Topological representation of sheaf cohomology of sites

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Abstract

For a site $\mathcal{S}$ (with enough points), we construct a topological space $X_{(\mathcal{S})}$ and a full embedding φ^* of the category of sheaves on $\mathcal{S}$ into those on $X_{(\mathcal{S})}$ (i.e., a morphism of toposes $\varphi: \mathrm{Sh}(X_{(\mathcal{S})}) \to \mathrm{Sh}(\mathcal{S})$). The embedding will be shown to induce a full embedding of derived categories, hence isomorphisms $\mathrm{H}^*(\mathcal{S}, A) = \mathrm{H}^*(X_{(\mathcal{S})}, \varphi^* A)$ for any abelian sheaf A on $\mathcal{S}$. As a particular case, this will give for any scheme Y a topological space $X_{(Y)}$ and a functorial isomorphism between the étale cohomology $\mathrm{H}^*(Y_{\text{ét}}, A)$ and the ordinary sheaf cohomology $\mathrm{H}^*(X_{(Y)}, \varphi^* A)$, for any sheaf A for the étale topology on Y.

1 Introduction and statement of the theorem

Many cohomology groups arising in geometry and topology are (or can be) defined as the cohomology groups of some topos; that is, as the sheaf cohomology groups of some site. This applies directly to étale and other cohomologies of schemes [1, 10], but also to many others such as Galois cohomology [12] and cyclic cohomology [2].

The purpose of this paper is to give a general construction which shows that all these cohomology groups are isomorphic to the ordinary sheaf cohomology groups of a topological space associated to the site or the topos. When the site is a group G (with associated topos of G–sets), our construction gives a model for the classifying space BG. In general, our result can be interpreted as the construction of a "classifying space" for any site (satisfying the following technical condition).

Our construction applies to topoi *with enough points*. We recall that a point p of a topos $\mathcal{T}$ is a topos morphism $p: \mathcal{S} \to \mathcal{T}$, from the topos $\mathcal{S}$ of sets into $\mathcal{T}$. Such a morphism can equivalently be described as a functor $p^*: \mathcal{T} \to \mathcal{S}$ which preserves colimits and finite limits, or as a morphism of sites $\mathbb{F}: \mathbb{C} \to \mathcal{S}$, where $\mathbb{C}$ is any site of definition for $\mathcal{T}$. The topos $\mathcal{T}$ is said to have enough points if for any sequence $A \to B \to C$ of abelian groups in $\mathcal{T}$ (i.e., sheaves of abelian groups on $\mathbb{C}$), the sequence is exact whenever for each point p of $\mathcal{T}$ the associated sequence $p^*A \to p^*B \to p^*C$ is an exact sequence of abelian groups. We hasten to point out that virtually all topoi arising in geometric practice have enough points. This applies, for example, to the presheaf topos $\hat{\mathbb{C}}$ on an arbitrary small category $\mathbb{C}$, and to the étale topos associated to a scheme. In fact, any "coherent" topos has enough points (Deligne, Appendix to Exposé VI in [1]).

For any topological space X, the category $\mathrm{Sh}(X)$ of sheaves on X is a topos (with enough points), whose cohomology groups are the ordinary sheaf cohomology groups of X [3, 6]. We will prove the following result:

Theorem. *Let $\mathcal{T}$ be a topos with enough points. There exists a topological space $X_{\mathcal{T}}$ and a topos morphism*

$$\varphi: \mathrm{Sh}(X_{\mathcal{T}}) \to \mathcal{T}$$

such that

(i) φ^* is a full and faithful embedding of $\mathcal{T}$ into $\mathrm{Sh}(X_{\mathcal{T}})$;

(ii) for any abelian group A in $\mathcal{T}$, the morphism φ induces isomorphisms

$$\mathrm{H}^*(\mathcal{T}, A) \xrightarrow{\simeq} \mathrm{H}^*(X_{\mathcal{T}}, \varphi^*(A)), \qquad n \geq 0.$$

Here $\mathrm{H}^*(X_{\mathcal{T}}, \varphi^*(A))$ denotes the sheaf cohomology of the space $X_{\mathcal{T}}$ with the sheaf $\varphi^*(A)$ as coefficients. We will give an explicit construction of this space $X_{\mathcal{T}}$ from $\mathcal{T}$, which depends not only on $\mathcal{T}$, but also on the choice of a site for $\mathcal{T}$. For this reason, the construction $\mathcal{T} \mapsto X_{\mathcal{T}}$ is only functorial in $\mathcal{T}$ in a weak sense (see Remark 2.4 below).

Note that, since the topos $\mathrm{Sh}(X_{\mathcal{T}})$ always has enough points, the (mild) assumption that $\mathcal{T}$ has enough points is a necessary one, being implied by part (i) of the theorem. For part (ii) of the theorem, we will actually prove that the derived functors $\mathrm{R}^q\varphi_*$ of the direct image functor $\varphi_*: \mathrm{Sh}(X_{\mathcal{T}}) \to \mathcal{T}$ have the property that

$$\mathrm{R}^q\varphi_*(\varphi^*A) = \begin{cases} A, & q = 0, \\ 0, & q > 0, \end{cases}$$

for any abelian group A in $\mathcal{T}$. This property states that $\varphi: \mathrm{Sh}(X_{\mathcal{T}}) \to \mathcal{T}$ is an *acyclic morphism*. It implies in particular that φ^* induces a full and faithful embedding of derived categories

$$D^+(\mathcal{T}) \hookrightarrow D^+(X_{\mathcal{T}}).$$

The same argument applies to ringed topoi: if $\mathcal{O}_{\mathcal{T}}$ is any ring in $\mathcal{T}$ and $D^+(\mathcal{T}, \mathcal{O}_{\mathcal{T}})$ is the associated derived category of complexes of $\mathcal{O}_{\mathcal{T}}$–modules [1], then φ^* induces a full and faithful embedding

$$D^+(\mathcal{T}, \mathcal{O}_{\mathcal{T}}) \hookrightarrow D^+(X_{\mathcal{T}}, \varphi^*(\mathcal{O}_{\mathcal{T}})).$$

The theorem, as well as the construction of the space $X_{\mathcal{T}}$, have been inspired by [8], where it is proved that any topos (not necessarily with enough points) is cohomologically equivalent to the topos of sheaves on a "locale". (A locale is an abstract notion of "topological space without points".) However, our theorem is not a consequence of this result of [8]. Furthermore, our proof is different. The proof in [8] made essential use of the "internal logic" of a topos and its behaviour under change–of–base. These methods cannot be applied to the topological space $X_{\mathcal{T}}$ constructed here.

2 Construction of the space $X_{\mathcal{T}}$ and of the map φ

In this section, $\mathcal{T}$ denotes a fixed topos with enough points. Recall [1, 9] that the latter means that the functors $p^*: \mathcal{T} \to \mathcal{S}$, for all points $p: \mathcal{S} \to \mathcal{T}$, are jointly conservative. Although the collection of all such points p is in general a proper class rather then a set, there will always be a *set* $\mathcal{P}$ of points p for which the functors p^*, for $p \in \mathcal{P}$, are already jointly conservative. We will fix such a set $\mathcal{P}$, and henceforth refer to points in this set as *small* points of $\mathcal{T}$. For a point p of $\mathcal{T}$ and an object (sheaf) E in $\mathcal{T}$, we will also use the common notation E_p for the set $p^*(E)$, and refer to E_p as "the stalk of E at p".

Next, we fix a sheaf G in $\mathcal{T}$ so that the collection of all subsheaves $C \subset G^n$, $n \geq 0$, generates $\mathcal{T}$. For example, G can be the disjoint sum (coproduct) of all the objects in some site of definition for $\mathcal{T}$. But often, there is a smaller and much more natural choice for G: the topos $\mathcal{T}$ will generally contain some "universal" structure U of a certain kind. For example, in the case of the étale topos, U is the universal strictly local ring [5]. More generally, if $\mathcal{T}$ is a classifying topos, U is the universal model for the theory classified by $\mathcal{T}$ (see [9], Chapter VIII). This object U will have the property required for G, namely that the subsheaves of finite products $U \times \cdots \times U$ generate $\mathcal{T}$.

Finally, we fix an infinite set I, which is big enough so that it surjects onto all the stalks G_p, for all small points p of $\mathcal{T}$; in other words,

$$\mathrm{card}(G_p) \leq \mathrm{card}(I).$$

The construction of the space $X_{\mathcal{T}}$ will depend on these choices, of the set $\mathcal{P}$ of points, of the sheaf G, and of the set I. (We come back to this dependence in Remark 2.4 below.)

The points of the space $X = X_{\mathcal{T}}$ are now defined to be equivalence classes of pairs

$$(p, \alpha)$$

where p is a small point of $\mathcal{T}$ and α is a function from a subset of I to G_p,

$$I \supset \mathrm{dom}(\alpha) \xrightarrow{\alpha} G_p,$$

with the property that $\alpha^{-1}(g)$ is infinite, for each $g \in G_p$. Two such pairs (p, α) and (q, β) are *equivalent* (i.e., define the same point $x \in X$), if there exists a natural isomorphism of functors $\theta: p^* \to q^*$ so that $\beta = \theta_G \circ \alpha$. We will often write $x = (p, \alpha)$ for a point $x \in X$, and not distinguish explicitly between such pairs (p, α) and their equivalence classes.

The topology on this set X of points is defined as follows: For any $n \geq 0$ and any subsheaf $C \subset G^n$, and any $i_1, \ldots, i_n \in I$, the set

$$U_{i_1,\ldots,i_n,C} = \{(p, \alpha) \mid i_1, \ldots, i_n \in \mathrm{dom}(\alpha) \text{ and } (\alpha(i_1), \ldots, \alpha(i_n)) \in C_p\} \qquad (1)$$

is to be a basic open set. Note that this set is well–defined on equivalence classes, i.e., $(p, \alpha) \in U_{i_1,\ldots,i_n,C}$ iff $(q, \beta) \in U_{i_1,\ldots,i_n,C}$. In the sequel, we will usually write i for

$i_1, \ldots, i_n$ and $\alpha(i)$ for $(\alpha(i_1), \ldots, \alpha(i_n))$, so that

$$U_{i,C} = \{(p, \alpha) \mid i \in \mathrm{dom}(\alpha) \text{ and } \alpha(i) \in C_p\}. \tag{2}$$

We remark that, by changing C, we can always assume that the sequence $i = (i_1, \ldots, i_n)$ does not contain repetitions. For example, $U_{i,i,C}$ for $C \subset G^2$ is equal to $U_{i,C'}$ for C' the pullback of C along the diagonal $\Delta: G \to G^2$. In the sequel we will often tacitly assume that a sequence i does not contain repetitions.

Lemma 2.1 *The sets $U_{i,C}$ form a basis for a topology on X.*

Proof. This is clear form the formula

$$U_{i,C} \cap U_{j,D} = U_{i,j,C \times D},$$

for any $C \subset G^n$, $D \subset G^m$, $i = (i_1, \ldots, i_n)$, $j = (j_1, \ldots, j_m)$, and i, j the concatenation of these two sequences. □

It can be shown that the space X thus defined is always a sober topological space ([1], IV.4.2.1), although it is not a Hausdorff space.

Next, we describe the morphism $\varphi: \mathrm{Sh}(X) \to \mathcal{T}$ occurring in the statement of the theorem. Recall that such a morphism of topoi is given by an inverse image functor $\varphi^*: \mathcal{T} \to \mathrm{Sh}(X)$ and a direct image functor $\varphi_*: \mathrm{Sh}(X) \to \mathcal{T}$, right adjoint to φ^*. The functor φ^* preserves colimits and finite limits, and these properties imply that φ^* has a right adjoint, unique up to isomorphism. So, to define φ, it suffices to define such a functor $\varphi^*: \mathcal{T} \to \mathrm{Sh}(X)$. For any sheaf E in $\mathcal{T}$, consider the set

$$\varphi^*(E) = \{(p, \alpha, e) \mid (p, \alpha) \in X, e \in E_p\},$$

with obvious projection $\pi: \varphi^*(E) \to X$. (Again, being more precise we should speak about equivalence classes of such triples, where (p, α, e) is equivalent to (q, β, g) if there exists a natural isomorphism of functors $\theta: p^* \to q^*$ so that $\beta = \theta_G \circ \alpha$ and $\theta_E(e) = g$.) The set $\varphi^*(E)$ carries a natural topology, with basic open sets

$$V_{i,C,f} = \{(p, \alpha, e) \mid (p, \alpha) \in U_{i,C} \text{ and } e = f(\alpha(i))\},$$

for any $i = (i_1, \ldots, i_n)$ and $C \subset G^n$ as above, and any morphism $f: C \to E$ in $\mathcal{T}$.

Lemma 2.2 *These sets $V_{i,C,f}$ form the basis for a topology on $\varphi^*(E)$, which makes the projection $\pi: \varphi^*(E) \to X$ into a local homeomorphism.*

Proof. Consider two such basic open sets $V_{i,C,f}$ and $V_{j,D,g}$. Let $h: C \times_E D \to E$ be the map from the pullback, $h = f \circ \pi_1 = g \circ \pi_2$. Then

$$V_{i,C,f} \cap V_{j,D,g} = V_{i,j,C \times_E D,h}.$$

Thus the sets $V_{i,C,f}$ form a basis for a well–defined topology on $\varphi^*(E)$. Furthermore, the sections

$$\sigma: U_{i,C} \to V_{i,C,f}, \qquad \sigma(p, \alpha) = f_p(\alpha(i))$$

are well–defined on equivalence classes and show that the projection $\pi: \varphi^*(E) \to X$ restricts to a homeomorphism $V_{i,C,f} \to U_{i,C}$. $\square$

Thus $\pi: \varphi^*(E) \to X$ is a sheaf on X. Note that for the stalk of this sheaf at a point (p, α) of X we have

$$\varphi^*(E)_{(p,\alpha)} = E_p. \tag{3}$$

Proposition 2.3 *The construction $E \mapsto \varphi^*(E)$ defines the inverse image functor of a topos morphism $\varphi: \mathrm{Sh}(X) \to \mathcal{T}$.*

Proof. We observe first that the construction is functorial in E. If $h: E \to F$ is a morphism in $\mathcal{T}$, the induced map

$$\varphi^*(h): \varphi^*(E) \to \varphi^*(F), \qquad (p, \alpha, e) \mapsto (p, \alpha, h_p(e))$$

is continuous for the topologies just defined. To see this, take any point (p, α, e) of $\varphi^*(E)$, and let $V_{i,C,f}$ be a basic open neighbourhood of $(p, \alpha, h_p(e))$ in $\varphi^*(F)$, where $f: C \to F$. Since the subsheaves of G^n generate $\mathcal{T}$, it follows that there is a $B \subset G^m$ and a map $u: B \to C \times_F E$ so that, for $c = \alpha(i)$, there exists a point $b \in B_p$ with $u_p(b) = (c, e) \in (C \times_F E)_p$. Choose $j = (j_1, \ldots, j_m)$ with $j_k \in I$, so that $b = \alpha(j) = (\alpha(j_1), \ldots, \alpha(j_m))$. Let $v = \pi_1 \circ u: B \to C$, and let $D = \mathrm{graph}(v) \subset B \times C \subset G^m \times G^n$. Then

$$W = V_{j,i,D,\pi_2 \circ u}$$

is a basic open set in $\varphi^*(E)$, such that $(p, \alpha, e) \in W$ and $\varphi^*(h)$ maps W into $V_{i,C,f}$.

This shows that φ^* is a functor. It remains to verify that φ^* preserves colimits and finite limits. But it suffices to show that this holds at the level of the stalks, where it is obvious from the identity (3). $\square$

Remark 2.4 (We recommend the reader to skip this remark, as we will make no future use of it in the present paper.) The construction of $X = X_{\mathcal{T}}$ depends on $\mathcal{P}$, G and I, in a functorial way. Clearly, for a larger set $\mathcal{P}' \supset \mathcal{P}$ of small points, there is a map $X(\mathcal{P}) \to X(\mathcal{P}')$ over $\mathcal{T}$. Similarly, it will be clear from §3 that a surjection $s: J \twoheadrightarrow I$ induces a map $s^*: X(I) \to X(J)$, while if $G' \supset G$ is a larger choice of an object so that the subsheaves of its finite powers generate, there is a restriction map $X(G') \to X(G)$. It is a consequence of our theorem that all these comparison maps induce isomorphisms in cohomology for abelian coefficients which come from $\mathcal{T}$, so that the dependence of X on $\mathcal{P}$, G and I is inessential in this sense.

If $f: \mathcal{T}_1 \to \mathcal{T}_2$ is a topos morphism, we can fix first the parameters $\mathcal{P}_1$ and I_1 for $\mathcal{T}_1$ and G_2 for $\mathcal{T}_2$, and then choose $\mathcal{P}_2$ large enough to include all composites $f \circ p$ for $p \in \mathcal{P}_1$, and $G_1 \supset f^*(G_2)$, and finally I_2 so large that there exists a surjection $I_2 \to I_1$. Then the constructed spaces X_1 and X_2 fit into a commutative diagram

$$\begin{tikzcd}
\mathrm{Sh}(X_1) \arrow[r, "\tilde{f}"] \arrow[d, "\varphi_1"'] & \mathrm{Sh}(X_2) \arrow[d, "\varphi_2"] \\
\mathcal{T}_1 \arrow[r, "f"'] & \mathcal{T}_2.
\end{tikzcd}$$

3 Enumeration spaces

The fibres of the morphism $\varphi\colon \mathrm{Sh}(X_{\mathcal{T}}) \to \mathcal{T}$ will turn out to be (approximated by) certain acyclic topological spaces, which we will discuss separately in this section.

Let I be a fixed infinite index set. For any set S, with cardinality $\mathrm{card}(S) \leq \mathrm{card}(I)$, the enumeration space

$$\mathrm{En}(S) \qquad (\text{or } \mathrm{En}_I(S))$$

has as points all functions $\alpha\colon D \to S$ defined on some subset $D = \mathrm{dom}(\alpha) \subset I$, and with the property that $\alpha^{-1}(s) \subset D$ is infinite for each $s \in S$. The basic open sets of $\mathrm{En}(S)$ are the sets of the form

$$V_{i_1,\ldots,i_n,s_1,\ldots,s_n} = \{\alpha \mid \alpha(i_k) = s_k, \text{ for } k = 1, \ldots, n\},$$

for any $i_1, \ldots, i_n \in I$ and $s_1, \ldots, s_n \in S$. It will be convenient to use a shorter notation, and write u for the finite partial function from I to S defined by $u(i_k) = s_k$ $(k = 1, \ldots, n)$, and write

$$V_u = \{\alpha \in \mathrm{En}(S) \mid u \subset \alpha\}$$

for the same basic open set. Note that for $n = 0$ (i.e., $u = \emptyset$) the entire space $\mathrm{En}(S)$ occurs among these basic open sets.

Notation 3.1 These finite partial functions u induce various continuous operations on $\mathrm{En}(S)$, which will be used in the sequel. For $\alpha \in \mathrm{En}(S)$, denote by $\alpha - u$ the restriction of α to $\mathrm{dom}(\alpha) - \mathrm{dom}(u)$. Furthermore, denote by $\alpha \cup u$ the union of these partial functions, defined only in case $\mathrm{dom}(\alpha) \cap \mathrm{dom}(u) = \emptyset$. Finally, we will use the notation (u/α) for $(\alpha - u) \cup u$, which is the function obtained by "writing u over α".

Remark 3.2 In relation to Remark 2.4, we note that if $S' \subset S$ is a subset, the restriction of $\alpha\colon D \to S$ to $\{i \in D \mid \alpha(i) \in S'\}$ defines a continuous map $\mathrm{res}\colon \mathrm{En}(S) \to \mathrm{En}(S')$. Furthermore, any surjection $t\colon J \to I$ defines by composition an obvious continuous map $t^*\colon \mathrm{En}_I(S) \to \mathrm{En}_J(S)$.

Lemma 3.3 *Each enumeration space* $\mathrm{En}(S)$ *is connected and locally connected; in fact, each basic open set* V_u *is connected.*

Proof. Fix an open set V_u, and let $V_u = O_1 \cup O_2$ be a cover by two non-empty open sets. Choose points $\alpha_1 \in O_1$ and $\alpha_2 \in O_2$, and basic open sets V_{u_1} and V_{u_2} with $\alpha_1 \in V_{u_1} \subset O_1$ and $\alpha_2 \in V_{u_2} \subset O_2$. These are given by finite partial functions u_1, u_2 with $u \subset u_1 \subset \alpha_1$ and $u \subset u_2 \subset \alpha_2$. Let $\beta = u_2/\alpha_1 \in O_2$ and $\gamma = (\alpha_1 - u_2) \cup u$. Thus $\gamma \subset \beta$ and $\gamma \subset \alpha_1$, hence β and α_1 belong to every open neighbourhood of γ in $\mathrm{En}(S)$. Now $\gamma \in V_u$, so $\gamma \in O_1$ or $\gamma \in O_2$. But if $\gamma \in O_1$, then $\beta \in O_1 \cap O_2$, and if $\gamma \in O_2$ then $\alpha_1 \in O_1 \cap O_2$. Thus $O_1 \cap O_2 \neq \emptyset$, showing that V_u is connected. $\square$

Next we consider Čech homology of $\mathrm{En}(S)$. The following proposition forms the crucial part of the proof of our theorem.

Proposition 3.4 *For any cover $\mathcal{U}$ of* $\mathrm{En}(S)$ *by basic open sets, we have*

$$\mathrm{H}_n(\mathcal{U},\mathbb{Z}) = 0 \qquad (n>0).$$

Proof. Let $\mathcal{U} = \{V_{u_\sigma} \mid \sigma \in \Sigma\}$ be such an open cover, indexed by a set Σ. To avoid too many indices, we will in this proof write σ for u_σ, and V_σ for V_{u_σ}. Let $C_\bullet(\mathcal{U})$ be the usual Čech complex, i.e., $C_n(\mathcal{U})$ is the free abelian group on the set $N_n(\mathcal{U}) = \{(\sigma_0,\dots,\sigma_n) \mid V_{\sigma_0}\cap\cdots\cap V_{\sigma_n} \neq \emptyset\}$. Note that $(\sigma_0,\dots,\sigma_n) \in N_n(\mathcal{U})$ iff the finite partial functions $\sigma_0,\dots,\sigma_n$ are compatible, in the sense that their union $\sigma_0\cup\cdots\cup\sigma_n$ (short for $u_{\sigma_0}\cup\cdots\cup u_{\sigma_n}$) is well–defined. We will show that this complex is chain contractible, by exhibiting an explicit chain homotopy h:

$$0 \leftarrow \mathbb{Z} \underset{h_{-1}}{\overset{\partial}{\leftrightarrows}} C_0(\mathcal{U}) \underset{h_0}{\overset{\partial}{\leftrightarrows}} C_1(\mathcal{U}) \underset{h_1}{\overset{\partial}{\leftrightarrows}} C_2(\mathcal{U}) \underset{h_2}{\overset{\partial}{\leftrightarrows}} \cdots$$

$$\partial\circ h_{-1} = \mathrm{id}, \qquad \partial h_n + h_{n-1}\partial = \mathrm{id}. \tag{4}$$

To define h, we fix a point $\alpha \in \mathrm{En}(S)$ and an index $\tau\in\Sigma$ with $\alpha \in V_{u_\tau}$. Furthermore, for each sequence $\boldsymbol{\sigma} = (\sigma_0,\dots,\sigma_n) \in N_n(\mathcal{U})$, we choose an index $f(\boldsymbol{\sigma})$ so that

$$\alpha - (\sigma_0\cup\cdots\cup\sigma_n\cup\tau) \in V_{f(\boldsymbol{\sigma})}. \tag{5}$$

The h_n are now defined by induction, by

$$\begin{aligned} h_{-1}(1) &= \tau \\ h_n(\boldsymbol{\sigma}) &= (-1)^{n+1}[\boldsymbol{\sigma} f(\boldsymbol{\sigma}) - h_{n-1}(\partial\boldsymbol{\sigma})f(\boldsymbol{\sigma})]. \end{aligned} \tag{6}$$

Here $\boldsymbol{\sigma}$ is the tuple $(\sigma_0,\dots,\sigma_n)$, $\boldsymbol{\sigma} f(\boldsymbol{\sigma}) = (\sigma_0,\dots,\sigma_n,f(\boldsymbol{\sigma}))$, and $h_{n-1}(\partial\boldsymbol{\sigma})f(\boldsymbol{\sigma})$ is the sum $\sum(-1)^i h_{n-1}(\sigma_0\dots\hat{\sigma}_i\dots\sigma_n)f(\boldsymbol{\sigma})$ obtained by adding $f(\boldsymbol{\sigma})$ to the end of every term in $h_{n-1}(\partial\boldsymbol{\sigma})$. For example,

$$\begin{aligned} h_0(\sigma_0) &= -(\sigma_0 f(\sigma_0) - \tau f(\sigma_0)) \\ h_1(\sigma_0\sigma_1) &= \sigma_0\sigma_1 f(\sigma_0\sigma_1) + \sigma_1 f(\sigma_1)f(\sigma_0\sigma_1) - \tau f(\sigma_1)f(\sigma_0\sigma_1) \\ &\quad - \sigma_0 f(\sigma_0)f(\sigma_0\sigma_1) + \tau f(\sigma_0)f(\sigma_0\sigma_1), \end{aligned}$$

etc. Let us observe first that $h_n(\boldsymbol{\sigma})$ is a well–defined element of $C_{n+1}(\mathcal{U})$; i.e., that for any sequence $\boldsymbol{\mu} = (\mu_0,\dots,\mu_{n+1})$ occurring in $h_n(\boldsymbol{\sigma})$, the corresponding basic open $V_{\boldsymbol{\mu}} = V_{\mu_0}\cap\cdots\cap V_{\mu_{n+1}}$ is non-empty. We will show by induction on n that for any generator $\boldsymbol{\mu}$ occurring in $h_n(\boldsymbol{\sigma})$, there exists a point $\beta = \beta_{\boldsymbol{\sigma}}(\boldsymbol{\mu})$ in $\mathrm{En}(S)$ such that

$$\beta \supset \alpha - (\sigma_0\cup\cdots\cup\sigma_n\cup\tau) \qquad \text{and} \qquad \beta \in V_{\boldsymbol{\mu}} = V_{\mu_0}\cap\cdots\cap V_{\mu_{n+1}} \tag{7}$$

For $n=0$, the two generators occurring in $h_0(\sigma_0)$ are $\sigma_0 f(\sigma_0)$ and $\tau f(\sigma_0)$ and, by (5), we can choose

$$\begin{aligned} \beta(\sigma_0 f(\sigma_0)) &= \alpha - (\sigma_0\cup\tau)\cup\sigma_0 \in V_{\sigma_0 f(\sigma_0)}, \\ \beta(\tau f(\sigma_0)) &= \alpha - (\sigma_0\cup\tau)\cup\tau \in V_{\tau f(\sigma_0)}. \end{aligned}$$

Suppose, then, that we have found a point β as in (7) for each $(\sigma_0, \dots, \sigma_n)$ and each generator $\boldsymbol{\mu}$ in $h_n(\boldsymbol{\sigma})$. Now consider a sequence $\boldsymbol{\sigma} = (\sigma_0, \dots, \sigma_{n+1}) \in N_{n+1}(\mathcal{U})$, with

$$h_n(\boldsymbol{\sigma}) = (-1)^{n+1}[\boldsymbol{\sigma} f(\boldsymbol{\sigma}) - h_{n-1}(\partial\boldsymbol{\sigma}) f(\boldsymbol{\sigma})] \tag{8}$$

as in (6). For the generator $\boldsymbol{\sigma} f(\boldsymbol{\sigma})$, we can take $\beta = (\alpha - (\sigma_0 \cup \cdots \cup \sigma_{n+1} \cup \tau)) \cup (\sigma_0 \cup \cdots \cup \sigma_{n+1}) = (\sigma_0 \cup \cdots \cup \sigma_{n+1})/(\alpha - \tau)$, since by (5), this β will satisfy $\beta \in V_{\boldsymbol{\sigma} f(\boldsymbol{\sigma})}$. Next consider $h_{n-1}(\partial\boldsymbol{\sigma}) f(\boldsymbol{\sigma})$. For a generator $\boldsymbol{\mu} = (\mu_0, \dots, \mu_{n+1})$ in $h(\sigma_0 \dots \hat{\sigma}_i \dots \sigma_n)$, we have by induction found a β_0 so that

$$\beta_0 \supset \alpha - (\sigma_0 \cup \dots \hat{\sigma}_i \dots \cup \sigma_n \cup \tau) \quad \text{and} \quad \beta_0 \in V_{\boldsymbol{\mu}}.$$

Also, $f(\boldsymbol{\sigma}) \subset \alpha - (\sigma_0 \cup \cdots \cup \sigma_n \cup \tau) \subset \alpha - (\sigma_0 \cup \cdots \hat{\sigma}_i \cdots \cup \sigma_n \cup \tau)$, so $\beta_0 \in V_{\boldsymbol{\mu} f(\boldsymbol{\sigma})}$. Thus β_0 is also a witness for the fact that the part $\boldsymbol{\mu} f(\boldsymbol{\sigma})$ occurring in $h_{n-1}(\partial\boldsymbol{\sigma}) f(\boldsymbol{\sigma})$ corresponds to a non-empty intersection of basic open sets.

It remains to prove the identities (4) for a chain homotopy. Clearly, $\partial h_{-1} = \mathrm{id}$, while for $\sigma_0 \in C_0(\mathcal{U})$,

$$\begin{aligned} \partial h_0(\sigma_0) + h_{-1}(\partial\sigma_0) &= -\partial(\sigma_0 f(\sigma_0)) + \partial(\tau f(\sigma_0)) + \tau \\ &= -(f(\sigma_0) + \sigma_0) + (f(\sigma_0) - \tau) + \tau \\ &= \sigma_0. \end{aligned}$$

We proceed by induction, and suppose the identity $\partial h_n + h_{n-1}\partial = \mathrm{id}$ has been proved. Consider, then, any generator $\sigma_0 \dots \sigma_{n+1} \in C_{n+1}(\mathcal{U})$. The induction hypothesis implies that

$$\partial h_n(\partial\boldsymbol{\sigma}) = \partial\boldsymbol{\sigma} - h_{n-1}(\partial^2\boldsymbol{\sigma}) = \partial\boldsymbol{\sigma},$$

whence

$$\partial h_n(\partial\boldsymbol{\sigma}) f(\boldsymbol{\sigma}) = (\partial\boldsymbol{\sigma}) f(\boldsymbol{\sigma}). \tag{9}$$

Thus, using the general identity

$$\partial(\mu_0 \dots \mu_n \rho) = \partial(\mu_0 \dots \mu_n)\rho + (-1)^{n+1}\mu_0 \dots \mu_n \tag{10}$$

we find

$$\begin{aligned} \partial h_{n+1}(\boldsymbol{\sigma}) &= \partial(-1)^n[\boldsymbol{\sigma} f(\boldsymbol{\sigma}) - h_n(\partial\boldsymbol{\sigma}) f(\boldsymbol{\sigma})] && \text{(by definition)} \\ &= (-1)^n[(\partial\boldsymbol{\sigma}) f(\boldsymbol{\sigma}) + (-1)^n\boldsymbol{\sigma} - \partial(h_n(\partial\boldsymbol{\sigma}) f(\boldsymbol{\sigma}))] && \text{(by (10))} \\ &= \boldsymbol{\sigma} + (-1)^n[(\partial\boldsymbol{\sigma}) f(\boldsymbol{\sigma}) - (\partial h_n(\partial\boldsymbol{\sigma})) f(\boldsymbol{\sigma}) - (-1)^{n+2} h_n(\partial\boldsymbol{\sigma})] && \text{(by (10))} \\ &= \boldsymbol{\sigma} - h_n(\partial\boldsymbol{\sigma}) + (-1)^n[(\partial\boldsymbol{\sigma}) f(\boldsymbol{\sigma}) - \partial h_n(\partial\boldsymbol{\sigma}) f(\boldsymbol{\sigma})] \\ &= \boldsymbol{\sigma} - h_n(\partial\boldsymbol{\sigma}). \end{aligned}$$

This completes the proof of the proposition. □

Proposition 3.5 *Let V be a basic open set in $\mathrm{En}(S)$, and let $\mathcal{U}$ be a cover of V by basic open sets. Then*

$$\mathrm{H}_n(\mathcal{U}, \mathbb{Z}) = 0 \qquad (n > 0).$$

Proof. This is proved in exactly the same way as the previous proposition. If $V = V_u$, then one modifies the proof by restricting all constructions to finite sequences v or points α with $u \subset v, \alpha$. □

4 Construction of $\varphi_!$ and a projection formula

The enumeration spaces $\mathrm{En}(S)$ are related to the space $X = X_{\mathcal{T}}$, constructed for a topos above, in the following way. For each small point $p: \mathcal{S} \to \mathcal{T}$, with stalk G_p of the special sheaf G, there is a continuous map

$$i_p: \mathrm{En}(G_p) \to X, \qquad i_p(\alpha) = (p, \alpha).$$

Denote by $\pi: \mathrm{En}(G_p) \to \mathrm{pt}$ the unique map into the one–point space. These two maps induce topos morphisms $\mathcal{S} \xleftarrow{\pi} \mathrm{Sh}(\mathrm{En}(G_p)) \xrightarrow{i_p} \mathrm{Sh}(X)$, which relate to the map $\varphi: \mathrm{Sh}(X) \to \mathcal{T}$ in the following way.

Lemma 4.1 *The square*

$$\begin{array}{ccc} \mathrm{Sh}(\mathrm{En}(G_p)) & \xrightarrow{i_p} & \mathrm{Sh}(X) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathcal{S} & \xrightarrow{p} & \mathcal{T} \end{array} \tag{11}$$

commutes up to natural isomorphism.

Proof. Let E be an object in $\mathcal{T}$, with sheaf $\varphi^*(E)$ on X as constructed in §2. Using the notation of the proof of Lemma 2.2, consider a canonical section

$$\sigma: U_{i,C} \to V_{i,C,f} \subset \varphi^*(E), \qquad \sigma(p, \alpha) = f_p(\alpha(i)),$$

of the sheaf $\varphi^*(E)$. The connected components of $i_p^{-1}(U_{i,C})$ are the basic open sets $V_g = \{\alpha \mid \alpha(i_1) = g_1, \dots, \alpha(i_n) = g_n\}$, for all $g = (g_1, \dots, g_n) \in C_p \subset G_p^n$. The section σ is constant on V_g, with value $f_p(g_1, \dots, g_n)$. This shows that $i_p^*\varphi^*(E)$ is a constant sheaf, with stalk E_p since $i_p^*\varphi^*(E)_{(p,\alpha)} = \varphi^*(E)_{(p,\alpha)} = E_p$. □

We note that the square (11) need not be a pullback of topoi, although it is very close to being one: $\mathrm{En}(G_p)$ is the space of points of the topos theoretic pullback.

Corollary 4.2 *Let $\sigma: U_{i,C} \to \varphi^*(E)$ be any section of the sheaf $\varphi^*(E)$, defined on the basic open set $U_{i,C}$. Then for any two points (p, α) and (p, β) in $U_{i,C}$,*

$$\alpha(i) = \beta(i) \quad \Rightarrow \quad \sigma(p, \alpha) = \sigma(p, \beta). \tag{12}$$

Proof. The section σ restricts along $i_p: \mathrm{En}(G_p) \to X$ to a section on $i_p^{-1}(U_{i,C})$ of the constant sheaf with stalk E_p. This section is constant on the connected components $V_g = \{\alpha \mid \alpha(i) = g\}$ of $i_p^{-1}(U_{i,C})$ already occurring in the proof of Lemma 4.1. Formula (12) follows. □

Recall that a topos morphism $\varphi: \mathcal{T}' \to \mathcal{T}$ consists of two particular functors φ^* and φ_*, with φ^* left exact and left adjoint to φ_*. The particular morphism $\varphi: \mathrm{Sh}(X) \to \mathcal{T}$ constructed above, has the following additional property.

Proposition 4.3 *There exists a functor $\varphi_!\colon \mathrm{Sh}(X) \to \mathcal{T}$ which is left adjoint to $\varphi^*\colon \mathcal{T} \to \mathrm{Sh}(X)$, i.e.,*

$$\mathrm{Hom}_{\mathcal{T}}(\varphi_!(F), E) \cong \mathrm{Hom}_{\mathrm{Sh}(X)}(F, \varphi^*(E)) \tag{13}$$

for any sheaf F on X and any object E of the topos $\mathcal{T}$.

Proof. For the proof of this proposition, we will construct for each sheaf F on X an object $\varphi_!(F)$ of the topos $\mathcal{T}$. Note that each basic open set $U_{i,C} \subset X$ can be viewed as a sheaf on X (where the sheaf projection is the inclusion $U_{i,C} \hookrightarrow X$). Furthermore, an arbitrary sheaf F is the colimit of such sheaves $U_{i,C}$ (the colimit being taken over the poset of sections of F defined on basic open sets). Thus, since the desired left adjoined $\varphi_!$ must necessarily commute with colimits, it suffices to construct $\varphi_!(U_{i,C})$ for each basic open set $U_{i,C}$ and prove the natural bijective correspondence of (13) in this special case:

$$\mathrm{Hom}(\varphi_!(U_{i,C}), E) \cong \Gamma(U_{i,C}, \varphi^*(E)) \tag{14}$$

We define

$$\varphi_!(U_{i,C}) =_{\mathrm{def}} C. \tag{15}$$

To prove (14) for this definition, we shall use the following two lemmas.

Lemma 4.4 *Let $U_{i,C}$ and $U_{j,B}$ be two basic open sets in X, and suppose $U_{j,B} \neq \emptyset$. Then $U_{j,B} \subset U_{i,C}$ iff the sequence $i = (i_1, \ldots, i_n)$ is a subsequence of $j = (j_1, \ldots, j_m)$, and the corresponding projection $G^m \to G^n$ maps B into C.*

Proof. The implication ($\Leftarrow$) is clear. For ($\Rightarrow$), choose a point $(p, \alpha) \in U_{j,B}$. If i_k is any index in i which does not occur among $(j_1, \ldots, j_m)$, let α' be the restriction of α to $\mathrm{dom}(\alpha) - \{i_k\}$. Then $(p, \alpha') \in U_{j,B}$ but $(p, \alpha') \notin U_{i,C}$. This shows that if $U_{j,B} \subset U_{i,C}$ then i must be a subsequence of j. Now consider the projection $\pi\colon G^m \to G^n$ coming from the fact that i is a subsequence of j. (Here we use that we can assume that both i and j do not contain repetitions, as explained just below (2).) To prove $\pi(B) \subset C$, it suffices to prove that, for each small point p,

$$\pi_p(B_p) \subset C_p,$$

(because the stalks at the small points are jointly conservative, by assumption).

Take $(g_1, \ldots, g_m) \in B_p$, and let $\alpha \in \mathrm{En}(G_p)$ be any enumeration with $\alpha(j_k) = g_k$ $(k = 1, \ldots, m)$. Then $(p, \alpha) \in U_{j,B} \subset U_{i,C}$, so $\pi_p(g_1, \ldots, g_m) = (\alpha(i_1), \ldots, \alpha(i_n)) = \alpha(i) \in C_p$. □

Lemma 4.5 *Let $U_{i,C}$ be a basic open set. Let $\{U_{j_\xi, B_\xi}\}$ be a family of non-empty basic open subsets of $U_{i,C}$, with associated projections $\pi_\xi\colon B_\xi \to C$ as in Lemma 4.4. Then $U_{i,C}$ is covered by $\{U_{j_\xi, B_\xi}\}$ in the space X iff $\{\pi_\xi\colon B_\xi \to C\}$ is an epimorphic family in $\mathcal{T}$.*

Proof. To simplify notation, we just treat the case where $i = i_1$ and $C \subset G$, while $j = (i_1, j_\xi)$ is a sequence of length 2 and $B_\xi \subset G^2$. By Lemma 4.3, the projection $\pi_2 : G^2 \to G$ maps each B_ξ into C, giving a map $\pi_\xi : B_\xi \to C$.

Suppose now that $U_{i,C} = \bigcup U_{j_\xi, B_\xi}$. To show that $\{\pi_\xi : B_\xi \to C\}$ is an epimorphic family, it suffices to prove, for each small point p,

$$C_p = \bigcup_\xi \pi_\xi(B_\xi)_p.$$

Take any $c \in C_p$, and choose an enumeration $\alpha \in \mathrm{En}(G_p)$ with $\alpha(i) = c$. Then $(p, \alpha) \in U_{i,C}$, hence for some ξ also $(p, \alpha) \in U_{j_\xi, B_\xi}$. Thus $j_\xi \in \mathrm{dom}(\alpha)$ and $b = (\alpha(i), \alpha(j_\xi)) \in (B_\xi)_p$, whence $c = \pi_\xi(b) \in \pi_\xi(B_\xi)_p$, as desired.

The converse is similar. $\square$

We now continue the proof of Proposition 4.3, and show the isomorphism (13) for $\varphi_!(U_{i,C}) = C$. In one direction, any map $f : C \to E$ in $\mathcal{T}$ defines a canonical section

$$\sigma_f : U_{i,C} \to \varphi^*(E), \qquad \sigma_f(p, \alpha) = f_p(\alpha(i)), \tag{16}$$

(as in the proof of Lemma 2.2).

In the other direction, suppose $\sigma : U_{i,C} \to \varphi^*(E)$ is an arbitrary section of $\varphi^*(E)$. Locally, σ must be a canonical section as described in §2. Thus, there is a cover

$$U_{i,C} = \bigcup_\xi U_{j_\xi, B_\xi} \tag{17}$$

and for each ξ a map

$$f_\xi : B_\xi \to E$$

so that

$$\sigma(p, \alpha) = (f_\xi)_p(\alpha(j_\xi)), \qquad \text{for } (p, \alpha) \in U_{j_\xi, B_\xi}. \tag{18}$$

By Lemma 4.5, the identity (17) implies that the B_ξ form a cover of C in the topos $\mathcal{T}$. Let us simplify the notation as in the proof of Lemma 4.5, and write $i = i_1$, $j = (i_1, j_\xi)$, $C \subset G$, $B_\xi \subset G^2$, and $\pi_\xi : B_\xi \to C$ for the restriction of the first projection $G^2 \to G$. We claim that the maps $f_\xi : B_\xi \to E$ form a compatible family for this cover $\{B_\xi \to C\}$, hence define a unique map $f : C \to E$ with $f \circ \pi_\xi = f_\xi$. For this, it needs to be shown, for any two indices ξ and ζ, that the square

$$\begin{tikzcd} B_\xi \times_C B_\zeta \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B_\zeta \arrow[d, "f_\zeta"] \\ B_\xi \arrow[r, "f_\xi"'] & E \end{tikzcd} \tag{19}$$

commutes in $\mathcal{T}$. It suffices to check that the corresponding diagram of stalks commutes for every small point p. Choose such a point p, and consider an element

$b \in (B_\xi \times_C B_\zeta)_p$. Write $\pi_1(b) = (c, b_\xi) \in (B_\xi)_p$ and $\pi_2(b) = (c, b_\zeta) \in (B_\zeta)_p$. Choose now two enumerations $\alpha, \beta \in \mathrm{En}(G_p)$, such that

$$\begin{array}{ll} \alpha(i) = c, & \alpha(j_\xi) = b_\xi, \\ \beta(i) = c, & \beta(j_\zeta) = b_\zeta. \end{array}$$

Then $(p, \alpha) \in U_{j_\xi, B_\xi}$ and $(p, \beta) \in U_{j_\zeta, B_\zeta}$, so

$$\begin{array}{rcll} (f_\xi \circ \pi_1)_p(b) & = & (f_\xi)_p(c, b_\xi) & \\ & = & (f_\xi)_p(\alpha(i), \alpha(j_\xi)) & \\ & = & \sigma(p, \alpha) & \qquad \text{(by (18))}, \end{array}$$

and similarly $(f_\zeta \circ \pi_2)_p(b) = \sigma(p, \beta)$. But $(p, \alpha), (p, \beta) \in U_{i,C}$, while $\alpha(i) = \beta(i)$, so $\sigma(p, \alpha) = \sigma(p, \beta)$ by Lemma 4.2. This proves that $(f_\xi \circ \pi_1)_p(b) = (f_\zeta \circ \pi_2)_p(b)$ for any $b \in (B_\xi \times_C B_\zeta)_p$, and hence that (19) commutes. Thus the f_ξ together uniquely determine a map $f = f_\sigma : C \to E$.

It is now straightforward to check that these constructions, of σ_f from f and of f_σ from σ, are mutually inverse, and prove the required bijection (14).

This completes the proof of Proposition 4.2. □

Let us reconsider the square (11) at the beginning of this section. Since $\mathrm{En}(G_p)$ is a locally connected space (Lemma 3.3) the inverse image functor $\pi^* : \mathcal{S} \to \mathrm{Sh}(\mathrm{En}(G_p))$, which sends a set to the constant sheaf, has a left adjoint $\pi_! : \mathrm{Sh}(\mathrm{En}(G_p)) \to \mathcal{S}$. For a sheaf F on $\mathrm{En}(G_p)$, $\pi_!(F)$ is simply the set of connected components of F, where F is viewed as an étale space over $\mathrm{En}(G_p)$.

Corollary 4.6 *For the square (11), the projection formula*

$$\pi_!(i_p)^* = p^* \varphi_!$$

holds.

Proof. First, a more precise formulation of this corollary should state that the canonical natural transformation

$$\pi_!(i_p)^*(F) \to p^* \varphi_!(F), \tag{20}$$

obtained from the isomorphism $i_p^* \varphi^* \cong \pi^* p^*$ and the adjunctions, is an isomorphism. Since the functors in (20) all preserve colimits, it suffices to check that (20) is an isomorphism in case F is (the sheaf corresponding to) a basic open set $U_{i,C}$. But $\pi_! i_p^*(U_{i,C})$ is the set of connected components of $i_p^{-1}(U_{i,C})$, and these are exactly the basic open sets $V_g = \{\alpha \mid \alpha(i_1) = g_1, \ldots, \alpha(i_n) = g_n\}$, for $g = (g_1, \ldots, g_n) \in C_p \subset G_p^n$, hence are in bijective correspondence with elements of $C_p = p^*(C) = p^* \varphi_!(U_{i,C})$ by (15). □

5 Proof of the theorem

We will now prove the theorem, stated in the introduction and repeated here:

Theorem 5.1 *For any sheaf of abelian groups A in $\mathcal{T}$, the morphism $\varphi\colon \mathrm{Sh}(X_{\mathcal{T}}) \to \mathcal{T}$ induces an isomorphism $\varphi^*\colon \mathrm{H}^n(\mathcal{T}, A) \to \mathrm{H}^n(X_{\mathcal{T}}, \varphi^* A)$, for any $n \geq 0$.*

For $n = 0$, this follows from

Lemma 5.2 *The inverse image functor $\varphi^*\colon \mathcal{T} \to \mathrm{Sh}(X_{\mathcal{T}})$ is full and faithful.*

Proof. The statement of the lemma is equivalent to the assertion that the counit of the adjunction $\varphi_!\varphi^*(E) \to E$ is an isomorphism, for every sheaf E on $\mathcal{T}$. It suffices to check this for the stalks at each small point p. But there we have

$$\begin{array}{rcll} \varphi_!\varphi^*(E)_p & = & p^*\varphi_!\varphi^*(E) & \\ & = & \pi_!(i_p)^*\varphi^*(E) & \text{(by Corollary 4.6)} \\ & = & \pi_!\pi^*(E_p) & \text{(by Lemma 4.1)} \\ & = & E_p, & \end{array}$$

the latter since $\mathrm{En}(G_p)$ is connected (Lemma 3.3). □

Latter, we will have to compare the Čech complex of an open cover in X to its inverse image along the map $i_p\colon \mathrm{En}(G_p) \to X$, where p is any small point of the topos $\mathcal{T}$. We will use the following simple observation:

Lemma 5.3 *Let $U_1, \ldots, U_n \subset U \subset X$ be basic open sets, and let $V \subset i_p^{-1}(U)$ be a connected component. Then the connected components of $i_p^{-1}(U_1 \cap \ldots \cap U_n)$ contained in V are the non–empty intersections $V_1 \cap \ldots \cap V_n$, where $V_i \subset V$ is a component of $i_p^{-1}(U_i)$.*

Proof. We already observed (e.g. in the proofs of 4.1 and 4.5) that for any basic open set $U \subset X$, the connected components of $i_p^{-1}(U)$ are basic open sets V in $\mathrm{En}(G_p)$. These basic open sets in $\mathrm{En}(G_p)$ are all connected (Lemma 3.3) and closed under intersection. The lemma follows immediately from this. □

Lemma 5.4 *Let I be any injective abelian sheaf in $\mathcal{T}$. Let $U \subset X$ be a basic open set, and let $\mathcal{U}$ be a cover of U by basic open sets. Then $\mathrm{H}^n(\mathcal{U}, \varphi^*(I) \restriction U) = 0$ for $n > 0$.*

Proof. Write $\mathcal{U} = \{U_\sigma \mid \sigma \in \Sigma\}$, and $N_n(\mathcal{U}) = \sum_{\sigma_0 \ldots \sigma_n} U_{\sigma_0 \ldots \sigma_n}$ where the sum is over all $(n+1)$–tuples of indices, and $U_{\sigma_0 \ldots \sigma_n} = U_{\sigma_0} \cap \ldots \cap U_{\sigma_n}$. Viewing each $U_{\sigma_0 \ldots \sigma_n}$ as an object of $\mathrm{Sh}(X)$, we see that $N_\bullet(\mathcal{U})$ is a simplicial object in $\mathrm{Sh}(X)$. The Čech complex $C^n(\mathcal{U}, \varphi^*(I) \restriction U)$ computing $\mathrm{H}^*(\mathcal{U}, \varphi^*(I) \restriction U)$ can now be described as

$$\begin{array}{rcl} C^n(\mathcal{U}, \varphi^*(I) \restriction U) & = & \mathrm{Hom}_{\mathrm{Sh}(X)}(N_n(\mathcal{U}), \varphi^*(I)) \\ & = & \mathrm{Hom}_{\mathcal{T}}(\varphi_! N_n(\mathcal{U}), I), \end{array}$$

the latter by the adjunction of 4.3. To prove the lemma, it thus suffices to show that the associated chain complex $\mathbb{Z}[\varphi_! N_\bullet(\mathcal{U})]$ of abelian groups in $\mathcal{T}$ is exact at $n > 0$. It is enough to check this for the stalk at each small point p. But

$$\begin{aligned} \mathbb{Z}[\varphi_! N_n(\mathcal{U})]_p &= \mathbb{Z}[\varphi_!(N_n(\mathcal{U}))_p] \\ &= \mathbb{Z}[\pi_!(i_p)^* N_n(\mathcal{U})], \qquad \text{(by Corollary 4.6)}, \end{aligned}$$

which is the chain complex of the simplicial set $\pi_! i_p^*(N_\bullet(\mathcal{U}))$. Now

$$\pi_! i_p^*(N_n(\mathcal{U})) = \{(\sigma_0 \ldots \sigma_n, W) \mid W \text{ a connected component of } i_p^{-1}(U_{\sigma_0 \ldots \sigma_n})\}.$$

For each connected component $V \subset i_p^{-1}(U)$, let $\mathcal{U}_V$ be the cover of V by connected components $W \subset i_p^*(U_\sigma)$, for all $\sigma \in \Sigma$. By Lemma 5.3, $\pi_! i_p^*(N_\bullet(\mathcal{U}))$ is the disjoint sum of the Čech nerves of these covers $\mathcal{U}_V$, and these nerves are acyclic by Proposition 3.5. Thus $\pi_! i_p^*(N_\bullet(\mathcal{U}))$ is acyclic also, and the lemma is proved. □

Proof of Theorem 5.1. By general homological algebra, it suffices to show that for any injective abelian group I in $\mathcal{T}$ the sheaf cohomology groups $\mathrm{H}^n(X, \varphi^*(I))$ vanish for $n > 0$. By Lemma 5.4, the sheaf $\varphi^*(I) \restriction U$ is 'Čech–acyclic' for each basic open set $U \subset X$. The result follows by applying Cartan's criterion [1], Proposition V.4.3, [3], Théorème 5.9.2. □

As stated in §1, the argument actually proves the somewhat stronger assertion that the higher right derived functors of $\varphi_*\colon \mathrm{Sh}(X) \to \mathcal{T}$ vanish. Before stating this as Corollary 5.6 below, we observe the following corollary.

Corollary 5.5 *Let E be any sheaf (of sets) in $\mathcal{T}$. Then in the pullback of topoi*

$$\begin{tikzcd} \mathrm{Sh}(\varphi^* E) \arrow[r, "\pi"] \arrow[d, "\varphi_E"'] & \mathrm{Sh}(X) \arrow[d, "\varphi"] \\ \mathcal{T}/E \arrow[r] & \mathcal{T} \end{tikzcd}$$

the map φ_E induces isomorphisms

$$\mathrm{H}^n(\mathcal{T}/E, A) \xrightarrow{\cong} \mathrm{H}^n(\varphi^*(E), \varphi_E^*(A)),$$

for any abelian sheaf A in $\mathcal{T}/E$.

Here $\mathcal{T}/E$ denotes the "induced topos" ([1], Exposé IV.5) of $\mathcal{T}$–objects over E, and $\mathcal{T}/E \to \mathcal{T}$ is the canonical morphism (*loc. cit.* (5.2.1)).

Proof. We claim that the map φ_E is again of the form $\varphi\colon \mathrm{Sh}(X_\mathcal{T}) \to \mathcal{T}$ so that Corollary 5.5 is actually a special case of Theorem 5.1. More precisely, $\varphi_E\colon \mathrm{Sh}(\varphi^* E) \to \mathcal{T}/E$ is precisely the map $\mathrm{Sh}(X_{(\mathcal{T}/E)}) \to \mathcal{T}/E$, for a suitable choice of the various parameters. Indeed, suppose $X_\mathcal{T}$ is defined using the set of small points $\mathcal{P}$, the object G so that subsheaves of G^n generate $\mathcal{T}$, and the index set I.

Then $H = (G \times E \to E)$ is an object of $\mathcal{T}/E$ so that subsheaves of H^n generate $\mathcal{T}/E$. Moreover, the points of $\mathcal{T}/E$ are in bijective correspondence with pairs (p, e), where p is a point of $\mathcal{T}$ and $e \in E_p$. For such a pair (p, e), the stalk of an object $(f: F \to E)$ at (p, e) is given by

$$(f: F \to E)_{(p,e)} = f_p^{-1}(e) \subset E_p.$$

In particular, $H_{(p,e)} = G_p$ for each e. Now for the set of small points of $\mathcal{T}/E$ we can take all these pairs (p, e) where $p \in \mathcal{P}$, and we can then take the same index set I.

The space $X_{(\mathcal{T}/E)}$ defined from these choices then is the space of triples (p, e, α), where p is a small point of $\mathcal{T}$, $e \in E_p$, and $\alpha \in \mathrm{En}(H_{(p,e)}) = \mathrm{En}(G_p)$. But this is exactly the space $\varphi^*(E)$ defined in §2. Further details are straightforward. $\square$

Corollary 5.6 *For any abelian sheaf A in $\mathcal{T}$, and any $n > 0$,*

$$(\mathrm{R}^n\varphi_*)(\varphi^* A) = 0.$$

Proof. As before, it suffices to prove this for A injective. For an arbitrary sheaf B on X, $\mathrm{R}^n\varphi_*(B)$ is the associated sheaf of the presheaf

$$E \mapsto \mathrm{H}^n(\varphi^*(E), \pi^*(B))$$

(where $\pi: \varphi^*(E) \to X$ is the sheaf projection); see [1], Proposition V.5.1 and [7], Lemma 8.18. For $B = \varphi^*(I)$ where I is injective, the result thus follows from Corollary 5.5. $\square$

6 Etale cohomology

By way of example, we will give an explicit description of the space $X_{\mathcal{T}}$ in the case where $\mathcal{T}$ is the étale topos over a scheme. The main reference for this section is Grothendieck's Exposé VIII in [1]. For basic properties of strictly henselian local rings and strict henselization, see [11].

Fix a ground field k, and a scheme Y (over k). Let $Y_{\text{ét}}$ be the étale site over Y, and let $\widetilde{Y_{\text{ét}}}$ be the associated étale topos. For a point $y \in Y$, $k(y)$ denotes the residue field of the local ring $\mathcal{O}_{Y,y}$, and $\overline{k(y)}$ its separable closure. The Galois group $\mathrm{Gal}(\overline{k(y)}/k(y))$ is denoted by π_y.

The functor A on $Y_{\text{ét}}$ which associates to each object $f: Z \to Y$ of the étale site the ring $\Gamma(Z, f^*(\mathcal{O}_Y))$ is a sheaf, and defines a local ring A in the topos $\widetilde{Y_{\text{ét}}}$. The functor A^{hs} on $Y_{\text{ét}}$ which associates to $f: Z \to Y$ the ring $\Gamma(Z, \mathcal{O}_Z)$ is again a sheaf, and a strictly henselian local ring in $\widetilde{Y_{\text{ét}}}$ [5]. The extension $A \to A^{\text{hs}}$ is a universal strict henselization of $\mathcal{O}_Y$ in the topos $\widetilde{Y_{\text{ét}}}$. The sheaf A^{hs} will play the role of the object G.

The étale topos has enough points. We recall from [1], Exposé VIII, that each point $y \in Y$ defines first a geometric point $\overline{y}: \mathrm{Spec}(\overline{k(y)}) \to Y$ of the scheme Y, and then a point of the topos $\widetilde{Y_{\text{ét}}}$, whose inverse image functor is the composition

$$\Gamma \circ \overline{y}^*: \widetilde{Y_{\text{ét}}} \to \widetilde{\mathrm{Spec}(\overline{k(y)})_{\text{ét}}} \to \mathcal{S},$$

and denoted $F \mapsto F_{\overline{y}}$. By *loc. cit.*, Corollaire VIII.3.6, the set of all these points is jointly conservative. So we can take this set of points $y \in Y$ for the set $\mathcal{P}$.

Consider again the extension $A \to A^{\mathrm{hs}}$ in the topos $\widetilde{Y_{\text{ét}}}$. As explained in [1], Exposé VIII.4, for any $y \in Y$ the stalk map $A_{\overline{y}} \to A^{\mathrm{hs}}_{\overline{y}}$ is a (the) strict henselization of $\mathcal{O}_{Y,y} = A_{\overline{y}}$, relative to the separable closure $k(y) \hookrightarrow \overline{k(y)}$. Thus we will write $\mathcal{O}^{\mathrm{hs}}_{Y,y}$ for $A^{\mathrm{hs}}_{\overline{y}}$, and we will identify $\mathcal{O}_{Y,y}$ with a subset of $\mathcal{O}^{\mathrm{hs}}_{Y,y}$.

By the universal property of the strict henselization [4], §18, [11], Section VIII.2, the group π_y acts on $\mathcal{O}^{\mathrm{hs}}_{Y,y}$, say from the left. The local ring $\mathcal{O}_{Y,y} \subset \mathcal{O}^{\mathrm{hs}}_{Y,y}$ is fixed under this action.

Let I be a set whose cardinality is at least as big as that of all these strict henselizations $\mathcal{O}^{\mathrm{hs}}_{Y,y}$.

We can now describe the space $X = X_{\mathcal{T}}$ of Theorem 5.1 in this special case where $\mathcal{T} = \widetilde{Y_{\text{ét}}}$. Let $y \in Y$, and consider all functions ("enumerations") $\alpha\colon \mathrm{dom}(\alpha) \to \mathcal{O}^{\mathrm{hs}}_{Y,y}$ defined on a subset $\mathrm{dom}(\alpha) \subset I$; and with the property that $\alpha^{-1}(b)$ is infinite for each $b \in \mathcal{O}^{\mathrm{hs}}_{Y,y}$. Call two such enumerations α and β equivalent, $\alpha \sim \beta$, if $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$, and if there is a $g \in \pi_y$ so that $g \cdot \alpha(i) = \beta(i)$ for each $i \in \mathrm{dom}(\alpha)$. The points of the space X are defined to be equivalence classes of pairs (y, α), with (y, α) equivalent to (z, β) iff $y = z$ and $\alpha \sim \beta$.

In this particular case, the topology of the space X, defined in general in §2, can be described more explicitly by using standard étale extensions. Fix for this an affine open $U = \mathrm{Spec}(R)$ of Y and (for some n) polynomials $p_1, \dots, p_n$ in $R[T_1, \dots, T_n]$ such that the determinant $\det(J)$ of the Jacobian $J = (\partial p_j / \partial T_k)_{j,k}$ is invertible in $R[T_1, \dots, T_n]/(p_1, \dots, p_n)$. Moreover, we fix a finite sequence of indices $i = (i_1, \dots, i_n)$. Together these data define the open set

$$\begin{aligned} V = \{(y, \alpha) \mid\ & y \in U,\ i_1, \dots, i_n \in \mathrm{dom}(\alpha), \\ & \text{and } p_k(\alpha(i_1), \dots, \alpha(i_n)) = 0 \text{ for } k = 1, \dots, n\ \}. \end{aligned}$$

Note that this makes sense, since each p_k has coefficients in R, and R maps to the localization $R_y = \mathcal{O}_{Y,y}$ and then to $\mathcal{O}^{\mathrm{hs}}_{Y,y}$. Thus p_k can be evaluated at the tuple $(\alpha(i_1), \dots, \alpha(i_n))$. These open sets of the form V generate the topology on X.

The construction of §2 gives for each étale sheaf $E \in \widetilde{Y_{\text{ét}}}$ a sheaf $\varphi^*(E)$ on this topological space X, with stalks

$$\varphi^*(E)_{(y,\alpha)} = E_{\overline{y}}.$$

Our theorem asserts that there is a natural isomorphism

$$\mathrm{H}^n(Y_{\text{ét}}, A) \cong \mathrm{H}^n(X, \varphi^* A),$$

for any abelian sheaf A on $Y_{\text{ét}}$ and any $n \geq 0$.

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