,t}$ exists a.e. in $\mathcal{L}^1_{\mathbb{R}}$.*

Another important tool is an analogue of the Blaschke type lemma ([15]) for multivalued functions with values in $\mathcal{P}_{wkc}(X)$ or $\mathcal{P}_{skc}(X)$.

Lemma 5.2 (Lemma 3.2, [3]) *a) For every $K \in \mathcal{P}_{wkc}(X)$ the subset $\mathcal{K} := \{C \in \mathcal{P}_{wkc}(X) : C \subset K\}$ of $\mathcal{P}_{wkc}(X)$ is metrizable and compact for the scalar convergence topology.*

b) For every $K \in \mathcal{P}_{skc}(X)$ the subset $\mathcal{K} := \{C \in \mathcal{P}_{skc}(X) : C \subset K\}$ of $\mathcal{P}_{skc}(X)$ is compact for the Hausdorff metric ρ_H.

Consider the following lemma, which will be often used in the sequel.

Lemma 5.3 *Let $(C_n) \subset \mathcal{P}_{wkc}(X)$. Suppose there exists an $R \in \mathcal{R}_w$ such that $C_n \subset R$ for all $n \in \mathbb{N}$. Suppose also that $(\|C_n\|)$ is bounded and that for all $x^* \in H$ the sequence $s(x^*, C_n)$ converges. Then there exists a $C_\infty \in \mathcal{P}_{wkc}(X)$ such that $C_n \to C_\infty$.*

Proof. By the assumption, $s(x^*, C_n)$ converges for all $x^* \in H$. Denote

$$\alpha_{x^*} := \lim_n s(x^*, C_n) \tag{2}$$

$$r := \sup_n \|C_n\|. \tag{3}$$

Notice that $C_n \subset K := R \cap B(0, r)$. Since $R \in \mathcal{R}_w$, K is compact and by application of part a) of lemma 5.2 there exists a subsequence $(C_{n_i}) \subset (C_n)$ which scalarly converges to some $C_\infty \in \mathcal{P}_{wkc}(X)$. Obviously $s(x^*, C_\infty) = \alpha_{x^*}$ for all $x^* \in H$. By (2), (1) and the fact that H is dense in X^*, C_∞ is the unique cluster point of the sequence (C_n) in the topology of scalar convergence. Therefore $C_n \to C_\infty$. □

PROOF OF THEOREM 3.1. It will be shown that for any $x^* \in H$, $s(x^*, F) := (s(x^*, F_{s,t}))$ and $\|F\| := (\|F_{s,t}\|)$ are subadditive superstationary real-valued processes. By subadditivity of F, for any $x^* \in H$, $s(x^*, F_{s,t}) \leq s(x^*, F_{s,u}) + s(x^*, F_{u,t})$ for all $s < u < t$. By superstationarity of F, for all $x^* \in H$, $u \in \mathbb{R}$, $(s,t) \in \Delta$ we have $P\left[s(x^*, F_{s,t}) > u\right] \geq P\left[s(x^*, F_{s+1,t+1}) > u\right]$. Assumption $i)$ and subadditivity yield

$$i') \quad \int_\Omega s(x^*, F_{0,t}) < \infty, \text{ for all } x^* \in H, t \in \mathbb{N}, \text{ and}$$

$$i'') \quad \inf_{s \geq 0} \int_\Omega s(x^*, F_{s,s+t}) \geq -\tilde{k}t, \text{ for all } x^* \in H, t \in \mathbb{N}.$$

Indeed

$$\int_\Omega s(x^*, F_{0,t}) \leq \sum_{u=0}^{t-1} \int_\Omega s(x^*, F_{u,u+1}) \leq \sum_{u=0}^{t-1} \int_\Omega \|F_{u,u+1}\| \leq \tilde{k}t < \infty$$

and

$$\int_\Omega s(x^*, F_{s,s+t}) \geq -\int_\Omega \|F_{s,s+t}\| \geq \sum_{u=0}^{t-1} \int_\Omega \|F(u,u+1)\| \geq -\tilde{k}t,$$

for all $s \in \mathbb{N}_0$, $t \in \mathbb{N}$, $x^* \in X^*$. $i'')$ follows by taking inf over all $s \in \mathbb{N}_0$. Now, for all $x^* \in H$, $s(x^*, F)$ is a real-valued subadditive superstationary process in the sense of [1] (see definition 5.1). Analogously it can be shown that $(\|F_{s,t}\|)$ is also a subadditive superstationary process. Abid's pointwise ergodic theorem implies that for all $x^* \in H$ there exist null sets N_{x^*} and M and functions $\varphi_{x^*}, \psi : \Omega \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{1}{t} s(x^*, F_{0,t}(\omega)) = \varphi_{x^*}(\omega) \text{ for all } \omega \in N_{x^*}^c \tag{4}$$

$$\lim_{t\to\infty} \frac{1}{t} \|F_{0,t}(\omega)\| = \psi_{x^*}(\omega) \text{ for all } \omega \in M^c. \tag{5}$$

Define the null set $N := \bigcup_{x^* \in H} N_{x^*} \cup M$. Fix any $\omega \in N^c$. Lemma 5.3, applied to $C_t := \frac{1}{t} F_{0,t}(\omega)$, yields the existence of a set C_∞^ω such that $\frac{1}{t} F_{0,t}(\omega) \to C_\infty^\omega$. Define

$$F_\infty(\omega) := \begin{cases} C_\infty^\omega, & \omega \in N^c \\ \{0\}, & \text{otherwise} \end{cases}$$

Obviously $\frac{1}{t} F_{0,t}(\omega) \to F_\infty(\omega)$ for all $\omega \in N^c$. Since for all $x \in X$

$$d(x, F_\infty(\omega)) = \sup_{x^* \in D} \left[\langle x, x^* \rangle - s(x^*, F_\infty(\omega))\right]$$

is measurable in ω (for $s(x^*, F_\infty(\cdot))$ is a limit of measurable functions), F_∞ is measurable (see [5], theorem III.9). Moreover, $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$. Indeed,

$$\int_\Omega \|F_\infty\| \leq \int_\Omega \liminf_t \|\frac{1}{t} F_{0,t}\| \leq \liminf_t \int_\Omega \|\frac{1}{t} F_{0,t}\| = \liminf_t \frac{1}{t} \int_\Omega \|F_{0,t}\| \leq \frac{1}{t} \cdot t\tilde{k} < \infty$$

The proof of part b) is an adaptation of the proof of part b) of theorem 2.1 of [3]. □

PROOF OF THEOREM 3.2. Denote $R : \Omega \to \mathcal{R}_s$ the function such that, $\bigcup_{t=1}^\infty \frac{1}{t} F_{0,t}(\omega) \subset R(\omega)$ a.e. As in the proof of theorem 3.1 one can show that there exists a function $F_\infty^w \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ such that

$$\frac{1}{t} F_{0,t} \to F_\infty^w \tag{6}$$

(Recall that $\rightharpoonup$ denotes scalar convergence). Also, as in the proof of theorem 3.1, $\|\frac{1}{t}F_{0,t}\|$ converges to some $\psi \in \mathcal{L}^1_{\mathbb{R}}$. Thus for sufficiently large t,

$$\frac{1}{t}F_{0,t}(\omega) \subset K(\omega) := R(\omega) \cap B(0, \psi(\omega) + 1),$$

where $K(\omega) \in \mathcal{P}_{skc}(X)$ (by the s-ball compactness of $R(\omega)$). Part b) of lemma 5.2 implies that there is a subsequence $(t_i) \subset (t)$ and $F_\infty \in \mathcal{P}_{skc}(X)$ such that

$$\rho_h(\frac{1}{t_i}F_{0,t_i}, F_\infty) \to 0 \text{ a.e.} \tag{7}$$

Now (6) and (7) imply that $F_\infty(\omega) = F^w_\infty(\omega)$ (a.e.) is a unique ρ_H-cluster point of the sequence $(\frac{1}{t}F_{0,t}(\omega))$. Thus a) is proved. To prove part b) one can adapt the proof of of theorem 2.1 b) in [3]. □

Proof of theorem 3.3. As in the proof of theorem 3.1, it can be shown that $(s(x^*, F_{s,t}))$ and $(\|F_{s,t}\|)$ are subadditive, superstationary families in Abid's sense. Thus there exist $\varphi_{x^*}, \psi \in \mathcal{L}^1_{\mathbb{R}}$ such that for all $x^* \in H$

$$\begin{aligned} \varphi_{x^*}(\omega) &= \lim_t \frac{1}{t} s(x^*, F_{0,t}(\omega)), \\ \psi(\omega) &= \lim_{t\to\infty} \frac{1}{t}\|F_{0,t}(\omega)\|. \end{aligned} \tag{8}$$

The sets $G_t(\omega) := \frac{1}{t}F_{0,t}(\omega)$ are uniformly bounded by $\psi(\omega) + 1$ (for sufficiently large t). Hence $\left(\frac{1}{t}s(x^*, F_{0,t}(\omega))\right)$ is equicontinuous on X^*, thus (8) holds for all $x^* \in X^*$. Denote $R_A := \operatorname{cl}\operatorname{co}\bigcup_{t=1}^\infty \frac{1}{t}\int_A F_{0,t}$. Define $\psi_A \colon H \to \mathbb{R}$ by $\psi_A(x^*) = \int_A \varphi_{x^*}$. The function $\psi_A(x^*)$ is subadditive on H and

$$\psi_A(x^*) \le s(x^*, R_A), \text{ for all } x^* \in H, \tag{9}$$

therefore ψ_A is τ-continuous on X^* and is also w^*-lower semicontinuous. Theorem II.16 of [5] implies that there exists a nonempty closed convex subset $M(A) \subset X$ such that $\psi_A(\cdot) = s(\cdot, M(A))$. Obviously, $s(x^*, M(A)) = \int_A \varphi_{x^*} \le \int_A \psi$, thus $\|M(A)\| = \sup_{x^*\in D} s(x^*, M(A)) \le \int_A \psi < \infty$. By (9), $M(A) \subset R_A$, thus it is w-compact. Now, as was done in the proof of theorem 2.5 in [3], it can be shown that $M \colon \mathcal{A} \to \mathcal{P}_{wkc}(X)$ is additive, absolutely continuous with respect to P, has bounded variation and $s(x^*, M(\cdot))$ is σ-additive for all $x^* \in X^*$. Thus the multivalued Radon-Nikodym theorem (see [6, Théorème 3] or [7, Théorème 8, p.III.31]) can be applied. It implies the existence of a multifunction $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ such that

$$M(A) = \int_A F_\infty dP, \text{ for all } A \in \mathcal{A}. \tag{10}$$

The multifunction F_∞ is defined uniquely up to a null set. Recalling ([11]) that for multivalued functions with $s(x^*, \int_A F) = \int_A s(x^*, F)$ for all $A \in \mathcal{A}$ part a) follows. The proof of part b) is the same as the one of theorem 2.5 b) in [3]. □

Proof of theorem 3.4. By theorem 3.1 $\frac{1}{t}s(x^*, F_{0,t})$ converges in $\mathcal{L}^1_{\mathbb{R}}$ (for all $x^* \in X^*$), therefore there exists a limit

$$\lim_t \frac{1}{t}\int_\Omega s(x^*, F_{0,t}) =: \tilde{\alpha}_{x^*} < \infty, \text{ for all } x^* \in X^*. \tag{11}$$

(Note that $\tilde{\alpha}_{x^*}$ does not depend on ω.) Following the proof in [9] (p.675), assumptions i), ii), iii) and (11) imply that

$$\lim_n \frac{1}{2^n m} s(x^*, F_{0,2^n m}(\omega)) = \tilde{\alpha}_{x^*} \text{ a.e. for all } x^* \in H,\ m \in \mathbb{N} \tag{12}$$

and

$$\lim_t \frac{1}{t} s(x^*, F_{0,t}) = \tilde{\alpha}_{x^*} \text{ in } \mathcal{L}^2_{\mathbb{R}} \text{ for all } x^* \in H. \tag{13}$$

As in the proof of theorem 3.1 it can be shown that $\left(\frac{1}{t}\|F_{0,t}(\omega)\|\right)$ is a.e. bounded. Thus by (12) and lemma 5.3 there exists a set $C \in \mathcal{P}_{wkc}(X)$ such that $\frac{1}{2^n m} F_{0,2^n m} \to C$ (for any m as $n \to \infty$). In view of theorem 3.1 $\frac{1}{t}F_{0,t}(\omega) \to C$ a.e. in Ω. Recalling that $s(x^*, C) = \tilde{\alpha}_{x^*}$ for all $x^* \in X^*$ and (13) we get b). □

Proof of theorem 3.5. Part b) holds by the argument used in the proof of theorem 3.4. Let us prove a). Theorem 3.4 implies that there exists $C^w \in \mathcal{P}_{wkc}(X)$ such that

$$\lim_t \frac{1}{t} s(x^*, F_{0,t}(\omega)) = s(x^*, C^w), \text{ a.e. for all } x^* \in X^*.$$

By theorem 3.2, there exists $F_\infty \in \mathcal{P}_{skc}(X)$ satisfying $\lim_t \rho_H(\frac{1}{t}F_{0,t}(\omega), F_\infty(\omega)) = 0$ a.e. thus also $\lim_t \frac{1}{t}s(x^*, F_{0,t}(\omega)) = s(x^*, F_\infty(\omega))$ a.e. for all $x^* \in X^*$. Therefore a) holds. □

Proof of theorem 3.6. The proof of theorem 3.4 can be adapted. It is enough to use theorem 3.3 where theorem 3.1 is used. □

Proof of proposition 3.1. By the diagonal method we extract a subsequence $(n_k) \subset (n)$ such that for all $x^* \in H$ there exists a limit

$$\varphi_{x^*}(\omega) = \lim_k s(x^*, F_{n_k}(\omega)) \text{ for all } \omega \in \Omega.$$

Let $\psi(\omega) := \lim_n \|F_n(\omega)\|$. Obviously $\psi \in \mathcal{L}^m_{\mathbb{R}}$. Thus $\|F_n(\omega)\| \le \psi(\omega) + 1$ for sufficiently large n. By $\mathcal{L}^m_{\mathbb{R}}$-boundness of ψ, for any $p \in \mathbb{N}$, there exists $\Omega_p \subset \Omega$ such that $(\|F_n(\omega)\|)$ is bounded on Ω_p and $P(\Omega_p^c) < \frac{1}{p}$. Lemma 5.3 applied to $\left(F_{n_k}|_{\Omega_p}\right)$ yields the existence of $F^p_\infty \in \mathcal{P}_{wkc}(X)$ such that for $k \to \infty$

$$F_{n_k}(\omega) \to F^p_\infty(\omega) \text{ a.e. on } \Omega_p. \tag{14}$$

Utilizing the diagonal method we can renumber the sequence (n_k) such that (14) holds for any $p \in \mathbb{N}$. Obviously on Ω_p, $F^p_\infty(\omega) = F^{p+1}_\infty(\omega)$. Define

$$F_\infty(\omega) := \begin{cases} F^p_\infty(\omega), & \omega \in \Omega_p \backslash \Omega_{p-1}, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Then $F_{n_k}(\omega) \to F_\infty(\omega)$ a.e. on Ω. Thus $F_n \to F_\infty$ in probability. This yields

$$\int_\Omega |s(x^*, F_n(\omega)) - s(x^*, F_\infty(\omega))|^m \to 0.$$

To show that $F_\infty \in \mathcal{L}^m_{\mathcal{P}_{wkc}(X)}$ recall that if $C_n \to C_\infty$ then $\|C_\infty\| \le \liminf_n \|C_n\|$. Therefore $\|F_\infty\|^m_m \le \liminf \|F_n\|^m_m < \infty$. □

Proof of theorem 3.7. Theorem 3.1 implies that there exists $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ such that $\frac{1}{t}s(x^*, F_{0,t})$ converges to $s(x^*, F_\infty)$ for any $x^* \in X^*$. Also, as it follows from the proof of theorem 3.1 there exists $\psi \in \mathcal{L}^1_{\mathbb{R}}$ such that $\lim_t \|F_{0,t}(\omega)\| = \psi(\omega)$. Now the result follows from proposition 3.1. □

Proof of proposition 3.2. As in the proof of proposition 3.1 there exists a subsequence $(n_k) \subset (n)$ and an $F^w_\infty \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ such that

$$s(x^*, F_{n_k}(\omega)) \to s(x^*, F^w_\infty(\omega)) \text{ a.e. for all } x^* \in X^*. \tag{15}$$

Let $\psi(\omega) := \lim_n \|F_n(\omega)\|$. Thus for sufficiently large n, $(\|F_n(\omega)\|)$ is majorized by $\psi(\omega) + 1$. Therefore, for all $p \in \mathbb{N}$ there exists $\Omega_p \subset \Omega$ such that $P(\Omega_p^c) < \frac{1}{p}$ and $(\|F_n(\omega)\|)_n$ is bounded for $\omega \in \Omega_p$. Thus by lemma 5.2 b) there exist a subsequence $(n_{k_i}) \subset (n_k)$ and an $F^p_\infty : \Omega_p \to \mathcal{P}_{skc}(X)$ satisfying

$$\lim_{i\to\infty} \rho_H\left(F_{n_{k_i}}(\omega), F^p_\infty(\omega)\right) = 0 \text{ on } \Omega_p. \tag{16}$$

By (15) and (16), $\lim_k \rho_H(F_{n_k}(\omega), F_\infty^p(\omega)) = 0$ on Ω_p. Proceeding as in the proof of proposition 3.1 we obtain $F_\infty: \Omega \to \mathcal{P}_{skc}(X)$ such that $\rho_H(F_n, F_\infty) \to 0$ in probability and, finally

$$\lim_{n\to\infty} \int_\Omega \rho_H^m(F_n, F_\infty) = 0.$$

To prove that $F_\infty \in \mathcal{L}^m_{\mathcal{P}_{skc}(X)}$ one can proceed as in the proof of proposition 3.1. □

Proof of theorem 3.8. By theorem 3.2 there exists an $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{skc}(X)}$ such that a.e. $\rho_H\left(\frac{1}{t}F_{0,t}(\omega), F_\infty(\omega)\right) \to 0$, thus $\left(\frac{1}{t}s(x^*, F_{0,t})\right)$, $\left(\frac{1}{t}\|F_{0,t}\|\right)$ converge in $\mathcal{L}^1_{\mathbb{R}}$. The result follows now from proposition 3.2. □

Proof of theorem 3.9. The existence of $C \in \mathcal{P}_{skc}(X)$ follows from theorem 3.5. The result follows by proposition 3.2, whose assumptions are satisfied by theorem 3.5. □

Acknowledgement. I would like to thank professor Erik Balder for stimulating discussions, helpful remarks and comments.

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GK: Mathematical Institute, University of Utrecht, P.O. Box 80.010, 3508 TA Utrecht, the Netherlands; e-mail: `krupa@math.ruu.nl`