The action of intertwining operators on spherical vectors in the minimal principal series of a reductive symmetric space

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Contents

Introduction

Let G be a connected real semisimple Lie group with finite center (or, more generally, a group of Harish-Chandra's class), σ an involution of G and H an open subgroup of the group G^σ of fixed points for σ. Moreover, let θ be a Cartan involution of G commuting with σ. In the Plancherel decomposition of $L^2(G/H)$, the most-continuous part is built from the minimal principal series for G/H. This is a series of parabolically induced representations $\pi_{\xi,\lambda} = \mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$, where $P = MAN$ is a minimal $\sigma\theta$-stable parabolic subgroup of G with the indicated Langlands decomposition, ξ a finite dimensional unitary representation of M and $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$, the space of complex characters ν of A with $\sigma\nu = -\nu$. Let $C^{-\infty}(P:\xi:\lambda)$ denote the space of generalized sections of the homogeneous vector bundle in which $\pi_{\xi,\lambda}$

is naturally realized. Then the subspace $C^{-\infty}(P:\xi:\lambda)^H$ of H-fixed generalized sections, also called the space of spherical vectors for $\pi_{\xi,\lambda}$, governs the contribution of $\pi_{\xi,\lambda}$ to the Plancherel decomposition. In the theory an important role is played by the standard intertwining operator

$$A(Q:P:\xi:\lambda):\ \ C^{-\infty}(P:\xi:\lambda) \to C^{-\infty}(Q:\xi:\lambda),$$

with Q a parabolic subgroup associated to P (i.e. its Langlands M, A-parts are the same). By equivariance the standard intertwining operator maps the spherical vectors for $\pi_{P,\xi,\lambda}$ to those for $\pi_{Q,\xi,\lambda}$.

In [1] we established the existence of a finite dimensional space Hilbert space $V(\xi)$, and a linear map

$$j(P:\xi:\lambda) : V(\xi) \to C^{-\infty}(P:\xi:\lambda)^H$$

depending meromorphically on $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$, and bijective for generic $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$. We also established the existence of a unique endomorphism $B(Q:P:\xi:\lambda)$ of $V(\xi)$, depending meromorphically on $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$, such that the following diagram commutes:

$$\begin{tikzcd} C^{-\infty}(P:\xi:\lambda)^H \arrow[r, "A(Q:P:\xi:\lambda)"] & C^{-\infty}(Q:\xi:\lambda)^H \\ V(\xi) \arrow[u, "j(P:\xi:\lambda)"] \arrow[r, "B(Q:P:\xi:\lambda)"] & V(\xi) \arrow[u, "j(Q:\xi:\lambda)"'] \end{tikzcd} \qquad (1)$$

Note that $B(Q:P:\xi:\lambda)$ thus essentially describes the action of the standard intertwining operators on the spherical vectors of the representations of the minimal principal series. In [1] we proved the following formula, where the suffix $*$ indicates the adjoint of an endomorphism of the Hilbert space $V(\xi)$:

$$B(Q:P:\xi:-\bar{\lambda})^* = B(P:Q:\xi:\lambda). \qquad (2)$$

For part of the argument leading to this formula we needed the restrictive assumptions that $H = G^\sigma$, the full fixed point group, and that all Cartan subgroups of G are abelian. The main result of the present paper is that (2) holds without restrictions on G, H.

The main difficulty in the proof is caused by the fact that the group $G_e \cap P$ need not be connected. Therefore a major part of the present paper is devoted to the description of connected components of parabolic subgroups and the action of σ on them.

The formula (2) plays a fundamental role in the harmonic analysis on G/H, since it lies at the heart of the Maass-Selberg relations for Eisenstein integrals related to the minimal principal series, see [2]. These Maass-Selberg relations in turn play a fundamental role in normalizations of Eisenstein integrals, see [4], and in the most-continuous part of the Plancherel decomposition for G/H, see [5].

The main result of this paper was (implicitly) announced some time ago in the survey paper [6] (cf. Theorem 11). In recent work ([7]) P. Delorme has established Maass-Selberg relations in the more general context of Eisenstein integrals for non-minimal $\sigma\theta$-stable parabolic subgroups, following a completely different method, involving the idea of truncation of eigenfunctions. The results of the present paper can now also be obtained by an application of Delorme's result (use the ideas of [2], Lemma 15.1 and Cor. 15.3).

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1 The minimal principal series

Throughout this paper G will be a group of Harish-Chandra's class, σ an involution of G and H an open subgroup of the group G^σ of fixed points for σ. Let θ be a Cartan involution of G commuting with σ. We denote the associated infinitesimal involutions of $\mathfrak{g}$, the Lie algebra of G, by the same symbols. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$ be the associated decompositions into $+1$ and -1 eigenspaces for θ and σ respectively. Then $\mathfrak{h}$ is the Lie algebra of H. In general we adopt the convention to denote Lie groups by Roman capitals and their Lie algebras by the corresponding lower case Gothic letters.

We extend the Killing form on $[\mathfrak{g}, \mathfrak{g}]$ to an invariant non-degenerate bilinear form B on $\mathfrak{g}$, which is negative definite on $\mathfrak{k}$, positive definite on $\mathfrak{p}$, and invariant under σ.

Let $\mathfrak{a}_{\mathrm{q}}$ be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. Then $\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ is a (possibly non-reduced) root system. By a σ-parabolic subgroup of G we will mean a parabolic subgroup that is invariant under the composition $\sigma\theta$. If P is a σ-parabolic subgroup and $P = M_P A_P N_P$ its Langlands decomposition, then M_P and A_P are both σ- and θ-invariant, and $\sigma P = \theta P$ is the opposite $\bar{P}$ of P. We write $M_{1P} = M_P A_P$, $A_{P\mathrm{h}} = A_P \cap H$, and

$$A_{P\mathrm{q}} = \{a \in A_P \mid \sigma a = a^{-1}\}.$$

Then $A_P \simeq A_{P\mathrm{h}} \times A_{P\mathrm{q}}$. Via the associated direct sum decomposition $\mathfrak{a}_P = \mathfrak{a}_{P\mathrm{h}} \oplus \mathfrak{a}_{P\mathrm{q}}$ we identify the complexified dual $\mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$ with a subspace of $\mathfrak{a}^*_{P\mathbb{C}}$.

Let $\mathcal{P}(A_{\mathrm{q}})$ denote the set of minimal σ-parabolic subgroups that contain $A_{\mathrm{q}} := \exp \mathfrak{a}_{\mathrm{q}}$. The components M_P and A_P are independent of the particular choice of $P \in \mathcal{P}(A_{\mathrm{q}})$. We therefore write M and A for these components. Moreover, $A_{P\mathrm{q}} = A_{\mathrm{q}}$. We write $M_1 = MA$ in accordance with the notations introduced before. Let $\Sigma(P)$ denote the set of roots of $\mathfrak{a}_{\mathrm{q}}$ in $\mathfrak{n}_P$. Then $P \mapsto \Sigma(P)$ defines a one-to-one correspondence from $\mathcal{P}(A_{\mathrm{q}})$ onto the set of positive root systems for Σ. The Weyl group $W = W(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ associated with the root system is naturally isomorphic with $N_K(\mathfrak{a}_{\mathrm{q}})/Z_K(\mathfrak{a}_{\mathrm{q}})$, the normalizer modulo the centralizer of $\mathfrak{a}_{\mathrm{q}}$ in K. Via conjugation it acts freely and transitively on the set $\mathcal{P}(A_{\mathrm{q}})$.

The group $N_K(\mathfrak{a}_{\mathrm{q}})$ normalizes M, and naturally induces an action of W on the set $\widehat{M}_{\mathrm{fu}}$ of (equivalence classes of) irreducible finite dimensional unitary representations of M. Let $\widehat{M}_H$ be the subset of $\xi \in \widehat{M}_{\mathrm{fu}}$ for which there exists a $w \in W$ such that $w\xi$ possesses a non-trivial $M \cap H$-fixed vector.

If $P \in \mathcal{P}(A_{\mathrm{q}})$, then the element $\rho_P = \frac{1}{2}\mathrm{tr}(\mathrm{ad}(\cdot)|\mathfrak{n}_P)$ of $\mathfrak{a}^*_P$ vanishes on $\mathfrak{a}_{P\mathrm{h}}$ hence defines an element of $\mathfrak{a}^*_{\mathrm{q}}$. If $\xi \in \widehat{M}_{\mathrm{fu}}$, let $\mathcal{H}_\xi$ be a fixed Hilbert space on which ξ is (unitarily) realized. If in addition $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$, then we define

$$C^{-\infty}(P : \xi : \lambda) \qquad (3)$$

to be the space of generalized functions $f : G \to \mathcal{H}_\xi$ transforming according to the rule:

$$f(manx) = a^{\lambda+\rho_P}\xi(m)f(x)$$

for $x \in G$, $m \in M$, $a \in A$, $n \in N_P$. We equip this space with the restriction of the right regular representation R of G. The Harish-Chandra module associated with the induced representation thus defined is denoted by $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$. The series of representations $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$ with $\xi \in \widehat{M}_H$, $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ is called the (minimal non-unitary) principal series for G/H.

In the following we shall describe the space $C^{-\infty}(P:\xi:\lambda)^H$ of H-fixed elements in (3). Let $W_{K\cap H}$ denote the canonical image of $N_{K\cap H}(\mathfrak{a}_\mathrm{q})$ in W. Fix a set $\mathcal{W}$ of representatives for the quotient space $W/W_{K\cap H}$ in $N_K(\mathfrak{a}_\mathrm{q})$. Then the map $w \mapsto PwH$ defines a one-to-one correspondence from $\mathcal{W}$ onto the set of open H-orbits on $P\backslash G$. If j is an H-fixed element of (3), then j is a smooth function on every open H-orbit. Therefore $j(w)$ is a well defined element of $\mathcal{H}_\xi$. In fact one readily verifies that $j(w)$ is $w(M\cap H)w^{-1}$-fixed. Thus we have a well defined evaluation map $\mathrm{ev}_w : j \to j(w)$ from the space of H-fixed elements in (3) to $\mathcal{V}(\xi, w) := (\mathcal{H}_\xi)^{w(M\cap H)w^{-1}}$. Equip $\mathcal{V}(\xi, w)$ with the restriction of the Hilbert structure of $\mathcal{H}_\xi$, and define the formal direct sum of Hilbert spaces:

$$V(\xi) = \oplus_{w\in\mathcal{W}}\mathcal{V}(\xi, w). \tag{4}$$

Then the direct sum ev of the evaluation maps ev_w, $w \in \mathcal{W}$ defines a linear map

$$\mathrm{ev} : C^{-\infty}(P:\xi:\lambda)^H \to V(\xi).$$

From [1], Corollary 5.3 and Theorem 5.10, we recall that for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ the map ev is a bijection. Here and in the following we will say that a statement holds for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, if it holds for λ in the complement of a countable union of complex hyperplanes of $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

The inverse to ev is given by the following result of [1]. For its appropriate formulation we need the compact picture of the induced representation $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$. In view of the decomposition $G = PK$, the restriction map $f \mapsto f|K$ induces a bijection from (3) onto the space

$$C^{-\infty}(K:\xi) \tag{5}$$

of generalized functions $f : K \to \mathcal{H}_\xi$ transforming according to the rule: $f(mk) = \xi(m)f(k)$ for all $m \in K_\mathrm{M} := K \cap M$ and $k \in K$. If $s \in \mathbb{N}$, then by $C^s(K:\xi)$ we denote the subspace of (5) consisting of the s-times continuously differentiable functions. This space carries a Banach topology since K is compact. If $s \in \mathbb{N}$, then by $C^{-s}(K:\xi)$ we denote the subspace of (5) consisting of the generalized functions of order at most s. Being the dual of $C^s(K:\xi^\vee)$, this space carries a Banach topology as well. Let $\Omega \subset \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ be an open subset, then a function $\varphi : \Omega \to C^{-\infty}(K:\xi)$ will be called holomorphic if for every $\lambda_0 \in \Omega$ there exists an open neighbourhood Ω_0 of λ_0 in Ω and a constant $s \in \mathbb{N}$ such that $\varphi(\Omega_0) \subset C^{-s}(K:\xi)$ and $\varphi|\Omega_0$ is a holomorphic map from Ω_0 into the Banach space $C^{-s}(K:\xi)$. A partially defined $C^{-\infty}(K:\xi)$-valued function on Ω will be called meromorphic if for every $\lambda_0 \in \Omega$ there exists a holomorphic $\mathbb{C}$ valued function ψ in a neigbourhood of λ_0 such that $\psi\varphi$ is holomorphic on this neighbourhood. The following result is essentially [1], Theorem 5.10.

Proposition 1.1 *Let $P \in \mathcal{P}(A_{\mathrm{q}})$, $\xi \in \widehat{M}_{\mathrm{fu}}$. Then there exists a unique meromorphic function $j(P:\xi:\cdot) : \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to V(\xi)^* \otimes C^{-\infty}(K:\xi)$ such that*

(a) *$j(P:\xi:\lambda)\eta \in C^{-\infty}(P:\xi:\lambda)^H$ for all $\eta \in V(\xi)$ and for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$;*

(b) *for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ we have $\mathrm{ev} \circ j(P:\xi:\lambda) = I_{V(\xi)}$.*

2 Intertwining operators

Let $P, Q \in \mathcal{P}(A_{\mathrm{q}})$. We recall that for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ the standard intertwining operator from $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$ to $\mathrm{Ind}_Q^G(\xi \otimes \lambda \otimes 1)$ uniquely extends to a (equivariant) continuous linear map

$$A(Q:P:\xi:\lambda) \; : \; C^{-\infty}(P:\xi:\lambda) \to C^{-\infty}(Q:\xi:\lambda).$$

In a suitable sense (cf. [1], §4) the intertwining operator depends meromorphically on $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. By equivariance it maps H-fixed elements to H-fixed elements. We recall from [1], Proposition 6.1, that there exists a unique $\mathrm{End}(V(\xi))$-valued meromorphic function $B(Q:P:\xi:\cdot)$ on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ such that the diagram (1) commutes for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. From Proposition 1.1 (b) and the commutativity of diagram (1) we see that the endomorphism B is also given by

$$B(Q:P:\xi:\lambda) = \mathrm{ev} \circ A(Q:P:\xi:\lambda) \circ j(P:\xi:\lambda). \tag{6}$$

The following result, which is the main result of this paper, was proved in [1] under the restrictive hypotheses that all Cartan subgroups of G are abelian and H equals the full group G^σ of fixed points for σ in G. If B is an endomorphism of $V(\xi)$, let B^* denote its adjoint with respect to the Hilbert structure defined above.

Theorem 2.1 *Let $P, Q \in \mathcal{P}(A_{\mathrm{q}})$. Then we have the following identity of $\mathrm{End}V(\xi)$-valued meromorphic functions on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$:*

$$B(Q:P:\xi:-\bar{\lambda})^* = B(P:Q:\xi:\lambda).$$

We will reduce the proof of this theorem to a particular case. The following lemma provides a first step in this reduction.

Lemma 2.2 *Let H_1, H_2 be open subgroups of G^σ such that $H_1 \subset H_2$. If Theorem 2.1 is true with $H = H_1$, then it is also true with $H = H_2$.*

Proof: We assume that Theorem 2.1 holds for $H = H_1$. For $k = 1, 2$, put $W_k = W_{K\cap H_k}$. Let $p : W/W_1 \to W/W_2$ be the natural projection map. Each of its fibers has $[W_2 : W_1]$ elements.

Let $\mathcal{W}_2$ be a set of representatives for W/W_2 in $N_K(\mathfrak{a}_{\mathrm{q}})$. For each $w_2 \in \mathcal{W}_2$ and every $s \in p^{-1}(\bar{w}_2)$ we select a representative $w_1(s) \in w_2 N_{K\cap H_2}(\mathfrak{a}_{\mathrm{q}}) \subset N_K(\mathfrak{a}_{\mathrm{q}})$ for s. Let $p_{21} : \mathcal{W}_1 \to \mathcal{W}_2$ be the map corresponding to p. Then the above choice comes down to

$$v \in p_{21}(v) N_{K\cap H_2}(\mathfrak{a}_{\mathrm{q}}) \quad \text{for each} \quad v \in \mathcal{W}_1.$$

Note that by [1], Remark 6.5 the choice of representatives does not affect the validity of Theorem 2.1.

Now that compatible choices for $\mathcal{W}_1, \mathcal{W}_2$ have been made, let for $k = 1, 2$ the objects $V_k(\xi)$, ev_k, $j_k(P\!:\!\xi\!:\!\lambda)$, $B_k(Q\!:\!P\!:\!\xi\!:\!\lambda)$ be defined as before, but with H_k in place of H.

For $k = 1, 2$, let $\mathcal{F}(\mathcal{W}_k, \mathcal{H}_\xi)$ denote the space of functions $\mathcal{W}_k \to \mathcal{H}_\xi$, equipped with the direct sum Hilbert structure. Define the map $i_k : \mathcal{V}_k(\xi, w) \to \mathcal{F}(\mathcal{W}_k, \mathcal{H}_\xi)$ by $i_k(\eta)(w) = \eta_w$ for $w \in \mathcal{W}_k$. Then i_k is an isometry from $V_k(\xi)$ into $\mathcal{F}(\mathcal{W}_k, \mathcal{H}_\xi)$. Via this isometry we shall view $V_k(\xi)$ as a subspace of $\mathcal{F}(\mathcal{W}_k, \mathcal{H}_\xi)$.

Let $p_{21}^* : \mathcal{F}(\mathcal{W}_2, \mathcal{H}_\xi) \to \mathcal{F}(\mathcal{W}_1, \mathcal{H}_\xi)$ be defined by pull-back: $p_{21}^*\varphi = \varphi \circ p_{21}$. Since each fiber of p_{21} has $[W_2 : W_1]$ elements, it follows that $[W_2 : W_1]^{-\frac{1}{2}} p_{21}^*$ is an isometric embedding. One readily checks that p_{21}^* maps $V_2(\xi)$ into $V_1(\xi)$. Let i_{12} be the restriction of p_{21}^* to $V_2(\xi)$, then it follows that

$$i_{12}^* \circ i_{12} = [W_2 : W_1]\, I_{V_2(\xi)}. \tag{7}$$

Let $I_{12} : C^{-\infty}(P : \xi : \lambda)^{H_2} \hookrightarrow C^{-\infty}(P : \xi : \lambda)^{H_1}$ be the inclusion map. Then one readily checks from the definitions that $\mathrm{ev}_1 \circ I_{12} = I_{12} \circ \mathrm{ev}_2$ and that

$$j_1(P\!:\!\xi\!:\!\lambda) \circ i_{12} = I_{12} \circ j_2(P\!:\!\xi\!:\!\lambda) \quad \text{on} \quad V_2(\xi).$$

Combining the last two formulas with (6) we infer that for all $P, Q \in \mathcal{P}(A_{\mathrm{q}})$ we have:

$$B_1(Q\!:\!P\!:\!\xi\!:\!\lambda) \circ i_{12} = i_{12} \circ B_2(Q\!:\!P\!:\!\xi\!:\!\lambda). \tag{8}$$

Taking adjoints of both sides of (8), applying i_{12} to the right and using (7) we obtain:

$$\begin{aligned} [W_2 : W_1]\, B_2(Q\!:\!P\!:\!\xi\!:\!\lambda)^* &= i_{21}^* \circ B_1(Q\!:\!P\!:\!\xi\!:\!\lambda)^* \circ i_{21} \\ &= i_{21}^* \circ B_1(P\!:\!Q\!:\!\xi\!:\!-\bar{\lambda}) \circ i_{21}, \end{aligned} \tag{9}$$

the last equality being a consequence of the hypothesis. In view of (8) with Q, P, ξ, λ replaced by $P, Q, \xi, -\bar{\lambda}$, it follows that the right hand side of (9) may be rewritten as

$$i_{21}^* \circ i_{21} \circ B_2(P\!:\!Q\!:\!P\!:\!\xi\!:\!-\bar{\lambda}) = [W_2 : W_1]\, B_2(P\!:\!Q\!:\!P\!:\!\xi\!:\!-\bar{\lambda}).$$

Hence Theorem 2.1 holds for $H = H_2$ as well. $\square$

Reduction of the proof of Theorem 2.1 to a particular case. Before we proceed with the proof of this theorem we discuss its reduction to a particular case that we did not succeed to handle in full generality in [1].

As we mentioned above, Theorem 2.1 was proved in [1] under the two assumptions that all Cartan subgroups of G are abelian and moreover that $H = G^\sigma$. The first of these assumptions was used to ensure that $M = M_e F$, with F a finite abelian subgroup. This fact was used at precisely two places in [1]: Lemma 5.4 and Lemma 6.16. The last assumption was not explicitly mentioned, but used in the proof of Lemma 6.16.

The first part of the proof of Theorem 2.1 consists of a reduction to the σ-split rank one case, i.e. $\dim \mathfrak{a}_\mathrm{q} = 1$. This reduction does not rely on any of the assumptions mentioned above, and can therefore be used in the present situation as well. Thus it suffices to prove Theorem 2.1 with $Q = \bar{P}$ in case $\dim \mathfrak{a}_\mathrm{q} = 1$. If $\Sigma = \emptyset$ then $\mathfrak{a}_\mathrm{q}$ is central in G, and there is nothing to prove. Therefore we may, and will, assume that $\Sigma \neq \emptyset$. Then the Weyl group W has order two, so that $W/W_{K\cap H}$ has either one or two elements. The proof in [1] of Theorem 2.1 in the latter case does not make any use of the assumptions mentioned above, and is valid without change in the present situation. Thus it remains to prove Theorem 2.1 in the case that $\dim \mathfrak{a}_\mathrm{q} = 1$ and $|W/W_{K\cap H}| = 1$. Moreover, by Lemma 2.2 we may in addition assume that H is connected. We call the resulting case, where these three assumptions are fulfilled, the reduced case. The proof of Theorem 2.1 in the reduced case will be given in Section 8.

3 Connected components of parabolic subgroups

This section is independent of the rest of the paper. Its purpose is to give a characterization of the possible connected components of parabolic subgroups of G when G is a *connected* group of Harish-Chandra's class.

Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}$, let Σ_0 be the system of roots of $\mathfrak{a}_0$ in $\mathfrak{g}$ and let Σ_0^+ be a choice of positive roots for this system.

Let $P_0 = M_0 A_0 N_0$ be the minimal parabolic subgroup of G associated with the pair $(\mathfrak{a}_0, \Sigma_0^+)$, and let P be any parabolic subgroup of G containing P_0. In this section we shall write $P = MAN$ for its Langlands decomposition, and $M_1 = MA$. Let

$$\Sigma_{0\mathrm{M}} := \{\alpha \in \Sigma_0 \mid \alpha = 0 \quad \text{on} \quad \mathfrak{a}\}.$$

Then $\Sigma_{0\mathrm{M}}$ equals the system of roots of $\mathfrak{a}_0$ in the centralizer $\mathfrak{m}_1$ of $\mathfrak{a}$ in $\mathfrak{g}$. Moreover, $\Sigma_{0\mathrm{M}}^+ := \Sigma_{0\mathrm{M}} \cap \Sigma_0^+$ is a choice of positive roots for this system.

Finally, we put $\Sigma_0(P) := \Sigma_0^+ \setminus \Sigma_{0\mathrm{M}}$. Then $\Sigma_0(P)$ equals the set of roots of $\mathfrak{a}_0$ in $\mathfrak{n}$. Let $\mathfrak{n}_\mathrm{M} = \mathfrak{n}_0 \cap \mathfrak{m}$. Then

$$\mathfrak{n}_0 = \mathfrak{n}_\mathrm{M} \oplus \mathfrak{n}.$$

Let $S_0, S_{0\mathrm{M}}$ denote the sets of simple roots of Σ_0^+ and $\Sigma_{0\mathrm{M}}^+$ respectively. Then it is well known that

$$S_{0\mathrm{M}} = S_0 \cap \Sigma_{0\mathrm{M}}. \tag{10}$$

Let $W_0, W_{0\mathrm{M}}$ denote the Weyl groups of the root systems Σ_0 and $\Sigma_{0\mathrm{M}}$ respectively. Then we have natural isomorphisms:

$$W_0 \simeq N_K(\mathfrak{a}_0)/Z_K(\mathfrak{a}_0), \qquad W_{0\mathrm{M}} \simeq N_{K\cap M}(\mathfrak{a}_0)/Z_{K\cap M}(\mathfrak{a}_0).$$

The following lemma is crucial for the purposes of this section. We recall that by the Bruhat decomposition the map $s \mapsto N_0 sP$ induces a bijection from $W_0/W_{0\mathrm{M}}$ onto the set $N_0\backslash G/P$ of N_0-orbits on G/P. Let

$${}_{\mathrm{M}}W_0 = \{v \in W \mid v(\Sigma_{0\mathrm{M}}^+) \subset \Sigma_0^+\}.$$

Then it is well known that the multiplication map ${}_{\mathrm{M}}W_0 \times W_{0\mathrm{M}} \to W_0$ is a bijection. Hence the map $s \mapsto N_0 sP$ defines a one-to-one correspondence from ${}_{\mathrm{M}}W_0$ onto $N_0\backslash G/P$.

Lemma 3.1 *Let $w \in N_K(\mathfrak{a}_0)$ be a representative for $s \in W_0$. Then the orbit $N_0 wP$ on G/P has dimension one if and only if $s \in s_\alpha W_{0\mathrm{M}}$, with $\alpha \in S_0$ a simple root such that*

(a) $\dim \mathfrak{g}_\alpha = 1$;

(b) $\alpha \notin \Sigma_{0\mathrm{M}}$.

Remark 3.2 Note that (a) implies that $2\alpha \notin \Sigma_0$.

Proof: If $\alpha \in S_0$ satisfies (b) then $s_\alpha \in {}_{\mathrm{M}}W_0$. We therefore assume that $s \in {}_{\mathrm{M}}W_0$; then it suffices to show that $N_0 sP$ has dimension 1 if and only if $s = s_\alpha$ with $\alpha \in S_0$ such that (a) and (b) hold. Let w be a representative of s in $N_K(\mathfrak{a}_0)$.

Recall that P has Lie algebra $\mathfrak{m}_1 + \mathfrak{n}$. By a standard computation of differentials one therefore readily checks that the orbit $N_0 sP$ has dimension one if and only if

$$\dim\left[\mathfrak{n}_0/\mathfrak{n}_0 \cap \mathrm{Ad}(w)(\mathfrak{m}_1 + \mathfrak{n})\right] = 1,$$

which in turn is equivalent to

$$\dim\left[\mathfrak{n}_0/\mathfrak{n}_0 \cap \mathrm{Ad}(w)(\bar{\mathfrak{n}}_{\mathrm{M}} + \mathfrak{n}_{\mathrm{M}} + \mathfrak{n})\right] = 1. \tag{11}$$

The hypothesis on s implies that $\mathfrak{n}_0 \cap \mathrm{Ad}(w)(\bar{\mathfrak{n}}_{\mathrm{M}}) = 0$, hence (11) is equivalent to the assertion that $\mathfrak{n}_0 \cap \mathrm{Ad}(w)(\mathfrak{n}_0) = \mathfrak{n}_0 \cap \mathrm{Ad}(w)(\mathfrak{n}_{\mathrm{M}} + \mathfrak{n})$ has codimension 1 in $\mathfrak{n}_0$. Since $\mathrm{Ad}(w)$ leaves $\mathfrak{n}_0 + \bar{\mathfrak{n}}_0$ invariant, the latter assertion is equivalent to

$$\dim[\bar{\mathfrak{n}}_0 \cap \mathrm{Ad}(w)\mathfrak{n}_0] = 1. \tag{12}$$

From (12) it follows that s is a Weyl group element of length 1, hence $s = s_\alpha$ with $\alpha \in S_0$. This implies that $\bar{\mathfrak{n}}_0 \cap \mathrm{Ad}(w)\mathfrak{n}_0 = \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$, hence (a). Moreover, (b) follows from the hypothesis that $s \in {}_{\mathrm{M}}W_0$.

Conversely assume that $\alpha \in S_0$ satisfies conditions (a) and (b) and put $s = s_\alpha$. Then $s \in {}_{\mathrm{M}}W_0$. Moreover, $\bar{\mathfrak{n}}_0 \cap \mathrm{Ad}(w)\mathfrak{n}_0 = \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$, and from (a) and Remark 3.2 it follows that the latter space has dimension 1, whence (12). $\square$

If P is a parabolic subgroup of G containing P_0, we write T_P for the set of roots $\alpha \in S_0$ such that conditions (a) and (b) of Lemma 3.1 are fulfilled. If Q is another such parabolic, then $P \supset Q \Rightarrow T_P \subset T_Q$. Put

$$T_0 = T_{P_0} = \{\alpha \in S_0 \mid \dim \alpha = 1\}. \tag{13}$$

Then in particular we have $T_P \subset T_0$.

For every $\alpha \in T_0$, let $H_\alpha \in \mathfrak{a}_0$ be the element orthogonal to $\ker \alpha$ determined by $\alpha(H_\alpha) = 2$. We define

$$\mathfrak{g}(\alpha) = \mathfrak{g}_{-\alpha} \oplus \mathbb{R}H_\alpha \oplus \mathfrak{g}_\alpha.$$

Obviously $\mathfrak{g}(\alpha)$ is a θ-invariant subalgebra of $\mathfrak{g}$. Since $\mathfrak{g}_\alpha$ is one dimensional, this subalgebra is isomorphic to $sl(2,\mathbb{R})$. We fix once and for all an element $X_\alpha \in \mathfrak{g}_\alpha$ such that $[X_\alpha, \theta X_\alpha] = -H_\alpha$. Since $\mathfrak{g}_\alpha$ is one dimensional, there are two possible choices of X_α, which differ by a minus sign.

Put $X_{-\alpha} = -\theta X_\alpha$. Then the triple $H_\alpha, X_\alpha, X_{-\alpha}$ is a standard $sl(2,\mathbb{R})$-triple. Let $i_\alpha : sl(2,\mathbb{R}) \to \mathfrak{g}(\alpha)$ be the Lie algebra isomorphism defined by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto H_\alpha, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto X_\alpha, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto X_{-\alpha}. \tag{14}$$

Then the pull-back of θ by i_α is the standard Cartan involution $X \mapsto -X^t$ of $sl(2,\mathbb{R})$.

For every $\alpha \in T_0$ we define

$$w_\alpha = \exp(\frac{\pi}{2}[X_\alpha + \theta X_\alpha]) \quad \text{and} \quad f_\alpha = w_\alpha^2. \tag{15}$$

Then w_α is a representative of the simple reflection s_α in $N_K(\mathfrak{a}_0)$, and f_α is contained in M_0, the centralizer of $\mathfrak{a}_0$ in K.

Let F_0 be the group generated by the elements f_α $(\alpha \in T_0)$. Then $F_0 \subset M_0$.

Lemma 3.3 *The group F_0 is a finite subgroup of M_0. Moreover,* $\mathrm{Ad}(F_0)$ *is abelian, consists of quadratic elements, and centralizes* $\mathfrak{m}_0$.

Proof: Let $G_{\mathbb{C}}$ be the connected complex automorphism group of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. Then $\mathrm{Ad}(G)$ is a subgroup of $G_{\mathbb{C}}$. The Lie algebra of $G_{\mathbb{C}}$ is naturally isomorphic with the complexification $(\mathrm{ad}\mathfrak{g})_{\mathbb{C}}$.

Fix $\alpha \in T_0$. Let $j_\alpha : sl(2,\mathbb{C}) \to (\mathrm{ad}\mathfrak{g})_{\mathbb{C}}$ be the complex linear extension of the monomorphism $\mathrm{ad} \circ i_\alpha$. Since $\mathrm{SL}(2,\mathbb{C})$ is simply connected, j_α lifts to a Lie group homomorphism $j_\alpha : \mathrm{SL}(2,\mathbb{C}) \to G_{\mathbb{C}}$. Let f be minus the identity matrix in $\mathrm{SL}(2,\mathbb{C})$. Then using (14) one readily sees that $j_\alpha(f) = \mathrm{Ad}(f_\alpha)$. From $f^2 = I$ it now follows that $\mathrm{Ad}(f_\alpha)^2 = I$. Let $F = \{k \in \mathrm{Ad}(K) | k^2 = I\}$. Then it is known that

$$F = \mathrm{Ad}(K) \cap \exp(i\mathrm{ad}\mathfrak{a}_0);$$

hence F is a finite abelian group that centralizes $\mathfrak{m}_0$ (cf. [9], p.435, Exercise A3). Since Ad maps the generators of F_0 into F, it follows that $\mathrm{Ad}(F_0) \subset F$, and all assertions follow. □

For every $\alpha \in T_P$ we define the smooth curve $c_\alpha : [-\frac{\pi}{2}, \frac{\pi}{2}] \to G/P$ by

$$c_\alpha(s) = w_\alpha \exp[s(X_\alpha + \theta X_\alpha)]\, P. \tag{16}$$

Then $c_\alpha(-\frac{\pi}{2}) = eP$ and $c_\alpha(\frac{\pi}{2}) = w_\alpha^2 P = f_\alpha P = eP$, hence c_α is a loop in G/P based at eP. Its class in the fundamental group $\Pi_1(G/P, eP)$ of G/P relative to the base point eP is denoted by $[c_\alpha]$.

Lemma 3.4 *The fundamental group $\Pi_1(G/P, eP)$ is generated by the classes $[c_\alpha]$, $\alpha \in T_P$.*

Proof: The manifold G/P is equipped with the structure of CW-complex whose cells are the orbits for the natural N_0-action. Let Σ_1 denote the union of the closures of the one dimensional cells, that is (Lemma 3.1):

$$\Sigma_1 = \cup_{\alpha \in T_P} \mathrm{cl}\,(N_0 w_\alpha P).$$

Then the inclusion map $i : \Sigma_1 \to G/P$ induces a surjective homomorphism of groups $i_* : \Pi_1(\Sigma_1, eP) \to \Pi_1(G/P, eP)$. We will finish the proof by showing that the loops c_α are all contained in Σ_1, and that $\Pi_1(\Sigma_1, eP)$ is the free group generated on the classes of the loops c_α, $\alpha \in T_P$.

For $\alpha \in T_P$ let $G(\alpha)$ be the analytic subgroup of G with Lie algebra $\mathfrak{g}(\alpha)$; recall that the latter algebra is isomorphic to $sl(2, \mathbb{R})$. The group $P(\alpha) = G \cap P$ is a (minimal) parabolic subgroup of $G(\alpha)$; hence $G(\alpha)/P(\alpha)$ is a circle, and the inclusion map $G(\alpha) \to G$ induces an embedding $\sigma_\alpha : G(\alpha)/P(\alpha) \to G/P$. The image of σ_α is denoted by S_α. By a straightforward $\mathrm{SL}(2, \mathbb{R})$ computation it follows that the class of c_α generates $\Pi_1(S_\alpha, e) \simeq \mathbb{Z}$. We will finish the proof by showing that the spheres S_α, $\alpha \in T_P$ form a bouquet with basepoint eP and union Σ_1.

If $\alpha \in T_P$, then $S_\alpha \setminus \{eP\}$ equals the image under σ_α of $G(\alpha)/P(\alpha)$ minus the origin $eP(\alpha)$. The Bruhat decomposition of $G(\alpha)/P(\alpha)$ consists of the two cells $eP(\alpha)$ and $N(\alpha)w_\alpha P(\alpha)$, where $N(\alpha) = \exp(\mathfrak{g}_\alpha)$ is the unipotent radical of $P(\alpha)$. We conclude that $S_\alpha \setminus \{eP\} = \sigma_\alpha(N(\alpha)w_\alpha P(\alpha)) = N(\alpha)w_\alpha P$. Thus if $\alpha, \beta \in T_P$ and the spheres S_α, S_β have a point besides eP in common, then $N(\alpha)w_\alpha P \cap N(\beta)w_\beta P \neq \emptyset$, hence $N_0 w_\alpha P \cap N_0 w_\beta P \neq \emptyset$, from which it follows that $s_\alpha W_{0\mathrm{M}} = s_\beta W_{0\mathrm{M}}$, hence $s_\alpha = s_\beta$. □

Remark 3.5 The above discussion is in [8], p. 335. It suggests the natural question whether the relations between the $[c_\alpha]$ can be determined explicitly in terms of root data, thus providing an explicit presentation of $\Pi_1(G/P, eP)$. Recently this question has been answered completely in [14].

We finish this section by a characterization of the connected components of P which will be of crucial importance later on.

Lemma 3.6 *Let P_e denote the identity component of P. Then $P = F_P P_e$.*

Proof: Consider the natural covering $p : G/P_e \to G/P$. The group $\Pi_1(G/P, eP)$ acts in a natural way transitively on the fiber $p^{-1}(eP)$. We shall describe this action in terms of the generators $[c_\alpha], \alpha \in T_P$. Fix $\alpha \in T_P$ and $xP_e \in p^{-1}(eP)$ (i.e. $x \in P$). Then by the unique lifting theorem for curves there exists a unique curve $\tilde{c}_\alpha : [-\frac{\pi}{2}, \frac{\pi}{2}] \to G/P_e$ with $\tilde{c}_\alpha(-\frac{\pi}{2}) = xP_e$ and $p \circ \tilde{c}_\alpha = c_\alpha$. From (16) one sees that the lifting is given by:

$$\tilde{c}_\alpha(s) = w_\alpha \exp[(X_\alpha + \theta X_\alpha)]xP_e.$$

The action of $[c_\alpha]$ on xP_e is now given by $[c_\alpha]xP_e = \tilde{c}_\alpha(\frac{\pi}{2})$, hence

$$[c_\alpha]xP_e = w_\alpha^2 xP_e = f_\alpha xP_e.$$

Since the $[c_\alpha]$ generate $\Pi_1(G/P, eP)$, it follows that the action of $\Pi_1(G/P, eP)$ on the fiber $p^{-1}(eP)$ preserves the image $F_P P_e$ of F_P in G/P_e (which is obviously a subset of the fiber). By transitivity of the action it now follows that $F_P P_e = p^{-1}(eP)$, whence $P = F_P P_e$. □

4 Lifting of involutions

In this section we assume that G is a connected group of Harish-Chandra's class. An involution of the Lie algebra $\mathfrak{g}$ need not lift to the group G; it does however lift to a finite covering of G (which is again of Harish-Chandra's class), provided a natural condition is fulfilled. More generally we shall formulate a result for finite groups of involutions of $\mathfrak{g}$. Note that such groups are necessarily abelian.

Lemma 4.1 *Let G be a connected group of Harish-Chandra's class and let L be the lattice in the center of $\mathfrak{g}$ consisting of $X \in \mathrm{center}(\mathfrak{g})$ with $\exp X = e$. Moreover, let $\mathcal{T}$ be a finite group of involutions of $\mathfrak{g}$ such that*

$$\mathrm{span}\,[\cap_{\tau \in \mathcal{T}}\ \tau(L)] = \mathrm{center}(\mathfrak{g}). \tag{17}$$

Then there exists a finite covering group G' of G such that every $\tau \in \mathcal{T}$ lifts to G'.

Proof: First assume that the result holds in the case that G is abelian as well as in the case that G is semisimple. In the second half of the proof we will establish these partial results; we start by showing that the general result follows from these partial results.

Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{c}$ be the decomposition of $\mathfrak{g}$ into its semisimple part and its center respectively. Let G_1, C be the analytic subgroups of G with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{c}$ respectively. Then obviously $G = G_1 C$. The groups G_1 and C are connected closed subgroups of Harish-Chandra's class; moreover, G_1 is semisimple and C is commutative. By the hypothesis we may select finite coverings $p_1 : G_1' \to G_1$ and $p_2 : C' \to C$ such that for every $\tau \in \mathcal{T}$ the involutions $\tau|\mathfrak{g}_1$ and $\tau|\mathfrak{c}$ lift to involutions τ_1' of G_1' and τ_2' of C' respectively. Then the group $G' = G_1' \times C'$ has Lie algebra $\mathfrak{g}_1 \oplus \mathfrak{c} = \mathfrak{g}$ and the involution τ lifts to the involution $\tau' = \tau_1' \times \tau_2'$ of G'. Moreover the group homomorphism

$p : G_1' \times C' \to G,\ (g, c) \mapsto p_1(g)p_2(c)$ is a local diffeomorphism onto G, hence a covering of G. Its kernel consists of the set of elements $(g, c) \in G_1' \times C'$ such that $p_1(g)p_2(c) = e$, or equivalently such that $g \in p_1^{-1}(G_1 \cap C)$, $c \in p_2^{-1}(G_1 \cap C)$ and $p_1(g)p_2(c) = e$. Now $G_1 \cap C$ is finite and p_1 and p_2 are finite coverings, hence $\ker p$ is finite and we see that p is a finite covering.

It remains to prove the result in the cases that G is abelian or semisimple. We first discuss the case that G is abelian. Then $\exp : \mathfrak{g} \to G$ is the universal covering. Let L be the kernel of exp, then by the assumption (17) we have that $L_0 = \cap_{\tau \in \mathcal{T}} \tau(L)$ is a spanning discrete subgroup of $\mathfrak{g}$; hence it is a sublattice of L of full dimension. This implies that the natural map $p : G' = \mathfrak{g}/L_0 \to G$ is a covering homomorphism which is finite since $\mathfrak{g}/L_0$ is compact. If $\tau \in \mathcal{T}$, then L_0 is τ-stable, hence the map τ factorizes to an involution of G', which is the lifting of τ.

We finally turn to the case that G is semisimple. Being of Harish-Chandra's class, G has finite center $Z(G)$. By [10], Theorem 3.1 and the remark before Theorem 3.7, there exists a maximal compact subgroup $\tilde{K}$ of $\mathrm{Aut}\,\mathfrak{g}$ containing $\mathcal{T}$. Let K be the preimage of $\tilde{K}$ under $\mathrm{Ad} : G \to \mathrm{Aut}\,(\mathfrak{g})$. Then K is maximally compact in G and its Lie algebra $\mathfrak{k}$ is $\mathcal{T}$-stable, i.e. invariant under every $\tau \in \mathcal{T}$. Let $\mathfrak{k}_{\mathrm{s}}$ be the semisimple part of $\mathfrak{k}$, and $\mathfrak{c}$ its center. Let K_{s}, C be the associated analytic subgroups of G, respectively. Let Λ be the lattice of $X \in \mathfrak{c}$ such that $\exp X = e$, and let Λ^0 be the lattice of $X \in \mathfrak{c}$ such that $e^{\mathrm{ad} X} = I$ (this is the Λ for the adjoint group $\mathrm{Ad}(G)$). Then $\Lambda \subset \Lambda_0$. We will show that

$$\mathrm{span}\,[\cap_{\tau \in \mathcal{T}} \tau(\Lambda)] = \mathfrak{c}. \tag{18}$$

To see that this holds, let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ be the decomposition of $\mathfrak{g}$ into its simple ideals. Put $\mathfrak{k}_j = \mathfrak{g}_j \cap \mathfrak{k}$ and $\mathfrak{c}_j = \mathfrak{g}_j \cap \mathfrak{c}$. Then we have decompositions $\mathfrak{k} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_n$ and $\mathfrak{c} = \mathfrak{c}_1 \oplus \cdots \oplus \mathfrak{c}_n$. Here each $\mathfrak{c}_j$ is either 0 or 1-dimensional. For $1 \le j \le n$, let Λ_j be the lattice of $X \in \mathfrak{c}_j$ with $\exp X = e$, and let Λ_j^0 be the lattice of $X \in \mathfrak{c}_j$ such that $e^{\mathrm{ad} X} = I$. Then $\Lambda_j \subset \Lambda_j^0$, and since $\dim \mathfrak{c}_j \le 1$, there exists a strictly positive integer p_j such that $\Lambda_j = p_j \Lambda_j^0$.

We obviously have

$$\Lambda \supset \Lambda_1 \oplus \cdots \oplus \Lambda_n,$$

and similarly $\Lambda^0 \supset \Lambda_1^0 \oplus \cdots \oplus \Lambda_n^0$. On the other hand, if $X \in \Lambda^0$, write $X = X_1 + \cdots + X_n$, with $X_j \in \mathfrak{c}_j$, and fix $1 \le k \le n$ for the moment. Then $e^{\mathrm{ad} X} = I$ on $\mathfrak{g}_k$ and $e^{\mathrm{ad} X_j} = I$ on $\mathfrak{g}_k$ for $j \ne k$. Hence $e^{\mathrm{ad} X_k} = I$ on $\mathfrak{g}_k$. Thus $X_k \in \Lambda_k^0$ for each $1 \le k \le n$. Hence:

$$\Lambda^0 = \Lambda_1^0 \oplus \cdots \oplus \Lambda_n^0.$$

Let p be a common multiple of the p_j, $1 \le j \le n$. Then it follows that $\Lambda_j = p_j \Lambda_j^0 \supset p \Lambda_j^0$. Hence

$$\Lambda \supset p\Lambda^0.$$

Now the lattice Λ^0 is invariant for every automorphism of $\mathfrak{g}$ which leaves $\mathfrak{k}$ invariant, hence for every $\tau \in \mathcal{T}$. This implies that the space on the left hand side of (18) contains $p\Lambda^0$ hence equals $\mathfrak{c}$.

The result having been established already for the commutative case, it follows from the claim that there exists a finite covering homomorphism $p_2 : C' \to C$ such that for every $\tau \in \mathcal{T}$ the restriction $\tau_2 = \tau|\mathfrak{c}$ lifts to an involution τ_2' of C'. Let $p_1 : K_{\mathrm{s}}' \to K_{\mathrm{s}}$ be the universal covering. It is finite since K_{s} is compact semisimple. Moreover for every $\tau \in \mathcal{T}$ the involution $\tau_2 = \tau|\mathfrak{k}_{\mathrm{s}}$ has a lifing to an involution τ_2' of K_{s}'. Now the map $p : K_{\mathrm{s}}' \times C' \to G$, $(k, c) \mapsto p_1(k)p_2(c)$ is a finite covering by the same argument as in the first part of the proof. Moreover, for every $\tau \in \mathcal{T}$ the map $\tau_1' \times \tau_2'$ is a lifting of $\tau|\mathfrak{k}$ to the finite covering group $K_{\mathrm{s}}' \times C'$. By the Cartan decomposition there exists a finite covering $G' \to G$ which over K is isomorphic to $K_{\mathrm{s}}' \times C' \to K$. By the above every $\tau \in \mathcal{T}$ lifts to G'. □

Recall the definition of T_0 from (13). We will end this section by attaching an involution to each root of T_0.

Lemma 4.2 *If $\alpha \in T_0$, then there exists a unique automorphism $\tau_\alpha : \mathfrak{g} \to \mathfrak{g}$ such that*

(a) $\tau_\alpha = I$ *on* $\mathfrak{m}_0 + \mathfrak{a}_0$;

(b) $\tau_\alpha = I$ *on* $\mathfrak{g}_\beta$ *for all* $\beta \in S_0 \setminus \{\alpha\}$;

(c) $\tau_\alpha = -I$ *on* $\mathfrak{g}_\alpha$.

The map τ_α is an involution which commutes with θ and with $\mathrm{Ad}(M_0)$. Finally, τ_α lifts to some finite covering group of G.

Proof: Let $\mathfrak{t}$ be a maximal torus in $\mathfrak{m}_0$, then $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}_0$ is a Cartan subalgebra of $\mathfrak{g}$. We denote the root system of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$ by Δ. Let Δ^+ be a system of positive roots compatible with Σ_0^+, and let Φ be the associated fundamental system. Let Δ_0 be the set of roots in Δ with zero restriction to $\mathfrak{a}_0$, and let Δ_{r} be its complement. Then Δ_0 may be naturally identified with the root system of $\mathfrak{t}$ in $\mathfrak{m}_{0\mathbb{C}}$. Moreover, restriction to $\mathfrak{a}_0$ induces a surjective map $r_0 : \Delta_{\mathrm{r}} \to \Sigma_0$. Put $\Phi_0 = \Phi \cap \Delta_0$ and $\Phi_{\mathrm{r}} = \Phi \cap \Delta_{\mathrm{r}}$. Then Φ_0 is a fundamental system for Δ_0, and r_0 maps Φ_{r} onto S_0. Since $\dim \mathfrak{g}_\alpha = 1$ the set $r_0^{-1}(\alpha)$ consists of a unique $\tilde{\alpha} \in \Delta_{\mathrm{r}}$ which belongs to Φ_{r}. It follows that $S_0 \setminus \{\alpha\}$ equals the image of $\Phi_{\mathrm{r}} \setminus \{\tilde{\alpha}\}$ under r_0. Let τ be an automorphism of $\mathfrak{g}$; we denote its complex linear extension to $\mathfrak{g}_{\mathbb{C}}$ by the same symbol. If τ satisfies conditions (a)–(c), then by the above observations it follows that

(i) $\tau = I$ on $\mathfrak{h}_{\mathbb{C}}$ and on the root spaces $(\mathfrak{g}_{\mathbb{C}})_\gamma, \gamma \in \Phi \setminus \{\tilde{\alpha}\}$;

(ii) $\tau = -I$ on $(\mathfrak{g}_{\mathbb{C}})_{\tilde{\alpha}}$.

The automorphism τ is uniquely determined by the properties (i) and (ii); this establishes uniqueness.

To establish existence we observe that, by Weyl's theorem, there exists a unique automorphism τ of $\mathfrak{g}_{\mathbb{C}}$ satisfying (i) and (ii). The automorphism is the identity on

$$\mathfrak{h}_{\mathbb{C}} \oplus \sum_{\gamma \in \Delta_0} (\mathfrak{g}_{\mathbb{C}})_\gamma.$$

This space is the centralizer of $\mathfrak{a}_0$ in $\mathfrak{g}_{\mathbb{C}}$, hence equals $(\mathfrak{m}_0 + \mathfrak{a}_0)_{\mathbb{C}}$.

On the other hand, if $\beta \in S_0$, then

$$(\mathfrak{g}_\beta)_{\mathbb{C}} = \sum_{\gamma \in r_0^{-1}(\beta)} (\mathfrak{g}_{\mathbb{C}})_\gamma. \tag{19}$$

If $\beta \neq \alpha$ then $r_0^{-1}(\beta)$ consists of roots in Δ^+ contained in a set of the form $\tilde{\beta} + \mathrm{span}\,(\Phi_0)$ with $\tilde{\beta} \in \Phi_{\mathrm{r}} \setminus \{\tilde{\alpha}\}$. It follows from condition (i) that $\tau = I$ on (19). If $\beta = \alpha$, then $r_0^{-1}(\beta) = \{\tilde{\alpha}\}$ and it follows from (ii) that $\tau = -I$ on $\mathfrak{g}_\beta$. Thus in all cases it follows that τ leaves the real subalgebra $\mathfrak{g}_\beta$ invariant. Any automorphism of $\mathfrak{g}_{\mathbb{C}}$ which leaves $\mathfrak{m}_0, \mathfrak{a}_0$ and $\mathfrak{g}_\beta$ $(\beta \in \Sigma_0^+)$ invariant leaves the root spaces $\mathfrak{g}_{-\beta}$ $(\beta \in \Sigma_0^+)$ invariant as well. Hence τ leaves the real subalgebra $\mathfrak{g}$ invariant. Moreover, in the above argument we have seen that $\tau_\alpha := \tau$ satisfies conditions (a) - (c). This establishes existence.

To establish the final assertions, note that $\tau_\alpha^2 = I$ on $\mathfrak{h}_{\mathbb{C}}$ and on all the root spaces for the simple roots in Φ. Hence $\tau_\alpha^2 = I$. If m is an element of M_0, then $\mathrm{Ad}(m)$ leaves $\mathfrak{m}_0, \mathfrak{a}_0$ and every root space $\mathfrak{g}_\beta$ $(\beta \in \Sigma_0)$ invariant. It follows that $\mathrm{Ad}(m) \circ \tau_\alpha = \tau_\alpha \circ \mathrm{Ad}(m)$ on $\mathfrak{m}_0 + \mathfrak{a}_0$ and on all $\mathfrak{g}_\beta$, $\beta \in S_0$. This implies that τ_α commutes with $\mathrm{Ad}(M_0)$. Finally, we notice that from (a) - (c) it follows that $\tau_\alpha = \epsilon(\beta) I$ on $\mathfrak{g}_\beta$, $\beta \in S_0$, where $\epsilon(\beta) = 1$ if $\beta \neq \alpha$, and where $\epsilon(\alpha) = -1$. It now follows that $\tau_\alpha \circ \theta = \theta \circ \tau_\alpha$ on $\mathfrak{m}_0 + \mathfrak{a}_0$ and on every root space $\mathfrak{g}_\beta$ $(\beta \in S_0)$. Hence τ_α commutes with θ.

By (a) we have that $\tau_\alpha = I$ on the center of $\mathfrak{g}$. Hence the final assertion follows by application of Lemma 4.1. □

5 Some results on σ-parabolic subgroups

In this section we assume that G is a group of Harish-Chandra's class. We select a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}$ containing $\mathfrak{a}_{\mathrm{q}}$ and put $A_0 = \exp \mathfrak{a}_0$ and $M_0 = Z_K(\mathfrak{a}_0)$.

Let $\Sigma_{0\mathrm{q}}$ be the set of roots $\alpha \in \Sigma_0 = \Sigma(\mathfrak{g}, \mathfrak{a}_0)$ such that $\alpha|\mathfrak{a}_{\mathrm{q}} \neq 0$. Let $\Sigma_{0\mathrm{M}}$ be the complement of $\Sigma_{0\mathrm{q}}$ in Σ_0. Then we may naturally identify $\Sigma_{0\mathrm{M}}$ with the root system of $\mathfrak{a}_0$ in $\mathfrak{m}_1$, the centralizer of $\mathfrak{a}_{\mathrm{q}}$ in $\mathfrak{g}$.

Restriction to $\mathfrak{a}_{\mathrm{q}}$ induces a surjective map from $\Sigma_{0\mathrm{q}}$ onto Σ, the root system of $\mathfrak{a}_{\mathrm{q}}$ in $\mathfrak{g}$. We select a positive system Σ^+ for Σ, and a compatible system Σ_0^+ of positive roots for Σ_0, i.e. if $\alpha \in \Sigma_{0\mathrm{q}}$, then $\alpha \in \Sigma_0^+ \iff \alpha|\mathfrak{a}_{\mathrm{q}} \in \Sigma^+$. Let $\mathfrak{n}$ be the sum of the positive root spaces for Σ^+, and put $N = \exp \mathfrak{n}$ and $M_1 = Z_G(\mathfrak{a}_{\mathrm{q}})$. Then $P = M_1 N$ belongs to $\mathcal{P}(A_{\mathrm{q}})$ (cf. Section 1). Let $P = MAN$ be its Langlands decomposition. Then $\mathfrak{a}_{\mathrm{q}} = \mathfrak{a} \cap \mathfrak{q}$.

Lemma 5.1 *If $\alpha \in \Sigma_{0\mathrm{M}}$ then $\mathfrak{g}_\alpha \subset \mathfrak{h}$.*

Proof: From $\sigma\alpha = \alpha$ it follows that σ leaves $\mathfrak{g}_\alpha$ invariant. Suppose that $X \in \mathfrak{g}_\alpha \cap \mathfrak{q}$. Then $\mathfrak{a}_{\mathrm{q}}$ centralizes X. Hence $X - \theta X$ belongs to $\mathfrak{p} \cap \mathfrak{q}$ and centralizes $\mathfrak{a}_{\mathrm{q}}$. Since $\mathfrak{a}_{\mathrm{q}}$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$, it follows that $X - \theta X \in \mathfrak{a}_{\mathrm{q}}$. This implies that $\mathfrak{a}_0$ centralizes $X - \theta X$, hence $X = 0$. We conclude that $\mathfrak{g}_\alpha \cap \mathfrak{q} = 0$ whence the result. □

Let $\mathfrak{m}_{\rm n}$ be the non-compact part of $\mathfrak{m}$, i.e. the smallest ideal in $\mathfrak{m}$ containing $\mathfrak{m} \cap \mathfrak{p}$. Then $M_{\rm n}$, the corresponding analytic subgroup of M, is invariant under both σ and θ. Moreover, $M_{\rm n}$ is a closed normal subgroup, and the quotient $M/M_{\rm n}$ is a compact Lie group.

The following lemma is well known.

Lemma 5.2 *Let $\xi \in \widehat{M}_{\rm fu}$. Then $\xi = 1$ on $M_{\rm n}$.*

Lemma 5.3 *The non-compact part $\mathfrak{m}_{\rm n}$ of $\mathfrak{m}$ equals the subalgebra of $\mathfrak{m}$ generated by $\mathfrak{a}_{0\rm M} = \mathfrak{a}_0 \cap \mathfrak{m}$ and the root spaces $\mathfrak{g}_\alpha$, $\alpha \in \Sigma_{0\rm M}$.*

Proof: Let $\mathfrak{v}$ denote the subalgebra generated by $\mathfrak{a}_{0\rm M} = \mathfrak{a}_0 \cap \mathfrak{m}$ and $\mathfrak{g}_\alpha$, $\alpha \in \Sigma_{0\rm M}$. Then $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{v}$ and since $\mathfrak{m}_0$ centralizes $\mathfrak{a}_0$ and normalizes the root spaces $\mathfrak{g}_\alpha$, we see that the algebra $\mathfrak{v}$ is an ideal in $\mathfrak{m}$. Moreover, from $\mathfrak{p} \subset \mathfrak{a}_0 + \sum_{\alpha \in \Sigma_0} \mathfrak{g}_\alpha$ it follows that $\mathfrak{m} \cap \mathfrak{p} \subset \mathfrak{v}$. Hence $\mathfrak{m}_{\rm n} \subset \mathfrak{v}$.

To see that the converse inclusion also holds, let $X \in \mathfrak{g}_\alpha$, $\alpha \in \Sigma_{0\rm M}$. Then $X - \theta X \in \mathfrak{m} \cap \mathfrak{p} \subset \mathfrak{m}_{\rm n}$. Select $H \in \mathfrak{a}_{0\rm M}$ such that $\alpha(H) = 1$. Then $X + \theta X = [H, X - \theta X]$ belongs to $\mathfrak{m}_{\rm n}$ as well, since $\mathfrak{m}_{\rm n}$ is an ideal; we conclude that $X \in \mathfrak{m}_{\rm n}$. Hence $\mathfrak{g}_\alpha \subset \mathfrak{m}_{\rm n}$ for every $\alpha \in \Sigma_{0\rm M}$ and it follows that $\mathfrak{v} \subset \mathfrak{m}_{\rm n}$. $\square$

Corollary 5.4 *The non-compact part $M_{\rm n}$ of M is contained in $(M \cap H)_e$.*

Proof: We have that $\mathfrak{a}_{0\rm M} \subset \mathfrak{a}_0 \cap \mathfrak{h}$. Moreover, $\mathfrak{g}_\alpha \subset \mathfrak{h}$ for every $\alpha \in \Sigma_{0\rm M}$, by Lemma 5.1. From the previous lemma it now follows that $\mathfrak{m}_{\rm n} \subset \mathfrak{m} \cap \mathfrak{h}$. $\square$

Let $s_{\rm M} \in {\rm Aut}\,(\mathfrak{a}_0)$ be the longest Weyl group element for the root system $\Sigma_{0\rm M}$, relative to the system $\Sigma_{0\rm M}^+ := \Sigma_{0\rm M} \cap \Sigma_0^+$ of positive roots.

Corollary 5.5 *The Weyl group element $s_{\rm M}$ has a representative in $M_{\rm n} \cap K$.*

Proof: This is an easy consequence of Lemma 5.3. $\square$

We now fix a representative $v_{\rm M}$ of $s_{\rm M}$ in $M_{\rm n} \cap K$. Note that by Corollary 5.4 we have

$$v_{\rm M} \in (M \cap H \cap K)_e.$$

Note also that by Lemma 5.2 we have that

$$\xi(v_{\rm M}) = 1 \quad \text{for every} \quad \xi \in \widehat{M}_{\rm fu}.$$

Let $W_{0\rm M}$ be the centralizer of $\mathfrak{a}_{\rm q}$ in W_0. The elements of $W_{0\rm M}$ normalize the decomposition $\mathfrak{a}_0 = \mathfrak{a}_{0\rm h} \oplus \mathfrak{a}_{\rm q}$, hence commute with σ and θ. In particular $s_{\rm M}$ commutes with σ, θ, hence the automorphism $s_{\rm M}\sigma\theta$ of $\mathfrak{a}_0$ is an involution. If φ is an automorphism of $\mathfrak{a}_0$, we denote its inverse transpose $\xi \mapsto \xi \circ \varphi^{-1}$, $\mathfrak{a}_0^* \to \mathfrak{a}_0^*$ by the same symbol. The accordingly defined involutions $\sigma, \theta, s_{\rm M}$ of $\mathfrak{a}_0^*$ obviously leave Σ_0 invariant.

Lemma 5.6 *The involution $s_{\mathrm{M}}\sigma\theta$ of $\mathfrak{a}_0^*$ leaves the sets $\Sigma_{0\mathrm{q}}, \Sigma_{0\mathrm{M}}, \Sigma_0^+, S_0, S_{0\mathrm{q}}, S_{0\mathrm{M}}$ and T_P invariant.*

Proof: If $\alpha \in \Sigma_0$, then $s_{\mathrm{M}}\sigma\theta\alpha|\mathfrak{a}_{\mathrm{q}} = \alpha|\mathfrak{a}_{\mathrm{q}}$, since $s_{\mathrm{M}}\sigma\theta$ centralizes $\mathfrak{a}_{\mathrm{q}}$. Hence $s_{\mathrm{M}}\sigma\theta$ leaves the sets $\Sigma_{0\mathrm{q}}$ and $\Sigma_{0\mathrm{q}}^+$ invariant. On the other hand, if $\alpha \in \Sigma_{0\mathrm{M}}^+$, then $s_{\mathrm{M}}\sigma\theta\alpha = -s_{\mathrm{M}}\alpha \in \Sigma_{0\mathrm{M}}^+$ and we see that $s_{\mathrm{M}}\sigma\theta$ leaves $\Sigma_{0\mathrm{M}}^+$ and $\Sigma_{0\mathrm{M}}$ invariant. Since $\Sigma_0^+ = \Sigma_{0\mathrm{M}}^+ \cup \Sigma_{0\mathrm{q}}^+$ it follows that $s_{\mathrm{M}}\sigma\theta$ leaves Σ_0^+ invariant. It is now obvious that $S_0, S_{0\mathrm{q}}, S_{0\mathrm{M}}$ are invariant as well. Finally let $\alpha \in T_P$, and put $\beta = s_{\mathrm{M}}\sigma\theta\alpha$. Then $\alpha \in S_0 \setminus \Sigma_{0\mathrm{M}} = S_{0\mathrm{q}}$, hence $\beta \in S_{0\mathrm{q}}$, and it follows that $\beta \in S_0 \setminus \Sigma_{0\mathrm{M}}$. Moreover, $\mathfrak{g}_\beta = \mathrm{Ad}(v_{\mathrm{M}})\sigma\theta\mathfrak{g}_\alpha$, hence $\dim \mathfrak{g}_\beta = \dim \mathfrak{g}_\alpha = 1$, and we conclude that $\beta \in T_P$. □

Finally, we define the compact part $\mathfrak{m}_{\mathrm{c}}$ of $\mathfrak{m}$ to be the orthocomplement of $\mathfrak{m}_{\mathrm{n}}$ in $\mathfrak{m}$, with respect to B (cf. Section 1). Then $\mathfrak{m} = \mathfrak{m}_{\mathrm{n}} \oplus \mathfrak{m}_{\mathrm{c}}$ as a direct sum of Lie algebras.

Lemma 5.7 *The compact part $\mathfrak{m}_{\mathrm{c}}$ of $\mathfrak{m}$ is contained in $\mathfrak{m}_0$.*

Proof: This is an immediate consequence of Lemma 5.3 □

6 Involutions of the root system Σ_0

We retain the assumptions and notations of the previous section. We shall need the following lemma relating the set S of simple roots for Σ^+ with the set S_0 of simple roots for Σ_0^+. Recall that $S_0 = S_{0\mathrm{M}} \cup S_{0\mathrm{q}}$ (disjoint union). Recall also that $P \in \mathcal{P}(A_{\mathrm{q}})$ and $\Sigma(P) = \Sigma^+$. Moreover, T_P is a subset from S_0 which is disjoint from $S_{0\mathrm{M}}$, hence $T_P \subset S_{0\mathrm{q}}$.

Lemma 6.1 *There exists a permutation ϑ of $S_{0\mathrm{q}}$ of order at most two such that for every $\alpha \in S_{0\mathrm{q}}$ we have $\sigma\theta\alpha \in \vartheta(\alpha) + \mathbb{N}S_{0\mathrm{M}}$. Let $r_0 : \Sigma_{0\mathrm{q}} \to \Sigma$ be the map induced by restriction to $\mathfrak{a}_{\mathrm{q}}$. Then r_0 maps $S_{0\mathrm{q}}$ onto S. Moreover, its fibers are precisely the orbits of the permutation ϑ.*

Proof: See [13]. □

In the rest of this section we assume that $\mathfrak{a}_{\mathrm{q}}$ is not central in $\mathfrak{g}$ (so that $\Sigma \neq \emptyset$) and that $\dim \mathfrak{a}_{\mathrm{q}} = 1$. Then S has one element, hence by the above lemma $S_{0\mathrm{q}}$ has either one or two elements. It follows that T_P has at most two elements.

We recall that s_{M} denotes the longest Weyl group element of the root system $\Sigma_{0\mathrm{M}}$ (relative to $S_{0\mathrm{M}}$) and that v_{M} is a representative of s_{M} in $(M_{\mathrm{n}} \cap K \cap H)_e$. It follows that the conjugation map $\mathcal{A}d(v_{\mathrm{M}}) : G \to G,\ x \mapsto v_{\mathrm{M}} x v_{\mathrm{M}}^{-1}$ commutes with both σ and θ.

If S is a system of simple roots for a root system R, we shall write $\mathcal{D}(\mathrm{S})$ for the associated Dynkin diagram; moreover, for brevity of speech we identify simple roots in S with the associated vertices in $\mathcal{D}(\mathrm{S})$. If $\mathrm{S}_1 \subset \mathrm{S}$ then $\mathcal{D}(\mathrm{S}_1)$ is the Dynkin diagram that arises from $\mathcal{D}(\mathrm{S})$ by omitting the roots from $\mathrm{S} \setminus \mathrm{S}_1$. If φ is an automorphism of the root sytem R which leaves S invariant, then φ induces an automorphism of the Dynkin diagram $\mathcal{D}(\mathrm{S})$, which we denote by φ again. If $\mathrm{S}_1 \subset \mathrm{S}$ is invariant under φ, then the automorphism φ of $\mathcal{D}(\mathrm{S})$ restricts to an automorphism of $\mathcal{D}(\mathrm{S}_1)$.

In particular it follows from Lemma 5.6 that $s_{\mathrm{M}}\sigma\theta$ induces involutions on $\mathcal{D}(S_0)$ and $\mathcal{D}(S_{0\mathrm{M}})$.

Lemma 6.2 *The involution $s_{\mathrm{M}}\sigma\theta$ leaves the connected components of the Dynkin diagram $\mathcal{D}(S_{0\mathrm{M}})$ invariant.*

Proof: Use that $s_{\mathrm{M}}\sigma\theta = -s_{\mathrm{M}}$ on $S_{0\mathrm{M}}$. $\square$

We recall from Lemma 5.6 that the involution $s_{\mathrm{M}}\sigma\theta$ leaves the sets $T_P \subset S_{0\mathrm{q}} \subset S_0$ invariant. The purpose of the following three lemmas is to distinguish cases, depending on the action of $s_{\mathrm{M}}\sigma\theta$ on T_P.

Lemma 6.3 *Assume that T_P contains a fixed point for $s_{\mathrm{M}}\sigma\theta$. Then $S_{0\mathrm{q}}$ consists of precisely one element. In particular $T_P = S_{0\mathrm{q}}$.*

Proof: Let $\alpha \in T_P$ be fixed under $s_{\mathrm{M}}\sigma\theta$. Then $\sigma\theta\alpha = s_{\mathrm{M}}\alpha$. Elements of $W_{0\mathrm{M}}$ leave the set $\alpha + \mathbb{Z}S_{0\mathrm{M}}$ invariant; hence $\sigma\theta\alpha \in \alpha + \mathbb{Z}S_{0\mathrm{M}}$. On the other hand we have $\sigma\theta\alpha \in \vartheta(\alpha) + \mathbb{N}S_{0\mathrm{M}}$, where ϑ is the permutation of $S_{0\mathrm{q}}$ defined in Lemma 6.1. By linear independence of the elements of S_0 it follows that $\vartheta(\alpha) = \alpha$. The desired result now follows from the last assertion of Lemma 6.1. $\square$

Before proceeding we prove a lemma that will be useful at a later stage.

Lemma 6.4 *Let the assumptions of Lemma 6.3 be fulfilled, and let $\alpha \in S_{0\mathrm{q}}$. Then the involution τ_α of $\mathfrak{g}$, defined in Lemma 4.2, is trivial on $\mathfrak{m}$ and commutes with $\mathrm{Ad}(v_{\mathrm{M}})$ and σ.*

Proof: The subalgebra $\mathfrak{m}_{\mathrm{n}}$ is generated by $\mathfrak{a}_0 \cap \mathfrak{m}$ and the root spaces $\mathfrak{g}_{\pm\gamma}$, where $\gamma \in S_{0\mathrm{M}}$. Since $S_{0\mathrm{M}} \cap T_P = \emptyset$ it follows that $\tau_\alpha = I$ on $\mathfrak{m}_{\mathrm{n}}$. Since τ_α is trivial on $\mathfrak{m}_0$ and $\mathfrak{m}_0 \supset \mathfrak{m}_{\mathrm{c}}$ by Lemma 5.7, it follows that $\tau_\alpha = I$ on $\mathfrak{m}$. Hence τ_α commutes with $\mathrm{Ad}(M_e)$, and in particular with $\mathrm{Ad}(v_{\mathrm{M}})$. Since τ_α commutes with θ as well, it suffices to show that τ_α commutes with $\psi := \mathrm{Ad}(v_{\mathrm{M}})\sigma\theta$. Now ψ leaves $\mathfrak{m}_0$ and $\mathfrak{a}_0$ invariant, commutes with θ, and permutes the roots of S_0. By the hypothesis it follows that $\psi\alpha = \alpha$. It now easily follows that $\psi \circ \tau_\alpha = \tau_\alpha \circ \psi$ on $\mathfrak{m}_0 \oplus \mathfrak{a}_0$ and on every $\mathfrak{g}_\beta$, $\beta \in S_0$. Hence ψ and τ_α commute. $\square$

Lemma 6.5 *Assume that T_P contains a root α such that α and $\beta := s_{\mathrm{M}}\sigma\theta\alpha$ are in different connected components of the Dynkin diagram of S_0. Then $S_{0\mathrm{q}} = \{\alpha, \beta\}$ and α and β are isolated vertices in the Dynkin diagram of S_0.*

Proof: From the hypothesis it follows that $\alpha \perp \beta$. Since $S_{0\mathrm{q}}$ contains T_P and consists of at most two elements we must have $S_{0\mathrm{q}} = \{\alpha, \beta\}$, hence $S_{0\mathrm{M}} = S_0 \setminus \{\alpha, \beta\}$. Now suppose that $\gamma \in S_{0\mathrm{M}}$ and $\gamma \not\perp \alpha$. Then $s_{\mathrm{M}}\sigma\theta\gamma \not\perp \beta$, hence γ and $s_{\mathrm{M}}\sigma\theta\gamma$ are in different connected components of the Dynkin diagram of S_0, contradicting Lemma 6.2. We conclude that α is isolated. By symmetry in the argument β is isolated as well. $\square$

If $\mathcal{D}$ is a Dynkin diagram, and α, β are simple roots in the same connected component of $\mathcal{D}$, then there exists a sequence of mutually different simple roots $\gamma_0, \dots, \gamma_n$ in $\mathcal{D}$ such that

(a) $\alpha = \gamma_0, \;\; \beta = \gamma_n$;

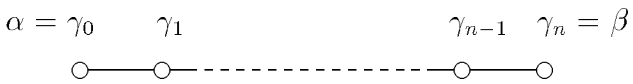
(b) $\gamma_j \not\perp \gamma_{j+1} \quad \text{for} \quad 0 \leq j < n.$

Since a Dynkin diagram cannot contain a closed circuit the above sequence is unique. We call it the sequence of roots in $\mathcal{D}$ connecting α and β.

Lemma 6.6 *Let $\alpha \in T_P$, and assume that the root $\beta = s_{\mathrm{M}}\sigma\theta\alpha$ is different from α, but contained in the same connected component of $\mathcal{D}(S_0)$. Then the sequence $\gamma_0 = \alpha, \dots, \gamma_n = \beta$ $(n \geq 1)$ of roots in $\mathcal{D}(S_0)$ connecting α and β has the following properties.*

(a) *$s_{\mathrm{M}}\sigma\theta\gamma_j = \gamma_{n-j}$ for all $0 \leq j \leq n$;*

(b) *the Dynkin diagram $\mathcal{D}(\gamma) := \mathcal{D}(\{\gamma_j \mid 0 \leq j \leq n\})$ is of type A_{n+1}, i.e. it is of the following form:*



(c) *$\gamma_j \in S_{0\mathrm{M}}$ for all $0 < j < n$;*

(d) *the Dynkin diagram $\mathcal{D}(\gamma)$ equals the connected component of $\mathcal{D}(S_0)$ containing α (and β).*

Proof: By applying $s_{\mathrm{M}}\sigma\theta$ to the sequence γ_j we obtain a sequence $s_{\mathrm{M}}\sigma\theta\gamma_j$ connecting β and α. Condition (a) follows by uniqueness of the connecting sequence.

Suppose that γ_j and γ_{j+1} are connected by a multiple link. Then so are their respective images γ_{n-j} and γ_{n-j-1} under $s_{\mathrm{M}}\sigma\theta$. Since a connected Dynkin diagram can contain at most one pair of roots connected by a multiple link, it follows that $j + 1 = n - j$. Hence $s_{\mathrm{M}}\sigma\theta\gamma_j = \gamma_{j+1}$ from which we see that γ_j and γ_{j+1} have equal length, contradicting the assumption that they are connected by a multiple link. Thus we see that all pairs γ_j, γ_{j+1} are connected by a single link, and (b) follows.

Since $S_{0\mathrm{q}}$ consists of at most two elements we must have $S_0 \setminus \{\alpha, \beta\} \subset S_{0\mathrm{M}}$. Hence the sequence satisfies condition (c).

Suppose now that (d) does not hold. Let $\gamma \in S_0$ be any root different from the γ_j such that $\gamma \not\perp \gamma_k$ for some $0 \leq k \leq n$. Then γ and $s_{\mathrm{M}}\sigma\theta\gamma$ are in the same connected component of the Dynkin diagram of $S_{0\mathrm{M}}$, hence these roots are connected by a sequence of roots from $S_{0\mathrm{M}}$. If $k = 0$ or $k = n$ this would imply the existence of a closed circuit in the Dynkin diagram of S_0 which is impossible. Hence $0 < k < n$ (and in particular n is at least 2). From the assumption on γ it follows that $s_{\mathrm{M}}\sigma\theta\gamma$ is not perpendicular to γ_{n-k}. By inspection of all possible connected Dynkin diagrams we now see that $k = n-k$, hence n is even and at least two, and $k = \frac{n}{2}$. Again by inspection of all Dynkin diagrams we see that either $n = 2$ or $n = 4$.

If $n = 2$, then the full connected component of the Dynkin diagram of S_0 must be of type D_l, with $l \geq 4$. Put $\gamma_3 = \gamma$ and let γ_j, $4 \leq j < l$ be determined by the requirement that the γ_j, $0 \leq j \leq l-1$, are mutually different, and that $\gamma_j \not\perp \gamma_{j+1}$ for all $3 \leq j < l-1$, see Figure 1.

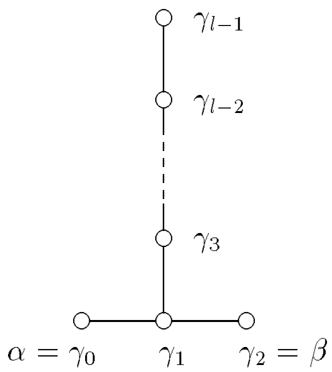


Figure 1.

Then obviously $s_{\mathrm{M}}\sigma\theta$ fixes the roots γ_j with $j \notin \{0, 2\}$. But these roots constitute a connected component of the Dynkin diagram of $S_{0\mathrm{M}}$, and $s_{\mathrm{M}}\sigma\theta$ acts on them as minus the associated longest Weyl group element. This contradicts the fact that in the root system A_{l-2} the longest Weyl group element does not equal minus the identity.

It follows that we must have $n = 4$. By inspecting all possible Dynkin diagrams we then see that the connected component of the Dynkin diagram of S_0 containing α is of type E_6 and consists of the roots $\gamma_0, \ldots, \gamma_4, \gamma$ (see Figure 2).

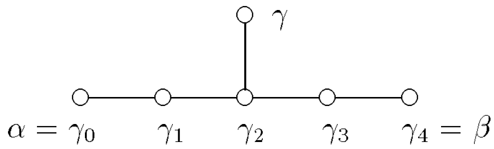


Figure 2.

The subdiagram consisting of the roots $\gamma_1, \gamma_2, \gamma_3, \gamma$ is of type D_4. Now $s_{\mathrm{M}}\sigma\theta$ induces an automorphism of this subdiagram which on the one hand is not the identity, and on the other hand equals minus the associated longest Weyl group element. The latter commutes with all automorphisms of the diagram hence must be the identity, contradiction.

Thus we see that the assumption that (d) does not hold leads to a contradiction. $\square$

Corollary 6.7 *Let $\alpha, \beta \in T_P$ be as in Lemma 6.6. With the notations of that lemma let $\Gamma = \Sigma_0 \cap \operatorname{span}\{\gamma_j,\ 0 \le j \le n\}$. Then:*

(a) *for every $\gamma \in \Gamma$ we have $\dim \mathfrak{g}_\gamma = 1$;*

(b) *the Lie subalgebra $\mathfrak{g}(\Gamma)$ of $\mathfrak{g}$ generated by the root spaces $\mathfrak{g}_\gamma,\ \gamma \in \Gamma$ is an ideal isomorphic to $sl(n+2, \mathbb{R})$;*

(c) $\sigma\theta\alpha = \gamma_1 + \ldots + \gamma_{n-1} + \beta$;

(d) *the Lie subalgebra generated by the space $\mathfrak{g}_\alpha$ and its images under $\sigma, \theta, \sigma\theta$ is isomorphic to $sl(3, \mathbb{R})$;*

(e) *for all $f \in F_P$ we have $f^2 \in H_e$.*

Proof: Let $\gamma \in \Gamma$ be an indivisible root. Then γ belongs to the reduced root system spanned by simple roots of $\mathcal{D}(\gamma)$. This root system is of type A_{n+1}, hence γ is Weyl conjugate to α. It follows that $\dim \mathfrak{g}_\gamma = \dim \mathfrak{g}_\alpha = 1$. Moreover, since $2\alpha \notin \Sigma_0$ it follows that 2γ is not a root. Hence all roots in Γ are reduced and of multiplicity one, and (a) follows.

Since $\mathcal{D}(\gamma)$ is a connected component of $\mathcal{D}(S_0)$, all roots from $\Sigma_0 \setminus \Gamma$ are orthogonal to Γ. Hence $\mathfrak{g}(\Gamma)$ is an ideal. Since all roots in Γ are reduced and of multiplicity 1, it follows that $\mathfrak{g}(\Gamma)$ is the normal real form of the complex simple algebra of type A_{n+1}. Hence $\mathfrak{g}(\Gamma) \simeq sl(n+2, \mathbb{R})$.

Now that (a) and (b) have been established, it follows by a straightforward computation in $sl(n+2, \mathbb{R})$ that $s_{\mathrm{M}}\beta = \beta + \gamma_1 + \cdots + \gamma_n$. But $\sigma\theta\alpha = s_{\mathrm{M}}\beta$ and (c) follows.

To establish assertion (d), put $\gamma = \sigma\theta\alpha$. Then $\gamma + \alpha$ is the longest root of the root system Γ. Hence the root system R generated by α and γ consists of $\alpha, \gamma, \alpha + \gamma$ and their inverses: it is therefore of type A_2. Since R is σ- and θ-invariant and consists of roots of multiplicity one, (d) follows.

To establish the last assertion we note that from (b) it follows that $\langle \alpha, \sigma\theta\alpha \rangle = \langle \alpha, \gamma_1 \rangle$. Put $\alpha_1 = \alpha, \alpha_2 = \sigma\theta\alpha$; then from (d) it follows that $\mathfrak{g}_{\alpha_j}, \mathfrak{g}_{-\alpha_j}$ $(j = 1, 2)$ generate a subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ which is stable under σ and θ and isomorphic to $sl(3, \mathbb{R})$. Let S be the corresponding analytic subgroup of G. Since $\mathrm{SL}(3, \mathbb{R})$ has trivial center and a universal cover which is twofold, it follows that S is isomorphic to either $\mathrm{SL}(3, \mathbb{R})$ or its double cover. It follows that the center $Z(S)$ of S has at most two elements. Now S is invariant under σ, and so is its center $Z(S)$. We claim that $Z(S) \subset H_e$. This is obvious if $Z(S)$ consists of one element. In the remaining case σ fixes the neutral element of $Z(S)$ hence it must also fix the second element. Hence $Z(S) \subset S^\sigma$. But in this case the group S is simply connected hence S^σ is connected (see [3]) and the claim follows.

Let $F(S)$ be the subgroup of S generated by the elements f_{α_j}, $j = 1, 2$. Then by a starightforward computation in $\mathrm{SL}(3, \mathbb{R})$ we see that $f^2 \in Z(S)$ for every $f \in F(S)$; hence $f^2 \in H_e$ for all $f \in F$.

To finish the proof we distinguish between the cases $n = 1$ and $n > 1$. If $n = 1$ then $F_P = F(S)$ and (e) follows.

If $n > 1$ then $\alpha \perp \beta$, hence f_α and f_β commute. Since the latter elements generate F_P, it follows that F_P is abelian, and it suffices to show that $f^2 \in H_e$ for $f \in \{f_\alpha, f_\beta\}$. By symmetry in the roles of α and β it suffices to show that $f_\alpha^2 \in H_e$. But this follows from the above since $f_\alpha \in F(S)$. □

7 The automorphism τ

The purpose of this section is to construct a special automorphism τ of $\mathfrak{g}$ that will be needed in the proof of the main result in the reduced case, which will be given in the next section.

We keep the assumptions and notations of the previous section. In particular we assume that that $\mathfrak{a}_{\mathrm{q}}$ has dimension one and is not central in $\mathfrak{g}$. Thus Σ is non-empty. We recall that $Q \mapsto \Sigma(Q)$ defines a bijection from $\mathcal{P}(A_{\mathrm{q}})$ onto the collection of positive sytems for $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$. Let $P \in \mathcal{P}(A_{\mathrm{q}})$ be determined by $\Sigma(P) = \Sigma^+$, and let $P = MAN$ be its Langlands decomposition. Then $\mathcal{P}(A_{\mathrm{q}}) = \{P, \bar{P}\}$, and any $Q \in \mathcal{P}(A_{\mathrm{q}})$ has the Langlands decomposition

$$Q = MAN_Q. \tag{20}$$

Put $\Omega_Q = QH_e$, and let the real analytic maps $a_Q : \Omega_Q \to A_{\mathrm{q}}$ and $m_Q : \Omega_Q \to M/M \cap H_e$ be defined by

$$x \in N_Q a_Q(x) m_Q(x) H_e \qquad (x \in \Omega_Q).$$

If φ is an automorphism of $\mathfrak{g}$ mapping $\bar{\mathfrak{n}}_Q$ onto $\mathfrak{n}_Q$, then its restriction to $\bar{\mathfrak{n}}_Q$ lifts to an isomorphism between the associated simply connected nilpotent groups, which we denote by the same symbol $\varphi : \bar{N}_Q \to N_Q$. Let the Haar measures of the nilpotent groups $N_Q, \bar{N}_Q$ be normalized as in [11], Section 4 (see also [1], p. 370). Finally we write $C^\omega(M \cap H_e \backslash M / M \cap H_e)$ for the space of bi-$(M \cap H_e)$-invariant real analytic functions on M.

Proposition 7.1 *There exists an automorphism τ of the Lie algebra $\mathfrak{g}$ with the following properties:*

(a) *τ commutes with σ, θ and $\mathrm{Ad}(v_{\mathrm{M}})$;*

(b) *τ leaves the subspaces $\mathfrak{a}_0$, $\mathfrak{a}$, $\mathfrak{a}_{\mathrm{q}}$ and $\mathfrak{m}$ invariant, and $\tau = -I$ on $\mathfrak{a}_{\mathrm{q}}$;*

moreover, if $Q \in \mathcal{P}(A_{\mathrm{q}})$, then

(c) *τ maps $\bar{\mathfrak{n}}_Q$ onto $\mathfrak{n}_Q$;*

(d) *the lifted isomorphism $\tau : \bar{N}_Q \to N_Q$ satisfies $\tau^*(dn_Q) = d\bar{n}_Q$;*

(e) *the lifted isomorphism τ maps $\bar{N}_Q \cap \Omega_Q$ into $N_Q \cap \Omega_{\bar{Q}}$, and for every function $\varphi \in C^\omega(M \cap H_e \backslash M / M \cap H_e)$ and all $\bar{n} \in \bar{N}_Q \cap \Omega_Q$ we have:*

$$\varphi(m_{\bar{Q}}(\tau \bar{n})) = \varphi(m_Q(\bar{n})^{-1}).$$

In the proof of this proposition we need the following lemma. Let $M_{\rm n}$ and $M_{\rm c}$ be the analytic subgroups of M with Lie algebras $\mathfrak{m}_{\rm n}$ and $\mathfrak{m}_{\rm c}$, respectively. Moreover, let $Z_M(M_{\rm c})$ denote the centralizer of $M_{\rm c}$ in M.

Lemma 7.2 *Let $\varphi \in C^\omega(M \cap H_e \backslash M / M \cap H_e)$. Then:*

(a) $\varphi(\sigma m) = \varphi(m^{-1}) \qquad (m \in M_e)$;

(b) $\varphi(my) = \varphi(ym) = \varphi(m) \qquad (m \in M,\ y \in M_{\rm n})$;

(c) $\varphi(fm) = \varphi(mf) \qquad (m \in M_e,\ f \in Z_M(M_{\rm c}))$.

Proof: (a) By analyticity of φ it suffices to prove the equation for m in a neighbourhood of e in M_e. Thus it suffices to prove the equation for $m = \exp Xh$ with $X \in \mathfrak{m} \cap \mathfrak{q}$, $h \in H_e$, and in fact we may as well assume that $h = e$ by the bi-H-invariance of φ. But for such m we have $m^{-1} = \sigma m$ and the equation follows.

For the remaining assertions we note that the natural map $K \cap M \to M/M \cap H_e$ is surjective. By the Peter-Weyl theorem it is therefore sufficient to prove (b) and (c) for a function φ of the form

$$\varphi(m) = \langle\, \eta_1 \mid \xi(m)\eta_2 \,\rangle, \tag{21}$$

with $\xi \in \widehat{M}_{\rm fu}$, and with η_1, η_2 vectors in $\mathcal{H}_\xi$ that are fixed under $\xi(M \cap H_e)$. For such a function φ the assertion (b) follows by application of Lemma 5.2.

For assertion (c) we first note that $M_e = M_{\rm c} M_{\rm n}$. Let $m \in M_e$ and $f \in Z_M(M_{\rm c})$. Write $m = xy$ with $x \in M_{\rm c}$, $y \in M_{\rm n}$. Then f centralizes $\mathfrak{m}_{\rm c}$, hence normalizes the complementary ideal $\mathfrak{m}_{\rm n}$ of $\mathfrak{m}$, and we see that $f^{-1}yf \in M_{\rm n}$. In view of (b) it now follows that

$$\varphi(fm) = \varphi(fxy) = \varphi(xf) = \varphi(xff^{-1}yf) = \varphi(mf).$$

□

Proof of Proposition 7.1: It follows from Corollaries 5.4 and 5.5 that $\sigma(v_{\rm M}) = v_{\rm M}$, hence σ commutes with $\mathrm{Ad}(v_{\rm M})$. It is now immediate that the properties (a) – (d) are all satified if τ is replaced by any of the automorphisms σ, θ. Since $v_{\rm M} \in (M \cap K)_e$, the automorphism $\mathrm{Ad}(v_{\rm M})$ centralizes $\mathfrak{a}$ and normalizes the spaces $\mathfrak{n}$ and $\bar{\mathfrak{n}}$, the action on the two latter spaces being by maps with determinant 1. Thus we see that (a) – (d) are also fulfilled with $\sigma \circ \mathrm{Ad}(v_{\rm M})$ in place of τ.

It remains to find an automorphism satisfying condition (e) in addition. For this we treat the case $T_P = \emptyset$ and the cases distinguished in Lemmas 6.3, 6.5, 6.6 separately.

Case (a). Assume that $T_P = \emptyset$. Then $M = M_P$ is connected by Lemma 3.6. We put $\tau = \sigma$. Then obviously σ maps $\bar{N}_Q \cap \Omega_Q$ onto $N_Q \cap \Omega_{\bar{Q}}$ and if $\bar{n} \in \bar{N}_Q \cap \Omega_Q$ then $m_{\bar{Q}}(\tau n) = \tau m_Q(\bar{n}) = \sigma m_Q(\bar{n})$. From Lemma 7.2 (a) we now see that τ satisfies (e).

If we are not in case (a), then $T_P \neq \emptyset$, and we distinguish between the case that $s_{\mathrm{M}}\sigma\theta$ has a fixed point in T_P (case (b)), and the remaining case that this is not so (case (c)).

In the latter case T_P consists of two different roots α and β, and $\beta = s_{\mathrm{M}}\sigma\theta\alpha$. We split this case in the subcases that either α and β are in different connected components of the Dynkin diagram of S_0 (case (c1)), or these roots are in the same connected component of the Dynkin diagram (case (c2)).

Case (b). Assume that $s_{\mathrm{M}}\sigma\theta$ has a fixed point α in T_P. Then by Lemma 6.3 we have $S_{0\mathrm{q}} = T_P = \{\alpha\}$. The root space $\mathfrak{g}_\alpha$ has dimension 1 and is invariant under the automorphism $\varphi = \mathrm{Ad}(v_{\mathrm{M}})\sigma\theta$. Thus φ acts by a scalar $c \in \mathbb{R}$ on $\mathfrak{g}_\alpha$. On the other hand, since v_{M}^2 centralizes $\mathfrak{a}_0$ it follows that $v_{\mathrm{M}}^2 \in M_0$, hence $\varphi^2 = \mathrm{Ad}(v_{\mathrm{M}})^2$ acts by the scalar ± 1 on $\mathfrak{g}_\alpha$. Hence $c^2 = \pm 1$ and it follows that $c = -(-1)^\epsilon$, with $\epsilon \in \{0, 1\}$. We define $\tau = \tau_\alpha^\epsilon \mathrm{Ad}(v_{\mathrm{M}})\sigma$. Before proceeding we note that with this definition:

$$\tau = -\theta \quad \text{on} \quad \mathfrak{g}_\alpha. \tag{22}$$

Combining the discussion at the beginning of this proof with Lemma 6.4 we see that conditions (a) – (d) are fulfilled. We claim that (e) is fulfilled as well.

One readily sees that it suffices to prove this for any finite covering group of G. In view of Lemma 4.1 we may therefore as well assume that the involutions σ and τ_α of $\mathfrak{g}$ lift to involutions of the group G. This will be assumed from now on.

Since τ leaves $\mathfrak{a}$ invariant by condition (b), its lifting τ leaves M invariant. By condition (a) the map τ leaves $M \cap H_e$ invariant, hence induces a diffeomorphism of $M/M \cap H_e$, which we denote by τ again. From (b) and (c) we see that τ maps Ω_Q onto $\Omega_{\bar{Q}}$, and that for all $x \in \Omega_Q$ we have:

$$m_{\bar{Q}}(\tau x) = \tau m_Q(x).$$

Therefore it suffices to prove that for any $\varphi \in C^\omega(M \cap H_e \backslash M/M \cap H_e)$ and all $\bar{n} \in \bar{N}_Q \cap \Omega_Q$ we have

$$\varphi(\tau m_Q(\bar{n})) = \varphi(m_Q(\bar{n})^{-1}).$$

For this it suffices to show that for all $m \in M$ we have:

$$\varphi(\tau m) = \varphi(m^{-1}). \tag{23}$$

By real analyticity and bi-$(M \cap H_e)$-invariance of φ, and in view of the fact that $M = F_P M_e$ by Lemma 3.6, it suffices to establish (23) for $m = f \exp X$, with $f \in F_P$ and $X \in \mathfrak{m} \cap \mathfrak{q}$.

In the present case F_P is the group generated by $f_\alpha = \exp(X_\alpha + \theta X_\alpha)$, where $X_\alpha \in \mathfrak{g}_\alpha$ is chosen as in (14). Using (22) and the fact that τ commutes with θ we see that

$$\tau f_\alpha = \exp(\tau(X_\alpha + \theta X_\alpha)) = \exp(-\theta X_\alpha - X_\alpha) = f_\alpha^{-1}.$$

It now follows that $\tau(f) = f^{-1}$ for all $f \in F_P$.

On the other hand, for $X \in \mathfrak{m} \cap \mathfrak{q}$ we have that

$$\tau(\exp X) = \exp(\tau X) = \exp(\tau_\alpha^\epsilon \mathrm{Ad}(v_{\mathrm{M}})\sigma X) = v_{\mathrm{M}} \exp(-\tau_\alpha^\epsilon X) v_{\mathrm{M}}^{-1}.$$

Since $\tau_\alpha = 1$ on $\mathfrak{m}$ by Lemma 6.4, it follows that

$$\tau(\exp X) = v_{\mathrm{M}} \exp(-X) v_{\mathrm{M}}^{-1}.$$

Combining these observations we obtain, for $f \in F_P$, $X \in \mathfrak{m} \cap \mathfrak{q}$, that

$$\tau(f \exp X) = f^{-1} v_{\mathrm{M}} \exp(-X) v_{\mathrm{M}}^{-1}.$$

Note that f centralizes $\mathfrak{m}_0$ by Lemma 3.3, hence $\mathfrak{m}_{\mathrm{c}}$ by Lemma 5.7. Moreover, $v_{\mathrm{M}} \in M_{\mathrm{n}}$. Hence for $\varphi \in C^\omega(M \cap H_e \backslash M / M \cap H_e)$ we have, by Lemma 7.2:

$$\varphi(\tau(f \exp X)) = \varphi(f^{-1} v_{\mathrm{M}} \exp(-X) v_{\mathrm{M}}^{-1}) = \varphi(v_{\mathrm{M}} \exp(-X) f^{-1}) = \varphi(\exp(-X) f^{-1}).$$

We thus see that (23) holds for $m = f \exp X$ with $f \in F_P$, $X \in \mathfrak{m} \cap \mathfrak{q}$. This completes the proof in case (b).

Case (c1). In this case we put $\tau = \sigma$. Then τ satisfies all properties (a) – (d), and it remains to establish property (e). Since σ maps $\Omega_{\bar{Q}} = \bar{N}_Q A M H_e$ onto $\Omega_Q = N_Q A M H_e$, it follows that for all $\bar{n} \in \bar{N}_Q \cap \Omega_Q$ we have:

$$m_{\bar{Q}}(\tau \bar{n}) = \tau m_Q(\bar{n}) = \sigma m_Q(\bar{n}).$$

Hence it suffices to show, for every $\varphi \in C^\omega(M \cap H_e \backslash M / M \cap H_e)$, that the identity

$$\varphi(\sigma m) = \varphi(m^{-1}) \tag{24}$$

holds for $m = m_Q(\bar{n})$.

By Lemma 6.5 we have that $T_P = S_{0\mathrm{q}} = \{\alpha, \beta\}$ with α, β isolated vertices in the Dynkin diagram $\mathcal{D}(S_0)$. From this and the fact that $2\alpha, 2\beta \notin \Sigma_0$ (cf. Remark 3.2) it follows that $\Sigma_{0\mathrm{q}}^+ = S_{0\mathrm{q}} = \{\alpha, \beta\}$. Moreover, s_{M} centralizes the roots α, β, and we conclude that $\beta = \sigma\theta\alpha$.

Let $\mathfrak{g}(T_P)$ be the Lie subalgebra of $\mathfrak{g}$ generated by the root spaces $\mathfrak{g}_{\pm\gamma}$, $\gamma \in T_P$. Then $\mathfrak{g}(T_P)$ is invariant under the involutions σ and θ. Now $\alpha + \beta$ is not a root, and it is immediate that $\mathfrak{g}(T_P) = \mathfrak{g}(\alpha) \oplus \mathfrak{g}(\beta)$ as Lie algebras; here we have written $\mathfrak{g}(\gamma)$ for the Lie algebra generated by $\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma}$, for $\gamma \in \{\alpha, \beta\}$. Select $X_\alpha \in \mathfrak{g}_\alpha$ as in (14); thus $H_\alpha, X_\alpha, X_{-\alpha} = -\theta(X_\alpha)$ is a standard $sl(2,\mathbb{R})$-triple. Put $X_\beta = \sigma\theta X_\alpha$. Then $H_\beta, X_\beta, X_{-\beta}$ are the respective images of $H_\alpha, X_\alpha, X_{-\alpha}$ under $\sigma\theta$. Let i_α, i_β be the associated embeddings of $sl(2,\mathbb{R})$ into $\mathfrak{g}$ defined as in (14). Then $i_\alpha \times i_\beta$ is an isomorphism from $\mathfrak{s} := sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$ onto $\mathfrak{g}(T_P)$; the respective pull-backs σ^*, θ^* of σ and θ under this isomorphism are given by $\theta^*(X, Y) = (-X^t, -Y^t)$ and $\sigma^*(X, Y) = (Y, X)$.

Since $\mathfrak{g}(\alpha)$ and $\mathfrak{g}(\beta)$ commute, it follows that f_α and f_β commute. Hence F_P is commutative. Put $U = X_\alpha + X_\beta$. Then U and θU are σ-stable, and hence $f_\alpha f_\beta = \exp(\pi(U + \theta U))$ belongs to H_e. Moreover, $\sigma(f_\alpha) = f_\beta$ and $\sigma(f_\beta) = f_\alpha$. Thus we see that

$f\sigma(f) \in H_e$ if f is any of the generators f_α, f_β of F_P. Since F_P is commutative we infer that $f\sigma(f) \in H_e$ for all $f \in F_P$. Let now $f \in F_P$. Then $\sigma(f)^{-1} = f(\sigma(f)f)^{-1} \in f(H_e \cap F)$. From this we see that (24) holds for every $m \in F_P$. It therefore suffices to show that m_Q maps $\bar{N}_Q \cap \Omega_Q$ into $F_P/F_P \cap H_e$.

In the above we inferred that $\Sigma_{0\mathrm{q}}^+ = \{\alpha, \beta\}$, hence N_Q and $\bar{N}_Q$ are contained in the analytic subgroup $G(T_P)$ of G with Lie algebra $\mathfrak{g}(T_P)$. It follows that m_Q maps $\bar{N}_Q \cap \Omega_Q$ into $M \cap G(T_P)/H_e \cap G(T_P)$. Since $M \cap G(T_P) \subset M_0 \cap G(T_P) \subset F_P$ it follows that m_Q maps $\bar{N}_Q \cap \Omega_Q$ into $F_P/F_P \cap H_e$. This completes the proof of (e) in case (c1).

Case (c2). This is the case occurring in Lemma 6.6. We define $\tau = \theta$. Then obviously τ satisfies conditions (a) – (d). We will finish the proof by showing that τ satisfies condition (e) as well. Since θ maps $\Omega_{\bar{Q}} = \bar{N}_Q AMH_e$ onto $\Omega_Q = N_Q AMH_e$, it follows that for all $\bar{n} \in \bar{N}_Q \cap \Omega_Q$ we have:

$$m_{\bar{Q}}(\tau\bar{n}) = \tau m_Q(\bar{n}) = \theta m_Q(\bar{n}).$$

Hence to prove (e) it suffices to show that for all $\bar{n} \in \bar{N}_Q \cap \Omega_Q$ we have:

$$\theta m_Q(\bar{n}) \in m_Q(\bar{n})^{-1}(M \cap H_e). \tag{25}$$

In the following we shall use the notations and conclusions of Lemma 6.6 and Corollary 6.7. From $\mathfrak{g}(\Gamma) \simeq sl\,(n+2, \mathbb{R})$ it follows that

$$\mathfrak{g}(\Gamma) = \mathfrak{a}_0(\Gamma) \oplus \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma,$$

where $\mathfrak{a}_0(\Gamma)$ is the ($(n+1)$-dimensional) vector sum of the lines $(\ker \gamma_j)^\perp$, $0 \le j \le n$. We note that $\mathfrak{a}_\mathrm{q} \subset \mathfrak{a}_0(\Gamma)$, hence

$$\mathfrak{a}_0(\Gamma) = \mathfrak{a}_{0\mathrm{h}}(\Gamma) \oplus \mathfrak{a}_\mathrm{q},$$

where $\mathfrak{a}_{0\mathrm{h}}(\Gamma) = \mathfrak{a}_0(\Gamma) \cap \mathfrak{h}$. Moreover, $\Sigma_{0\mathrm{q}} = \{\alpha \in \Sigma_0 \mid \alpha|\mathfrak{a}_\mathrm{q} \neq 0\}$ is contained in $\Sigma_0(\Gamma)$. Hence for $Q \in \mathcal{P}(A_\mathrm{q})$ we have that $N_Q, \bar{N}_Q \subset G(\Gamma)$, where the latter denotes the analytic subgroup of G with Lie algebra $\mathfrak{g}(\Gamma)$.

Let $M_1(\Gamma) = M_1 \cap G(\Gamma)$. Then $M_1(\Gamma)$ is the centralizer of $\mathfrak{a}_\mathrm{q}$ in $G(\Gamma)$, and we see that $Q(\Gamma) = M_1(\Gamma)N_Q$ is a minimal σ-parabolic subgroup of $G(\Gamma)$. Moreover, we readily see that $Q(\Gamma) = Q \cap G(\Gamma)$.

Let $H(\Gamma) = H \cap G(\Gamma)$. Then $H(\Gamma)$ is an open subgroup of $G(\Gamma)^\sigma$. The root α restricts to a root ν of $\mathfrak{a}_\mathrm{q}$ in $\mathfrak{g}(\Gamma)$. Let $\mathfrak{g}(\Gamma)_\nu$ be the associated root space. If $X \in \mathfrak{g}_\alpha \setminus \{0\}$, then $X + \sigma\theta X$ is a non-trivial $\sigma\theta$-stable element of $\mathfrak{g}(\Gamma)_\nu$. Hence $\Sigma(\mathfrak{g}(\Gamma)^{\sigma\theta}, \mathfrak{a}_\mathrm{q}) \neq \emptyset$ and it follows that the set $\Omega_{Q(\Gamma)} := Q(\Gamma)H(\Gamma)_e$ is open and dense in $G(\Gamma)$ (use [1], Appendix B). This implies that $\bar{N}_Q \cap \Omega_{Q(\Gamma)}$ is open and dense in $\bar{N}_Q \cap \Omega_Q$.

Next we observe that

$$\mathfrak{m}_1(\Gamma) = \mathfrak{a}_0(\Gamma) \oplus \bigoplus_{\gamma \in \Gamma \cap \Sigma_{0\mathrm{M}}} \mathfrak{g}_\gamma. \tag{26}$$

Let $M(\Gamma)$ be the Langlands M of $Q(\Gamma)$. Then $\mathfrak{m}(\Gamma)$ is contained in the orthocomplement of $\mathfrak{a}_\mathrm{q}$ in $\mathfrak{m}_1(\Gamma)$, with respect to B. From (26) we now see that $\mathfrak{m}(\Gamma) \subset \mathfrak{h}(\Gamma)$, in view of

Lemma 5.1. Hence $M(\Gamma)_e \subset H(\Gamma)_e$, and by Lemma 3.6 the inclusion $F_P \to M(\Gamma)$ induces a surjective map from F_P onto the discrete space $M(\Gamma)/M(\Gamma) \cap H(\Gamma)_e$.

Let $m_{Q(\Gamma)} : \Omega_{Q(\Gamma)} \to M(\Gamma)/M(\Gamma) \cap H(\Gamma)_e$ be the map defined by

$$x \in N_Q A_{\mathrm{q}} m_{Q(\Gamma)}(x) H(\Gamma)_e.$$

Then by compatibility of decompositions it is obvious that for $\bar{n} \in \bar{N}_Q \cap \Omega_{Q(\Gamma)}$ the element $m_{Q(\Gamma)}(\bar{n}) \in M(\Gamma)/M(\Gamma) \cap H(\Gamma)_e$ has canonical image $m_Q(\bar{n})$ in $M/M \cap H_e$. It follows that $m_Q(\bar{N}_Q \cap \Omega_{Q(\Gamma)})$ is contained in the canonical image of F_P in $M/M \cap H_e$. By density and continuity it finally follows that for every $\bar{n} \in \bar{N}_Q \cap \Omega_Q$ there exists a $f(\bar{n}) \in F_P$ such that

$$m_Q(\bar{n}) = f(\bar{n})(M \cap H_e).$$

Hence

$$\theta m_Q(\bar{n}) = f(\bar{n})(M \cap H_e) = f(\bar{n})^{-1}(M \cap H_e) \in m_Q(\bar{n})^{-1}(M \cap H_e),$$

from which (25) follows. □

For $Q \in \mathcal{P}(A_{\mathrm{q}})$, let $h_Q : G \to A$ be the real analytic map defined by

$$x \in N_Q h_Q(x) M_Q K \qquad (x \in G). \tag{27}$$

Then we have the following consequence of Proposition 7.1.

Lemma 7.3 *Let τ be an automorphism of $\mathfrak{g}$ which satifies conditions (a) – (e) of Proposition 7.1. Moreover, let $Q \in \mathcal{P}(A_{\mathrm{q}})$. Then for every $\nu \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ and all $\bar{n} \in \bar{N}_Q$ we have:*

$$h_{\bar{Q}}(\tau\bar{n})^{\nu} = h_Q(\bar{n})^{-\nu} \tag{28}$$

Moreover, for every $\mu \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ and all $\bar{n} \in \bar{N}_Q \cap \Omega_Q$ we have $\tau\bar{n} \in N_Q \cap \Omega_{\bar{Q}}$ and

$$a_{\bar{Q}}(\tau\bar{n})^{\mu} = a_Q(\bar{n})^{-\mu} \tag{29}$$

Proof: It suffices to prove the assertions for any covering group of G to which σ has a lifting. By replacing G by a suitable finite covering we may as well assume that both σ and τ have a lifting from $\mathfrak{g}$ to G (use Lemma 4.1).

The automorphism τ of G commutes with θ by condition (a) of Proposition 7.1 and therefore stabilizes K. Moreover, τ preserves $\mathfrak{a}$, hence M_1 and M. It maps $\bar{N}_Q$ onto N_Q. Thus if $\bar{n} \in \bar{N}_Q$ then from $\bar{n} \in N_Q h_Q(\bar{n}) MK$ it follows that $\tau\bar{n} \in N_Q \tau h_Q(\bar{n}) MK$, and we conclude that $h_{\bar{Q}}(\tau\bar{n}) = \tau h_Q(\bar{n})$. Since $\tau = -I$ on $\mathfrak{a}_{\mathrm{q}}$ this implies (28).

The automorphism τ of G commutes with σ as well, hence leaves H_e invariant. It therefore maps the decomposition $\Omega_Q = N_Q AMH_e$ onto $\Omega_{\bar{Q}} = \bar{N}_Q AMH_e$. The assertion (29) now follows by an argument similar to the one above. □

In the proof of the main result in the next section, a key role is played by the following corollary.

Corollary 7.4 *Let τ be an automorphism of $\mathfrak{g}$ satisfying conditions (a) – (e) of Proposition 7.1, and let $Q \in \mathcal{P}(A_{\rm q})$. Let $\xi \in \widehat{M}_{\rm fu}$, and assume that $\eta_1, \eta_2 \in \mathcal{H}_\xi^H$. Then for all $\lambda, \nu \in \mathfrak{a}_{{\rm q}\mathbb{C}}^*$ and all $\bar{n} \in \bar{N}_Q \cap \Omega_Q$ we have*

$$\begin{aligned}\langle\, h_Q(\bar{n})^\nu a_Q(\bar{n})^{\lambda-\nu+\rho_Q} \xi(m_Q(\bar{n}))\, \eta_1\,,\, \eta_2 \rangle \\ = \;\; \langle\, \eta_1\,,\, h_{\bar{Q}}(\tau\bar{n})^{-\bar{\nu}} a_{\bar{Q}}(\tau\bar{n})^{-\bar{\lambda}+\bar{\nu}+\rho_{\bar{Q}}} \xi(m_{\bar{Q}}(\tau\bar{n})\, \eta_2\rangle\,. \end{aligned} \tag{30}$$

Proof: By unitarity of ξ the expression in the left hand side of (30) equals

$$h_Q(\bar{n})^\nu a_Q(\bar{n})^{\lambda-\nu+\rho_Q} \langle\, \eta_1\,,\, \xi(m_Q(\bar{n}))^{-1} \eta_2 \rangle \tag{31}$$

The function $m \mapsto \langle\, \eta_1\,,\, \xi(m)\eta_2\rangle$ belongs to $C^\omega(M\cap H_e\backslash M/M\cap H_e)$. Applying Proposition 7.1 and Lemma 7.3 we may therefore rewrite (31) as

$$h_{\bar{Q}}(\tau\bar{n})^{-\nu} a_{\bar{Q}}(\tau\bar{n})^{-\lambda+\nu+\rho_{\bar{Q}}} \langle\, \eta_1\,,\, \xi(m_{\bar{Q}}(\tau\bar{n}))\, \eta_2\rangle$$

which in turns equals the right hand side of (30). □

8 Proof of the main result in the reduced case

In this section we work under the same assumptions as in the previous one. Thus $\mathfrak{a}_{\rm q}$ has dimension one and is not central in $\mathfrak{g}$. Moreover, P is a minimal σ-parabolic subgroup containing $A_{\rm q}$. Thus $\mathcal{P}(A_{\rm q}) = \{P, \bar{P}\}$.

We assume that the group H is connected. Then $\Omega_Q = QH_e = QH$ for every $Q \in \mathcal{P}(A_{\rm q})$.

Let $\xi \in \widehat{M}_{\rm fu}$ be fixed from now on. In the course of this section we shall prove the following result.

Proposition 8.1 *Let $\eta_1, \eta_2 \in \mathcal{H}_\xi^{M\cap H}$. Then*

$$\langle\, B(\bar{P}\,{:}\,P\,{:}\,\xi\,{:}\,\lambda)\, \eta_1\,,\, \eta_2\rangle \; = \langle\, \eta_1\,,\, B(P\,{:}\,\bar{P}\,{:}\,\xi\,{:}\,-\bar{\lambda})\, \eta_2\rangle$$

as an identity of meromorphic functions of $\lambda \in \mathfrak{a}_{{\rm q}\mathbb{C}}^*$.

Before beginning with the proof of this proposition we will first derive Theorem 2.1 from it.

Proof of Theorem 2.1 in the reduced case. Assume in addition that $|W/W_{K\cap H}| = 1$. Then we are in the reduced case where Theorem 2.1 still needs to be proved. In this case the direct sum (4) has one term, so that $V(\xi) \simeq \mathcal{H}_\xi^{M\cap H}$. The above proposition therefore implies Theorem 2.1 with $Q = \bar{P}$. This is sufficient since $\mathcal{P}(A_{\rm q}) = \{P, \bar{P}\}$. □

To explain the idea of the proof of Proposition 8.1, we will first discuss a sequence of equalities which hold in a formal sense. Later these equalities will be interpreted by means of a meromorphic continuation.

Let $\mathrm{pr}_1 : V(\xi) \to \mathcal{H}_\xi^{M\cap H}$ be the projection onto the $w = 1$ component in the decomposition (4). Let $Q \in \mathcal{P}(A_\mathrm{q})$, $\eta \in \mathcal{H}_\xi^{M\cap H}$. Our first goal is to obtain a formal expression for $\mathrm{pr}_1 \circ B(\bar{Q}\,{:}\,Q\,{:}\,\xi\,{:}\,\lambda)\eta$. Using [1], Theorem 5.10 and Proposition 6.1, we obtain that

$$\mathrm{pr}_1 \circ B(\bar{Q}\,{:}\,Q\,{:}\,\xi\,{:}\,\lambda)\eta = \mathrm{ev}_1 \circ A(\bar{Q}\,{:}\,Q\,{:}\,\xi\,{:}\,\lambda)j(Q\,{:}\,\xi\,{:}\,\lambda)\eta \tag{32}$$

as an identity of meromorphic functions in $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. For $Q \in \mathcal{P}(A_\mathrm{q})$ and $R \in \mathbb{R}$ we put:

$$\mathfrak{a}^*_\mathrm{q}(Q, R) = \{\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \mid \langle \operatorname{Re}\lambda\,,\,\alpha\rangle < R,\ \ \forall\,\alpha \in \Sigma(Q)\}.$$

Then by [1], Proposition 4.1, there exists a constant $C > 0$ such that for all $Q \in \mathcal{P}(A_\mathrm{q})$ and all $\lambda \in \mathfrak{a}^*_\mathrm{q}(\bar{Q}, C)$ the intertwining operator is given by an absolutely converging integral on continuous functions:

$$[A(\bar{Q}\,{:}\,Q\,{:}\,\xi\,{:}\,\lambda)f](x) = \int_{\bar{N}_Q} f(\bar{n}x)d\bar{n} \qquad (x \in G) \tag{33}$$

for $f \in C(Q\,{:}\,\xi\,{:}\,\lambda)$.

On the other hand, we have the following result about the continuity of $j(\xi\,{:}\,\lambda)$.

Lemma 8.2 *Let $Q \in \mathcal{P}(A_\mathrm{q})$, and $\eta \in \mathcal{H}_\xi^{M\cap H}$. Then for $\lambda + \rho_Q \in \mathfrak{a}^*_\mathrm{q}(Q, 0)$ the H-fixed generalized function $j(Q\,{:}\,\xi\,{:}\,\lambda\,{:}\,\eta)$ is continuous and given by the formula*

$$\begin{aligned} j(Q\,{:}\,\xi\,{:}\,\lambda\,{:}\,\eta)(x) &= a_Q(x)^{\lambda+\rho_Q}\xi(m_Q(x))\eta \quad &&\text{for} \quad x \in \Omega_Q = QH, \\ &= 0 &&\text{elsewhere.} \end{aligned} \tag{34}$$

Proof: This follows from [1], Proposition 5.6 and the display preceding Lemma 5.7. □

Combining (32), (33) and (34) we obtain

$$\mathrm{pr}_1 \circ B(\bar{Q}\,{:}\,Q\,{:}\,\xi\,{:}\,\lambda)\eta = \int_{\bar{N}_Q \cap \Omega_Q} a_Q(\bar{n})^{\lambda+\rho_Q}\,\xi(m_Q(\bar{n}))\eta\; d\bar{n} \tag{35}$$

for λ in the intersection of $\mathfrak{a}_\mathrm{q}(\bar{Q}, C)$ and $\mathfrak{a}_\mathrm{q}(Q, 0)$. Unfortunately this intersection is empty, so that we can only interpret (35) in a formal sense. Continuing in this formal fashion, let $\eta_1, \eta_2 \in \mathcal{H}_\xi^{M\cap H}$. Let τ be an automorphism of $\mathfrak{g}$ satisfying conditions (a) – (e) of Proposition 7.1. Then applying Corollary 7.4 with $\nu = 0$ we obtain (formally):

$$\begin{aligned} \langle\, B(\bar{P}\,{:}\,P\,{:}\,\xi\,{:}\,\lambda)\eta_1\,,\,\eta_2\rangle &= \int_{\bar{N}_P\cap\Omega_P} \langle\, a_P(\bar{n})^{\lambda+\rho_P}\xi(m_P(\bar{n}))\eta_1\,,\,\eta_2\rangle\; d\bar{n} &(36)\\ &= \int_{\bar{N}_P\cap\Omega_P} \langle\, \eta_1\,,\, a_{\bar{P}}(\tau\bar{n})^{-\bar{\lambda}+\rho_{\bar{P}}}\xi(m_{\bar{P}}(\tau\bar{n}))\eta_2\rangle\; d\bar{n}. &(37) \end{aligned}$$

We now make the substitution of variables $n = \tau \bar{n}$. Since $\tau : \bar{n} \mapsto \tau n,\ \bar{N}_P \to N_P = \bar{N}_{\bar{P}}$ is a diffeomorphism with $\tau^*(dn) = d\bar{n}$ (see Proposition 7.1 (d)), the integral in (37) becomes:

$$\int_{N_P \cap \Omega_{\bar{P}}} \langle\, \eta_1\,,\, a_{\bar{P}}(n)^{-\lambda + \rho_{\bar{P}}} \xi(m_P(n))\eta_2 \rangle \; dn = \langle\, \eta_1\,,\, B(P\,{:}\,\bar{P}\,{:}\,\xi\,{:}\,-\bar{\lambda})\eta_2 \rangle\,. \tag{38}$$

We will interpret the above sequence of equalities by a meromorphic continuation, involving an additional parameter $\nu \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. The particular continuation is based on an idea that goes back to [12], and was also applied in [1], Sections 7 and 8.

Let $Q \in \mathcal{P}(A_{\mathrm{q}})$, $\eta \in \mathcal{H}_\xi^{M\cap H}$. Then for every $(\lambda, \nu) \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \times \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ such that $j(Q\,{:}\,\xi\,{:}\,\cdot\,{:}\,\eta)$ is regular at $\lambda - \nu$, we define an element of $C^{-\infty}(Q\,{:}\,\xi\,{:}\,\lambda)$ by:

$$J(Q\,{:}\,\xi\,{:}\,\lambda\,{:}\,\nu\,{:}\,\eta) = h_Q^\nu \, j(Q\,{:}\,\xi\,{:}\,\lambda - \nu\,{:}\,\eta), \tag{39}$$

where h_Q is defined by (27). Then viewed as a $\mathrm{Hom}(\mathcal{H}_\xi^{M\cap H}, C^{-\infty}(K\,{:}\,\xi))$-valued function, $J(Q\,{:}\,\xi\,{:}\,\lambda\,{:}\,\nu)$ depends meromorphically on $(\lambda, \nu) \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \times \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$.

From [1], Corollary 4.14, it follows that $A(\bar{Q} : Q : \xi : \lambda) J(Q : \xi : \lambda : \nu : \eta)$ restricts to a smooth function on $\bar{Q}H$, whenever (λ, ν) is not a singular point for any of the meromorphic factors involved. In particular the expression may then be evaluated at the identity element. Put

$$\mathbf{B}(\bar{Q}\,{:}\,Q\,{:}\,\xi\,{:}\,\lambda\,{:}\,\nu)\eta = \mathrm{ev}_e \circ A(\bar{Q}\,{:}\,Q\,{:}\,\xi\,{:}\,\lambda) J(Q\,{:}\,\xi\,{:}\,\lambda\,{:}\,\nu\,{:}\,\eta). \tag{40}$$

Lemma 8.3 *The linear map*

$$\mathbf{B}(\bar{Q}\,{:}\,Q\,{:}\,\xi\,{:}\,\lambda\,{:}\,\nu) \in \mathrm{Hom}(\mathcal{H}_\xi^{M\cap H}, \mathcal{H}_\xi) \tag{41}$$

depends meromorphically on $(\lambda, \nu) \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \times \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. Moreover, if $\lambda_0 \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ is not a singularity for any of the meromorphic maps $\lambda \mapsto A(\bar{Q}\,{:}\,Q\,{:}\,\xi\,{:}\,\lambda)$ or $\lambda \mapsto j(Q\,{:}\,\xi\,{:}\,\lambda)$, then $(\lambda_0, 0)$ is not a singularity for (41), and

$$\mathbf{B}(\bar{Q}\,{:}\,Q\,{:}\,\xi\,{:}\,\lambda_0\,{:}\,0) = \mathrm{pr}_1 \circ B(\bar{Q}\,{:}\,Q\,{:}\,\xi\,{:}\,\lambda_0)|\mathcal{H}_\xi^{M\cap H}.$$

Proof: From the assumptions it follows that the right hand side and hence the left hand side of (39) depends holomorphically on (λ, ν) in a neighborhood of $(\lambda_0, 0)$. Combining this observation with (40) and applying [1], Cor. 4.14, we see that (41) is regular at $(\lambda_0, 0)$. Moreover, if $\eta \in \mathcal{H}_\xi^{M\cap H}$, then

$$\begin{aligned} B(\bar{Q}\,{:}\,Q\,{:}\,\xi\,{:}\,\lambda_0\,{:}\,0)\eta &= \mathrm{ev}_e \circ A(\bar{Q}\,{:}\,Q\,{:}\,\xi\,{:}\,\lambda_0) J(Q\,{:}\,\xi\,{:}\,\lambda_0\,{:}\,0\,{:}\,\eta) \\ &= \mathrm{ev}_e \circ A(\bar{Q}\,{:}\,Q\,{:}\,\xi\,{:}\,\lambda_0) j(Q\,{:}\,\xi\,{:}\,\lambda_0\,{:}\,\eta) \\ &= \mathrm{pr}_1 \circ B(\bar{Q}\,{:}\,Q\,{:}\,\xi\,{:}\,\lambda_0)\eta. \end{aligned}$$

□

In view of Lemma 8.3, Proposition 8.1 is now a straightforward consequence of the following result.

Proposition 8.4 *Let $\eta_1, \eta_2 \in \mathcal{H}_\xi^{M\cap H}$. Then we have the following identity of meromorphic functions of $(\lambda, \nu) \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \times \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$:*

$$\langle\, \mathbf{B}(\bar{P}\!:\!P\!:\!\xi\!:\!\lambda\!:\!\nu)\eta_1\,,\, \eta_2\rangle \;=\langle\, \eta_1\,,\, \mathbf{B}(P\!:\!\bar{P}\!:\!\xi-\bar{\lambda}\!:\!-\bar{\nu})\eta_2\rangle\,. \tag{42}$$

Proof: We will establish this proposition by the reasoning indicated in equations (36) - (38). We start with a useful lemma. If $Q \in \mathcal{P}(A_{\mathrm{q}})$, let

$$\mathcal{A}_Q = \{(\lambda,\nu) \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \times \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \mid \lambda \in \mathfrak{a}^*_{\mathrm{q}}(\bar{Q}, C) \quad \text{and} \quad \lambda - \nu + \rho_Q \in \mathfrak{a}^*_{\mathrm{q}}(Q, 0)\}$$

Then obviously $\mathcal{A}_Q$ is a non-empty open subset of $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \times \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$.

Lemma 8.5 *Let $Q \in \mathcal{P}(A_{\mathrm{q}})$ and $(\lambda,\nu) \in \mathcal{A}_Q$. Then for all $\eta \in \mathcal{H}_\xi^{M\cap H}$ we have:*

$$\mathbf{B}(\bar{Q}\!:\!Q\!:\!\xi\!:\!\lambda\!:\!\nu)\eta = \int_{\bar{N}_Q\cap\Omega_Q} h_Q(\bar{n})^\nu\, a_Q(\bar{n})^{\lambda-\nu+\rho_Q}\, \xi(m_Q(\bar{n}))\, \eta\, d\bar{n}, \tag{43}$$

the integral being absolutely convergent.

Proof: Consider equation (40). Since $\lambda - \nu + \rho_Q \in \mathfrak{a}^*_{\mathrm{q}}(Q, 0)$, the occurring function J is continuous (use Lemma 8.2 and the continuity of h_Q), and since $\lambda \in \mathfrak{a}^*_{\mathrm{q}}(\bar{Q}, C)$, the integral for the intertwining operator A converges absolutely. Therefore the left hand side of (43) equals:

$$\int_{\bar{N}_Q} J(Q\!:\!\xi\!:\!\lambda\!:\!\nu\!:\!\eta)(\bar{n})\, d\bar{n}.$$

Using (39) and (34) we may rewrite this integral as the right hand side of (43). □

Completion of the proof of Proposition 8.4. By meromorphy it suffices to establish (42) for (λ,ν) contained in the non-empty open set $\mathcal{A}_P$. Then by Lemma 8.5 with $Q = P$ the left hand side of (42) may be written as the absolutely convergent integral:

$$\int_{\bar{N}_P\cap\Omega_P} \langle\, a_P(\bar{n})^{\lambda-\nu+\rho_P} h_P(\bar{n})^\nu \xi(m_P(\bar{n}))\eta_1\,,\, \eta_2\rangle\; d\bar{n}.$$

Let τ be an automorphism of $\mathfrak{g}$ satisfying conditions (a) – (e) of Proposition 7.1. Then applying Corollary 7.4 we see that the above integral equals

$$\int_{\bar{N}_P\cap\Omega_P} \langle\, \eta_1\,,\, a_{\bar{P}}(\tau\bar{n})^{-\bar{\lambda}+\bar{\nu}+\rho_{\bar{P}}} h_{\bar{P}}(\tau\bar{n})^{-\bar{\nu}} \xi(m_{\bar{P}}(\tau\bar{n}))\, \eta_2\rangle\; d\bar{n}$$

Using the substitution of variables $n = \tau\bar{n}$ we may rewrite this integral as

$$\int_{N_P\cap\Omega_{\bar{P}}} \langle\, \eta_1\,,\, a_{\bar{P}}(n)^{-\bar{\lambda}+\bar{\nu}+\rho_{\bar{P}}} h_{\bar{P}}(n)^{-\bar{\nu}} \xi(m_{\bar{P}}(n))\, \eta_2\rangle\; dn. \tag{44}$$

Now $(-\bar{\lambda}, -\bar{\nu}) \in \mathcal{A}_{\bar{P}}$, so that we may use Lemma 8.5 with $Q = \bar{P}$ to infer that (44) equals $\langle\, \eta_1\,,\, \mathbf{B}(P\!:\!\bar{P}\!:\!\xi\!:\!-\bar{\lambda}\!:\!-\bar{\nu})\, \eta_2)\rangle$. This establishes (42). □

References

[1] *E.P. van den Ban,* The principal series for a reductive symmetric space I. H-fixed distribution vectors. Ann. sci. Ec. Norm. Sup. **21** (1988), 359-412.

[2] *E.P. van den Ban,* The principal series for a reductive symmetric space II. Eisenstein integrals. J. Func. Anal. **109** (1992), 331-441.

[3] *E.P. van den Ban and H. Schlichtkrull,* Multiplicities in the Plancherel decomposition for a semisimple symmetric space. Contemporary Math. **145** (1993), 163-180.

[4] *E.P. van den Ban and H. Schlichtkrull,* Fourier transforms on a semisimple symmetric space. Preprint no. 888, Universiteit Utrecht, November 1994.

[5] *E.P. van den Ban and H. Schlichtkrull,* The most-continuous part of the Plancherel decomposition for a reductive symmetric space. Preprint no. 921, Universiteit Utrecht, August 1995. To appear in Ann. Math.

[6] *E. van den Ban, M. Flensted-Jensen and H. Schlichtkrull,* Basic harmonic analysis on pseudo-Riemannian symmetric spaces. pp. 69 - 101 in: Noncompact Lie Groups and Some of Their Applications, E.A. Tanner and R. Wilson (eds.), Kluwer Academic Publishers, Dordrecht, 1994.

[7] *P. Delorme,* Troncature pour les espaces symétriques réductif. Preprint, Faculté des Sciences de Luminy, 1996.

[8] *J.J. Duistermaat, J.A.C. Kolk and V.S. Varadarajan,* Functions, flows and oscillatory integrals on flag manifolds and conjugacy classes in real semisimple Lie groups. Compositio Math. **49** (1983), 309-398.

[9] *S. Helgason,* Differential Geometry, Lie Groups, and Symmetric Spaces. Academic Press, New York, 1978.

[10] *G. Hochschild,* The structure of Lie groups. Holden-Day, Inc., San Francisco, 1965.

[11] *A.W. Knapp and E.M. Stein,* Intertwining operators for semisimple Lie groups, II. Invent. math. **60** (1980), 9-84.

[12] *T. Oshima and J. Sekiguchi,* Eigenspaces of invariant differential operators on a semisimple symmetric space. Invent. Math. **57** (1980), 1-81.

[13] *H. Schlichtkrull,* Hyperfunctions and Harmonic Analysis on Symmetric Spaces. Birkhäuser Verlag, Boston, 1984.

[14] *M. Wiggerman,* The fundamental group of a real flag manifold. Report No. 9544, University of Nijmegen, October 1995.

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