CENTRAL LIMIT THEOREM FOR A WEAKLY INTERACTING RANDOM POLYMER

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Abstract

The Domb-Joyce model in one dimension is a transformed path measure for simple random walk on $\mathbb{Z}$ in which an n-step path gets a penalty $e^{-2\beta_n}$ for every self-intersection. Here β_n is the strength of repellence, which may depend on n. We prove a central limit theorem for the end-to-end distance of the path in the case where $\beta_n \to 0$ and $n^{\frac{3}{2}}\beta_n \to \infty$ as $n \to \infty$. It turns out that the mean grows like $b^*\beta_n^{\frac{1}{3}}n$ and the standard deviation like $c^*\sqrt{n}$, where b^* and c^* are constants that can be identified in terms of a Sturm-Liouville problem. The asymptotic mean shows an interpolation between ballistic behavior ($\beta_n \equiv \beta$) and diffusive behavior ($\beta_n = \beta n^{-\frac{3}{2}}$). Strikingly, the asymptotic standard deviation is *independent* of β_n. Our result is closely related to the central limit theorem for the Edwards model (the continuous space-time analogue of the Domb-Joyce model based on Brownian motion on $\mathbb{R}$), which is proved in a separate paper.

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0 Introduction and main results

0.1 Model and motivation

A polymer is a long chain of molecules with two characteristic properties: an irregular shape and a certain stiffness. One way of describing a polymer is the following model based on a *random walk with self-repellence.*

Let $(S_i)_{i\in\mathbb{N}_0}$ be simple random walk on $\mathbb{Z}$, starting at the origin. Let P be its law and let E be expectation w.r.t. P. For $n \in \mathbb{N}$, define a new path law Q_n^β by setting

$$\frac{dQ_n^\beta}{dP}\Big((S_i)_{i\in\mathbb{N}_0}\Big) = \frac{1}{Z_n^\beta}\exp\Big[-\beta\sum_{\substack{i,j=0\\ i\neq j}}^{n} 1_{\{S_i=S_j\}}\Big], \tag{0.1}$$

where Z_n^β is the normalizing constant

$$Z_n^\beta = E\Big(\exp\Big[-\beta\sum_{\substack{i,j=0\\ i\neq j}}^{n} 1_{\{S_i=S_j\}}\Big]\Big) \tag{0.2}$$

and $\beta \in \mathbb{R}^+ = (0,\infty)$ is a parameter. The law Q_n^β is called the *n-polymer measure with strength of repellence* β. Eqs. (0.1-0.2) define what is called the Domb-Joyce model for 'soft polymers' (see Madras and Slade (1993) Subsection 10.1). The path gets a penalty $e^{-2\beta}$ for every self-intersection until time n. This causes an effective self-repellence, which tends to spread out the walk. The limiting cases $\beta = 0$ and $\beta = \infty$ correspond to simple random walk resp. self-avoiding walk.

In this paper we shall be concerned with the end-to-end distance S_n under the measure

$$Q_n^{\beta_n} \text{ with } \beta_n \to 0 \text{ and } n^{\frac{3}{2}}\beta_n \to \infty \text{ as } n \to \infty. \tag{0.3}$$

This is a weak interaction limit where the strength of repellence decreases with the length of the path. Our main results are Theorems 4 and 5 in Subsections 0.3 and 0.4. The regime in (0.3) is mathematically interesting because the weak interaction limit is singular, in the sense that it cannot be obtained as a small perturbation of simple random walk. It is also physically interesting as a family of models describing a polymer with fixed n, β lying on some curve $n \mapsto \beta_n$ fitting the constraints.

Earlier work concerned the following cases:

(a) ('ballistic') $\beta_n \equiv \beta \in \mathbb{R}^+$: Greven and den Hollander (1993) and König (1996),

(b) ('weakly ballistic') $\beta_n \equiv \beta \downarrow 0$: van der Hofstad and den Hollander (1995),

(c) ('diffusive') $\beta_n = \beta n^{-\frac{3}{2}}$ with $\beta \in \mathbb{R}^+$: Brydges and Slade (1995).

The results from these papers will be described in Subsections 0.2, 0.3 resp. 0.5. The regime in (0.3) will provide an interpolation between (a-c). Technically this regime is difficult because the polymer has intricate scaling properties.

The problem addressed in this paper was brought to our attention by David Brydges and Gordon Slade. They were interested in whether a law of large numbers (LLN) could be proved when $\beta_n = \beta n^{-p}(\beta \in \mathbb{R}^+, p \in (0, \frac{3}{2}))$ and in what way the result would be related to what was known for cases (b) and (c). This point will be clarified in Subsection 0.4, where we state a central limit theorem (CLT) for S_n under the law $Q_n^{\beta_n}(\,\cdot \mid S_n > 0)$ in (0.3).

0.2 Case $\beta_n \equiv \beta \in \mathbb{R}^+$

Theorems 1 and 2 below are a LLN resp. a CLT for S_n under the law $Q_n^{\beta}(\,\cdot \mid S_n > 0)$. In the formulation of these results several quantities appear that will play a crucial role in later sections.

Theorem 1 (A) *(Greven and den Hollander (1993)) For every $\beta \in \mathbb{R}^+$ there exists $\theta^*(\beta) \in (0,1)$ such that*

$$\lim_{n\to\infty} Q_n^{\beta}\left(\left|\frac{1}{n}S_n - \theta^*(\beta)\right| \leq \varepsilon \;\middle|\; S_n > 0\right) = 1 \textit{ for every } \varepsilon > 0. \tag{0.4}$$

The quantity $\theta^*(\beta)$ in (0.4) is the *speed of the polymer.*[2] A recipe to compute $\theta^*(\beta)$ can be given in terms of the following objects. For $r \in \mathbb{R}, \beta \in \mathbb{R}^+$ define the matrix $A_{r,\beta}$ by

$$A_{r,\beta}(i,j) = e^{r(i+j-1)-\beta(i+j-1)^2} P(i,j) \qquad (i,j \in \mathbb{N}), \tag{0.5}$$

[2]By symmetry, the LLN implies that $Q_n^{\beta}(S_n/n)^{-1} \Longrightarrow^w \frac{1}{2}[\delta_{\theta^*(\beta)} + \delta_{-\theta^*(\beta)}]$ as $n \to \infty$, where δ_θ is the Dirac point measure in θ, $\Longrightarrow^w$ denotes weak convergence, and $\mu(X)^{-1}$ denotes the distribution of a random variable X under a measure μ.

where P is the stochastic matrix

$$P(i,j) = \binom{i+j-2}{i-1} \left(\frac{1}{2}\right)^{i+j-1}. \tag{0.6}$$

Define $\lambda(r,\beta)$ to be the unique largest eigenvalue of $A_{r,\beta}$ in $l^2(\mathbb{N})$.[3] Furthermore, for fixed β, let $r^*(\beta)$ be the unique solution of the equation $\lambda(r,\beta) = 1$, i.e.,

$$\lambda(r^*(\beta), \beta) = 1. \tag{0.7}$$

Theorem 1 (B) *(Greven and den Hollander (1993)) For every $\beta \in \mathbb{R}^+$,*

$$\theta^*(\beta) = \left[\frac{\partial}{\partial r}\lambda(r,\beta)\right]^{-1}_{r=r^*(\beta)}. \tag{0.8}$$

Theorem 1 was proved via a large deviation analysis of the double sum in (0.1). The result was later supplemented as follows:

Theorem 2 *(König (1996)) For every $\beta \in \mathbb{R}^+$,*

$$\lim_{n\to\infty} Q_n^\beta\Big(\frac{S_n - \theta^*(\beta)n}{\sigma^*(\beta)\sqrt{n}} \le C \;\Big|\; S_n > 0\Big) = \mathcal{N}((-\infty, C]) \text{ for every } C \in \overline{\mathbb{R}}, \tag{0.9}$$

where $\mathcal{N}$ is the standard normal distribution and $\sigma^*(\beta)$ is given by

$$\sigma^*(\beta)^2 = \theta^*(\beta)^3 \left[\frac{\partial^2}{\partial r^2}\lambda(r,\beta) - \frac{1}{\theta^{*2}(\beta)}\right]_{r=r^*(\beta)}. \tag{0.10}$$

The quantity $\sigma^*(\beta)$ is the *spread of the polymer*. Theorem 2 was proved via a higher order large deviation analysis of (0.1).

From (0.8), (0.10) and footnote 3 it follows that $\beta \mapsto \theta^*(\beta)$ and $\beta \mapsto \sigma^*(\beta)$ are analytic on $\mathbb{R}^+$. Furthermore, it can be shown that $\lim_{\beta\downarrow 0}\theta^*(\beta) = 0$ and $\lim_{\beta\to\infty}\theta^*(\beta) = 1$.

0.3 Case $\beta_n \equiv \beta \downarrow 0$

The quantities introduced in (0.7-0.8) have the following behavior for small β.

Theorem 3 (A) *(van der Hofstad and den Hollander (1995)) There exist $a^*, b^* \in \mathbb{R}^+$ such that as $\beta \downarrow 0$,*

$$\beta^{-\frac{2}{3}} r^*(\beta) \;\to\; a^*, \tag{0.11}$$

$$\beta^{-\frac{1}{3}} \theta^*(\beta) \;\to\; b^*. \tag{0.12}$$

[3] $A_{r,\beta} : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is positive, self-adjoint and compact for all $r \in \mathbb{R}, \beta \in \mathbb{R}^+$. Moreover, $(r,\beta) \mapsto \lambda(r,\beta)$ is analytic. Furthermore, $r \mapsto \lambda(r,\beta)$ is strictly increasing and log-convex, $\lambda(0,\beta) < 1$ and $\lim_{r\to\infty}\lambda(r,\beta) = \infty$ for every $\beta \in \mathbb{R}^+$ (see Greven and den Hollander (1993)).

Theorem 3(A) was proved via a scaling analysis of $\lambda(r,\beta)$, which we shall describe now ($\mathbb{R}_0^+ = [0,\infty)$). For $a \in \mathbb{R}$, let $\mathcal{L}^a$ be the differential operator defined by

$$\begin{aligned}&(\mathcal{L}^a x)(u) = (2au - 4u^2)x(u) + x'(u) + ux''(u),\\ &(x \in L^2(\mathbb{R}_0^+) \cap C^2(\mathbb{R}_0^+), \textstyle\int_0^\infty \{u^2[x(u)]^2 + u[x'(u)]^2\} du < \infty),\end{aligned} \tag{0.13}$$

and let $\rho(a)$ be the unique largest eigenvalue of $\mathcal{L}^a$ in its domain.[4] Then we have the following identification of the constants a^*, b^*.

Theorem 3 (B) *(van der Hofstad and den Hollander (1995)) As $\beta \downarrow 0$, uniformly in a on compacts in $\mathbb{R}$,*

$$\beta^{-\frac{1}{3}}\left[\lambda(a\beta^{\frac{2}{3}},\beta) - 1\right] \quad\rightarrow\quad \rho(a), \tag{0.14}$$

$$\beta^{-\frac{1}{3}}\frac{\partial}{\partial a}\lambda(a\beta^{\frac{2}{3}},\beta) \quad\rightarrow\quad \rho'(a). \tag{0.15}$$

Consequently,

$$a^* \textit{ is the unique solution of } \rho(a) = 0, \tag{0.16}$$

$$b^* = \frac{1}{\rho'(a^*)}. \tag{0.17}$$

Note that (0.7), (0.14) explain (0.11), (0.16) (see footnotes 3 and 4). In a similar way (0.8), (0.15) explain (0.12), (0.17).

We shall need that also the second derivative of λ has a scaling as in (0.14-0.15). This is formulated in the next theorem, which identifies the behavior for small β of the quantity $\sigma^*(\beta)$ introduced in (0.10).

Theorem 4 *As $\beta \downarrow 0$, uniformly in a on compacts in $\mathbb{R}$,*

$$\beta^{-\frac{1}{3}}\frac{\partial^2}{\partial a^2}\lambda(a\beta^{\frac{2}{3}},\beta) \rightarrow \rho''(a). \tag{0.18}$$

Consequently, as $\beta \downarrow 0$,

$$\sigma^*(\beta) \rightarrow c^*, \tag{0.19}$$

with

$$c^{*2} = \frac{\rho''(a^*)}{\rho'(a^*)^3}. \tag{0.20}$$

[4] $a \mapsto \rho(a)$ is analytic, strictly convex and strictly increasing, with $\rho(0) < 0$, $\lim_{a\to-\infty}\rho(a) = -\infty$ and $\lim_{a\to\infty}\rho(a) = \infty$ (see van der Hofstad and den Hollander (1995)).

Note that (0.10-0.12), (0.17-0.18) explain (0.19-0.20).

We shall see that Theorems 3 and 4 are the key technical results underlying our central limit theorem, Theorem 5 in Subsection 0.4. The proof of Theorem 4 is given in Section 7. The proof uses an extension of (0.14), Proposition 3 in Subsection 3.2, which states that all eigenvalues of $A_{r,\beta}$ have a scaling as in (0.14). The proof of Proposition 3 is in Section 8.

Numerical values for the constants a^* and b^* were obtained by estimating $\rho(a)$ for a range of a-values (van der Hofstad and den Hollander (1995) Subsection 0.3). This was done by estimating the l.h.s. of (0.14) for $\beta = 10^{-2} - 10^{-6}$ and extrapolating for $\beta \downarrow 0$. The computation was based on a 300×300 truncation of the matrix $A_{a\beta^{\frac{2}{3}},\beta}$ defined in (0.5) and a standard iteration method to find the largest eigenvalue of the truncated matrix. The result is $a^* = 2.19 \pm 0.01$ and $b^* = 1.11 \pm 0.01$. The same data produce the estimate $c^* = 0.7 \pm 0.1$.

Rigorous bounds on a^*, b^*, c^* appear in van der Hofstad (in preparation).

0.4 Case $\beta_n \to 0$ and $n^{\frac{3}{2}}\beta_n \to \infty$

In terms of the objects defined in Subsections 0.2 and 0.3, we can now state the main result of our paper. Recall (0.8) and (0.10). Define

$$\theta_n = \theta^*(\beta_n) \tag{0.21}$$

$$\sigma_n = \sigma^*(\beta_n). \tag{0.22}$$

Theorem 5 *If $\beta_n \to 0$ and $\beta_n n^{\frac{3}{2}} \to \infty$, then*

$$\lim_{n\to\infty} Q_n^{\beta_n}\left(\frac{S_n - \theta_n n}{\sigma_n\sqrt{n}} \leq C \;\middle|\; S_n > 0\right) = \mathcal{N}((-\infty, C]) \text{ for every } C \in \mathbb{R}. \tag{0.23}$$

The proof of Theorem 5 will be given in Sections 1-8. The essence of Theorem 5 is that the CLT holds with $\theta_n = \theta^*(\beta_n)$ and $\sigma_n = \sigma^*(\beta_n)$, i.e., the interaction parameter β_n is simply substituted into the quantities $\theta^*(\beta)$ and $\sigma^*(\beta)$ appearing in Theorems 1 and 2 for $\beta_n \equiv \beta$. Thus the weak interaction behavior is *uniform* in the regime under consideration.

0.5 Case $\beta_n = \beta n^{-\frac{3}{2}}$ $(\beta \in \mathbb{R}^+)$

Let $(B_t)_{t\geq 0}$ be standard Brownian motion on $\mathbb{R}$, starting at the origin. For $T > 0$, formally define a path measure $\widehat{Q}_T^\beta$ by

$$\frac{d\widehat{Q}_T^\beta}{d\widehat{P}}\left((B_t)_{t\geq 0}\right) = \frac{1}{\widehat{Z}_T^\beta} \exp\left[-\beta \int_0^T ds \int_0^T dt \; \delta(B_s - B_t)\right], \tag{0.24}$$

where $\beta \in \mathbb{R}^+$, $\widehat{Z}_T^\beta$ is the normalizing constant and $\widehat{P}$ is the Wiener measure. The double integral in (0.24) is called the self-intersection local time and can be rigorously defined in terms of Brownian local times (see e.g. Brydges and Slade (1995)). The law $\widehat{Q}_T^\beta$ is called the *T-polymer measure with strength of repellence* β. Eq. (0.24) defines what is called the Edwards model and is the continuous space-time analogue of (0.1-0.2).

Theorem 6 *(Brydges and Slade (1995)) For $\beta \in \mathbb{R}^+$ and $\beta_n = \beta n^{-\frac{3}{2}}$,*

$$\lim_{n\to\infty} Q_n^{\beta_n}\left(\frac{S_n}{\sqrt{n}} \le C\right) = \widehat{Q}_1^\beta(B_1 \le C) \text{ for every } C \in \mathbb{R}. \tag{0.25}$$

The key to the proof of Theorem 6 is that as $n \to \infty$ (recall footnote 2)

$$P\left(\frac{1}{n^{\frac{3}{2}}}\sum_{i,j=0}^{n} 1_{\{S_i=S_j\}}, \frac{1}{\sqrt{n}}S_n\right)^{-1} \Longrightarrow^w \widehat{P}\left(\int_0^1 ds \int_0^1 dt\ \delta(B_s - B_t), B_1\right)^{-1}. \tag{0.26}$$

0.6 Discussion

Using (0.12) and (0.19), we see that the key quantities in Theorem 5 behave as

$$\begin{aligned} \theta_n n &\sim b^*\beta_n^{\frac{1}{3}} n & (0.27)\\ \sigma_n\sqrt{n} &\sim c^*\sqrt{n}. & (0.28)\end{aligned}$$

If $\beta_n = \beta n^{-p}$ then $\theta_n n \sim b^*\beta^{\frac{1}{3}} n^{1-\frac{p}{3}}$. The exponent $1-\frac{p}{3}$ is seen to be a linear interpolation between the boundary cases 1 ($p=0$) and $\frac{1}{2}$ ($p=\frac{3}{2}$), corresponding to ballistic resp. diffusive behavior (compare with Theorems 1 and 6 above). This exponent was recently conjectured by Brydges and Slade (1995).

The fact that the fluctuations of S_n are asymptotically Gaussian and are of order $\sqrt{n}$ means that the CLT is robust under the weak interaction limit, as was perhaps to be expected. However, the fact that the standard deviation $\sigma_n\sqrt{n} \sim c^*\sqrt{n}$ is asymptotically *independent* of the parameter β_n is rather striking. This has to do with scaling properties of Brownian motion. Indeed, the constants b^*, c^* also appear in the CLT for the Edwards model proved in van der Hofstad, den Hollander and König (preprint 1995). Namely, the speed $\hat{\theta}^*(\beta)$ and the spread $\hat{\sigma}^*(\beta)$ of the T-polymer in (0.24) in the limit as $T \to \infty$ turn out to be

$$\begin{aligned} \hat{\theta}^*(\beta) &= b^*\beta^{\frac{1}{3}} \\ \hat{\sigma}^*(\beta) &= c^* \qquad (\beta \in \mathbb{R}^+). \end{aligned} \tag{0.29}$$

Thus the weak interaction limit of the Domb-Joyce model connects up nicely with the Edwards model.

Despite this connection, the proof of our CLT is rather involved. In fact, we shall need to develop the full scaling picture of the polymer measure, which is difficult because of the global nature of the path interaction. Unfortunately the weak interaction Domb-Joyce model and the Edwards model cannot be coupled nicely on one probability space. Therefore we shall be able to benefit very little from what we know for the Brownian case.

Our proof uses a higher order large deviation analysis of $Z_n^{\beta_n}$ (see (0.2)) as $n \to \infty$, namely up to and including $\mathcal{O}(1)$. It turns out that the $\mathcal{O}(1)$-term is structurally *different* from the one appearing in the analysis of $\widehat{Z}_T^{\beta}$ as $T \to \infty$ (see (0.24)) given in van der Hofstad, den Hollander and König (preprint 1995). This is another indication that the models are hard to compare directly.

We have $\theta^*(0) = 0, \sigma^*(0) = 1$ since $Q_n^0 = P$ for all n. Hence the speed $\theta^*(\beta)$ is continuous at $\beta = 0$ by (0.12). However, the spread $\sigma^*(\beta)$ is not continuous at $\beta = 0$, because of (0.19) and the numerical estimate $c^* < 1$ at the end of Subsection 0.3. This once more shows that the weak interaction limit is singular.

Finally, we have no doubt that $\Big((S_{\lfloor nt \rfloor} - \theta_n nt)/\sigma_n\sqrt{n}\Big)_{0\leq t\leq 1}$ under $Q_n^{\beta_n}(\,\cdot\,|S_n > 0)$ converges to Brownian motion. The convergence of the finite-dimensional distributions should run along the lines of the present paper. The tightness, however, will require additional arguments.

0.7 Outline

Section 1 gives a Markovian description of the local times of simple random walk (Knight's theorem). We use this description to write the moment generating function of S_n under $Q_n^{\beta_n}(\,\cdot\,|S_n > 0)$ as the expectation of an exponential functional of three Markov chains. These Markov chains correspond to the local times in the areas $(-\infty, 0)$, $[0, S_n]$ and (S_n, ∞).

In Section 2 we absorb the exponential functional into the transition kernels of the Markov chains and rewrite the moment generating function as a correlation function involving three scaled continuous-time processes.

In Sections 4-6 we show that, in the limit as $n \to \infty$, the correlation function factorizes into a product of three parts. The part corresponding to $[0, S_n]$ gives the CLT in Theorem 5, the parts corresponding to $(-\infty, 0)$ and (S_n, ∞) give rise to constants that drop out in the normalization.

In Section 3 we formulate an important tool used in Sections 4-6: a scaling limit assertion for the spectrum of the transition kernels introduced in Section 2. The limit is the spectrum of the operator $\mathcal{L}^a$ defined in (0.13), which determines the constants in our CLT. The proof of the limit assertion appears in Section 8, the proof of Theorem 4 in Section 7.

1 Reformulation of the problem

In Subsection 1.1 we formulate our main proposition, Proposition 1 below, implying Theorem 5. In Subsections 1.2 and 1.3 we apply Knight's description of the local times of simple random walk to get a convenient representation, Lemma 1 below, of the key quantity appearing in Proposition 1.

1.1 The main proposition: Proposition 1

Define the n-step local times

$$\ell_n(x) = \#\{0 \le i \le n : S_i = x\} \qquad (n \in \mathbb{N}_0, x \in \mathbb{Z}). \tag{1.1}$$

Then

$$\sum_{\substack{i,j=0 \\ i \neq j}}^{n} 1_{\{S_i = S_j\}} = \sum_{x \in \mathbb{Z}} \ell_n^2(x) - (n+1), \tag{1.2}$$

and so (0.1) can be rewritten as

$$\frac{dQ_n^\beta}{dP} = \frac{1}{\widetilde{Z}_n^\beta} \exp\left[-\beta \sum_{x \in \mathbb{Z}} \ell_n^2(x)\right] \tag{1.3}$$

with $\widetilde{Z}_n^\beta = Z_n^\beta \exp[-\beta(n+1)]$.

Next, in addition to $\theta_n = \theta^*(\beta_n), \sigma_n = \sigma^*(\beta_n)$ introduced in (0.21-0.22), define (see (0.7))

$$r_n = r^*(\beta_n). \tag{1.4}$$

For future reference, we recall here the limiting behavior of r_n, θ_n and σ_n (see (0.11), (0.12) and (0.19)):

$$\beta_n^{-\frac{2}{3}} r_n \to a^*, \qquad \beta_n^{-\frac{1}{3}} \theta_n \to b^*, \qquad \sigma_n \to c^*. \tag{1.5}$$

The rest of this paper is devoted to the proof of the following main proposition.

Proposition 1 *There is an $L \in \mathbb{R}^+$ such that for every $\mu \in \mathbb{R}$,*

$$\lim_{n\to\infty} e^{r_n n} \beta_n^{-\frac{1}{3}} E\left(e^{-\beta_n \sum_x \ell_n^2(x)}\, e^{\mu \frac{S_n - \theta_n n}{\sigma_n \sqrt{n}}} 1_{\{S_n > 0\}}\right) = L e^{\frac{\mu^2}{2}}. \tag{1.6}$$

Proposition 1 implies that under the law $Q_n^{\beta_n}(\,\cdot \mid S_n > 0)$ the moment generating function of $(S_n - \theta_n n)/\sigma_n \sqrt{n}$ converges pointwise to the one of $\mathcal{N}$ as $n \to \infty$ (divide the l.h.s. of (1.6) by the same expression for $\mu = 0$ and use (1.3)). Therefore it implies the central limit theorem as stated in Theorem 5.

1.2 Knight's description of the local times

This subsection provides an important tool in the proof of Proposition 1, namely, a family of Markov chains that describes the local times of simple random walk (recall (1.1)) at certain stopping times, viewed as a process in the spatial parameter. The following material is taken from Knight (1963) and is the discrete space-time analogue of the Ray-Knight theorems for local times of Brownian motion.

Fix $d \in \mathbb{N}_0$. Define the successive times at which the walker makes steps $d \to d+1$ and $d+1 \to d$, by putting $T^{\uparrow}_{0,d} = T^{\downarrow}_{0,d} = 0$ and for $k \in \mathbb{N}$,

$$\begin{aligned} T^{\uparrow}_{k,d} &= \inf\{i > T^{\uparrow}_{k-1,d} : S_{i-1} = d, S_i = d+1\}, \\ T^{\downarrow}_{k,d} &= \inf\{i > T^{\downarrow}_{k-1,d} : S_{i-1} = d+1, S_i = d\}. \end{aligned} \tag{1.7}$$

By discarding null sets we may and shall assume that all these stopping times are finite (simple random walk is recurrent). Note that $T^{\uparrow}_{k,d} < T^{\downarrow}_{k,d} < T^{\uparrow}_{k+1,d}$. Recall the definition of the stochastic $\mathbb{N} \times \mathbb{N}$ matrix P in (0.6), and introduce a stochastic $\mathbb{N}_0 \times \mathbb{N}_0$ matrix $P^{\star}$ by putting

$$P^{\star}(i,j) = 1_{\{i \neq 0\}} P(i, j+1) + 1_{\{i=0\}} 1_{\{j=0\}} \qquad (i, j \in \mathbb{N}_0). \tag{1.8}$$

Let

$$\{m(x)\}_{x \in \mathbb{N}_0} \qquad \text{and} \qquad \{m^{\star}(x)\}_{x \in \mathbb{N}_0} \tag{1.9}$$

be the Markov chains with transition kernel P resp. $P^{\star}$. Later we shall need that both $\{m(x)\}_{x \in \mathbb{N}_0}$ and $\{m^{\star}(x)\}_{x \in \mathbb{N}_0}$ are critical branching processes with a geometric offspring distribution with parameter $\frac{1}{2}$, where $\{m(x)\}_{x \in \mathbb{N}_0}$ has one immigrant per time unit and $\{m^{\star}(x)\}_{x \in \mathbb{N}_0}$ has none. The point 0 is therefore absorbing for $\{m^{\star}(x)\}_{x \in \mathbb{N}_0}$.

In terms of these Markov chains, we can describe the distribution of the local times of simple random walk at the stopping times $T^{\uparrow}_{k,d}$ resp. $T^{\downarrow}_{k,d}$ as follows. ($\stackrel{\mathcal{L}}{=}$ means equality in law.)

Knight's Theorem *Fix $k, d \in \mathbb{N}$. Let $\{m(x)\}_{x \in \mathbb{N}_0}$ start at $m(0) = k$. Let $\{m^{\star}_1(x)\}_{x \in \mathbb{N}_0}$ and $\{m^{\star}_2(x)\}_{x \in \mathbb{N}_0}$ be two independent copies of $\{m^{\star}(x)\}_{x \in \mathbb{N}_0}$ starting at $m^{\star}_1(0) = m(0)$ resp. $m^{\star}_2(0) = m(d)$. Then*

$$\begin{aligned} \left\{\ell_{T^{\uparrow}_{k,d}}(d+1-x)\right\}_{x=1,\dots,d} &\stackrel{\mathcal{L}}{=} \{m(x) + m(x-1) - 1\}_{x=1,\dots,d}, \\ \left\{\ell_{T^{\uparrow}_{k,d}}(d+x)\right\}_{x \in \mathbb{N}} &\stackrel{\mathcal{L}}{=} \{m^{\star}_1(x) + m^{\star}_1(x-1)\}_{x \in \mathbb{N}}, \\ \left\{\ell_{T^{\uparrow}_{k,d}}(1-x)\right\}_{x \in \mathbb{N}} &\stackrel{\mathcal{L}}{=} \{m^{\star}_2(x) + m^{\star}_2(x-1)\}_{x \in \mathbb{N}}. \end{aligned} \tag{1.10}$$

The three processes in the l.h.s. of (1.10) are conditionally independent given $m(0)$ *and* $m(d)$. *Furthermore,*

$$\ell_{T^{\downarrow}_{k,d}}(x) = \begin{cases} \ell_{T^{\uparrow}_{k,d}}(x) + 1_{\{x=d\}} & \text{if } x \le d, \\ \ell_{T^{\uparrow}_{k+1,d}}(x) - 1_{\{x=d+1\}} & \text{otherwise.} \end{cases} \tag{1.11}$$

Proof. Fix $k, d \in \mathbb{N}$. Define the number of steps $x \to x+1$ until time $T^{\uparrow}_{k,d}$ by

$$m_{k,d}(x) = \#\{0 < i \le T^{\uparrow}_{k,d} : S_{i-1} = x, S_i = x+1\} \qquad (x \in \mathbb{Z}). \tag{1.12}$$

From Theorem 1.1 and Corollary 1.1 in Knight (1963) it follows that the random processes $\{m_{k,d}(d+x)\}_{x\in\mathbb{N}_0}$ and $\{m_{k,d}(d-x)\}_{x\in\mathbb{N}_0}$ are independent Markov chains, both starting at k. Furthermore, $\{m_{k,d}(d-x)\}_{x\in\{0,\dots,d\}}$ is homogeneous and $\mathbb{N}$-valued with transition kernel P, while $\{m_{k,d}(d+x)\}_{x\in\mathbb{N}_0}$ and $\{m_{k,d}(-x)\}_{x\in\mathbb{N}_0}$ are homogeneous and $\mathbb{N}_0$-valued both with transition kernel $P^{\star}$.

Use the relation

$$\begin{aligned} &\#\{0 < i \le T^{\uparrow}_{k,d} : S_{i-1} = x, S_i = x-1\} \\ &\qquad = \begin{cases} m_{k,d}(x-1) - 1 & \text{if } x \in \{1,\dots,d\} \\ m_{k,d}(x-1) & \text{otherwise} \end{cases} \end{aligned} \tag{1.13}$$

and recall (1.1) to get

$$\ell_{T^{\uparrow}_{k,d}}(x) = \begin{cases} m_{k,d}(x) + m_{k,d}(x-1) - 1 & \text{if } x \in \{1,\dots,d\}, \\ m_{k,d}(x) + m_{k,d}(x-1) & \text{otherwise.} \end{cases} \tag{1.14}$$

Hence (1.10) and the conditional independence assertion follow from the previous remarks. The reader easily verifies (1.11). □

In the sequel $\mathbb{P}_k$ and $\mathbb{P}^{\star}_k$ will denote the laws of the two Markov chains in (1.9) starting in $k \in \mathbb{N}$ resp. $k \in \mathbb{N}_0$. We write $\mathbb{E}_k$ and $\mathbb{E}^{\star}_k$ for expectation w.r.t. $\mathbb{P}_k$ resp. $\mathbb{P}^{\star}_k$.

1.3 The distribution of $(\{\ell_n(x)\}_{x\in\mathbb{Z}}, S_n)$

The description of the local times given in Knight's theorem has the disadvantage that the local times are observed at certain stopping times. For the description of the polymer we need to go back to the fixed time n. One of the problems we consequently have to deal with is the global restriction $\sum_{x\in\mathbb{Z}} \ell_n(x) = n+1$.

Fix d, $n \in \mathbb{N}$. In this subsection we derive a representation for the expression

$$E\big(e^{-\beta_n \sum_{x\in\mathbb{Z}} \ell_n^2(x)} 1_{\{S_n=d\}}\big) \tag{1.15}$$

in terms of the Markov chains introduced in the preceding subsection. The idea is to sum over the number of steps $0 \to 1$, $d \to d+1$ (resp. $d+1 \to d$), and over the amount of time the walker spends in the three areas $-\mathbb{N}_0$, $\{1, \dots, d\}$ and $\{d+1, d+2, \dots\}$ until time n.

Define the functionals

$$
\begin{aligned}
U_d &= \sum_{x=1}^{d} [m(x) + m(x-1) - 1], && (1.16)\\
V_d &= \sum_{x=1}^{d} [m(x) + m(x-1) - 1]^2, && (1.17)\\
U^\star &= \sum_{x=1}^{\infty} [m^\star(x) + m^\star(x-1)], && (1.18)\\
V^\star &= \sum_{x=1}^{\infty} [m^\star(x) + m^\star(x-1)]^2. && (1.19)
\end{aligned}
$$

In terms of these new objects we may write:

Lemma 1 *For all n, $d \in \mathbb{N}$,*

$$
\begin{aligned}
&E\Big(e^{-\beta_n \sum_{x\in\mathbb{Z}} \ell_n^2(x)} 1_{\{S_n = d+1, S_{n-1} = d\}}\Big)\\
&= \sum_{k_1, n_1 \in \mathbb{N}} \sum_{k_2, n_2 \in \mathbb{N}} \prod_{i=1}^{2} \mathbb{E}^\star_{k_i} \left(e^{-\beta_n V^\star} 1_{\{U^\star = n_i\}}\right) \qquad (1.20)\\
&\quad \times \mathbb{E}_{k_1}\left(e^{-\beta_n V_d} 1_{\{U_d = n - n_1 - n_2 + 1\}} 1_{\{m(d) = k_2\}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&E\Big(e^{-\beta_n \sum_{x\in\mathbb{Z}} \ell_n^2(x)} 1_{\{S_n = d, S_{n-1} = d+1\}}\Big)\\
&= \sum_{k_1 \in \mathbb{N}\setminus\{1\}, n_1 \in \mathbb{N}_0} \sum_{k_2, n_2 \in \mathbb{N}} \prod_{i=1}^{2} \mathbb{E}^\star_{k_i} \left(e^{-\beta_n [V^\star - \delta_i]} 1_{\{U^\star = n_i\}}\right) \qquad (1.21)\\
&\quad \times \mathbb{E}_{k_1 - 1}\Big(e^{-\beta_n [V_d + \delta_3]} 1_{\{U_d = n - n_1 - n_2 + 1\}} 1_{\{m(d) = k_2\}}\Big),
\end{aligned}
$$

with

$$
\delta_1 = 2m^\star(1), \qquad \delta_2 = 0, \qquad \delta_3 = 2m(1). \qquad (1.22)
$$

Proof. Observe that $\sum_{x\in\mathbb{Z}} \ell_t(x) = t+1$ for any (random or fixed) $t \in \mathbb{N}_0$. Split the class of paths under the indicator in the l.h.s. of (1.20) according to the amount of time the walker spends in the three areas $-\mathbb{N}_0$, $\{1,\dots,d\}$ and $\{d+1,d+2,\dots\}$ and to the number of steps $0\to 1$, $d\to d+1$ until time n :

$$\{S_n = d+1, S_{n-1} = d\} = \bigcup_{k\in\mathbb{N}} \Big\{T^{\uparrow}_{k,d} = n\Big\} = \bigcup_{k_1,k_2,n_1,n_2\in\mathbb{N}} \Bigg\{\sum_{x=1}^{\infty} \ell_{T^{\uparrow}_{k_1,d}}(d+x) = n_1,$$
$$\sum_{x=1}^{d} \ell_{T^{\uparrow}_{k_1,d}}(d+1-x) = n-n_1-n_2+1, \sum_{x=1}^{\infty} \ell_{T^{\uparrow}_{k_1,d}}(1-x) = n_2, m_{k_1,d}(0) = k_2\Bigg\}. \tag{1.23}$$

Furthermore, write the exponent in the l.h.s. of (1.20) as

$$\sum_{x\in\mathbb{Z}} \ell_n^2(x) = \sum_{x=1}^{\infty} \ell_n^2(d+x) + \sum_{x=1}^{d} \ell_n^2(d+1-x) + \sum_{x=1}^{\infty} \ell_n^2(1-x). \tag{1.24}$$

Combine (1.23) and (1.24), use Knight's theorem and substitute (1.16–1.19) to arrive at (1.20).

In order to prove (1.21), split

$$\{S_n = d, S_{n-1} = d+1\} = \bigcup_{k\in\mathbb{N}} \{T^{\downarrow}_{k,d} = n\} = \bigcup_{k_1,k_2,n_1,n_2\in\mathbb{N}} \Bigg\{\sum_{x=1}^{\infty} \ell_{T^{\downarrow}_{k,d}}(d+x) = n_1,$$
$$\sum_{x=1}^{d} \ell_{T^{\downarrow}_{k_1,d}}(d+x-1) = n-n_1-n_2+1, \sum_{x=1}^{\infty} \ell_{T^{\downarrow}_{k_1,d}}(1-x) = n_2, m_{k_1,d}(0) = k_2\Bigg\}. \tag{1.25}$$

Now substitute (1.11) and proceed analogously. Along the way, use that $\{\ell_{T^{\uparrow}_{k_1+1,d}}(d+x)\}_{x\in\mathbb{N}}$ and $\{\ell_{T^{\uparrow}_{k_1,d}}(d-x)\}_{x\in\mathbb{N}_0}$ are conditionally independent given $m_{k_1,d}(0)$, and shift the sums over k_1 and n_1 by one. $\square$

In the proof of Proposition 1 we shall focus on the contribution coming from the r.h.s. of (1.20). It will be argued at the end of Subsection 2.6 that (1.21) behaves in the same manner as (1.20) as $n\to\infty$, i.e., the small perturbations are harmless.

The role of Lemma 1 is that we have rewritten the key quantity of Proposition 1 in terms of expectations of exponential functionals of the two Markov chains defined in (1.9). We can henceforth forget about the underlying random walk. It is important that in Lemma 1 we have *products* of expectations.

2 Structure of the proof of Proposition 1

In this section we explain the main steps in the proof of Proposition 1. Our approach is a variation on the method used in van der Hofstad, den Hollander and König (preprint

1995). In Subsections 2.1 and 2.2 we introduce transformed and time-changed Markov chains that are specially adapted to our problem. In Subsections 2.3 and 2.4 we introduce several quantities that are needed to rewrite Lemma 1 in a more appropriate form, Lemmas 2 and 3 below. In Subsection 2.5 this leads to a key proposition, Proposition 2 below, that is the technical core of the argument. In Subsection 2.6 we finish the proof of Proposition 1 subject to Proposition 2. The proof of Proposition 2 follows in Sections 5-8.

2.1 A transformed Markov chain

In this subsection we define a transformation of the Markov chain $\{m(x)\}_{x\in\mathbb{N}_0}$ introduced in Subsection 1.3. The goal of this transformation is to absorb the random variable $e^{-\beta_n V_d}$ (see (1.17)) into the new transition probabilities.

Recall (0.5-0.7) and fix $r\in\mathbb{R}$ and $\beta\in\mathbb{R}^+$. As was pointed out in Subsection 0.2, the matrix $A_{r,\beta}$ has a unique largest eigenvalue $\lambda(r,\beta)$. We shall denote the associated positive $l^2(\mathbb{N})$-normalized eigenvector by $\tau_{r,\beta}$. Consequently,

$$P_{r,\beta}(i,j)=\frac{A_{r,\beta}(i,j)}{\lambda(r,\beta)}\,\frac{\tau_{r,\beta}(j)}{\tau_{r,\beta}(i)}\qquad (i,j\in\mathbb{N}) \tag{2.1}$$

defines a stochastic matrix $P_{r,\beta}$. We shall write $\mathbb{P}_k^{r,\beta}$ to denote the law of the Markov chain $\{m(x)\}_{x\in\mathbb{N}_0}$ starting at $k\in\mathbb{N}$ and having $P_{r,\beta}$ as its transition kernel. We write $\mathbb{E}_k^{r,\beta}$ for the corresponding expectation. Note that this chain is positive recurrent with invariant distribution $\{\tau_{r,\beta}^2(i)\}_{i\in\mathbb{N}}$. We write $\mathbb{P}^{r,\beta},\mathbb{E}^{r,\beta}$ when the chain starts in its invariant distribution.

2.2 A time-changed Markov chain

Since it will turn out that the transformed Markov chain $\{m(x)\}_{x\in\mathbb{N}_0}$ needs to be evaluated at the random times at which the additive functional $\{U_d\}_{d\in\mathbb{N}}$ in (1.16) exceeds certain values, we must introduce some more notation. For $l\in\mathbb{N}$ define

$$T_l=\inf\{\,d\in\mathbb{N}_0: U_d\ge l\,\} \tag{2.2}$$

and

$$X_l=U_{T_l}-l,\qquad Y_l=m(T_l),\qquad Z_l=m(T_l-1). \tag{2.3}$$

The triple

$$\Gamma_l=(X_l,Y_l,Z_l) \tag{2.4}$$

is a random member of the set

$$\Sigma=\{\,(i,j,k)\in\mathbb{N}_0\times\mathbb{N}^2: i\le j+k-2\,\}. \tag{2.5}$$

Fix $r \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$. For any $k \in \mathbb{N}$, under the law $\mathbb{P}_k^{r,\beta}$ the process $\{\Gamma_l\}_{l\in\mathbb{N}_0}$ is a Markov renewal process with transition kernel $Q_{r,\beta}$ on Σ given by

$$\begin{aligned} &Q_{r,\beta}\left((i_1,j_1,k_1),(i_2,j_2,k_2)\right) \\ &\qquad = 1_{\{i_1=0,i_2=j_2+k_2-2,k_2=j_1\}} P_{r,\beta}(j_1,j_2) + 1_{\{i_2=i_1-1,j_2=j_1,k_2=k_1\}} \end{aligned} \tag{2.6}$$

and starting at $\Gamma_0 = (0,k,k)$. It is easily checked that the probability distribution $\nu_{r,\beta}$ on Σ defined by

$$\nu_{r,\beta}(i,j,k) = \tau_{r,\beta}(j)\frac{A_{r,\beta}(j,k)}{\partial_r\lambda(r,\beta)}\tau_{r,\beta}(k) \tag{2.7}$$

is the associated invariant distribution (∂_r denotes the partial derivative w.r.t. r).[5]

We write $\widetilde{\mathbb{P}}^{r,\beta}$ and $\widetilde{\mathbb{E}}^{r,\beta}$ to denote probability and expectation w.r.t. the Markov chain $\{\Gamma_l\}_{l\in\mathbb{N}_0}$ starting in its invariant distribution $\nu_{r,\beta}$.

2.3 Unscaled representation

We are going to reformulate the r.h.s. of (1.20) in terms of $\{\Gamma_l\}_{l\in\mathbb{N}_0}$, since this is the natural object for our analysis. First we need some more notation. For $r \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$, define the function $w_{r,\beta} : \mathbb{N}_0^2 \to \mathbb{R}_0^+$ by (see (1.18-1.19))

$$w_{r,\beta}(k,l) = \mathbb{E}_k^\star\left(e^{-\beta V^\star + rU^\star} 1_{\{U^\star=l\}}\right) = e^{rl} w_{0,\beta}(k,l) \tag{2.8}$$

and the functions $f_{r,\beta}^+$ and $f_{r,\beta}^- : \Sigma \times \mathbb{N}_0 \to \mathbb{R}^+$ by

$$f_{r,\beta}^\pm((i,j,k);l) = \frac{w_{r,\beta}(j, l\pm i)}{(j+k-1)\tau_{r,\beta}(j)}. \tag{2.9}$$

Our reformulation of the l.h.s. of (1.6) in Proposition 1 (up to some factors and an indicator) now reads as follows.

Lemma 2 *For $\mu \in \mathbb{R}$ and $n \in \mathbb{N}$,*

$$\begin{aligned} &e^{(n+1)r_n^*} E\Big(e^{-\beta_n \sum_x \ell_n^2(x)} e^{\mu\frac{S_n}{\sigma_n\sqrt{n}}} 1_{\{0\le S_{n-1}<S_n\}}\Big) \\ &\quad = \frac{1}{\theta_n^*}\sum_{n_1,n_2\in\mathbb{Z}} \widetilde{\mathbb{E}}^{r_n^*,\beta_n}\left(f_{r_n^*,\beta_n}^+(\Gamma_0;n_1) 1_{\{X_0\le n-n_1-n_2+1\}} f_{r_n^*,\beta_n}^-(\Gamma_{n-n_1-n_2+1};n_2)\right), \end{aligned} \tag{2.10}$$

where $r_n^* = r_n^*(\mu)$ and $\theta_n^* = \theta_n^*(\mu)$ are given by

$$\lambda(r_n^*,\beta_n) = e^{-\frac{\mu}{\sigma_n\sqrt{n}}}, \tag{2.11}$$

$$\theta_n^* = [\partial_r\lambda(r_n^*,\beta_n)]^{-1}. \tag{2.12}$$

[5]To see that $\nu_{r,\beta}$ is normalized, differentiate the relation $\lambda(r,\beta) = \langle\tau_{r,\beta}, A_{r,\beta}\tau_{r,\beta}\rangle_{l^2}$ w.r.t. r and use that $\partial_r A_{r,\beta}(j,k) = (j+k-1)A_{r,\beta}(j,k)$, $A_{r,\beta}\tau_{r,\beta} = \lambda(r,\beta)\tau_{r,\beta}$ and $\partial_r\langle\tau_{r,\beta},\tau_{r,\beta}\rangle_{l^2} = 0$.

Proof. Begin by observing that for every $k_1, k_2, n_1, n_2, d \in \mathbb{N}, r \in \mathbb{R}$ and r_n^* as in (2.11),

$$e^{(n+1)r}[\lambda(r_n^*,\beta_n)]^d \mathbb{E}_{k_1}\Big(e^{-\beta_n V_d} e^{\mu \frac{d}{\sigma_n\sqrt{n}}} 1_{\{U_d=n-n_1-n_2+1\}} 1_{\{m(d)=k_2\}}\Big)$$
$$= e^{(n_1+n_2)r}\,[\lambda(r,\beta_n)]^d\,\mathbb{P}_{k_1}^{r,\beta_n}\Big(U_d = n-n_1-n_2+1, m(d)=k_2\Big)\tfrac{\tau_{r,\beta_n}(k_1)}{\tau_{r,\beta_n}(k_2)}. \quad (2.13)$$

This identity is a straightforward consequence of (0.5-0.6), (1.16-1.17), (2.1) and (2.11). Insert (2.13) for $r = r_n^*$ into (1.20), use (0.7), (1.4) and (2.8), and write the abbreviations $w_n = w_{r_n^*,\beta_n}$, $\tau_n = \tau_{r_n^*,\beta_n}$, $\mathbb{P}_k^n = \mathbb{P}_k^{r_n^*,\beta_n}$ and $\mathbb{E}_k^n = \mathbb{E}_k^{r_n^*,\beta_n}$, to obtain

$$\begin{aligned}
&\text{l.h.s. of (2.10)}\\
&= \sum_{k_1,k_2\in\mathbb{N}}\sum_{n_1,n_2\in\mathbb{N}} w_n(k_1,n_1)w_n(k_2,n_2)\\
&\quad\times \sum_{d\in\mathbb{N}} \mathbb{P}_{k_1}^n\Big(U_d=n-n_1-n_2+1, m(d)=k_2\Big)\tfrac{\tau_n(k_1)}{\tau_n(k_2)}\\
&= \sum_{k_1,k_2\in\mathbb{N}}\sum_{n_1\in\mathbb{N}} w_n(k_1,n_1)\tfrac{\tau_n(k_1)}{\tau_n(k_2)}\\
&\quad\times \sum_{n_2\in\mathbb{N}}\sum_{d\in\mathbb{N}} \mathbb{E}_{k_1}^n\,\Big(1_{\{U_d=n-n_1-n_2+1\}}1_{\{m(d)=k_2\}}w_n\left(k_2,n-n_1-U_d+1\right)\Big).
\end{aligned} \quad (2.14)$$

Interchange the sum over d and n_2 and carry out the sum over n_2 to see that

$$\begin{aligned}
&\text{last line of (2.14)}\\
&= \sum_{d\in\mathbb{N}} \mathbb{E}_{k_1}^n\,\Big(1_{\{U_d\le n-n_1\}}1_{\{m(d)=k_2\}}w_n\left(k_2,n-n_1-U_d+1\right)\Big)\\
&= \mathbb{E}_{k_1}^n\,\left(\sum_{d=1}^{T_{n-n_1}} 1_{\{m(d)=k_2\}}w_n\left(k_2,n-n_1-U_d+1\right)\right) \qquad (2.15)\\
&= \mathbb{E}_{k_1}^n\,\left(\sum_{k=1}^{n-n_1} 1_{\{m(T_k)=k_2\}}\frac{w_n\big(k_2,n-n_1-U_{T_k}+1\big)}{m(T_k)+m(T_k-1)-1}\right),
\end{aligned}$$

where the last equality holds because for every $d \in \mathbb{N}$ there are precisely $m(d)+m(d-1)-1$ numbers k such that $T_k = d$ (recall (1.16) and (2.2)). Now write $n_2 = n-n_1-k$

in (2.15) and use the notation in (2.3), to get

$$
\begin{aligned}
&\text{r.h.s. of (2.15)}\\
&= \sum_{n_2=0}^{n-n_1-1} \mathbb{E}^n_{k_1}\left(1_{\{m(T_{n-n_1-n_2})=k_2\}} \frac{w_n\left(k_2, n-n_1-U_{T_{n-n_1-n_2}}+1\right)}{m(T_{n-n_1-n_2})+m(T_{n-n_1-n_2}-1)-1}\right)\\
&= \sum_{n_2=0}^{n-n_1-1} \mathbb{E}^n_{k_1}\left(1_{\{Y_{n-n_1-n_2}=k_2\}} \frac{w_n(k_2, n_2-X_{n-n_1-n_2}+1)}{Y_{n-n_1-n_2}+Z_{n-n_1-n_2}-1}\right).
\end{aligned}
\tag{2.16}
$$

Substitute (2.16) into (2.14), change the starting measure into $\{\tau_n(k_1)^2\}_{k_1\in\mathbb{N}}$, use that $Z_1 = m(0)$ since $T_1 = 1$, carry out the sums over k_1 and k_2 and recall (2.9), to arrive at

$$
\text{l.h.s. of (2.10)} = \sum_{n_1\in\mathbb{N}} \sum_{n_2=0}^{n-n_1-1} \mathbb{E}^n\left(\frac{w_n(Z_1,n_1)}{\tau_n(Z_1)} f_n^-\left(\Gamma_{n-n_1-n_2}, n_2+1\right)\right), \tag{2.17}
$$

where we abbreviate $\mathbb{E}^n = \mathbb{E}^{r_n^*,\beta_n}$ and $f_n^\pm = f^\pm_{r_n^*,\beta_n}$.

Now let $\overline{\mathbb{P}}^n$ be the distribution of the Markov chain $\{\Gamma_l\}_{l\in\mathbb{N}_0}$ on Σ with transition kernel $Q_{r_n^*,\beta_n}$ and initial distribution

$$
\overline{\mathbb{P}}^n\left(\Gamma_0=(i,j,k)\right) = 1_{\{i=0\}}\frac{1}{\theta_n^*}\nu_{r_n^*,\beta_n}(i,j,k). \tag{2.18}
$$

Since the distribution of Γ_1 is the same under $\overline{\mathbb{P}}^n$ as under $\mathbb{P}^n$, we can write $\overline{\mathbb{E}}^n$ instead of $\mathbb{E}^n$ in (2.17). Moreover, $Z_1 = Y_0$ under $\overline{\mathbb{P}}^n$. Therefore (2.18) allows us to change the starting measure from $\overline{\mathbb{P}}^n$ to $\widetilde{\mathbb{P}}^n = \widetilde{\mathbb{P}}^{r_n^*,\beta_n}$ and obtain

$$
\text{l.h.s. of (2.10)} = \frac{1}{\theta_n^*}\sum_{n_1\in\mathbb{N}} \sum_{n_2=0}^{n-n_1-1} \widetilde{\mathbb{E}}^n\left(\frac{w_n(Y_0,n_1)}{\tau_n(Y_0)} 1_{\{X_0=0\}} f_n^-\left(\Gamma_{n-n_1-n_2}, n_2+1\right)\right). \tag{2.19}
$$

Next, formally extend the time range of the Markov chain $\{\Gamma_l\}_{l\in\mathbb{N}_0}$ to the negative integers by putting

$$
(X_l, Y_l, Z_l) = (-l, Y_0, Z_0) \qquad (l = -(Y_0+Z_0-2), \dots, 0), \tag{2.20}
$$

on $\{X_0 = 0\}$. Note that $\{\Gamma_l\}_{l\geq -(Y_0+Z_0-2)}$ is still a Markov chain with transition kernel $Q_{r_n^*,\beta_n}$. In (2.19) we can now use (2.20) to replace

$$
\frac{1}{\theta_n^*}\frac{w_n(Y_0,n_1)}{\tau_n(Y_0)}1_{\{X_0=0\}} \tag{2.21}
$$

by

$$
\frac{1}{\theta_n^*}\sum_{i\in\mathbb{N}_0}\frac{w_n(Y_{-i},n_1)}{(Y_{-i}+Z_{-i}-1)\tau_n(Z_{-i})}1_{\{X_{-i}=i\}}. \tag{2.22}
$$

Substitute this into the r.h.s. of (2.17) to obtain

$$\text{l.h.s. of (2.10)}$$

$$= \frac{1}{\theta_n^*} \sum_{n_1 \in \mathbb{N}} \sum_{n_2=0}^{n-n_1-1} \sum_{i \in \mathbb{N}_0} \widetilde{\mathbb{E}}^n \left(\frac{w_n(Y_{-i}, n_1)}{(Y_{-i}+Z_{-i}-1)\tau_n(Z_{-i})} 1_{\{X_{-i}=i\}} f_n^-(\Gamma_{n-n_1-n_2}, n_2+1) \right) \tag{2.23}$$

Finally, use that $\{\Gamma_l\}_{l \geq -(Y_0+Z_0-2)}$ is stationary under $\widetilde{\mathbb{P}}^n$ to shift the time by i. Then shift the sum over n_1 by i. Carry out the sum over i and shift the sum over n_2 by one to obtain the r.h.s. of (2.10). □

In the r.h.s. of (2.10) appears a *correlation function*. In the sequel we shall prove that the first and the last factor in this correlation function are asymptotically independent as $n \to \infty$. The indicator on $\{X_0 \leq n - n_1 - n_2 + 1\}$ will be harmless, as X_0 will turn out to be of order $\beta_n^{-\frac{1}{3}} = o(\sqrt{n})$ and n_1 and n_2 of order $\beta_n^{-\frac{2}{3}} = o(n)$ (see (2.26) below).

The sums over n_1 and n_2 in (2.10) range only formally over $\mathbb{Z}$, since they are restricted by the conditions $n - n_1 - n_2 + 1 \geq 0$, $n_1 + X_0 \geq 0$ and $n_2 - X_{n-n_1-n_2+1} \geq 0$ (see (2.8-2.9)).

2.4 Scaled representation

The limiting behavior of the r.h.s. of (2.10) will come out of a scaling analysis. We shall turn the sums over $(i,j,k) \in \Sigma$ and $n_1, n_2 \in \mathbb{Z}$ into integrals over

$$(u,v,w) \in S = \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R} \quad \text{and} \quad t_1, t_2 \in \mathbb{R} \tag{2.24}$$

using the substitutions

$$(i,j,k) = (\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} + w\beta^{-\frac{1}{6}} \rceil) = (u,v,w)_\beta \tag{2.25}$$

and

$$n_1 = \lceil t_1 \beta^{-\frac{2}{3}} \rceil, \; n_2 = \lceil t_2 \beta^{-\frac{2}{3}} \rceil. \tag{2.26}$$

Fix $r \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$. Define a scaled version of the measure $\nu_{r,\beta}$ defined in (2.7) by

$$\overline{\nu}_{r,\beta}(u,v,w) = \beta^{-\frac{5}{6}} 1_{\{u \leq 2v + w\beta^{\frac{1}{6}}\}} \nu_{r,\beta}((u,v,w)_\beta). \tag{2.27}$$

Here the power of β is chosen so that $\overline{\nu}_{r,\beta}$ is a Lebesgue probability density on S.

Next, we need scaled versions of the functions $f_{r,\beta}^\pm$ defined in (2.9). Let

$$R = S \times \mathbb{R}^2. \tag{2.28}$$

Define $\overline{f}^-_{r,\beta,\delta}$, $\overline{f}^+_{r,\beta,\delta}$ and $\overline{g}^{n,-}_{r,\beta,\delta} : R \to \mathbb{R}^+_0$ by

$$\overline{f}^+_{r,\beta,\delta}((u,v,w),t_1,t_2) = e^{\delta(t_1-t_2)}\beta^{-\frac{5}{6}}\overline{f}^+_{r,\beta}((u,v,w)_\beta; \lceil t_1\beta^{-\frac{2}{3}}\rceil)1_{\{u\le n\beta^{\frac{2}{3}}-t_1-t_2\}} \quad (2.29)$$

$$\overline{f}^-_{r,\beta,\delta}((u,v,w),t_1,t_2) = e^{\delta(t_1-t_2)}\beta^{-\frac{5}{6}}\overline{f}^-_{r,\beta}((u,v,w)_\beta; \lceil t_1\beta^{-\frac{2}{3}}\rceil) \quad (2.30)$$

$$\begin{aligned}\overline{g}^{n,-}_{r,\beta,\delta}((u,v,w),t_1,t_2) &= e^{\delta(t_2-t_1)}\beta^{-\frac{5}{6}}\widetilde{\mathbb{E}}^{r,\beta}\left(f^-_{r,\beta}(\Gamma_{t_n^\beta(t_1,t_2)}; \lceil t_2\beta^{-\frac{2}{3}}\rceil)\,\Big|\,\Gamma_0=(u,v,w)_\beta\right)\\ &= e^{\delta(t_2-t_1)}\beta^{-\frac{5}{6}}\left(Q^{t_n^\beta(t_1,t_2)}_{r,\beta} f^-_{r,\beta}(\,\cdot\,; \lceil t_2\beta^{-\frac{2}{3}}\rceil)\right)((u,v,w)_\beta) \quad (2.31)\end{aligned}$$

for $(u,v,w)\in S$ and $t_1,t_2\ge 0$, where

$$t_n^\beta(t_1,t_2) = n - \lceil t_1\beta^{-\frac{2}{3}}\rceil - \lceil t_2\beta^{-\frac{2}{3}}\rceil, \quad (2.32)$$

$Q^t_{r,\beta}$ is the t'th power of the transition kernel $Q_{r,\beta}$ defined in (2.6), and $\delta>0$ is an auxiliary parameter that will turn out to be convenient.

We may and shall regard $\overline{\nu}_{r,\beta}$ as a function on R that does not depend on the last two coordinates.

In terms of the scaled objects introduced above we have the following representation for the l.h.s. of (1.6) appearing in Proposition 1 (up to an indicator).

Lemma 3 *For $\delta>0$, $\mu\in\mathbb{R}$ and $n\in\mathbb{N}$,*

$$\begin{aligned}&e^{r_n n}\beta_n^{-\frac{1}{3}}E\left(e^{-\beta_n\sum_x \ell_n^2(x)}e^{\mu\frac{S_n-\theta_n n}{\sigma_n\sqrt{n}}}1_{\{0\le S_{n-1}<S_n\}}\right)\\ &= e^{n\left(r_n-r_n^*-\mu\frac{\theta_n}{\sigma_n\sqrt{n}}\right)}\frac{1}{\beta_n^{-\frac{1}{3}}\theta_n^*}\left\langle\sqrt{\overline{\nu}_{r_n^*,\beta_n}}\,\overline{f}^+_{r_n^*,\beta_n,\delta}, \sqrt{\overline{\nu}_{r_n^*,\beta_n}}\,\overline{g}^{n,-}_{r_n^*,\beta_n,\delta}\right\rangle_{L^2(R)}\end{aligned} \quad (2.33)$$

where $r_n^*=r_n^*(\mu)$ and $\theta_n^*=\theta_n^*(\mu)$ are given in (2.11-2.12).

Proof. Substitute (2.27) and (2.29-2.31) for $r=r_n^*$ and $\beta=\beta_n$ into (2.10). □

2.5 A key proposition: Proposition 2

Lemma 3 gives us the final representation for the quantity appearing in Proposition 1. We are now ready to state the main technical ingredient needed for the proof of Proposition 1.

Proposition 2 *There exists an integrable function $\gamma : (\mathbb{R}^+_0)^2 \to \mathbb{R}^+$ such that for $\delta>0$ sufficiently small and any sequence $r'_n = \beta_n^{\frac{2}{3}}(a^*+o(1))$,*

$$\lim_{n\to\infty}\left\langle\sqrt{\overline{\nu}_{r'_n,\beta_n}}\,\overline{f}^+_{r'_n,\beta_n,\delta}, \sqrt{\overline{\nu}_{r'_n,\beta_n}}\,\overline{g}^{n,-}_{r'_n,\beta_n,\delta}\right\rangle_{L^2(R)} = b^*\int_0^\infty dt_1\int_0^\infty dt_2\,\gamma(t_1,t_2). \quad (2.34)$$

The function γ will be identified in Subsection 3.3. The proof of Proposition 2 is given in Sections 4-8.

2.6 Proof of Proposition 1

In this subsection we finish the proof of Proposition 1 subject to Proposition 2. In fact, we show that Proposition 1 holds with

$$L = 2\int_0^\infty dt_1 \int_0^\infty dt_2\, \gamma(t_1,t_2). \tag{2.35}$$

Fix $\mu \in \mathbb{R}$. First we analyze the asymptotics of the exponential in the r.h.s. of (2.33).

STEP 1 $\lim_{n\to\infty} \left\{ n \left(r_n - r_n^*(\mu) - \mu \frac{\theta_n}{\sigma_n \sqrt{n}} \right) \right\} = \frac{\mu^2}{2}$.

Proof. We write $s \mapsto \lambda^{-1}(s,\beta)$ for the inverse of $r \mapsto \lambda(r,\beta)$ for fixed β, and we write $\partial_s \lambda^{-1}$ for the partial derivative of λ^{-1} w.r.t. its first argument. Expand $\lambda^{-1}(s,\beta_n)$ in a Taylor series around $s=1$. Abbreviate $\mu_n = \frac{\mu}{\sigma_n\sqrt{n}}$. Then, from (2.11), we obtain the existence of some number ξ_n in between 1 and $e^{-\mu_n}$ such that

$$\begin{aligned} r_n^*(\mu) &= \lambda^{-1}(e^{-\mu_n},\beta_n) \\ &= \lambda^{-1}(1,\beta_n) + (e^{-\mu_n}-1)\,\partial_s\lambda^{-1}(1,\beta_n) + \tfrac{1}{2}\left(e^{-\mu_n}-1\right)^2 \partial_s^2 \lambda^{-1}(\xi_n,\beta_n) \\ &= r_n + (e^{-\mu_n}-1)\,\theta_n + \tfrac{1}{2}\left(e^{-\mu_n}-1\right)^2 \partial_s^2\lambda^{-1}(\xi_n,\beta_n). \end{aligned} \tag{2.36}$$

Here the last equality follows from (0.7-0.8).

Next, we calculate

$$\begin{aligned} \partial_s^2\lambda^{-1}(\xi_n,\beta_n) &= \left[\partial_s \left[\partial_r \lambda(r,\beta_n)\right]^{-1}_{r=\lambda^{-1}(s,\beta_n)}\right]_{s=\xi_n} \\ &= -\left[\frac{\partial_r^2\lambda(r,\beta_n)}{\{\partial_r\lambda(r,\beta_n)\}^3}\right]_{r=\lambda^{-1}(\xi_n,\beta_n)} \\ &= -\left[\frac{\beta_n^{-\frac{1}{3}}\,\partial_a^2\lambda\left(a\beta_n^{\frac{2}{3}},\beta_n\right)}{\left\{\beta_n^{-\frac{1}{3}}\,\partial_a\lambda\left(a\beta_n^{\frac{2}{3}},\beta_n\right)\right\}^3}\right]_{a=\beta_n^{-\frac{2}{3}}\lambda^{-1}(\xi_n,\beta_n)}. \end{aligned} \tag{2.37}$$

Equation (0.14), together with the fact that $\mu_n = o(\beta_n^{\frac{1}{3}})$ (recall (0.3) and (1.5)), implies that

$$\lambda^{-1}(\xi_n,\beta_n) - \lambda^{-1}(1,\beta_n) = o(\beta_n^{\frac{2}{3}}). \tag{2.38}$$

Hence, (0.15) and (0.18) give that the numerator in the r.h.s. of (2.37) converges to $\rho''(a^*)$ and the denominator to $\rho'(a^*)^3$. Thus we obtain $\lim_{n\to\infty} \partial_s^2 \lambda^{-1}(\xi_n, \beta_n) = c^{*2}$ with c^{*2} as in (0.20). Substituting (2.37) into (2.36), and noting that $e^{-\mu_n} - 1 = -\frac{\mu}{\sigma_n\sqrt{n}} + \mathcal{O}(\frac{1}{n})$, we get

$$\begin{aligned} r_n^*(\mu) &= r_n + \left(-\tfrac{\mu}{\sigma_n\sqrt{n}} + \mathcal{O}(\tfrac{1}{n})\right)\theta_n + \tfrac{1}{2}\left(\tfrac{\mu}{\sigma_n\sqrt{n}} + \mathcal{O}(\tfrac{1}{n})\right)^2 c^{*2}(1+o(1)) \\ &= r_n - \mu\tfrac{\theta_n}{\sigma_n\sqrt{n}} + \tfrac{1}{2}\mu^2\tfrac{c^{*2}}{\sigma_n^2 n}(1+o(1)). \end{aligned} \tag{2.39}$$

This together with (1.5) implies the claim. □

STEP 2 $\lim_{n\to\infty} r_n^* \beta_n^{-\frac{2}{3}} = a^*$.

Proof. From Step 1 we have

$$\lim_{n\to\infty}(r_n - r_n^*)\beta_n^{-\frac{2}{3}} = \lim_{n\to\infty}\left(\mu\frac{\theta_n}{\sigma_n\sqrt{n}\beta_n^{\frac{2}{3}}} + \mu^2\frac{1}{2n\beta_n^{\frac{2}{3}}}\right). \tag{2.40}$$

Use (0.3) and (1.5) to obtain that the r.h.s. of (2.40) vanishes as $n \to \infty$. Now use (1.5) once more to get the claim. □

STEP 3 *Conclusion of the proof.*

Proof. Because of Step 2, we may apply Proposition 2 for $r_n' = r_n^*$ and $\theta_n' = \theta_n^*$ and obtain that the inner product in the r.h.s. of (2.33) tends to $b^*\frac{L}{2}$, where L is given in (2.35). Furthermore, Step 1 says that the exponential in the r.h.s. of (2.33) converges towards $e^{\frac{\mu^2}{2}}$ as $n \to \infty$, while (0.15), (0.17) and (2.12) yield that $\beta_n^{-\frac{1}{3}}\theta_n^* \to b^*$ as $n \to \infty$.

Summarizing, we have now proved (1.6) with the additional indicator on the event $\{0 \le S_{n-1} < S_n\}$ in the l.h.s. and the additional factor $\frac{1}{2}$ in the r.h.s. However, the limit assertion remains true with $1_{\{0\le S_{n-1}<S_n\}}$ replaced by $1_{\{0\le S_n<S_{n-1}\}}$, since (1.21) is only a small perturbation of (1.20). Indeed, $m^\star(1)$ and $m(1)$ are of order $\beta_n^{-\frac{1}{3}} = o(\beta_n^{-1})$ (see also (3.23) and (3.27) below). The details are left to the reader.

Adding the two limit assertions, we end up with (1.6). □

3 Preparatory tools for the proof of Proposition 2

In this section we collect some tools that will be needed in the remaining sections for the proofs of Theorem 4 and Proposition 2. We shall frequently refer to van der Hofstad and den Hollander (1995) and to van der Hofstad, den Hollander and König (preprint 1995). Henceforth we abbreviate these papers as HH resp. HHK. The quantities appearing below require some patience of the reader, as their full meaning will only become clear later on.

3.1 Spectral properties

In this subsection we describe spectral properties of some operators involved in the proof of Proposition 2, since we later shall need to do some eigenvalue expansions. We are able to characterize the spectra of $\mathcal{L}^a$, $A_{r,\beta}$ and $\frac{1}{\mathrm{id}}\mathcal{L}^{a^*}$ completely, as well as a large part of the spectrum of $Q_{r,\beta}$. The latter will be needed in Section 5 to identify the l.h.s. of (2.34), and will turn out to approximate the spectrum of $\frac{1}{\mathrm{id}}\mathcal{L}^{a^*}$ in a certain sense.

$\mathcal{L}^a$: For any $a \in \mathbb{R}$, the differential operator $\mathcal{L}^a$ defined in (0.13) is a Sturm-Liouville operator on $L^2(\mathbb{R}_0^+)$. For $l \in \mathbb{N}_0$, let $\rho^{(l)}(a)$ be the l'th eigenvalue of $\mathcal{L}^a$ (arranged in decreasing order) with corresponding eigenfunction $x_a^{(l)} \in L^2(\mathbb{R}_0^+)$, normed such that $\|x_a^{(l)}\|_{L^2(\mathbb{R}_0^+)} = 1$. From general Sturm-Liouville theory it follows that $\rho^{(l)}(a)$ is simple, $x_a^{(l)}$ is a real-analytic function on $\mathbb{R}_0^+$ and

$$\{x_a^{(l)}\}_{l\in\mathbb{N}_0} \text{ is an orthonormal basis of } L^2(\mathbb{R}_0^+). \tag{3.1}$$

From HH Lemma 20 it follows that $x_a^{(l)}$ has a subexponentially small tail at infinity. The principal eigenvalue $\rho^{(0)}(a) = \rho(a)$ and corresponding eigenvector $x_a^{(0)} = x_a$ will play a key role in the sequel. Since x_a has no zeroes on $\mathbb{R}_0^+$, we may and shall pick the sign such that $x_a(u) > 0$ for all $u \geq 0$. Note that $\mathcal{L}^{a^*} x_{a^*} = 0$, because $\rho(a^*) = 0$ (see (0.16)).

$A_{r,\beta}$: For any $r \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$, the matrix $A_{r,\beta}$ defined in (0.5) is a symmetric Hilbert-Schmidt operator on $l^2(\mathbb{N})$. For $l \in \mathbb{N}_0$, let $\lambda^{(l)}(r,\beta)$ be the l'th eigenvalue of $A_{r,\beta}$ (arranged in decreasing order of absolute values) with corresponding eigenvector $\tau_{r,\beta}^{(l)}$, normed such that $\|\tau_{r,\beta}^{(l)}\|_{l^2(\mathbb{N})} = 1$. Note that $\lambda^{(0)}(r,\beta) = \lambda(r,\beta)$ and $\tau_{r,\beta}^{(0)} = \tau_{r,\beta}$ as defined in Subsection 2.1. Differentiate the formula $\lambda^{(l)}(r,\beta) = \langle \tau_{r,\beta}^{(l)}, A_{r,\beta}\tau_{r,\beta}^{(l)}\rangle_{l^2(\mathbb{N})}$ w.r.t. r to obtain that

$$\partial_r \lambda^{(l)}(r,\beta) = \lambda^{(l)}(r,\beta) \sum_{i\in\mathbb{N}} (2i-1)\, \tau_{r,\beta}^{(l)}(i)^2. \tag{3.2}$$

Thus, $\lambda^{(l)}(\,\cdot\,,\beta)$ maps $\mathbb{R}$ either onto $-\mathbb{R}^+$, $\{0\}$ or $\mathbb{R}^+$. Since $\lambda^{(l)}(r,0) > 0$ for all $r < 0$,[6] the continuity of $\beta \mapsto \lambda^{(l)}(-1,\beta)$ in zero and (3.2) imply that $\lambda^{(l)}(r,\beta) > 0$ for all $r \in \mathbb{R}$ and all $\beta \in (0,\beta_0(l))$ for some $\beta_0(l) > 0$. Thus, the map $r \mapsto \lambda^{(l)}(r,\beta)$ is strictly increasing and has limits 0 resp. ∞ as $r \to -\infty$ resp. $r \to \infty$ for those β.

$\frac{1}{\mathrm{id}}\mathcal{L}^{a^*}$: Introduce the weighted L^2-space

$$L^{2,\circ}(\mathbb{R}_0^+) = \left\{ f : \mathbb{R}_0^+ \to \mathbb{R} \text{ measurable} : \int_0^\infty dh\, h f(h)^2 < \infty \right\} \tag{3.3}$$

[6] To see why, use (0.5-0.6) and the Gamma-integral representation for $(i+j-2)!$ to write $2\langle \tau, A_{r,0}\tau\rangle_{l^2} = e^r \int_0^\infty dt\, e^{-t} \left(\sum_{i\in\mathbb{N}} \frac{\tau(i)}{(i-1)!} (\frac{te^r}{2})^{i-1} \right)^2 > 0$ for any $\tau \in l^2(\mathbb{N}), \tau \neq 0$.

and the operator $\mathcal{M}^{a^*}$ on $L^{2,\circ}(\mathbb{R}^+) \cap C^2(\mathbb{R}^+)$ given by

$$\left(\mathcal{M}^{a^*} x\right)(u) = \frac{\left(\mathcal{L}^{a^*} x\right)(u)}{u} \qquad (u > 0). \tag{3.4}$$

From the symmetry of $\mathcal{L}^{a^*}$ on $L^2(\mathbb{R}_0^+)$ it follows that $\mathcal{M}^{a^*}$ is symmetric w.r.t. the natural inner product $\langle f, g\rangle_{L^{2,\circ}(\mathbb{R}_0^+)} = \int_0^\infty dh\, h f(h) g(h)$ on $L^{2,\circ}(\mathbb{R}_0^+)$. Differentiate the formula $\rho^{(l)}(a) = \langle x_a^{(l)}, \mathcal{L}^a x_a^{(l)}\rangle_{L^2(\mathbb{R}_0^+)}$ w.r.t. a to obtain

$$\frac{d}{da}\rho^{(l)}(a) = 2\|x_a^{(l)}\|^2_{L^{2,\circ}(\mathbb{R}_0^+)} > 0. \tag{3.5}$$

Thus, $\rho^{(l)} : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing. Moreover, $\lim_{a\to\pm\infty} \rho^{(l)}(a) = \pm\infty$. Therefore, for every $l \in \mathbb{N}_0$ we may define $\alpha^{(l)} \in \mathbb{R}$ by

$$\rho^{(l)}(a^* - \alpha^{(l)}) = \rho(a^*) = 0. \tag{3.6}$$

Let

$$y^{(l)}(v) = \frac{x^{(l)}_{a^*-\alpha^{(l)}}(v)}{\|x^{(l)}_{a^*-\alpha^{(l)}}\|_{L^{2,\circ}(\mathbb{R}_0^+)}}. \tag{3.7}$$

Then $y^{(l)}$ is a normed element of $L^{2,\circ}(\mathbb{R}_0^+)$. As explained in HHK Step 1 in the proof of Proposition 3, $\{y^{(l)}\}_{l\in\mathbb{N}_0}$ is the set of *all* eigenvectors of $\mathcal{M}^{a^*}$ (up to multiples) and

$$\{y^{(l)}\}_{l\in\mathbb{N}_0} \text{ is an orthonormal basis of } L^{2,\circ}(\mathbb{R}_0^+). \tag{3.8}$$

$Q_{r,\beta}$: Fix $l \in \mathbb{N}$ and $\beta \in \mathbb{R}^+$ so small that $\lambda^{(l)}(r,\beta) > 0$ for all $r \in \mathbb{R}$. Define $\alpha^{(l)}(r,\beta) \in \mathbb{R}$ by

$$\lambda^{(l)}(r - \alpha^{(l)}(r,\beta), \beta) = \lambda^{(0)}(r,\beta). \tag{3.9}$$

Note that $\alpha^{(l)}(r,\beta) < \alpha^{(0)}(r,\beta) = 0$ because the map $r \mapsto \lambda^{(l)}(r,\beta)$ is strictly increasing. Define a vector $\nu^{(l)}_{r,\beta}$ by

$$\begin{aligned} \nu^{(l)}_{r,\beta}(i,j,k) &= e^{-\alpha^{(l)}(r,\beta)(j+k-i)}\tau_{r,\beta}(j) A_{r,\beta}(j,k)\tau^{(l)}_{r-\alpha^{(l)}(r,\beta),\beta}(k) \\ &\times \left(\partial_r \lambda(r,\beta)\, \partial_r \lambda^{(l)}(r - \alpha^{(l)}(r,\beta), \beta)\right)^{-\frac{1}{2}} \quad ((i,j,k) \in \Sigma). \end{aligned} \tag{3.10}$$

Note that $\nu^{(0)}_{r,\beta} = \nu_{r,\beta}$ defined in (2.7). A straightforward calculation shows that

$$\left(\nu^{(l)}_{r,\beta} Q_{r,\beta}\right)(i,j,k) = e^{\alpha^{(l)}(r,\beta)} \nu^{(l)}_{r,\beta}(i,j,k) \qquad ((i,j,k) \in \Sigma), \tag{3.11}$$

i.e., $\nu^{(l)}_{r,\beta}$ is a left-eigenvector of $Q_{r,\beta}$ with eigenvalue $e^{\alpha^{(l)}(r,\beta)}$. (In order to derive (3.11), we distinguish between the cases $i = j + k - 2$ and $i < j + k - 2$, use the eigenvalue property of $\tau^{(l)}_{r,\beta}$ for $A_{r,\beta}$ and the symmetry of $A_{r,\beta}$, and observe that $A_{r,\beta}(j,k)e^{-\alpha(j+k-1)} = A_{r-\alpha,\beta}(j,k)$ by (0.5).)

Next, introduce

$$y^{(l)}_{r,\beta} = \frac{\nu^{(l)}_{r,\beta}}{\sqrt{\nu^{(0)}_{r,\beta}}} : \Sigma \to \mathbb{R}. \tag{3.12}$$

This quantity will later turn out to play an analogous role as $y^{(l)}$ defined in (3.7). However, $Q_{r,\beta}$ is not reversible, so we cannot expect that $\{y^{(l)}_{r,\beta}\}_{l\in\mathbb{N}_0}$ is a basis of $l^2(\Sigma)$.

In the sequel we shall suppress $\mathbb{R}^+_0$ resp. $\mathbb{N}$ from the notation for the spaces L^2 and $L^{2,\circ}$ resp. l^2.

3.2 Eigenvector scaling limits: Proposition 3

Proposition 3 below relates the eigenvalues and the eigenvectors of $\mathcal{L}^a$ and $A_{r,\beta}$. For $\beta \in \mathbb{R}^+$, define scaled L^2-versions of vectors $\tau_\beta \in l^2(\mathbb{N})$ by putting

$$\overline{\tau}_\beta(h) = \beta^{-\frac{1}{6}}\tau_\beta(\lceil h\beta^{-\frac{1}{3}}\rceil) \qquad (h > 0) \tag{3.13}$$

and $\overline{\tau}_\beta(0) = \overline{\tau}_\beta(0+)$. Here the power of β is chosen in such a way that $\|\tau_\beta\|_{l^2} = \|\overline{\tau}_\beta\|_{L^2}$. We have the following scaling limit result extending Theorem 3(B).

Proposition 3 *For all $a \in \mathbb{R}$, as $\beta \downarrow 0$,*

(i) for all $l \in \mathbb{N}_0$,

$$\begin{aligned} \beta^{-\frac{1}{3}}\left[\lambda^{(l)}(a\beta^{\frac{2}{3}},\beta) - 1\right] &\to \rho^{(l)}(a) \\ \overline{\tau}^{(l)}_{a\beta^{\frac{2}{3}},\beta} &\to x^{(l)}_a \quad \textit{in } L^{2,\circ} \textit{ and in } L^2. \end{aligned} \tag{3.14}$$

(ii) $\overline{\tau}^{(0)}_{a\beta^{\frac{2}{3}},\beta} = \overline{\tau}_{a\beta^{\frac{2}{3}},\beta}$ converges to $x^{(0)}_a = x_a$ uniformly on $\mathbb{R}^+_0$, provided $|\rho(a)| < 1$.

The proof of Proposition 3 is given in Section 8.

3.3 The function γ

In this subsection we introduce the function $\gamma : (\mathbb{R}_0^+)^2 \to \mathbb{R}^+$ that appears in the formulation of Proposition 2.

Denote by $X^\star = \{X^\star(\sigma)\}_{\sigma \geq 0}$ the zero-dimensional squared Bessel process with generator

$$(G^\star f)(u) = 2uf''(u). \tag{3.15}$$

With a slight abuse of notation (see the end of Subsection 1.2), we denote the distribution of $X^\star$ conditioned on starting at $v \geq 0$ by $\mathbb{P}_v^\star$ and the corresponding expectation by $\mathbb{E}_v^\star$. The point 0 is absorbing for $X^\star$ and is reached almost surely in finite time.

Put $F_a(u) = u^2 - au$ for $a, u \in \mathbb{R}$. For $v, t \geq 0$ and $a \in \mathbb{R}$ define

$$w_a(v,t) = \mathbb{E}_v^\star \left(e^{-\int_0^\infty F_a(X^\star(\sigma))\,d\sigma} \;\middle|\; \int_0^\infty X^\star(\sigma)\,d\sigma = t \right) \psi_v(t) = e^{at} w_0(v,t), \tag{3.16}$$

where (see HHK Lemma 7)

$$\psi_v(t) = \frac{\mathbb{P}_v^\star(\int_0^\infty X^\star(\sigma)\,d\sigma \in dt)}{dt} = \frac{v}{\sqrt{2\pi t^3}} e^{-\frac{v^2}{2t}}. \tag{3.17}$$

It is shown in HHK Lemmas 5 and 6 that there is a critical $a_c \in (2^{\frac{1}{3}} a^*, \infty)$ such that for every $a < a_c$ the function z_a defined by

$$z_a(v) = \int_0^\infty w_a(v,t)\,dt = \mathbb{E}_v^\star \left(e^{-\int_0^\infty F_a(X^\star(\sigma))\,d\sigma} \right) \qquad (v \geq 0) \tag{3.18}$$

is real-analytic on $\mathbb{R}_0^+$ and has a subexponentially small tail at infinity.

Define a function $\gamma : (\mathbb{R}_0^+)^2 \to \mathbb{R}^+$ by

$$\gamma(t_1,t_2) = \frac{b^*}{2} \langle w_{a^*}(\cdot,t_1), x_{a^*} \rangle_{L^2} \langle w_{a^*}(\cdot,t_2), x_{a^*} \rangle_{L^2}. \tag{3.19}$$

Note that

$$e^{\delta(t_1+t_2)} \gamma(t_1,t_2) = \frac{b^*}{2} \langle w_{a^*+\delta}(\cdot,t_1), x_{a^*} \rangle_{L^2} \langle w_{a^*+\delta}(\cdot,t_2), x_{a^*} \rangle_{L^2}. \tag{3.20}$$

Hence, since $a^* < a_c$, the r.h.s. of (3.20) is integrable for $\delta > 0$ small enough. This implies in particular that $\gamma(t_1,t_2)$ decays exponentially fast towards zero as $t_1 \to \infty$ or $t_2 \to \infty$. From (3.17) it can be easily deduced that $\gamma(t_1,t_2)$ is of order $\mathcal{O}(t_i^{-\frac{1}{2}})$ for $t_i \downarrow 0$ $(i = 1,2)$. Consequently, γ is integrable on $(\mathbb{R}_0^+)^2$.

3.4 Convergence of the function $w_{r,\beta}$: Lemmas 4 - 6

For the proof of Proposition 2 we next isolate the appropriate convergence assertion for the function $w_{r,\beta}$ defined in (2.8). Recall that $\{m^\star(x)\}_{x\in\mathbb{N}_0}$ is the Markov chain on $\mathbb{N}_0$ with transition kernel $P^\star$ that was introduced in Subsection 1.3, and $\mathbb{P}^\star_k$ is its distribution when started at $k \in \mathbb{N}_0$.

Fix $r \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$. Define a scaled version of $w_{r,\beta}$ by putting

$$\overline{w}_{r,\beta}(v,t) = \beta^{-\frac{2}{3}} w_{r,\beta}(\lceil v\beta^{-\frac{1}{3}}\rceil, \lceil t\beta^{-\frac{2}{3}}\rceil) \qquad (v,t \geq 0). \tag{3.21}$$

We also need to introduce the sum of $w_{r,\beta}$ over its second argument and its scaled version, namely

$$\begin{aligned} z_{r,\beta}(k) &= \sum_{l\in\mathbb{N}_0} w_{r,\beta}(k,l) = \mathbb{E}^\star_k\left(e^{rU^\star-\beta V^\star}\right) && (k\in\mathbb{N}_0) \\ \overline{z}_{r,\beta}(v) &= \int_0^\infty \overline{w}_{r,\beta}(v,t)\,dt = z_{r,\beta}(\lceil v\beta^{-\frac{1}{3}}\rceil) && (v\geq 0). \end{aligned} \tag{3.22}$$

The reader gains more insight into these quantities once they are expressed in terms of the scaled continuous-time process

$$\left\{X^\star_\beta(\sigma)\right\}_{\sigma\geq 0} = \left\{\beta^{\frac{1}{3}}\left(m^\star(\lceil\sigma\beta^{-\frac{1}{3}}\rceil) + m^\star(\lceil\sigma\beta^{-\frac{1}{3}}\rceil - 1)\right)\right\}_{\sigma\geq 0}. \tag{3.23}$$

Indeed, for any $v \geq 0$, denote the distribution of the process $X^\star_\beta = \{X^\star_\beta(\sigma)\}_{\sigma\geq 0}$ under $\mathbb{P}^\star_{\lceil v\beta^{-\frac{1}{3}}\rceil}$ by $\mathbb{P}^{\star,\beta}_v$ and the corresponding expectation by $\mathbb{E}^{\star,\beta}_v$. Then (see (1.18-1.19))

$$\begin{aligned} U^\star &= \beta^{-\frac{2}{3}}\int_0^\infty X^\star_\beta(\sigma)\,d\sigma, \\ V^\star &= \beta^{-1}\int_0^\infty X^\star_\beta(\sigma)^2\,d\sigma. \end{aligned} \tag{3.24}$$

Thus, with the abbreviation $F^{(\beta)}_r(u) = u^2 - r\beta^{-\frac{2}{3}}u$, we have

$$\overline{w}_{r,\beta}(v,t) = \mathbb{E}^{\star,\beta}_v\left(e^{-\int_0^\infty F^{(\beta)}_r(X^\star_\beta(\sigma))\,d\sigma}\,\Big|\int_0^\infty X^\star_\beta(\sigma)\,d\sigma = \lceil t\beta^{-\frac{2}{3}}\rceil\beta^{\frac{2}{3}}\right)\psi^{(\beta)}_v(t), \tag{3.25}$$

where (see (3.24))

$$\psi^{(\beta)}_v(t) = \beta^{-\frac{2}{3}}\mathbb{P}^{\star,\beta}_v\left(U^\star = \lceil t\beta^{-\frac{2}{3}}\rceil\right). \tag{3.26}$$

In Subsection 4.2 we shall identify $\psi^{(\beta)}_v$.

Recall that $\{m^\star(x)\}_{x\in\mathbb{N}_0}$ is a branching process whose offspring distribution has mean one and variance two. From Ethier and Kurtz (1986) Theorem 9.1.3 it therefore follows that

$$\mathbb{P}^{\star,\beta}_{v_\beta} \Longrightarrow \mathbb{P}^\star_v \quad \text{if} \quad v_\beta \to v \in \mathbb{R}^+_0 \quad \text{and} \quad \beta \downarrow 0. \tag{3.27}$$

In view of this, the following assertions are plausible. Their proofs are deferred to Section 6.

Lemma 4 *For every $a < a_c$, $r'_n = \beta_n^{\frac{2}{3}}(a + o(1))$ and compact interval $I \subset \mathbb{R}^+$,*

$$\limsup_{n\to\infty} \int_I dt \int_0^\infty dv \, \frac{\overline{w}_{r'_n,\beta_n}(v,t)^2}{v} \le \int_I dt \int_0^\infty dv \, \frac{w_a(v,t)^2}{v}. \tag{3.28}$$

Lemma 5 *For every $a < a_c$ and $r'_n = \beta_n^{\frac{2}{3}}(a + o(1))$ there are $q \in (0,1)$ and $C > 0$ such that for sufficiently large n,*

$$\overline{z}_{r'_n,\beta_n}(v) \le C q^v \qquad (v \in \mathbb{R}_0^+). \tag{3.29}$$

Lemma 6 *For every $a < a_c$, $r'_n = \beta_n^{\frac{2}{3}}(a + o(1))$ and any interval $I \subset \mathbb{R}_0^+$,*

$$\int_I dt \; \overline{w}_{r'_n,\beta_n}(\,\cdot\,,t) \longrightarrow^{L^2} \int_I dt \; w_a(\,\cdot\,,t) \quad \text{as} \quad n \to \infty. \tag{3.30}$$

4 Proof of Proposition 2

In this section we begin the proof of Proposition 2. The assertion we have to prove is of the form $\int_{\mathbb{R}}\int_{\mathbb{R}} \gamma_n \to \int_0^\infty \int_0^\infty \gamma$ for certain functions γ_n, γ. We shall prove this assertion by splitting the integrals into the boundary pieces near 0 resp. ∞ and the main piece in the middle, and showing that the boundary pieces give small contributions. In Subsection 4.1 we formulate the program in three lemmas. In Subsections 4.2 and 4.3 we deal with the boundary pieces. The convergence of the main piece is proved in Section 5.

4.1 Splitting the integrals: Lemmas 7 - 9

Fix some sequence $r'_n = \beta_n^{\frac{2}{3}}(a^* + o(1))$, put $\theta'_n = [\partial_r \lambda(r'_n, \beta_n)]^{-1}$ and observe from (0.15) that

$$b'_n = \beta_n^{-\frac{1}{3}} \theta'_n \to b^* \qquad (n \to \infty). \tag{4.1}$$

Furthermore, fix $\delta \in (0, a_c 2^{-\frac{1}{3}} - a^*)$, abbreviate $\overline{f}^{\pm}_{n,\delta} = \overline{f}^{\pm}_{r'_n,\beta_n,\delta}$, $\overline{g}^{-}_{n,\delta} = \overline{g}^{n,-}_{r'_n,\beta_n,\delta}$ and $\overline{\nu}_n = \overline{\nu}_{r'_n,\beta_n}$, and introduce the abbreviation (see the l.h.s. of (2.33))

$$\gamma_n(t_1,t_2) = \frac{1}{b'_n} \left\langle \sqrt{\overline{\nu}_n}\; \overline{f}^{+}_{n,\delta}(\cdot,t_1,t_2), \sqrt{\overline{\nu}_n}\; \overline{g}^{-}_{n,\delta}(\cdot,t_1,t_2) \right\rangle_{L^2(S)} \qquad (t_1,t_2 \in \mathbb{R}). \tag{4.2}$$

Observe from (2.29-2.31) that $\overline{f}^{+}_{n,\delta}(u,v,w,t_1,t_2) = 0$ for $t_1 < -u\beta_n^{\frac{1}{3}}$ and $\overline{g}^{-}_{n,\delta}(\cdot,t_1,t_2) = 0$ for $t_2 < 0$ (see also the end of Subsection 2.3) . Thus $\lim_{n\to\infty} \gamma_n(t_1,t_2) = 0$ for $t_1 < 0$ and $\gamma_n(t_1,t_2) = 0$ for $t_2 < 0$.

According to Lemma 3, Proposition 2 states that

$$\lim_{n\to\infty}\int_{\mathbb{R}} dt_1 \int_{\mathbb{R}} dt_2\, \gamma_n(t_1,t_2) = \int_0^\infty dt_1 \int_0^\infty dt_2\, \gamma(t_1,t_2), \tag{4.3}$$

where γ has been introduced in Subsection 3.3.

We split each of the two integrals in (4.3) into $\int_{-\infty}^{\varepsilon} + \int_{\varepsilon}^{N} + \int_N^\infty$. Lemmas 7-8 below state that the mixed contributions coming from the first and the third integrals are small, uniformly in n, when $\varepsilon > 0$ is small and $N < \infty$ is large. The precise assertions are the following.

Lemma 7 *For any* $0 < \varepsilon < \infty$*,*

$$\lim_{N\to\infty}\limsup_{n\to\infty}\int_\varepsilon^\infty dt_1 \int_N^\infty dt_2\, \gamma_n(t_1,t_2) = 0 = \lim_{N\to\infty}\limsup_{n\to\infty}\int_N^\infty dt_1 \int_\varepsilon^\infty dt_2\, \gamma_n(t_1,t_2). \tag{4.4}$$

Lemma 8

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\int_{-\infty}^\varepsilon dt_1 \int_{-\infty}^\infty dt_2\, \gamma_n(t_1,t_2) = 0 = \lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\int_{-\infty}^\infty dt_1 \int_0^\varepsilon dt_2\, \gamma_n(t_1,t_2). \tag{4.5}$$

Since γ is integrable on $(\mathbb{R}_0^+)^2$, Proposition 2 directly follows from Lemmas 7-8 and the following lemma, which states the convergence of the main piece.

Lemma 9 *For any* $0 < \varepsilon < N < \infty$*,*

$$\lim_{n\to\infty}\int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, \gamma_n(t_1,t_2) = \int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, \gamma(t_1,t_2). \tag{4.6}$$

4.2 Proof of Lemma 7: cutting away large t_1, t_2

We shall give the proof for the second equality in (4.4) only, since the proof for the first is similar.

Recall (4.2) and use the Cauchy-Schwarz inequality to estimate

$$\begin{aligned}
&\textstyle\int_N^\infty dt_1 \int_\varepsilon^\infty dt_2\, \gamma_n(t_1,t_2)\\
&\quad = \textstyle\int_N^\infty dt_1 \int_\varepsilon^\infty dt_2 \int_S ds\; \overline{\nu}_n(s)\overline{f}^{+}_{n,\delta}(s,t_1,t_2)\overline{g}^{-}_{n,\delta}(s,t_1,t_2)\\
&\quad \le \left(\textstyle\int_N^\infty dt_1 \int_\varepsilon^\infty dt_2 \int_S ds\; \overline{\nu}_n(s)\overline{f}^{+}_{n,\delta}(s,t_1,t_2)^2\right)^{\frac12}\\
&\qquad \times \left(\textstyle\int_N^\infty dt_1 \int_\varepsilon^\infty dt_2 \int_S ds\; \overline{\nu}_n(s)\overline{g}^{-}_{n,\delta}(s,t_1,t_2)^2\right)^{\frac12}.
\end{aligned} \tag{4.7}$$

In Steps 3 and 4 below we give the respective estimates for the two factors in the r.h.s. of (4.7). First we make two intermediate steps, the first of which identifies the function $\psi_v^{(\beta)}$ defined in (3.26). Recall that P denotes the distribution of simple random walk $(S_k)_{k\in\mathbb{N}_0}$.

STEP 1 *For any $v, t, \beta \in \mathbb{R}^+$,*

$$\psi_v^{(\beta)}(t) = \frac{\lceil v\beta^{-\frac{1}{3}}\rceil}{\lceil t\beta^{-\frac{2}{3}}\rceil \beta^{\frac{1}{3}}} \beta^{-\frac{1}{3}} P\left(S_{\lceil t\beta^{-\frac{2}{3}}\rceil} = \lceil v\beta^{-\frac{1}{3}}\rceil\right). \tag{4.8}$$

Proof. Since $\{m^\star(x)\}_{x\in\mathbb{N}_0}$ is a critical branching process, Theorem (2.11.2) in Jagers (1975) implies that

$$\mathbb{P}_k^\star\left(\sum_{x=0}^{\infty} m^\star(x) = j\right) = \frac{k}{j} P^\star(j, j-k) \qquad (j \geq k). \tag{4.9}$$

Note that $U^\star = 2\sum_{x=0}^{\infty} m^\star(x) - k$ (recall (1.18)) $\mathbb{P}_k^\star$-a.s., and so (4.9) implies for $l \geq k$,

$$\mathbb{P}_k^\star(U^\star = l) = \frac{2k}{l+k} P^\star\left(\frac{l+k}{2}, \frac{l-k}{2}\right) = \frac{k}{l}\left(\frac{1}{2}\right)^l \binom{l}{\frac{1}{2}(l+k)} = \frac{k}{l} P(S_l = k). \tag{4.10}$$

Substitute $k = \lceil v\beta^{-\frac{1}{3}}\rceil$ and $l = \lceil t\beta^{-\frac{2}{3}}\rceil$ and recall (3.26) to arrive at (4.8). □

STEP 2 *There is a $C > 0$ such that for sufficiently small $\beta \in \mathbb{R}^+$,*

$$\frac{\overline{w}_{r,\beta}(v,t)^2}{\beta^{\frac{1}{3}}\lceil v\beta^{-\frac{1}{3}}\rceil} \leq \frac{C}{t^{\frac{3}{2}}} \overline{w}_{2r,2\beta}(2^{\frac{1}{3}}v, 2^{\frac{2}{3}}t) \qquad (r \in \mathbb{R}, v > 0, t > 0). \tag{4.11}$$

Proof. Use (3.21) and (3.25) to rewrite

$$\overline{w}_{r,\beta}(v,t) = e^{r\beta^{-\frac{2}{3}}t} \mathbb{E}^\star_{\lfloor v\beta^{-\frac{1}{3}}\rfloor}\left(e^{-\beta V^\star} \,\middle|\, U^\star = \lceil t\beta^{-\frac{2}{3}}\rceil\right) \psi_v^{(\beta)}(t). \tag{4.12}$$

Use the Cauchy-Schwarz inequality and Step 1 to find

$$\begin{aligned} \overline{w}_{r,\beta}(v,t)^2 &\leq e^{2r(2\beta)^{-\frac{2}{3}}t2^{\frac{2}{3}}} \mathbb{E}^\star_{\lfloor 2^{\frac{1}{3}}v(2\beta)^{-\frac{1}{3}}\rfloor}\left(e^{-2\beta V^\star} \,\middle|\, U^\star = \lceil t2^{\frac{2}{3}}(2\beta)^{-\frac{2}{3}}\rceil\right) \psi_v^{(\beta)}(t)^2 \\ &= \overline{w}_{2r,2\beta}(2^{\frac{1}{3}}v, 2^{\frac{2}{3}}t) \frac{\psi_v^{(\beta)}(t)^2}{\psi^{(2\beta)}_{2^{\frac{1}{3}}v}(2^{\frac{2}{3}}t)} = \overline{w}_{2r,2\beta}(2^{\frac{1}{3}}v, 2^{\frac{2}{3}}t) 2^{\frac{1}{3}} \psi_v^{(\beta)}(t). \end{aligned} \tag{4.13}$$

Use Step 1 and Stirling's formula to arrive at (4.11). □

STEP 3 *There is a $C > 0$ such that for all $N, \varepsilon > 0$,*

$$\limsup_{n\to\infty} \int_N^\infty dt_1 \int_\varepsilon^\infty dt_2 \int_S ds\ \overline{\nu}_n(s) \overline{f}^{\pm}_{n,\delta}(s, t_1, t_2)^2 \leq \frac{C}{N^{\frac{3}{2}}} e^{-\varepsilon\delta}. \tag{4.14}$$

Proof. Abbreviate $\overline{\tau}_n = \overline{\tau}_{r'_n,\beta_n}$ and $\overline{w}_n^{(\delta)} = \overline{w}_{r'_n+\delta\beta_n^{\frac{2}{3}},\beta_n}$. Use (2.29-2.30), (3.21), (2.25) and (2.9) to see that for $(u,v,w)\in S$,

$$\overline{f}^{\pm}_{n,\delta}((u,v,w),t_1,t_2) \le \frac{\overline{w}_n^{(\delta)}(v,t_1\pm u\beta_n^{\frac{1}{3}})}{\beta^{\frac{1}{3}}\lceil(2v+w\beta_n^{\frac{1}{6}})\beta^{-\frac{1}{3}}\rceil\overline{\tau}_n(v)}e^{\delta u\beta_n^{\frac{1}{3}}}e^{-\delta t_2}. \tag{4.15}$$

Introduce for $v,\tilde{v}\in\mathbb{R}$ the notation

$$\overline{A}_n(v,\tilde{v}) = \begin{cases}\beta_n^{-\frac{1}{6}}A_{r'_n,\beta_n}(\lceil v\beta_n^{-\frac{1}{3}}\rceil,\lceil\tilde{v}\beta_n^{-\frac{1}{3}}\rceil) & \text{if } v,\tilde{v}>0\\ 0 & \text{otherwise.}\end{cases} \tag{4.16}$$

Then for any $(u,v,w)\in S$,

$$\overline{\nu}_n(u,v,w) = b'_n\overline{\tau}_n(v)\overline{A}_n(v,v+w\beta_n^{\frac{1}{6}})\overline{\tau}_n(v+w\beta_n^{\frac{1}{6}}). \tag{4.17}$$

Next write $\int_S ds$ as $\int_0^\infty dv\int_{\mathbb{R}}dw\int_0^{2v+w\beta_n^{\frac{1}{6}}}du$, substitute (4.15) and (4.17) into the l.h.s. of (4.14) and carry out the integral over t_2, to get

$$\begin{aligned}&\int_N^\infty dt_1\int_\varepsilon^\infty dt_2\int_S ds\;\overline{\nu}_n(s)\overline{f}^{\pm}_{n,\delta}(s,t_1,t_2)^2\\ &\le \tfrac{b'_n}{2\delta}e^{-\varepsilon\delta}\int_N^\infty dt_1\int_0^\infty dv\int_{\mathbb{R}}dw\,\overline{A}_n(v,v+w\beta_n^{\frac{1}{6}})\tfrac{\overline{\tau}_n(v+w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)}\\ &\quad\times\int_0^{2v+w\beta_n^{\frac{1}{6}}}du\,\tfrac{e^{2\delta u\beta_n^{\frac{1}{3}}}}{2v+w\beta_n^{\frac{1}{6}}}\;\tfrac{\overline{w}_n^{(\delta)}(v,t_1\pm u\beta_n^{\frac{1}{3}})^2}{\beta^{\frac{1}{3}}\lceil(2v+w\beta_n^{\frac{1}{6}})\beta^{-\frac{1}{3}}\rceil}.\end{aligned} \tag{4.18}$$

By (4.16) we may let the w-integral range over $[-v\beta_n^{-\frac{1}{6}},\infty)$ only, where we estimate $2v+w\beta_n^{\frac{1}{6}}\ge v$. Now use Step 2 for $\beta=\beta_n, r=r'_n+\delta\beta_n^{\frac{2}{3}}$ and $t=t_1\pm u\beta_n^{\frac{1}{3}}$, estimate $(t_1+u\beta_n^{\frac{1}{3}})^{\frac{3}{2}}\ge N^{\frac{3}{2}}$ (the case "$-$" requires a further standard cutting argument for the u-integral, this is left to the reader), carry out the t_1-integral and the u-integral, to obtain

$$\begin{aligned}&\text{l.h.s. of (4.18)}\\ &\le \tfrac{C}{N^{\frac{3}{2}}}\tfrac{b'_n}{2\delta}e^{-\varepsilon\delta}\int_0^\infty dv\int_{\mathbb{R}}dw\,\overline{A}_n(v,v+w\beta_n^{\frac{1}{6}})\tfrac{\overline{\tau}_n(v+w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)}e^{2v+w\beta_n^{\frac{1}{6}}}\overline{z}_n^{(\delta)}(v2^{\frac{1}{3}}),\end{aligned} \tag{4.19}$$

where we abbreviated $\overline{z}_n^{(\delta)} = \overline{z}_{2(r'_n+\delta\beta_n^{\frac{2}{3}}),2\beta_n}$ (see (3.22)).

Split the w-integral into $\int_{-v\beta_n^{-\frac{1}{6}}}^{v\beta_n^{-\frac{1}{6}}}+\int_{v\beta_n^{-\frac{1}{6}}}^\infty$. In the first part, estimate $e^{\beta_n^{\frac{1}{3}}(2v+w\beta_n^{\frac{1}{6}})}\le e^{3v\beta_n^{\frac{1}{3}}}$ and use the following scaled form of the eigenvector relation:

$$\int_{-v\beta_n^{-\frac{1}{6}}}^\infty dw\,\overline{A}_n(v,v+w\beta_n^{\frac{1}{6}})\frac{\overline{\tau}_n(v+w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)} = 1. \tag{4.20}$$

In the second part, use that

$$c_1 = \sup_{n\in\mathbb{N}} \sup_{\widetilde{v}\geq v\geq 0} \frac{\overline{\tau}_n(v+\widetilde{v})}{\overline{\tau}_n(v)} < \infty \tag{4.21}$$

because $\overline{\tau}_n$ converges to x_{a^*} uniformly on $\mathbb{R}^+$ (see Proposition 3) and is decreasing on $[\frac{1}{2}r'_n\beta_n^{-\frac{2}{3}}, \infty)$ (see HH Lemma 12). Furthermore, from HH Lemma 3(i) it follows that there exists some $c_2 > 0$ such that

$$\overline{A}_n(v, v+w\beta_n^{\frac{1}{6}}) \leq \beta_n^{-\frac{1}{6}} c_2 \exp\left[-c_2 \frac{w^2}{2v+w\beta_n^{\frac{1}{6}}}\right] \qquad (v, w > 0, n \in \mathbb{N}). \tag{4.22}$$

This bound is smaller than $c_2\beta_n^{-\frac{1}{6}} e^{-\frac{1}{3}c_2 w\beta_n^{-\frac{1}{6}}}$ for $v < w\beta_n^{\frac{1}{6}}$. Therefore we can estimate

$$\begin{aligned} &\textstyle\int_{v\beta_n^{-\frac{1}{6}}}^{\infty} dw\, \overline{A}_n(v, v+w\beta_n^{\frac{1}{6}}) \frac{\overline{\tau}_n(v+w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)} e^{\beta_n^{\frac{1}{3}}(2v+w\beta_n^{\frac{1}{6}})} \\ &\quad \leq c_1 e^{2v\beta_n^{\frac{1}{3}}} \textstyle\int_{v\beta_n^{-\frac{1}{3}}}^{\infty} d\widetilde{w}\, e^{-\widetilde{w}(\frac{c_2}{3}-\beta_n^{\frac{2}{3}})} \\ &\quad \leq c_3 e^{-c_3 v\beta_n^{-\frac{1}{3}}} \end{aligned} \tag{4.23}$$

for large n and some $c_3 > 0$.

Collecting all these estimates and substituting them into the r.h.s. of (4.19), we get that, for some $\widetilde{C} > 0$,

$$\text{l.h.s. of (4.19)} \leq \frac{\widetilde{C}}{N^{\frac{3}{2}}} b'_n e^{-\varepsilon\delta} \int_0^\infty dv\, e^{3v\beta_n^{\frac{1}{3}}} \overline{z}_n^{(\delta)}(v2^{\frac{1}{3}}) + o(1). \tag{4.24}$$

Now use (4.1) and Lemma 5 for $2\beta_n$ instead of β_n and for $a = 2^{\frac{1}{3}}a^* + \delta$. □

STEP 4 *There is a $C > 0$ such that for all $N, \varepsilon > 0$ and $k \in \mathbb{N}$,*

$$\limsup_{n\to\infty} \sup_{k\in\mathbb{N}} \int_N^\infty dt_1 \int_\varepsilon^\infty dt_2 \int_S ds\; \overline{\nu}_n(s) \overline{g}^{k,-}_{r'_n,\beta_n,\delta}(s, t_1, t_2)^2 \leq \frac{C}{\varepsilon^{\frac{3}{2}}} e^{-N\delta}. \tag{4.25}$$

Proof. Fix $t_1, t_2 \geq 0$, recall (2.32) and abbreviate $t_k^{\beta_n} = t_k^{\beta_n}(t_1, t_2)$. Apply the Cauchy-Schwarz inequality to the expectation w.r.t. the stochastic matrix $Q_{r'_n,\beta_n}$ in (2.31) (recall (2.6)), use (2.25) and (2.30-2.31) and recall that ν_n is invariant for $Q_{r'_n,\beta_n}$, to see that

$$\begin{aligned} &\textstyle\int_S ds\; \overline{\nu}_n(s) \overline{g}^{n,-}_{r'_n,\beta_n,\delta}(s, t_1, t_2)^2 \\ &\quad \leq e^{2\delta(t_2-t_1)} \textstyle\int_S ds\; \overline{\nu}_n(s) \beta_n^{-\frac{5}{3}} \left(Q^{t_k^{\beta_n}}_{r'_n,\beta_n} f^-_{r'_n,\beta_n}(\,\cdot\,; \lceil t_2\beta_n^{-\frac{2}{3}} \rceil)^2 \right) ((u,v,w)_{\beta_n}) \\ &\quad = \textstyle\int_S ds\; \overline{\nu}_n(s) \overline{f}^-_{r'_n,\beta_n,\delta}(s, t_2, t_1)^2. \end{aligned} \tag{4.26}$$

Now use Step 3 with the roles of N and ε reversed. □

4.3 Proof of Lemma 8: cutting away small t_1, t_2

In this subsection we prove Lemma 8 subject to Lemma 9. We shall prove the second assertion in (4.5) only, the proof of the first is similar. For the proof it will be expedient to return to the underlying random walk picture that we left behind at the end of Section 1.

First we need some abbreviations. For $k, n \in \mathbb{N}$ and $\varepsilon > 0$, let

$$\begin{aligned} K_{k,\varepsilon}^{(n)} &= e^{r_n' k} E\left(e^{-\beta_n \sum_{x\in\mathbb{Z}} \ell_k^2(x)} 1_{\left\{\sum_{x>S_k} \ell_k(x) \le \varepsilon \beta_n^{-\frac{2}{3}}\right\}} 1_{\{S_k>0\}} \right) \\ &= e^{r_n' k} E\left(e^{-\beta_n \sum_{x\in\mathbb{Z}} \ell_k^2(x)} 1_{\left\{\sum_{x<0} \ell_k(x) \le \varepsilon \beta_n^{-\frac{2}{3}}\right\}} 1_{\{S_k>0\}} \right) \end{aligned} \tag{4.27}$$

(the last equality holds by reversibility of the random walk). Let

$$L_{k,\varepsilon}^{(n)} = e^{r_n' k} E\left(e^{-\beta_n \sum_{x\in\mathbb{Z}} \ell_k^2(x)} 1_{\left\{\sum_{x>S_k} \ell_k(x) > \varepsilon \beta_n^{-\frac{2}{3}}, \sum_{x<0} \ell_k(x) > \varepsilon \beta_n^{-\frac{2}{3}}\right\}} 1_{\{S_k>0\}} \right) \tag{4.28}$$

and

$$Z_k^{(n)} = e^{r_n' k} E\left(e^{-\beta_n \sum_{x\in\mathbb{Z}} \ell_k^2(x)} 1_{\{S_k>0\}} \right). \tag{4.29}$$

The proof of Lemma 8 is now divided into five steps. In the first step we shall estimate $\int_{\mathbb{R}} dt_1 \int_0^\varepsilon dt_2\, \gamma_n(t_1,t_2)$ above by $K_{n,\varepsilon}^{(n)}$. Then we shall prove that $\limsup_{n\to\infty} K_{n,\varepsilon}^{(n)}$ vanishes as $\varepsilon \downarrow 0$.

STEP 1 *For all $n \in \mathbb{N}$, $\varepsilon > 0$,*

$$\int_{\mathbb{R}} dt_1 \int_0^\varepsilon dt_2\, \gamma_n(t_1,t_2) \le K_{n,\varepsilon}^{(n)}. \tag{4.30}$$

Proof. Tracing back the steps from Lemma 1 to Lemma 3, it is seen that t_2 plays the role of the scaled amount of time the random walk $(S_0, \ldots, S_n)$ spends below 0. Indeed, recall (4.2), (2.27) and (2.9) and use Lemma 2, to see that

$$\begin{aligned} &\int_{\mathbb{R}} dt_1 \int_0^\varepsilon dt_2\, \gamma_n(t_1,t_2) \\ &\le \sum_{n_1\in\mathbb{N}} \sum_{n_2=1}^{\lceil \varepsilon \beta_n^{-\frac{2}{3}} \rceil} \widetilde{\mathbb{E}}^{r_n',\beta_n} \left(f_{r_n',\beta_n}^{+}(\Gamma_0; n_1)\; f_{r_n',\beta_n}^{-}(\Gamma_{n-n_1-n_2+1}; n_2) \right). \end{aligned} \tag{4.31}$$

Then use Knight's Theorem to see that

$$\begin{aligned} &\int_0^\infty dt_1 \int_0^\varepsilon dt_2\, \gamma_n(t_1,t_2) \\ &\le e^{r_n' n} E\left(e^{-\beta_n \sum_{x\in\mathbb{Z}} \ell_n^2(x)} 1_{\left\{\sum_{x<0} \ell_n(x) \le \varepsilon \beta_n^{-\frac{2}{3}}\right\}} 1_{\{0 \le S_{n-1} < S_n\}} \right) \le K_{n,\varepsilon}^{(n)}. \end{aligned} \tag{4.32}$$

□

Define

$$p_\varepsilon^{(n)} = P\left(\sum_{x<0} \ell_{\lfloor \beta_n^{-\frac{2}{3}} \rfloor}(x) < \varepsilon \beta_n^{-\frac{2}{3}}\right). \tag{4.33}$$

In terms of this quantity we have the following bound for $K_{m,\varepsilon}^{(n)}$.

STEP 2 *For all $m \geq \beta_n^{-\frac{2}{3}}$,*

$$K_{m,\varepsilon}^{(n)} \leq 2e^{r_n' \beta_n^{-\frac{2}{3}}} p_\varepsilon^{(n)} Z^{(n)}_{m-\lfloor \beta_n^{-\frac{2}{3}} \rfloor}. \tag{4.34}$$

Proof. Obviously, for all $k \leq m$,

$$1_{\left\{\sum\limits_{x>S_m} \ell_m(x) \leq \varepsilon \beta_n^{-\frac{2}{3}}\right\}} e^{-\beta_n \sum\limits_{x\in\mathbb{Z}} \ell_m^2(x)} 1_{\{S_m>0\}} \leq 1_{\left\{\sum\limits_{x>S_m} [\ell_m-\ell_k](x) \leq \varepsilon \beta_n^{-\frac{2}{3}}\right\}} e^{-\beta_n \sum\limits_{x\in\mathbb{Z}} \ell_k^2(x)}. \tag{4.35}$$

But, $\{[\ell_m - \ell_k](x+S_k)\}_{x\in\mathbb{Z}}$ is independent of $\{\ell_k(x)\}_{x\in\mathbb{Z}}$ and has the same distribution as $\{\ell_{m-k}(x)\}_{x\in\mathbb{Z}}$. Multiply both sides with $e^{r_n' m}$, take expectations on both sides of (4.35) and pick $k = m - \lfloor \beta_n^{-\frac{2}{3}} \rfloor$, to arrive at

$$\begin{aligned} K_{m,\varepsilon}^{(n)} &\leq e^{r_n' m} P\left(\sum_{x>S_{m-k}} \ell_{m-k}(x) \leq \varepsilon \beta_n^{-\frac{2}{3}}\right) E\left(e^{-\beta_n \sum\limits_{x\in\mathbb{Z}} \ell_k^2(x)}\right) \\ &= 2e^{r_n' \lfloor \beta_n^{-\frac{2}{3}} \rfloor} p_\varepsilon^{(n)} \; Z^{(n)}_{m-\lfloor \beta_n^{-\frac{2}{3}} \rfloor}. \end{aligned} \tag{4.36}$$

□

STEP 3

$$\limsup_{n\to\infty} p_\varepsilon^{(n)} = \mathcal{O}(\sqrt{\varepsilon}) \quad as \quad \varepsilon \downarrow 0. \tag{4.37}$$

Proof. From the Arcsine law (see Spitzer (1976) Section 20) it follows that

$$\lim_{n\to\infty} p_\varepsilon^{(n)} = \frac{2}{\pi} \int_0^\varepsilon \frac{dx}{\sqrt{x(1-x)}}. \tag{4.38}$$

□

In view of (4.34) in order to prove Lemma 8 it suffices to prove that $\left\{Z^{(n)}_{n-\lfloor \beta_n^{-\frac{2}{3}} \rfloor}\right\}_{n\in\mathbb{N}}$ is bounded. We shall do so by using a recursive chain of estimates on $Z_k^{(n)}$.

STEP 4 *For sufficiently small $\varepsilon' > 0$ and all $\lfloor \beta_n^{-\frac{2}{3}} \rfloor < m \leq n$,*

$$Z_m^{(n)} \leq \frac{1}{2} Z^{(n)}_{m-\lfloor \beta_n^{-\frac{2}{3}} \rfloor} + L^{(n)}_{m,\varepsilon'}. \tag{4.39}$$

Proof. Use that

$$Z_m^{(n)} \leq 2K^{(n)}_{m,\varepsilon'} + L^{(n)}_{m,\varepsilon'} \tag{4.40}$$

(recall (4.27-4.28)). Then use Steps 2-3 and $r_n' = a^* \beta_n^{\frac{2}{3}}(1+o(1))$. □

STEP 5 *Proof of Lemma 8.*

Proof. Use Steps 1 - 2 for $m = n$ and Step 3 to get that $\int_{\mathbb{R}} dt_1 \int_0^\varepsilon dt_2 \, \gamma_n(t_1,t_2) \leq C\sqrt{\varepsilon} Z^{(n)}_{n-\lfloor \beta_n^{-\frac{2}{3}} \rfloor}$ for some $C > 0$, all $n \in \mathbb{N}$ and all $\varepsilon \in (0, \frac{1}{2})$, say. Define

$$C^{(n)}_{\varepsilon'} = \sup_{k \in \mathbb{N}} L^{(n)}_{k,\varepsilon'}. \tag{4.41}$$

Choose ε' according to Step 4 and apply (4.39) repeatedly to obtain

$$Z^{(n)}_{n-\lfloor \beta_n^{-\frac{2}{3}} \rfloor} \leq C^{(n)}_{\varepsilon'} \sum_{l=0}^{\lfloor n\beta_n^{\frac{2}{3}} \rfloor - 1} \left(\frac{1}{2}\right)^l + \left(\frac{1}{2}\right)^{\lfloor n\beta_n^{\frac{2}{3}} \rfloor} \sup_{1 \leq k \leq n} Z_k^{(n)} \leq 2C^{(n)}_{\varepsilon'} + e^{r_n' \beta_n^{-\frac{2}{3}}}. \tag{4.42}$$

Like in Step 1, for all $\varepsilon' > 0$,

$$L^{(n)}_{k,\varepsilon'} \leq 2\int_{\varepsilon'}^\infty dt_1 \int_{\varepsilon'}^\infty dt_2 \int_S ds \, \nu_n(s) \, f^+_{n,\delta}(s,t_1,t_2) g^{k,-}_{r_n',\beta_n,\delta}(s,t_1,t_2) + o(1). \tag{4.43}$$

(Here we use that the expressions in (1.20) and (1.21) have the same limiting behavior.) From (4.43), (4.7) and Steps 3-4 in Subsection 4.2 it follows that

$$\limsup_{n\to\infty} C^{(n)}_{\varepsilon'} < \infty \text{ for all } \varepsilon' > 0. \tag{4.44}$$

Finally, use (1.5) to get the boundedness of $\left\{ Z^{(n)}_{n-\lfloor \beta_n^{-\frac{2}{3}} \rfloor} \right\}_{n\in\mathbb{N}}$. □

5 Proof of Lemma 9: intermediate t_1, t_2

In this section we give the proof of Lemma 9 subject to Lemmas 4-6. The latter will be proved in Section 6.

Our strategy is the following. In Subsection 5.1 we show strong convergence of the left argument of the inner product in (4.2), defining the function γ_n, and weak relative compactness of the right argument. Consequently, $\int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, \gamma_n(t_1,t_2)$ converges along certain subsequences. The limit turns out to be independent of the subsequence and is identified in Subsection 5.2 with the help of a certain eigenvalue expansion.

For the remainder of this section, fix some sequence $r'_n = \beta_n^{\frac{2}{3}}(a^* + o(1))$, some $0 < \varepsilon < N < \infty$ and some $\delta > 0$ sufficiently small. Abbreviate

$$R' = S \times [\varepsilon, N] \times [\varepsilon, N] \subset R \tag{5.1}$$

and write $L^2(R')$ for the space of the square integrable functions on R'. Regard this as a subspace of $L^2(R)$. Recall the notations and abbreviations introduced at the beginning of Subsection 4.1. We introduce an operator Φ mapping functions $x : \mathbb{R}^+ \to \mathbb{R}$ to functions $\Phi x : S \to \mathbb{R}$ as

$$(\Phi x)(u,v,w) = x(v) 1_{\{u \le 2v\}} \sqrt{\phi_{2v}(w)}, \tag{5.2}$$

where ϕ_{2v} is the normal density with mean 0 and variance $2v$. Note that $\|\Phi x\|^2_{L^2(S)} = 2\|x\|^2_{L^{2,\circ}}$ for $x \in L^{2,\circ}$.

5.1 Convergence along subsequences

This subsection is devoted to the proof of the following.

Lemma 10 *The sequence $\{\sqrt{\overline{\nu}_n}\overline{g}^-_{n,\delta}\}_{n\in\mathbb{N}}$ is weakly relatively compact in $L^2(R')$. Furthermore, for every subsequence in n there exists a further subsequence along which*

$$\int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, \gamma_n(t_1,t_2) \to \langle \mu_\delta, g_\delta \rangle_{L^2(R')}, \tag{5.3}$$

where g_δ is the weak limit of $\sqrt{\overline{\nu}_n}\overline{g}^-_{n,\delta}$ along this subsequence, and

$$\mu_\delta(s,t_1,t_2) = e^{\delta(t_1-t_2)}\sqrt{b^*}\left(\Phi \frac{w_{a^*}(\cdot,t_1)}{2\mathrm{id}}\right)(s). \tag{5.4}$$

Proof. The proof is divided into four steps. First we formulate a general functional analytic statement that will be needed in the proof.

STEP 1 *Let $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ be measurable. If $x, x_1, x_2, \ldots \in L^2(\Omega)$ satisfy*
(i) $x_n \to x$ pointwise on Ω
(ii) $\limsup_{n\to\infty} \|x_n\|_{L^2(\Omega)} \le \|x\|_{L^2(\Omega)}$,
then $x_n \to x$ strongly in $L^2(\Omega)$.

Proof. By Condition (ii), every subsequence has a further subsequence that weakly converges. The weak limit must be x by Condition (i). Since strong convergence is equivalent to weak convergence and Condition (ii), the claim follows. $\square$

For $(u,v,w) \in S$ and $n \in \mathbb{N}$ define

$$\begin{aligned} h_n(u,v,w) &= \tfrac{x_{a^*}(v)}{\overline{\tau}_n(v)} \sqrt{\tfrac{\overline{\nu}_n(u,v,w)}{2v+\beta_n^{\frac{1}{6}} w}}, \\ h(u,v,w) &= \tfrac{\sqrt{b^*}}{2} \left(\Phi \tfrac{x_{a^*}}{\sqrt{\mathrm{id}}} \right)(u,v,w). \end{aligned} \tag{5.5}$$

STEP 2 *$h_n \to h$ strongly in $L^2(S)$.*

Proof. Recall (4.16) and (4.17). Apply Step 1 for $\Omega = S$, $x_n = h_n$ and $x = h$. Condition (i) is satisfied by the uniform convergence of $\overline{\tau}_n$ to x_{a^*} (see Proposition 3(ii)) and by the fact that

$$\lim_{n\to\infty} \overline{A}_n(v, v + w\beta_n^{\frac{1}{6}}) = \phi_{2v}(w) \qquad (v > 0, w \in \mathbb{R}) \tag{5.6}$$

(which follows from HH Lemma 3(i), (0.5) and (1.5)). In order to show that Condition (ii) is satisfied, use the scaled eigenvalue relation (4.20) to calculate

$$\|h_n\|_{L^2(S)}^2 = b_n' \int_0^\infty dv\, x_{a^*}^2(v) \int_{-\infty}^\infty dw\, \overline{A}_n(v, v + w\beta_n^{\frac{1}{6}}) \frac{\overline{\tau}_n(v + w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)} = b_n', \tag{5.7}$$

where we use that x_{a^*} is L^2-normalized. Since $\|h\|_{L^2(S)} = b^* \|\frac{x_{a^*}}{\sqrt{\mathrm{id}}}\|_{L^{2,\circ}} = b^*$, the proof is finished via (4.1). $\square$

Next abbreviate $\overline{w}_n = \overline{w}_{r_n', \beta_n}$ and define

$$q_{n,\delta}((u,v,w), t_1, t_2) = \frac{\overline{w}_n(v, t + u\beta_n^{\frac{1}{3}})}{x_{a^*}(v)\sqrt{2v + \beta_n^{\frac{1}{6}} w}} e^{-\delta(t_2 - t_1)} \tag{5.8}$$

and $\mu_{n,\delta} = q_{n,\delta} h_n \in L^2(R')$ (where $h_n$is regarded as an element of $L^2(R')$).

STEP 3 *$\mu_{n,\delta} \to \mu_\delta$ strongly in $L^2(R')$.*

Proof. It suffices to handle the case $\delta = 0$, since the dependence of $\mu_{n,\delta}$ and μ_δ on δ is very simple. Since h_n is uniformly bounded on R', in view of Step 2 it is enough to show that $q_{n,0} h \to \mu_0$ strongly in $L^2(R')$.

First we prove the weak convergence. To that end we want to show that $\langle q_{n,0} h, z \rangle_{L^2(R')} \to \langle \mu_0, z \rangle_{L^2(R')}$ for any $z \in L^2(R')$. We need to do this for functions

$z(s,t_1,t_2) = y(s,t_2)\, 1_{a\le t_1\le b}$ with $y \in L^2(S\times[\varepsilon,N])$ and $[a,b]\subset[\varepsilon,N]$ only, since the class of these functions is dense in $L^2(R')$. For such z we have (abbreviating $s=(u,v,w)$),

$$
\begin{aligned}
\langle q_{n,0}h, z\rangle_{L^2(R')} &= \int_\varepsilon^N dt_2 \int_0^\infty dv \int_{-\infty}^\infty dw \int_0^{2v+w\beta_n^{\frac16}} du\, \frac{\int_a^b dt_1\, \overline{w}_n(v,t_1+u\beta_n^{\frac13})}{x_{a^*}(v)\sqrt{2v+\beta_n^{\frac16}w}} h(s)y(s,t_2) \\
&\stackrel{n\to\infty}{\longrightarrow} \int_\varepsilon^N dt_2 \int_0^\infty dv \int_{-\infty}^\infty dw \int_0^{2v} du\, \frac{\int_a^b dt_1\, w_{a^*}(v,t_1)}{x_{a^*}(v)\sqrt{2v}} h(s)y(s,t_2) \\
&= \langle \mu_0, z\rangle_{L^2(R')},
\end{aligned}
\tag{5.9}
$$

where we used Lemmas 5-6 and the dominated convergence theorem.

In order to show the strong convergence, we estimate with the help of Lemma 4,

$$
\begin{aligned}
&\limsup_{n\to\infty} \|q_{n,0}h\|^2_{L^2(R')} \\
&= (N-\varepsilon)\limsup_{n\to\infty} \int_\varepsilon^N dt_1 \int_0^\infty dv \int_{-\infty}^\infty dw \int_0^{2v+w\beta_n^{\frac16}} du\, \frac{\overline{w}_n(v,t_1+u\beta_n^{\frac13})^2}{2v+w\beta_n^{\frac16}} \frac{b^*}{2v}\phi_{2v}(w) \\
&\le b^*(N-\varepsilon)\limsup_{n\to\infty} \int_\varepsilon^N dt_1 \int_0^\infty dv \frac{\overline{w}_n(v,t_1)^2}{2v} \\
&\le b^*(N-\varepsilon) \int_\varepsilon^N dt_1 \int_0^\infty dv \frac{w_{a^*}(v,t_1)^2}{2v} \\
&= \|\mu_0\|^2_{L^2(R')}.
\end{aligned}
\tag{5.10}
$$

Fatou's lemma, together with the weak convergence, implies that $\|q_{n,0}h\|^2_{L^2(R')} \to \|\mu_0\|^2_{L^2(R')}$. Weak convergence together with convergence of norms implies strong convergence. $\square$

STEP 4 *Proof of Lemma 10.*

Proof. From (4.25) it follows that $\sup_{n\in\mathbb{N}} \|\sqrt{\overline{\nu}_n}\overline{g}^-_{n,\delta}\|_{L^2(R')} < \infty$, and so the first assertion in Lemma 10 follows. Given any subsequence in n, choose some further subsequence along which $\sqrt{\overline{\nu}_n}\overline{g}^-_{n,\delta}$ converges weakly towards some $g_\delta \in L^2(R')$. Note that $\mu_{n,\delta} = \sqrt{\overline{\nu}_n}\overline{f}^+_{n,\delta}$. Then the second assertion follows from Step 3 by

$$
\text{l.h.s. of (5.3)} = \left\langle \mu_{n,\delta}, \overline{g}^-_{n,\delta}\sqrt{\overline{\nu}_n}\right\rangle_{L^2(R')} \stackrel{n\to\infty}{\to} \langle \mu_\delta, g_\delta\rangle_{L^2(R')}\,. \tag{5.11}
$$

$\square$

$\square$

5.2 Identification of the limit

The proof of Lemma 9 is finished by the following assertion.

Lemma 11 *For all weak accumulation points $g_\delta \in L^2(R')$ of $\{\sqrt{\nu_n}\,\overline{g}_{n,\delta}^-\}_n$,*

$$\langle \mu_\delta, g_\delta\rangle_{L^2(R')} = b^* \int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\ \gamma(t_1,t_2). \tag{5.12}$$

Proof. The proof is divided into four steps. Recall Subsections 3.1 and 3.2. For $l \in \mathbb{N}_0$, let

$$w^{(l)}(v,t) = \frac{1}{2}\left\langle y^{(l)}, w_{a^*}(\cdot,t_1)\right\rangle_{L^2} y^{(l)}(v). \tag{5.13}$$

Recall (5.2).

STEP 1 *For all $g \in L^2(R')$,*

$$\langle \mu_\delta, g\rangle_{L^2(R')} = \sum_{l=0}^{\infty} \sqrt{b^*} \int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, e^{\delta(t_1-t_2)} \left\langle \left(\Phi w^{(l)}(\cdot,t_1)\right)(\cdot), g(\cdot,t_1,t_2)\right\rangle_{L^2(S)}. \tag{5.14}$$

Proof. Observe from (5.13) and (3.8) that $\sum_{l=0}^k w^{(l)}(\cdot,t_1)$ is the $L^{2,\circ}$-projection of $\frac{w_{a^*}(\cdot,t_1)}{2\mathrm{id}}$ onto the subspace spanned by $y^{(0)},\dots,y^{(k)}$. According to (3.8), it therefore converges in $L^{2,\circ}$ to $\frac{w_{a^*}(\cdot,t_1)}{2\mathrm{id}}$ as $k\to\infty$, for any $t_1\in[\varepsilon,N]$. Thus we also have $\sum_{l=0}^k \left(\Phi w^{(l)}(\cdot,t_1)\right) \to \left(\Phi \frac{w_{a^*}(\cdot,t_1)}{2\mathrm{id}}\right)$ in $L^2(S)$. In view of (5.4), the dominated convergence theorem yields the assertion since we have the following integrable majorant:

$$\begin{aligned}
&\left| e^{\delta(t_1-t_2)} \textstyle\sum_{l=0}^k \left\langle \left(\Phi w^{(l)}(\cdot,t_1)\right)(\cdot), g(\cdot,t_1,t_2)\right\rangle_{L^2(S)}\right| \\
&\le \|g(\cdot,t_1,t_2)\|_{L^2(S)} e^{\delta N} \left\| \textstyle\sum_{l=0}^k w^{(l)}(\cdot,t_1)\right\|_{L^{2,\circ}} \\
&\le \|g(\cdot,t_1,t_2)\|_{L^2(S)} e^{\delta N} \left\| \tfrac{w_{a^*}(\cdot,t_1)}{2\mathrm{id}}\right\|_{L^{2,\circ}},
\end{aligned} \tag{5.15}$$

where the second inequality is Bessel's inequality. Now use the estimate $w_{a^*}(v,t) \le e^{a^*t}\psi_v(t)$ (see (3.16)) to get the bound. $\square$

Later we shall apply Step 1 for g a weak accumulation point of $\{\sqrt{\nu_n}\overline{g}_{n,\delta}^-\}_{n\in\mathbb{N}}$ to identify each summand in the r.h.s. of (5.14). For this we shall use an approximation of $\Phi y^{(l)}$, where $y^{(l)}$ appears in the definition of $w^{(l)}$ in (5.13), in terms of the left-eigenvectors of Q_{r_n',β_n} introduced in Subsection 3.1. It will in fact turn out that every summand in the r.h.s. of (5.14) is equal to zero with the exception of the 0'th one, which is equal to the r.h.s. of (5.12). As we already mentioned in Subsection 3.1, we suspect that it is not possible to expand $\sqrt{\overline{\nu}_n}\overline{g}^-_{n,\delta}$ directly in terms of $\{y^{(l)}_{r'_n,\beta_n}\}_{l\in\mathbb{N}_0}$ (see (3.12)).

Fix $l \in \mathbb{N}_0$ and choose n so large that $\lambda^{(l)}(r,\beta_n) > 0$ for all $r \in \mathbb{R}$. In the sequel we shall abbreviate $\alpha^{(l)}_n = \alpha^{(l)}(r'_n,\beta_n)$ and $\nu^{(l)}_n = \nu^{(l)}_{r'_n,\beta_n}$, $y^{(l)}_n = y^{(l)}_{r'_n,\beta_n}$, and we introduce $b^{(l)}_n = \beta_n^{\frac{1}{3}}[\partial_r\lambda^{(l)}(r'_n - \alpha^{(l)}_n,\beta_n)]^{-1}$. Note that from Proposition 3 and the monotonicity of $r \mapsto \lambda^{(l)}(r,\beta_n)$ and $a \mapsto \rho^{(l)}(a)$ we have

$$\lim_{n\to\infty} \alpha^{(l)}_n \beta_n^{-\frac{2}{3}} = \alpha^{(l)} \tag{5.16}$$

(see also HH (6.3-6.4)). Recall (3.10) and (3.12). The eigenvector property of $\nu^{(l)}_n$ leads to the following. Recall (3.12) and (2.25) and define $\overline{y}^{(l)}_n(u,v,w) = \beta_n^{-\frac{5}{12}} y^{(l)}_n((u,v,w)_{\beta_n})$ for $(u,v,w) \in S$.

STEP 2 *For large $n \in \mathbb{N}$ and all $t_1,t_2 \in [\varepsilon,N]$,*

$$\left\langle \overline{y}^{(l)}_n, \sqrt{\overline{\nu}_n}\overline{g}^-_{n,\delta}(\cdot,t_1,t_2)\right\rangle_{L^2(S)} = e^{\alpha^{(l)}_n t_n(t_1,t_2)} \left\langle \overline{y}^{(l)}_n, \sqrt{\overline{\nu}_n}\overline{f}^-_{n,\delta}(\cdot,t_1,t_2)\right\rangle_{L^2(S)}, \tag{5.17}$$

where $t_n(t_1,t_2) = n - \lceil t_1\beta_n^{-\frac{2}{3}}\rceil - \lceil t_2\beta_n^{-\frac{2}{3}}\rceil$.

Proof. Note that the term $\sqrt{\overline{\nu}_n}$ cancels in both inner products. From (2.30-2.31) and (2.9) it can be seen that (5.17) is nothing but the inner product of the scaled version of (3.11) with $f^-_{n,\delta}(\cdot,\lceil t_1\beta_n^{-\frac{2}{3}}\rceil,\lceil t_2\beta_n^{-\frac{2}{3}}\rceil)$. $\square$

Recall the notation in (5.2).

STEP 3 *For any $l \in \mathbb{N}_0$,*

$$\lim_{n\to\infty} \left\|\overline{y}^{(l)}_n - \Phi y^{(l)}\right\|_{L^2(S)} = 0. \tag{5.18}$$

Proof. Recall that $\overline{\tau}_n = \overline{\tau}_{r'_n,\beta_n}$ (see also (4.16)). Define for $(u,v,w) \in S$

$$\begin{aligned}\Phi_n(u,v,w) &= 1_{\{u\le 2v+w\beta_n^{\frac{1}{6}}\}} e^{-\alpha^{(l)}_n\beta_n^{-\frac{1}{3}}(2v+w\beta_n^{\frac{1}{6}}-u)} \\ &\quad\times\sqrt{\overline{A}_n(v,v+w\beta_n^{\frac{1}{6}})\frac{\overline{\tau}_n(v)}{\overline{\tau}_n(v+w\beta_n^{\frac{1}{6}})}}.\end{aligned} \tag{5.19}$$

Then it is clear that

$$\overline{y}^{(l)}_n(u,v,w) = \Phi_n(u,v,w)\sqrt{b^{(l)}_n}\,\overline{\tau}^{(l)}_{r'_n-\alpha^{(l)}_n,\beta_n}(v+w\beta_n^{\frac{1}{6}}) \qquad ((u,v,w)\in S). \tag{5.20}$$

As an intermediate step, we show first that

$$\lim_{n\to\infty} \|\widehat{y}_n^{(l)} - y_n^{(l)}\|_{L^2(S)}^2 = 0, \tag{5.21}$$

where

$$\widehat{y}_n^{(l)}(u,v,w) = \Phi_n(u,v,w) y^{(l)}(v + w\beta_n^{\frac{1}{6}}) \qquad ((u,v,w) \in S). \tag{5.22}$$

To this end, we write $\int_S ds = \int_{\mathbb{R}} dw \int_0^\infty dv \int_0^{2v+w\beta_n^{\frac{1}{6}}} du$, shift the v-integral by $w\beta_n^{\frac{1}{6}}$ and evaluate the u-integral, to get

$$\begin{aligned}
&\|\widehat{y}_n^{(l)} - \overline{y}_n^{(l)}\|_{L^2(S)}^2 \\
&= \int_{\mathbb{R}} dw \int_{w\beta_n^{\frac{1}{6}}}^\infty dv \int_0^{2v-w\beta_n^{\frac{1}{6}}} du\, \Phi_n(u, v - w\beta_n^{\frac{1}{6}}, w)^2 \left(\sqrt{b_n^{(l)}}\overline{\tau}^{(l)}_{r_n'-\alpha_n^{(l)},\beta_n}(v) - y^{(l)}(v)\right)^2 \\
&= \int_{\mathbb{R}} dw \int_{w\beta_n^{\frac{1}{6}}}^\infty dv\, \frac{e^{-\alpha_n^{(l)}\beta_n^{-\frac{1}{3}}(2v-w\beta_n^{\frac{1}{6}})}-1}{-2\alpha_n^{(l)}\beta_n^{-\frac{1}{3}}} \overline{A}_n(v - w\beta_n^{\frac{1}{6}}, v) \frac{\overline{\tau}_n(v-w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)} \\
&\quad \times \left(\sqrt{b_n^{(l)}}\overline{\tau}^{(l)}_{r_n'-\alpha_n^{(l)},\beta_n}(v) - y^{(l)}(v)\right)^2 \qquad\qquad (5.23) \\
&\le \int_0^\infty dv \int_{-\infty}^{v\beta_n^{-\frac{1}{6}}} dw\, \overline{A}_n(v - w\beta_n^{\frac{1}{6}}, v) \frac{\overline{\tau}_n(v-w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)} \\
&\quad \times (v - \tfrac{w}{2}\beta_n^{\frac{1}{6}}) \left(\sqrt{b_n^{(l)}}\overline{\tau}^{(l)}_{r_n'-\alpha_n^{(l)},\beta_n}(v) - y^{(l)}(v)\right)^2.
\end{aligned}$$

Similarly as in the proof of Step 3 in Lemma 7 in Subsection 4.2, split the w-integral into $\int_{-\infty}^{-v\beta_n^{-\frac{1}{6}}} + \int_{-v\beta_n^{-\frac{1}{6}}}^{v\beta_n^{-\frac{1}{6}}}$. Use (4.22) to see that the first part vanishes as $n \to \infty$. In the second part, estimate $v - \frac{w}{2}\beta_n^{\frac{1}{6}} \le 3v$, carry out the w-integral and use (4.20), to see that

$$\|\widehat{y}_n^{(l)} - \overline{y}_n^{(l)}\|_{L^2(S)}^2 \le o(1) + 3\|\sqrt{b_n^{(l)}}\overline{\tau}^{(l)}_{r_n'-\alpha_n^{(l)},\beta_n} - y^{(l)}\|_{L^{2,\circ}}. \tag{5.24}$$

Now (5.21) follows from Proposition 3(i) together with (5.16).

In order to show (5.18), it is now enough to show that $\widehat{y}_n^{(l)} \to \Phi y^{(l)}$ in $L^2(S)$. To do this, we shall apply Step 1 in Subsection 5.1. First, in (5.19) we see that Φ_n converges pointwise towards $\Phi = \Phi 1$ on S. Indeed, use (5.16), (0.15) and (5.6) as well as the uniform convergence of $\overline{\tau}_n$ (see Proposition 3) to derive the pointwise convergence of Φ_n to Φ. Since $y^{(l)}$ is continuous on $\mathbb{R}^+$, clearly $\widehat{y}_n^{(l)}$ converges towards $\Phi y^{(l)}$ pointwise on S (see (5.22)). Thus, Condition (i) of Step 1 in Subsection 5.1 holds. Next, in order to show that Conditon (ii) is satisfied, we derive as in (5.23),

$$\|\widehat{y}_n^{(l)}\|_{L^2(S)}^2 = \int_0^\infty dv \int_{-\infty}^{v\beta_n^{-\frac{1}{6}}} dw\, \overline{A}_n(v - w\beta_n^{\frac{1}{6}}, v) \frac{\overline{\tau}_n(v - w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)} (v - \frac{w}{2}\beta_n^{\frac{1}{6}}) y^{(l)}(v)^2. \tag{5.25}$$

This time, split the w-integral into $\int_{-\infty}^{-2pv\beta_n^{-\frac{1}{6}}} + \int_{-2pv\beta_n^{-\frac{1}{6}}}^{v\beta_n^{-\frac{1}{6}}}$ for some small $p > 0$. Proceed as in (5.23-5.24) to arrive at

$$\limsup_{n\to\infty} \|\widehat{y}_n^{(l)}\|_{L^2(S)}^2 \leq (1+p)\|y^{(l)}\|_{L^{2,\circ}}^2 = 1+p. \tag{5.26}$$

Letting $p \downarrow 0$, we see that also Condition (ii) holds. □

STEP 4 *Proof of Lemma 11.*

Proof. Let g_δ be any accumulation point of $\{\sqrt{\overline{\nu}_n}\overline{g}_{n,\delta}^-\}_{n\in\{\in\mathbb{N}}$ in $L^2(R')$. We may and shall assume that $\sqrt{\overline{\nu}_n}\overline{g}_{n,\delta}^- \to g_\delta$ weakly in $L^2(R')$. We apply Step 1 to $g = g_\delta$. We shall show that the l-th summand in the r.h.s. of (5.14) is equal to zero for $l \geq 1$ and equal to $\int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, \gamma(t_1,t_2)$ for $l = 0$. To this end, we recall that $(\Phi w^{(l)}(\cdot,t_1))(\cdot) = \frac{1}{2}\left\langle y^{(l)}, w_{a^*}(\cdot,t_1)\right\rangle_{L^2}(\Phi y^{(l)})(\cdot)$ and use Step 3 to see that the l-th summand is equal to

$$\begin{aligned}
&\sqrt{b^*}\int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, e^{\delta(t_1-t_2)}\tfrac{1}{2}\left\langle y^{(l)}, w_{a^*}(\cdot,t_1)\right\rangle_{L^2}\left\langle(\Phi y^{(l)})(\cdot), g_\delta(\cdot,t_1,t_2)\right\rangle_{L^2(S)} \\
&= \sqrt{b^*}\lim_{n\to\infty}\int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, e^{\delta(t_1-t_2)}\tfrac{1}{2}\left\langle y^{(l)}, w_{a^*}(\cdot,t_1)\right\rangle_{L^2}\left\langle y_n^{(l)}, \sqrt{\overline{\nu}_n}\overline{g}_{n,\delta}^-(\cdot,t_1,t_2)\right\rangle_{L^2(S)},
\end{aligned} \tag{5.27}$$

since the map $(t_1,t_2) \mapsto e^{\delta(t_1-t_2)}\frac{1}{2}\left\langle y^{(l)}, w_{a^*}(\cdot,t_1)\right\rangle_{L^2}$ is bounded on $[\varepsilon, N]^2$. Use Step 2 to get

$$\begin{aligned}
\text{r.h.s. of (5.27)} = \sqrt{b^*}\lim_{n\to\infty} &\int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, e^{\delta(t_1-t_2)}\tfrac{1}{2}\left\langle y^{(l)}, w_{a^*}(\cdot,t_1)\right\rangle_{L^2} \\
&\times e^{\alpha_n^{(l)}t_n(t_1,t_2)}\left\langle \overline{y}_n^{(l)}, \sqrt{\overline{\nu}_n}\overline{f}_{n,\delta}^-(\cdot,t_1,t_2)\right\rangle_{L^2(S)}.
\end{aligned} \tag{5.28}$$

For $l \geq 1$, we estimate

$$\begin{aligned}
&|\text{r.h.s. of (5.28)}| \\
&\leq \sqrt{b^*}\limsup_{n\to\infty}\; e^{\alpha_n^{(l)}(n-N\beta_n^{-\frac{2}{3}})}\left|\int_0^\infty dt_1\, \tfrac{1}{2}\left\langle y^{(l)}, w_{a^*}(\cdot,t_1)\right\rangle_{L^2}\right| \\
&\qquad\times\left|\int_0^\infty dt_2\,\left\langle \overline{y}_n^{(l)}, \sqrt{\overline{\nu}_n}\overline{f}_{r_n',n}^-(\cdot,t_2)\right\rangle_{L^2(S)}\right| \qquad (5.29) \\
&\leq \sqrt{b^*}\limsup_{n\to\infty}\; e^{\alpha_n^{(l)}(n-N\beta_n^{-\frac{2}{3}})}\tfrac{1}{2}\left|\langle y^{(l)}, z_{a^*}\rangle_{L^2}\left\langle\sqrt{b_n^{(l)}}\overline{\tau}^{(l)}_{r_n'-\alpha_n^{(l)},\beta_n}, \overline{z}_{r_n',\beta_n}\right\rangle_{L^{2,\circ}}\right|\sqrt{b_n^{(0)}}.
\end{aligned}$$

The r.h.s. is zero by (5.16), because $\alpha^{(l)} < 0$, $n\beta_n^{\frac{2}{3}} \to \infty$ and $(\|\overline{\tau}^{(l)}_{r_n',\beta_n}\|_{L^{2,\circ}}\|\overline{z}_{r_n',\beta_n}\|_{L^{2,\circ}})_n$ is bounded (see Proposition 3 and Lemma 5).

For $l=0$, recall $y^{(0)}=\sqrt{b^*}x_{a^*}$ and use Proposition 3 and Lemma 6 to get

$$
\begin{aligned}
&\text{r.h.s. of (5.28)}\\
&= \sqrt{b^*}\lim_{n\to\infty}\int_\varepsilon^N dt_1\int_\varepsilon^N dt_2\,\tfrac{1}{2}\left\langle y^{(0)},w_{a^*}(\cdot,t_1)\right\rangle_{L^2}\langle\sqrt{\overline{\nu}_n},\sqrt{\overline{\nu}_n}\overline{f}_{n,0}(\cdot,t_2)\rangle_{L^2(S)}\\
&= \sqrt{b^*}\lim_{n\to\infty} b_n^{(0)}\int_\varepsilon^N dt_1\int_\varepsilon^N dt_2\,\tfrac{1}{2}\left\langle y^{(0)},w_{a^*}(\cdot,t_1)\right\rangle_{L^2}\langle\overline{\tau}_n,\overline{w}_{r_n',\beta_n}(\cdot,t_2)\rangle_{L^2}\\
&= \tfrac{b^{*2}}{2}\int_\varepsilon^N dt_1\int_\varepsilon^N dt_2\,\langle x_{a^*},w_{a^*}(\cdot,t_1)\rangle_{L^2}\langle x_{a^*},w_{a^*}(\cdot,t_2)\rangle_{L^2}\\
&= b^*\textstyle\int_\varepsilon^N dt_1\int_\varepsilon^N dt_2\,\gamma(t_1,t_2).
\end{aligned}
\tag{5.30}
$$

□

□

6 Proof of Lemmas 4 - 6

6.1 Proof of Lemma 4: properties of $\overline{w}_{r_n,\beta_n}$

In this subsection we prove Lemma 4. Our first step is a pointwise asymptotic estimate. Fix $a<a_c, r_n'=a\beta_n^{\frac{2}{3}}(1+o(1))$ and $v,t\in\mathbb{R}^+$. Pick a sequence $t_n\to t$ such that $\lceil t_n\beta_n^{-\frac{2}{3}}\rceil+\lceil v\beta_n^{-\frac{1}{3}}\rceil$ is even (otherwise $w_{r_n',\beta_n}(v,t_n)=0$).

STEP 1 $\limsup_{n\to\infty}\overline{w}_{r_n',\beta_n}(v,t_n)\le w_a(v,t)$.

Proof. Note that (4.8) implies, with the help of Stirling's formula, for every $\eta>0$,

$$
\lim_{n\to\infty}\sup\left\{\left|\psi_w^{(\beta_n)}(s)-\psi_w(s)\right|: w\in\mathbb{R}^+, s\ge\eta, \lceil s\beta_n^{-\frac{2}{3}}\rceil+\lceil w\beta_n^{-\frac{1}{3}}\rceil \text{ even}\right\}=0.
\tag{6.1}
$$

Therefore it suffices to show that (see (4.12))

$$
\begin{aligned}
&\limsup_{n\to\infty}\mathbb{E}_v^{\star,\beta_n}\left(e^{-\beta_n V^\star}\,\middle|\,U^\star=\lceil t_n\beta_n^{-\frac{2}{3}}\rceil\right)\\
&\qquad\le\mathbb{E}_v^\star\left(e^{-\int_0^\infty X^\star(\sigma)^2\,d\sigma}\,\middle|\,\int_0^\infty X^\star(\sigma)\,d\sigma=t\right).
\end{aligned}
\tag{6.2}
$$

To this end, first note that for every $N \in \mathbb{N}$ and $\delta > 0$,

$$\mathbb{E}_v^{\star,\beta_n}\left(e^{-\beta_n V^\star}\,\Big|\,U^\star = \lceil t_n\beta_n^{-\frac{2}{3}}\rceil\right) \le \mathbb{E}_v^{\star,\beta_n}\left(e^{-\frac{1}{N}\sum_{k=1}^{N^2\wedge\xi_\delta^n} Y_n^\star(k)^2}\,\Big|\,U^\star = \lceil t_n\beta_n^{-\frac{2}{3}}\rceil\right), \tag{6.3}$$

where

$$\begin{aligned} \xi_\delta^n &= \beta_n^{\frac{1}{3}}\inf\{m\in\mathbb{N} : \int_0^{m\beta_n^{\frac{1}{3}}} X_{\beta_n}^\star(\sigma)\,d\sigma \ge t_n-\delta\} \\ Y_n^\star(k) &= \inf\left\{X_{\beta_n}^\star(\sigma)\,\Big|\,\tfrac{k-1}{N}\le\sigma\le\tfrac{k}{N}\right\}. \end{aligned} \tag{6.4}$$

Our next aim is to show that the distribution of $(Y_n^\star(k\wedge(N\xi_\delta^n)))_{k=1,\dots,N^2}$ under $\mathbb{P}_v^{\star,\beta_n}(\,\cdot\,|\,U^\star = \lceil t_n\beta_n^{-\frac{2}{3}}\rceil)$ converges towards the one of $(Y^\star(k\wedge(N\xi_\delta)))_{k=1,\dots,N^2}$ under $\mathbb{P}_v^\star(\,\cdot\,|\int_0^\infty X^\star(\sigma)\,d\sigma = t)$, where

$$\begin{aligned} \xi_\delta &= \inf\{s>0 : \textstyle\int_0^s X^\star(\sigma)\,d\sigma \ge t-\delta\} \\ Y^\star(k) &= \inf\left\{X^\star(\sigma)\,\Big|\,\tfrac{k-1}{N}\le\sigma\le\tfrac{k}{N}\right\}. \end{aligned} \tag{6.5}$$

To this end, pick Borel sets $A_1,\dots,A_{N^2}\subset\mathbb{R}_0^+$ and use the strong Markov property for the Markov chain $\{m^\star(x)\}_{x\in\mathbb{N}_0}$ at time $(N_n\wedge\xi_\delta^n)\beta_n^{\frac{1}{3}}$ where $N_n = \beta_n^{\frac{1}{3}}\lceil N\beta_n^{-\frac{1}{3}}\rceil$, to obtain

$$\begin{aligned} &\mathbb{P}_v^{\star,\beta_n}\left(\cap_{k=1}^{N^2}\{Y_n^\star(k\wedge(N\xi_\delta^n))\in A_k\}\,\Big|\,U^\star = \lceil t_n\beta_n^{-\frac{2}{3}}\rceil\right)\psi_v^{(\beta_n)}(t_n) \\ &= \mathbb{E}_v^{\star,\beta_n}\left(\left[\textstyle\prod_{k=1}^{N^2} 1_{\{Y_n^\star(k\wedge(N\xi_\delta^n))\in A_k\}}\right]\psi^{(\beta_n)}_{X^\star_{\beta_n}(N_n\wedge\xi_\delta^n)}\left(t_n - \int_0^{N_n\wedge\xi_\delta^n} X_{\beta_n}^\star(\sigma)\,d\sigma\right)\right) \end{aligned} \tag{6.6}$$

(recall (1.18)). Since, by (6.4),

$$t_n - \int_0^{N_n\wedge\xi_\delta^n} X_{\beta_n}^\star(\sigma)\,d\sigma \ge \delta - \beta_n^{\frac{1}{3}}X_{\beta_n}^\star(N\wedge\xi_\delta^n), \tag{6.7}$$

we may insert the indicator on the event $\{t_n - \int_0^{N_n\wedge\xi_\delta^n} X_{\beta_n}^\star(\sigma)\,d\sigma \ge \frac{1}{2}\delta\}$ in the expectation in the r.h.s. of (6.6). Indeed, on the complement of this set, we have $X_{\beta_n}^\star(N_n\wedge\xi_\delta^n)\ge\frac{1}{2}\delta\beta_n^{-\frac{1}{3}}$, hence the ψ-term in the expectation in the r.h.s. of (6.6) is equal to zero by (4.8).

Then use (6.1) to see that

$$\begin{aligned} &\lim_{n\to\infty}\ \text{r.h.s. of (6.6)} \\ &= \lim_{n\to\infty}\mathbb{E}_v^{\star,\beta_n}\left(\left[\textstyle\prod_{k=1}^{N^2} 1_{\{Y_n^\star(k\wedge\xi_\delta^n)\in A_k\}}\right]\psi_{X^\star_{\beta_n}(N\wedge\xi_\delta^n)}\left(t - \textstyle\int_0^{N\wedge\xi_\delta^n} X_{\beta_n}^\star(\sigma)\,d\sigma\right)\right) \end{aligned} \tag{6.8}$$

Since $(v,t) \mapsto \psi_v(t)$ is a bounded continuous function on $\mathbb{R}_0^+ \times [\frac{1}{2}\delta, \infty)$ and $X^\star \mapsto \xi_\delta$ is a continuous functional, the map

$$X^\star \mapsto \psi_{X^\star(N\wedge\xi_\delta)}\left(t - \int_0^{N\wedge\xi_\delta} X^\star(\sigma)\,d\sigma\right) \tag{6.9}$$

is a bounded continuous functional. Hence, we get from (3.27) that

$$\begin{aligned}
&\lim_{n\to\infty} \text{l.h.s. of (6.6)} \\
&= \mathbb{E}_v^\star\left(\left[\prod_{k=1}^{N^2} 1_{\{Y^\star(k\wedge(N\xi_\delta))\in A_k\}}\right]\psi_{X^\star(N\wedge\xi_\delta)}\left(t - \textstyle\int_0^{N\wedge\xi_\delta} X^\star(\sigma)\,d\sigma\right)\right) \\
&= \mathbb{P}_v^\star\left(\cap_{k=1}^{N^2}\{Y^\star(k\wedge(N\xi_\delta))\in A_k\} \mid \textstyle\int_0^\infty X^\star(\sigma)\,d\sigma = t\right)\psi_v(t),
\end{aligned} \tag{6.10}$$

where we used the strong Markov property at time $N\wedge\xi_\delta$. Therefore, for any $k = 1,\ldots,N^2$, the distribution of $Y_n^\star(k\wedge(N\xi_\delta^n))$ under $\mathbb{P}_v^{\star,\beta_n}(\,\cdot\,|\,U^\star = \lceil t_n\beta_n^{-\frac{2}{3}}\rceil)$ converges towards that of $Y^\star(k\wedge(N\xi_\delta))$ under $\mathbb{P}_v^\star(\,\cdot\mid\int_0^\infty X^\star(\sigma)\;d\sigma = t)$. So we obtain for every $N\in\mathbb{N}$ and $\delta>0$ that

$$\text{l.h.s. of (6.2)} \le \mathbb{E}_v^\star\left(e^{-\frac{1}{N}\sum_{k=1}^{N^2\wedge\xi_\delta} Y^\star(k)^2}\;\middle|\;\int_0^\infty X^\star(\sigma)\;d\sigma = t\right). \tag{6.11}$$

Now, let $\delta\to 0$ and $N\to\infty$ to get (6.2) by the dominated convergence theorem. $\square$

STEP 2 *Conclusion of the proof of Lemma 4*

Proof. Let $I\subset\mathbb{R}^+$ be a compact interval. Since

$$\int_I dt\int_0^\infty dv\frac{\sup_n \overline{w}_{r_n',\beta_n}(v,t)^2}{v} < \infty \tag{6.12}$$

(use (3.25) and (4.8)), we may apply the reversed Fatou inequality, and so by Step 1 the assertion follows. $\square$

6.2 Preparations for the proof of Lemmas 5-6

In this subsection we shall start the proof of Lemmas 5-6. Their proofs will be finished in the next subsection. Recall Subsection 3.4.

We need some more notation. Let

$$\xi_n = \inf\{\sigma > 0 : X_{\beta_n}^\star(\sigma) = 0\} \tag{6.13}$$

be the absorption time of $X_{\beta_n}^\star$. Furthermore, define for $l>0$

$$K_l^{(n)} = \sup_{v\in\mathbb{R}_0^+} \overline{z}_{r_n',\beta_n,l}(v), \tag{6.14}$$

where

$$\overline{z}_{r'_n,\beta_n,l}(v) = \mathbb{E}_v^{\star,\beta_n}\left(e^{-\int_0^\infty F_{r'_n}^{(\beta_n)}(X^\star_{\beta_n}(\sigma))\,d\sigma}1_{\{\xi_n\le l\}}\right). \tag{6.15}$$

Clearly, $K_l^{(n)}$ is finite for all l and n.

STEP 1 *There exists $N>0$ such that for all $l>0$ and $n\in\mathbb{N}$,*

$$K_l^{(n)} = \sup_{v\in[0,N]} \overline{z}_{r'_n,\beta_n,l}(v). \tag{6.16}$$

Proof. Pick N so large that $F_{r'_n}^{(\beta_n)}$ is positive and increasing on $[\frac{1}{2}N,\infty)$ for all $n\in\mathbb{N}$. Use the strong Markov property for the Markov chain $\{m^\star(x)\}_{x\in\mathbb{N}_0}$ at time $\beta_n^{-\frac{1}{3}}$ where

$$\tau_N^{(n)} = \beta_n^{\frac{1}{3}}\inf\left\{t\in\mathbb{N} : X^\star_{\beta_n}(t\beta_n^{\frac{1}{3}}) \le \frac{1}{2}N\right\} \tag{6.17}$$

to estimate

$$\begin{aligned}
\overline{z}_{r'_n,\beta_n,l}(v) \quad &\le \mathbb{E}_v^{\star,\beta_n}\left(e^{-\int_0^{\tau_N^{(n)}} F_{r'_n}^{(\beta_n)}(X^\star_{\beta_n}(\sigma))\,d\sigma}1_{\{\xi_n\le l\}}\right)\sup_{u\in[0,\frac{1}{2}N]}\overline{z}_{r'_n,\beta_n,l}(u)\\
&\le \sup_{u\in[0,N]}\mathbb{E}_v^{\star,\beta_n}\left(e^{-\tau_N^{(n)}F_{r'_n}^{(\beta_n)}(\frac{1}{2}N)}\right)\overline{z}_{r'_n,\beta_n,l}(u)\\
&\le \sup_{u\in[0,N]}\overline{z}_{r'_n,\beta_n,l}(u),
\end{aligned}\tag{6.18}$$

where in the first inequality we use the monotonicity of $\overline{z}_{r'_n,\beta_n,l}(u)$ in l. □

In Step 2 below, we derive a recursive upper bound for $K_l^{(n)}$. For $\varepsilon>0$, define

$$\eta_{k,N}^{(n)} = \sup_{v\in[0,N]}\mathbb{P}_v^{\star,\beta_n}(\xi_n>k) \tag{6.19}$$

$$C_{\varepsilon,k,N}^{(n)} = \sup_{v\in[0,N]}\mathbb{E}_v^{\star,\beta_n}\left(e^{-(1+\varepsilon)\int_0^k F_{r'_n}^{(\beta_n)}(X^\star_{\beta_n}(\sigma))\,d\sigma}\right). \tag{6.20}$$

STEP 2 *For all l,k,ε, $N>0$ and $n\in\mathbb{N}$,*

$$K_l^{(n)} \le C_{0,k,N}^{(n)} + K_l^{(n)}\left[C_{\varepsilon,k,N}^{(n)}\right]^{\frac{\varepsilon}{1+\varepsilon}}\left[\eta_{k,N}^{(n)}\right]^{\frac{1}{1+\varepsilon}}. \tag{6.21}$$

Proof. Use the monotonicity of $K_l^{(n)}$ in l, the Markov property at time k, and Hölder's inequality to obtain

$$
\begin{aligned}
K_l^{(n)} &\le K_{l+k}^{(n)} \\
&= \sup_{v\in[0,N]} \Big\{ \mathbb{E}_v^{\star,\beta_n} \left(e^{-\int_0^k F_{r_n'}^{(\beta_n)}(X^\star_{\beta_n}(\sigma))\,d\sigma} 1_{\{\xi_n \le k\}} \right) \\
&\quad + \int_{\mathbb{R}^+} d\tilde{v}\, \mathbb{E}_v^{\star,\beta_n} \left(e^{-\int_0^k F_{r_n'}^{(\beta_n)}(X^\star_{\beta_n}(\sigma))\,d\sigma} 1_{\{\xi_n > k\}} ; X^\star_{\beta_n}(k) \in d\tilde{v} \right) \overline{z}_{r_n',\beta_n,l}(\tilde{v}) \Big\} \\
&\le C_{0,k,N}^{(n)} + \sup_{v\in[0,N]} \mathbb{E}_v^{\star,\beta_n} \left(e^{-\int_0^k F_{r_n'}^{(\beta_n)}(X^\star_{\beta_n}(\sigma))\,d\sigma} 1_{\{\xi_n > k\}} \right) K_l^{(n)} \\
&\le C_{0,k,N}^{(n)} + K_l^{(n)} \left[\sup_{v\in[0,N]} \mathbb{E}_v^{\star,\beta_n} \left(e^{-(1+\varepsilon)\int_0^k F_{r_n'}^{(\beta_n)}(X^\star_{\beta_n}(\sigma))\,d\sigma} \right) \right]^{\frac{\varepsilon}{1+\varepsilon}} \\
&\qquad \times \left[\sup_{v\in[0,N]} \mathbb{P}_v^{\star,\beta_n}(\xi_n > k) \right]^{\frac{1}{1+\varepsilon}} \\
&= C_{0,k,N}^{(n)} + K_l^{(n)} \left[C_{\varepsilon,k,N}^{(n)} \right]^{\frac{\varepsilon}{1+\varepsilon}} \left[\eta_{k,N}^{(n)} \right]^{\frac{1}{1+\varepsilon}} .
\end{aligned}
\tag{6.22}
$$

□

STEP 3 *For any $N > 0$, $\limsup_{k\to\infty} \limsup_{n\to\infty} \eta_{k,N}^{(n)} = 0$.*

Proof. For all $n, k \in \mathbb{N}$,

$$
\begin{aligned}
\eta_{k,N}^{(n)} &= \sup_{v\in[0,N]} \left[1 - \mathbb{P}_v^{\star,\beta_n}(X^\star_{\beta_n}(k) = 0) \right] \\
&= \sup_{v\in[0,N]} \left[1 - \left(1 + \frac{1}{\lceil k\beta_n^{-\frac{1}{3}} \rceil} \right)^{-\lceil v\beta_n^{-\frac{1}{3}} \rceil} \right] \\
&= 1 - \left(1 + \frac{1}{\lceil k\beta_n^{-\frac{1}{3}} \rceil} \right)^{-\lceil N\beta_n^{-\frac{1}{3}} \rceil} .
\end{aligned}
\tag{6.23}
$$

The second equality is taken from Knight (1963) Theorem 1.2. Since $\beta \mapsto (1 + \frac{1}{k\beta})^{-N\beta}$ decreases towards $e^{-\frac{N}{k}}$, the claim follows. □

STEP 4 *For every $N > 0$ and any $\varepsilon \geq 0$ such that $a + \varepsilon < a_c$,*

$$\limsup_{k\to\infty} \limsup_{n\to\infty} C^{(n)}_{\varepsilon,k,N} < \infty. \tag{6.24}$$

Proof. For all k,

$$\limsup_{n\to\infty} C^{(n)}_{\varepsilon,k,N} \leq \sup_{v\in[0,N]} \mathbb{E}^\star_v \left(e^{-\int_0^k F_{(a+\varepsilon)}(X^\star_\sigma)\, d\sigma} \right) \tag{6.25}$$

by the following argument. Pick $v_n \in [0, N]$ to be the maximizer in (6.20). Choose a subsequence (n_l) along which v_{n_l} converges towards some $v \in [0, N]$ and

$$\limsup_{n\to\infty} C^{(n)}_{\varepsilon,k,N} = \lim_{l\to\infty} \mathbb{E}^{\star,\beta_{n_l}}_{v_{n_l}} \left(e^{-(1+\varepsilon)\int_0^k F^{(n_l)}_{a_{n_l}}(X^\star_{n_l}(\sigma))\, d\sigma} \right). \tag{6.26}$$

Then, by (3.27) and the convergence of $F^{(n_l)}_{a_{n_l}}$ towards F_a,

$$\limsup_{n\to\infty} C^{(n)}_{\varepsilon,k,N} \leq \mathbb{E}^\star_v \left(e^{-\int_0^k F_{a+\varepsilon}(X^\star(\sigma))\, d\sigma} \right). \tag{6.27}$$

The r.h.s. of (6.27) converges to $z_{a+\varepsilon}(v)$ as $k \to \infty$ (see HHK Lemmas 5 and 8). $\square$

6.3 Proof of Lemmas 5 and 6

STEP 5 *For all $r \in \mathbb{R}$, $\beta > 0$ and $k_1, k_2 \in \mathbb{N}_0$,*

$$z_{r,\beta}(k_1 + k_2) \leq z_{r,\beta}(k_1) z_{r,\beta}(k_2). \tag{6.28}$$

Proof. Let $\{m^\star_1(x)\}_{x\in\mathbb{N}_0}$ and $\{m^\star_2(x)\}_{x\in\mathbb{N}_0}$ be two independent copies of the Markov chain $\{m^\star(x)\}_{x\in\mathbb{N}_0}$ in (1.9). Then the distribution of $\{m^\star(x)\}_{x\in\mathbb{N}_0} = \{m^\star_1(x) + m^\star_2(x)\}_{x\in\mathbb{N}_0}$ is equal to $\mathbb{P}^\star_{k_1+k_2}$ (since they are branching processes). Now recall (1.18) and estimate

$$-\beta U^\star \leq -\beta \sum_{x\in\mathbb{N}_0} [m^\star_1(x) + m^\star_1(x-1)]^2 - \beta \sum_{x\in\mathbb{N}_0} [m^\star_2(x) + m^\star_2(x-1)]^2, \tag{6.29}$$

recall (1.19) and (3.22), and use the independence. $\square$

Let $a < a_c$ and choose some sequence $r'_n = \beta_n(a + o(1))$. First we derive a uniform bound on the function $\overline{z}_{r'_n,\beta_n}$ defined in (3.22).

STEP 6 $\sup_{n\in\mathbb{N}} \sup_{v\geq 0} \overline{z}_{r'_n,\beta_n}(v) < \infty$.

Proof. Pick N as in Step 1 and $\varepsilon > 0$ with $a + \varepsilon < a_c$. Then, according to Steps 3 and 4, for all sufficiently large k and sufficiently large n (depending on k) we may conclude from Step 2 that

$$K_l^{(n)} \le \frac{C_{0,k,N}^{(n)}}{1 - \left[C_{\varepsilon,k,N}^{(n)}\right]^{\frac{\varepsilon}{1+\varepsilon}} \left[\eta_{k,N}^{(n)}\right]^{\frac{1}{1+\varepsilon}}}. \tag{6.30}$$

Letting $l \to \infty$ and using the monotone convergence theorem, we obtain that $\sup_{v\ge 0} \overline{z}_{r'_n,\beta_n}(v)$ is bounded above by the r.h.s. of (6.30). Letting first $n \to \infty$ and then $k \to \infty$, we get the claim via Step 4. □

STEP 7 *Conclusion of the Proof of Lemma 5*

Proof. Pick some $\varepsilon > 0$ and choose $N > 0$ such that

$$z_a(N) \le 1 - 2\varepsilon. \tag{6.31}$$

Lemma 6 with $I = \mathbb{R}_0^+$ states

$$\lim_{n\to\infty} \overline{z}_{r'_n,\beta_n}(v) = z_a(v) \qquad (v \ge 0). \tag{6.32}$$

Consequently, for all sufficiently large $n \in \mathbb{N}$,

$$\overline{z}_{r'_n,\beta_n}(N) < 1 - \varepsilon. \tag{6.33}$$

Recall (3.22) and use Step 5 repeatedly to conclude that for all $v \ge 0$,

$$\overline{z}_{r'_n,\beta_n}(v) \le (1-\varepsilon)^{\lfloor \frac{v}{N} \rfloor} \sup_{u\ge 0} \overline{z}_{r'_n,\beta_n}(u). \tag{6.34}$$

Now the assertion follows with $q = (1-\varepsilon)^{\frac{1}{N}}$ and some C chosen according to Step 4. □

STEP 8 *Proof of Lemma 6.*

Proof. First we derive the distributional convergence of

$$D_{r'_n}^{(n)} = \exp\left(-\int_0^\infty F_{r'_n}^{(\beta_n)}(X_{\beta_n}^\star(\sigma))\, d\sigma\right) 1_{\{\int_{\mathbb{R}_0^+} X_{\beta_n}^\star(\sigma)\, d\sigma \in I\}} \tag{6.35}$$

under $\mathbb{P}_v^{\star,\beta_n}$ where $I \subset \mathbb{R}_0^+$ is any interval. We do this for $I = \mathbb{R}_0^+$ only; the general case is similar. To this end, pick $\delta > 0$ and choose k so large such that $\mathbb{P}_v^\star(\xi > k) \le \delta$ (where $\xi = \inf\{t > 0 : X^\star(t) = 0\}$ denotes the absorption time of $X^\star$) and such that

$\mathbb{P}_v^{\star,\beta_n}(\xi_n > k) \le \delta$ for all $n \in \mathbb{N}$ (this is possible by Step 3). Then, for every $\gamma \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{P}_v^{\star,\beta_n}\left(\int_0^\infty F_{r'_n}^{(\beta_n)}(X^\star_{\beta_n}(\sigma))\,d\sigma \ge \gamma\right)\\
&\quad\le \limsup_{n\to\infty}\left\{\mathbb{P}_v^{\star,\beta_n}\left(\int_0^k F_{r'_n}^{(\beta_n)}(X^\star_{\beta_n}(\sigma))\,d\sigma \ge \gamma\right) + \mathbb{P}_v^{\star,\beta_n}(\xi_n > k)\right\}\\
&\quad\le \mathbb{P}_v^\star\left(\int_0^k F_a(X^\star(\sigma))\,d\sigma \ge \gamma\right) + \delta\\
&\quad\le \mathbb{P}_v^\star\left(\int_0^\infty F_a(X^\star(\sigma))\,d\sigma \ge \gamma\right) + 2\delta.
\end{aligned} \tag{6.36}$$

Let $\delta \downarrow 0$ to get the upper bound. The lower bound is derived in a similar way.

Thus, we have derived the distributional convergence of $D_{r'_n}^{(n)}$ under $\mathbb{P}_v^{\star,\beta_n}$ towards that of $\exp(-\int_0^\infty F_a(X^\star(\sigma))\,d\sigma)1_{\{\int_{\mathbb{R}_0^+} X^\star(\sigma)\,d\sigma \in I\}}$ under $\mathbb{P}_v^\star$. In order to derive the convergence in the L^1-norm, it is sufficient to prove uniform integrability of the sequence $(D_{r'_n}^{(n)})$. This is done by simply noting that $(D_{r'_n}^{(n)})^{1+\varepsilon} \le D_{b_n}^{(n)}$ with $b_n = (1+\varepsilon)r'_n = (1+\varepsilon)a\beta_n^{\frac{2}{3}}$ and by applying Step 6 to b_n instead of r'_n for some $\varepsilon > 0$ sufficiently small.

Summarizing, we have proved the pointwise convergence in (3.30). Using the bound in Lemma 5, we conclude that Lemma 6 holds, with the help of the dominated convergence theorem. □

7 Proof of Theorem 4

In this section, we prove Theorem 4 subject to Proposition 3. Fix $a \in \mathbb{R}$. In the sequel, when $r = a\beta^{\frac{2}{3}}$, we shall only indicate the β-dependence and shall write $\tau_\beta = \tau_{a\beta^{\frac{2}{3}},\beta}$, $A_\beta = A_{a\beta^{\frac{2}{3}},\beta}$ and $\lambda(\beta) = \lambda(a\beta^{\frac{2}{3}},\beta)$ and so on. Recall Subsections 3.1-3.2. We shall make repeated use of the scaling notation (3.13). The proof of Theorem 4 is divided into four steps.

STEP 1 *For all* $\beta \in \mathbb{R}^+$,

$$\beta^{-\frac{1}{3}}\partial_a^2\lambda(\beta) = \beta^{-\frac{1}{3}}[\partial_a\lambda(\beta)]^2 + \langle \partial_a\overline{\tau}_\beta, \overline{g}_\beta\rangle_{L^2}, \tag{7.1}$$

where

$$g_\beta(i) = \beta^{\frac{1}{3}}(2i-1)\tau_\beta(i) \qquad (i \in \mathbb{N}). \tag{7.2}$$

Proof. Differentiate (3.2) for $l = 0$ w.r.t. a. $\square$

STEP 2 $\limsup_{\beta \downarrow 0} \|\partial_a \overline{\tau}_\beta\|_{L^2} < \infty$.

Proof. By differentiating the relation $\tau_\beta = \frac{1}{\lambda(\beta)} A_\beta \tau_\beta$ componentwise w.r.t. a, we have

$$\partial_a \tau_\beta(i) = -\frac{\partial_a \lambda(\beta)}{\lambda(\beta)^2} \tau_\beta(i) + \frac{\beta^{\frac{1}{3}} h_\beta(i)}{\lambda(\beta)} + \frac{(A_\beta \partial_a \tau_\beta)(i)}{\lambda(\beta)}, \tag{7.3}$$

where

$$h_\beta(i) = \beta^{\frac{1}{3}} \left((\partial_a A_\beta) \tau_\beta \right)(i) \qquad (i \in \mathbb{N}). \tag{7.4}$$

Multiply (7.3) by $\beta^{-\frac{1}{3}} \partial_a \overline{\tau}_\beta(i)$, sum over $i \in \mathbb{N}$ and use the notation (8.5) below and the fact that $\langle \partial_a \overline{\tau}_\beta, \overline{\tau}_\beta \rangle_{L^2} = 0$, to obtain

$$F_\beta(\partial_a \overline{\tau}_\beta) = \beta^{-\frac{1}{3}} [\lambda(\beta) - 1] \|\partial_a \overline{\tau}_\beta\|_{L^2}^2 - \langle \overline{h}_\beta, \partial_a \overline{\tau}_\beta \rangle_{L^2}, \tag{7.5}$$

Use (8.12) below for $y = \partial_a \overline{\tau}_\beta / \|\partial_a \overline{\tau}_\beta\|_{L^2}$ and note that F_β is homogeneous of order two and that $\langle \partial_a \overline{\tau}_\beta, \overline{\tau}_\beta \rangle_{L^2} = 0$, to get

$$F_\beta(\partial_a \overline{\tau}_\beta) \le \beta^{-\frac{1}{3}} [\lambda^{(1)}(\beta) - 1] \|\partial_a \overline{\tau}_\beta\|_{L^2}^2. \tag{7.6}$$

Combine this with (7.5) and use the Cauchy-Schwarz inequality to obtain

$$\|\partial_a \overline{\tau}_\beta\|_{L^2} \le \frac{\|\overline{h}_\beta\|_{L^2}}{\beta^{-\frac{1}{3}} [\lambda(\beta) - \lambda^{(1)}(\beta)]}. \tag{7.7}$$

Now use that $\limsup_{\beta \downarrow 0} \|\overline{h}_\beta\|_{L^2} < \infty$ (see HH Lemma 11(i)) and that $\beta^{-\frac{1}{3}}(\lambda(\beta) - \lambda^{(1)}(\beta)) \to \rho(a) - \rho^{(1)}(a) > 0$ by Proposition 3(i) to get the claim. $\square$

STEP 3 *$\partial_a \overline{\tau}_\beta$ converges to $\partial_a x_a$ weakly in L^2 as $\beta \downarrow 0$.*

Proof. By Step 2, every subsequence of $\{\partial_a \overline{\tau}_\beta\}_{\beta > 0}$ has a further subsequence that converges weakly in L^2 as $\beta \downarrow 0$. Denote the weak limit along such a subsequence by y_a. We shall prove that $y_a = \partial_a x_a$ independently of the subsequence involved. By (3.1), it suffices to prove for all $l \in \mathbb{N}_0$ that

$$\langle y_a, x_a^{(l)} \rangle_{L^2} = \langle \partial_a x_a, x_a^{(l)} \rangle_{L^2}. \tag{7.8}$$

This is easily derived for $l = 0$, since

$$\langle \partial_a x_a, x_a^{(0)} \rangle_{L^2} = 0 = \langle \partial_a \overline{\tau}_\beta, \overline{\tau}_\beta^{(0)} \rangle_{L^2} \to \langle y_a, x_a^{(0)} \rangle_{L^2} \tag{7.9}$$

along this subsequence since $\overline{\tau}_\beta^{(0)} \to^{L^2} x_a$ according to HH Theorem 5.

In order to derive (7.8) for $l \geq 1$, differentiate the relation $\mathcal{L}^a x_a = \rho(a) x_a$ w.r.t. a to get with $f_a(u) = 2u x_a(u)$

$$f_a + \mathcal{L}^a \partial_a x_a = \rho(a)\partial_a x_a + \rho'(a) x_a. \tag{7.10}$$

Now, take the inner product with $x_a^{(l)}$ and use the L^2-symmetry of $\mathcal{L}^a$, $\langle x_a, x_a^{(l)}\rangle_{L^2} = 0$ and the eigenvalue relation $\mathcal{L}^a x_a^{(l)} = \rho^{(l)}(a) x_a^{(l)}$ to get for $l \geq 1$

$$\langle \partial_a x_a, x_a^{(l)}\rangle_{L^2} = \frac{\langle f_a, x_a^{(l)}\rangle_{L^2}}{\rho(a) - \rho^{(l)}(a)}. \tag{7.11}$$

By Proposition 3(i), $\overline{\tau}_\beta^{(l)}$ converges converges strongly to $x_a^{(l)}$ and therefore

$$\langle \partial_a \overline{\tau}_\beta, \overline{\tau}_\beta^{(l)}\rangle_{L^2} \to \langle y_a, x_a^{(l)}\rangle_{L^2} \tag{7.12}$$

along the subsequence. To investigate the l.h.s. of (7.12) for $l \geq 1$, multiply (7.3) by $\tau_\beta^{(l)}(i)$ and sum over $i \in \mathbb{N}$, to get

$$\langle \partial_a \overline{\tau}_\beta, \overline{\tau}_\beta^{(l)}\rangle_{L^2} = \lambda^{(l)}(\beta) \frac{\langle \overline{h}_\beta, \overline{\tau}_\beta^{(l)}\rangle_{L^2}}{\beta^{-\frac{1}{3}}(\lambda(\beta) - \lambda^{(l)}(\beta))} \tag{7.13}$$

(recall (7.4)). Now use Proposition 3(i) and the fact that $\overline{h}_\beta \to^{L^2} f_a$ to see that

$$\langle \partial_a \overline{\tau}_\beta, \overline{\tau}_\beta^{(l)}\rangle_{L^2} \to \text{r.h.s. of (7.11)}. \tag{7.14}$$

Finally, combine (7.11), (7.12) and (7.14) to arrive at (7.8) for $l \geq 1$. $\Box$

STEP 4 *Conclusion of the proof of Theorem 4.*

Proof. Use (0.14-0.15) to see that the first summand in the r.h.s. of (7.1) vanishes as $\beta \downarrow 0$. Since $\overline{g}_\beta \to^{L^2} f_a$, we therefore conclude from Step 1 that

$$\lim_{\beta \downarrow 0} \beta^{-\frac{1}{3}} \partial_a^2 \lambda(\beta) = \langle \partial_a x_a, f_a\rangle_{L^2}. \tag{7.15}$$

Differentiate (3.5) w.r.t. a to see that the r.h.s. of (7.15) is equal to $\rho''(a)$. $\Box$

8 Proof of Proposition 3

In this section we prove Proposition 3. In Subsection 8.1 we shall introduce the notion of epi-convergence. In Subsection 8.2 we use the Rayleigh formula to derive a variational representation for the spectral gap of $A_{a\beta^{\frac{2}{3}},\beta}$. In Subsection 8.3 we shall use the notion of epi-convergence to prove the convergence of eigenvalues and eigenfunctions as stated in Proposition 3(i). In Subsection 8.4 we shall prove uniform convergence of the scaled largest eigenvector as stated in Proposition 3(ii).

8.1 Epi-convergence

We start with the notion of epi-convergence. Let (X,τ) be a metrizable topological space and let $Y \subset X$ be dense in X. Let

$$\begin{array}{lll} G_\beta & : X \to \mathbb{R} & (\beta \in \mathbb{R}^+) \\ G & : X \to \overline{\mathbb{R}}. & \end{array} \tag{8.1}$$

Definition 1 *The family $(G_\beta)_{\beta\in\mathbb{R}^+}$ is said to be epi-convergent to G on Y, written*

$$e - \lim_{\beta\downarrow 0} G_\beta = G \text{ on } Y, \tag{8.2}$$

if the following properties hold:

$$\begin{array}{ll} (i) & \forall x_\beta \to^\tau x \text{ in } Y : \limsup_{\beta\downarrow 0} G_\beta(x_\beta) \le G(x) \\ (ii) & \exists x_\beta \to^\tau x \text{ in } Y : \liminf_{\beta\downarrow 0} G_\beta(x_\beta) \ge G(x). \end{array} \tag{8.3}$$

The importance of the notion of epi-convergence is contained in the following proposition.

Proposition 4 *Suppose that*

$(A1)$ $\quad e - \lim_{\beta\downarrow 0} G_\beta = G$ *on* Y

$(A2)$ $\quad G_\beta$ *has a maximizer* $x_\beta^* \in X$

$(A3)$ $\quad \exists K \subset Y$ *such that*

(i) $\quad K$ *is τ-relatively compact in* X

(ii) $\quad G$ *has a unique maximizer* $x^* \in \overline{K}$

(iii) $\quad \exists (x_\beta)_{\beta\in\mathbb{R}^+} \subset \overline{K}$ *such that* $x_\beta - x_\beta^* \to^\tau 0$ *and* $G_\beta(x_\beta) - G_\beta(x_\beta^*) \to 0$.

Then as $\beta \downarrow 0$

$$\sup_{x\in X} G_\beta(x) \to \sup_{x\in X} G(x) \qquad \text{and} \qquad x_\beta^* \to^\tau x^*. \tag{8.4}$$

Proof. See Attouch (1984) Theorem 1.10 and Proposition 1.14. □

8.2 Proof of Proposition 3(i): variational representations

In this subsection we derive a variational formula for the spectral gap of $A_{a\beta^{\frac{2}{3}},\beta}$. As in Section 7, we suppress the dependence on a in the notations of various objects we are dealing with. Fix $\beta \in \mathbb{R}^+$ so small that $\lambda^{(1)}(\beta) > 0$ (see below (3.2)).

Rayleigh's formula for $(\lambda^{(0)}(\beta), \overline{\tau}_\beta^{(0)})$ resp. $(\lambda^{(1)}(\beta), \overline{\tau}_\beta^{(1)})$ reads as follows. Define the functional $F_\beta : L^2 \to \overline{\mathbb{R}}$ as

$$F_\beta(x) = \beta^{-\frac{2}{3}} \int_0^\infty du \int_0^\infty dv \; x(u)x(v)A_\beta\Big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\Big) - \beta^{-\frac{1}{3}}\|x\|_{L^2}^2. \tag{8.5}$$

Lemma 12

$$(i) \quad \beta^{-\frac{1}{3}}\Big[\lambda^{(0)}(\beta)-1\Big] = \max_{x\in L^2, x\geq 0, \|x\|_{L^2}=1} F_\beta(x)$$

$$\overline{\tau}_\beta^{(0)} \text{ is the unique maximizer} \tag{8.6}$$

$$(ii) \quad \beta^{-\frac{1}{3}}\Big[\lambda^{(1)}(\beta)-1\Big] = \max_{x\in L^2, \|x\|_{L^2}=1, \langle x, \overline{\tau}_\beta^{(0)}\rangle_{L^2}=0} F_\beta(x)$$

$$\overline{\tau}_\beta^{(1)} \text{ is a maximizer.} \tag{8.7}$$

Proof. For (i), see HH Lemma 1. For (ii), note that by the positivity of $\lambda^{(1)}(\beta)$ (see footnote 6) and Rayleigh's formula, we have

$$\lambda^{(1)}(\beta) = \max_{x\in l^2, \|x\|_{l^2}=1, \langle x, \tau_\beta^{(0)}\rangle_{l^2}=0} \langle x, A_\beta x\rangle_{l^2}. \tag{8.8}$$

Now see HH Proof of Lemma 1. □

Proposition 3(i) for $l=0$ was proved in HH by applying Proposition 4 to the representation in Lemma 12(i) for the following choices:

$$\begin{aligned} X &= \{x\in L^2 : x\geq 0, \|x\|_{L^2}=1\} \\ Y &= X\cap C^1(\mathbb{R}_0^+) \\ \tau &= \text{ topology induced by } \|\cdot\|_{L^2} \\ G_\beta &= F_\beta \\ G &= F \\ K &= K_C = \{x\in Y : F(x)\geq -C\} \text{ for some } C \text{ large enough,} \end{aligned} \tag{8.9}$$

where $F: L^2\to\overline{\mathbb{R}}$ is the functional defined as

$$F(x) = \int_0^\infty \Big\{(2au-4u^2)[x(u)]^2 - u[x'(u)]^2\Big\}du \tag{8.10}$$

(with the understanding that $F(x)=-\infty$ if the integral is not defined). The link between F and the differential operator $\mathcal{L}^a$ defined in (0.13) is that for $x\in C^2(\mathbb{R}_0^+)\cap L^2(\mathbb{R}_0^+)$ satisfying $\lim_{u\to\infty} ux'(u)x(u)=0$,

$$F(x) = \langle x, \mathcal{L}^a x\rangle_{L^2}. \tag{8.11}$$

We now want to follow the same scheme for the representation in Lemma 12(ii). In order to prepare for this, we first rewrite the maximum in (8.7) in such a way as to remove the β-dependence from the set over which the maximum is taken. After that we subtract the maximum in (8.6) to get the spectral gap.

Lemma 13

$$-\beta^{-\frac{1}{3}}\Big[\lambda^{(0)}(\beta)-\lambda^{(1)}(\beta)\Big] = \max_{y\in L^2, \|y\|_{L^2}=1} \frac{F_\beta(y)-\beta^{-\frac{1}{3}}\Big[\lambda^{(0)}(\beta)-1\Big]}{1-\langle y,\overline{\tau}^{(0)}_\beta\rangle^2_{L^2}} \tag{8.12}$$

$\overline{\tau}^{(1)}_\beta$ *is a maximizer.*

Proof. First, since F_β is quadratic and $x\mapsto x-\langle x,\overline{\tau}^{(0)}_\beta\rangle_{L^2}\overline{\tau}^{(0)}_\beta$ is surjective from L^2 to $\{x\in L^2 : \langle x,\overline{\tau}^{(0)}_\beta\rangle_{L^2}=0\}$, we may write

$$\text{r.h.s. of (8.7)} = \max_{y\in L^2, \|y\|_{L^2}=1} \frac{F_\beta\Big(y-\langle y,\overline{\tau}^{(0)}_\beta\rangle_{L^2}\overline{\tau}^{(0)}_\beta\Big)}{1-\langle y,\overline{\tau}^{(0)}_\beta\rangle^2_{L^2}} \tag{8.13}$$

(where the functional is defined to be $-\infty$ when $y=\overline{\tau}^{(0)}_\beta$). Next, define the bilinear form

$$F^{(2)}_\beta(x,y) = \beta^{-\frac{2}{3}}\int_0^\infty du\int_0^\infty dv\ x(u)y(v)A_\beta\Big(\lceil u\beta^{-\frac{1}{3}}\rceil,\lceil v\beta^{-\frac{1}{3}}\rceil\Big)-\beta^{-\frac{1}{3}}\langle x,y\rangle_{L^2}. \tag{8.14}$$

Note that $F^{(2)}_\beta(x,x)=F_\beta(x)$. Moreover,

$$F^{(2)}_\beta(x,\overline{\tau}^{(0)}_\beta) = \beta^{-\frac{1}{3}}\Big[\lambda^{(0)}(\beta)-1\Big]\langle x,\overline{\tau}^{(0)}_\beta\rangle_{L^2} \tag{8.15}$$

because $\overline{\tau}^{(0)}_\beta$ is the scaled eigenvector of A_β associated with $\lambda(\beta)$. Hence

$$\begin{aligned} &F_\beta\left(y-\langle y,\overline{\tau}^{(0)}_\beta\rangle_{L^2}\overline{\tau}^{(0)}_\beta\right) \\ &= F^{(2)}_\beta(y,y)-2\langle y,\overline{\tau}^{(0)}_\beta\rangle_{L^2}F^{(2)}_\beta(y,\overline{\tau}^{(0)}_\beta)+\langle y,\overline{\tau}^{(0)}_\beta\rangle^2_{L^2}F^{(2)}_\beta(\overline{\tau}^{(0)}_\beta,\overline{\tau}^{(0)}_\beta). \end{aligned} \tag{8.16}$$

Combine Lemma 12 with (8.13) and (8.15-8.16) to get the claim. □

Note that the numerator in the r.h.s. of (8.12) is negative for any $y\neq\overline{\tau}^{(0)}_\beta$ with $\|y\|_{L^2}=1$ by Lemma 12(i), and that the denominator is maximal for $y=\overline{\tau}^{(1)}_\beta$ because $\langle\overline{\tau}^{(1)}_\beta,\overline{\tau}^{(0)}_\beta\rangle_{L^2}=0$.

8.3 Proof of Proposition 3(i): convergence of $\overline{\tau}_\beta^{(l)}$ and $\lambda^{(l)}(\beta)$

In this subsection we use the variational representation in Lemma 13 to prove Proposition 3(i) for $l = 1$ following the patterns of the proof of Proposition 3(i) for $l = 0$, given in HH. The proof for general l can then easily be completed using induction on l, as is explained in the end of this subsection.

In what follows convergence is studied for $\beta \downarrow 0$ and fixed $a \in \mathbb{R}$. However, all arguments remain valid when a is replaced by $a(\beta)$ with $\lim_{\beta\downarrow 0} a(\beta) = a$, i.e., the convergence is uniform for a in compacts. As in Section 7, a is suppressed from the notation when $r = a\beta^{\frac{2}{3}}$ and we only indicate the β-dependence.

We shall apply Proposition 4 to the maximum in the r.h.s. of (8.12), this time with the following choices:

$$
\begin{aligned}
X &= \{x \in L^2 : \|x\|_{L^2} = 1\} \\
Y &= X \cap C^1(\mathbb{R}_0^+) \\
\tau &= \text{ topology induced by } \|\cdot\|_{L^2} \\
G_\beta(x) &= \frac{F_\beta(x) - \beta^{-\frac{1}{3}}\left[\lambda^{(0)}(\beta) - 1\right]}{1 - \langle x, \overline{\tau}_\beta^{(0)}\rangle_{L^2}^2} \\
G(x) &= \frac{F(x) - \rho^{(0)}(a)}{1 - \langle x, x_a^{(0)}\rangle_{L^2}^2} \\
K &= K_C = \{x \in Y : F(x) \geq -C\} \text{ for some } C \text{ large enough.}
\end{aligned}
\tag{8.17}
$$

STEP 1 *If assumptions (A1-A3) in Proposition 4 hold for the choice in (8.17), then Proposition 3(i) for $l = 1$ follows.*

Proof. Proposition 4 then implies that as $\beta \downarrow 0$

$$
\begin{aligned}
\beta^{-\frac{1}{3}}\left[\lambda^{(0)}(\beta) - \lambda^{(1)}(\beta)\right] &\to \max_{x\in X} G(x) \\
\overline{\tau}_\beta^{(1)} &\to^{L^2} \text{ unique maximizer of } G.
\end{aligned}
\tag{8.18}
$$

Repeat the argument in the proof of Lemma 13 to see that

$$
\begin{aligned}
&\rho^{(1)}(a) = \max_{x\in X, \langle x, x_a^{(0)}\rangle_{L^2}=0} F(x) = \rho^{(0)}(a) + \max_{x\in X} G(x) \\
&\text{with unique maximizer } x_a^{(1)}
\end{aligned}
\tag{8.19}
$$

(recall Subsection 3.1 for the definition of $x_a^{(1)}$). This completes the proof of the scaling of $\lambda^{(1)}(\beta)$ and the L^2-convergence of $\overline{\tau}_\beta^{(1)}$. The $L^{2,\circ}$-convergence of $\overline{\tau}_\beta^{(1)}$ follows from the L^2-convergence and Lemma 15(i) below. □

STEP 2 *Proof of (A1-A3) for the choice in (8.17).*

Proof. Proof of (A1)

We know from HH Lemmas 5-8 that $e - \lim_{\beta \downarrow 0} F_\beta = F$. Moreover, $\overline{\tau}_\beta^{(0)} \to^{L^2} x_a^{(0)}$ by Proposition 3 for $l = 0$. Hence, for all $x_\beta \to^{L^2} x$ we have

$$\langle x_\beta, \overline{\tau}_\beta^{(0)} \rangle_{L^2} \;\to\; \langle x, x_a^{(0)} \rangle_{L^2}. \tag{8.20}$$

Since $\beta^{-\frac{1}{3}}\left[\lambda^{(0)}(\beta) - 1\right] \to \rho^{(0)}(a)$ by Proposition 3(i) for $l = 0$, the claim follows.

Proof of (A2)

See Lemma 12(ii).

Proof of (A3)

(A3)(i) is proved in HH Lemma 17.

(A3)(ii) follows from (8.19).

The proof of (A3)(iii) requires a minor adaptation of the proof of the corresponding statement for $\overline{\tau}_\beta^{(0)}$ in HH Lemma 9-11. The point is to construct a relatively compact sequence of approximate maximizers of G_β approximating $\overline{\tau}_\beta^{(1)}$ in L^2. For this sequence we shall pick the following linear and renormalized interpolation of $\tau_\beta^{(1)}$. For sequences $\{\tau(i)\}_{i \in \mathbb{N}}$ introduce the notation

$$\Delta\tau(i) = \tau(i+1) - \tau(i) \qquad (i \in \mathbb{N}) \tag{8.21}$$

and define

$$\begin{aligned} \tilde{\tau}_\beta^{(1)} &= \hat{\tau}_\beta^{(1)} \|\hat{\tau}_\beta^{(1)}\|_{L^2}^{-1} \\ \hat{\tau}_\beta^{(1)}(u) &= \overline{\tau}_\beta^{(1)}(u) + \beta^{-\frac{1}{6}}(u\beta^{-\frac{1}{3}} - i)\Delta\tau_\beta^{(1)}(i-1) \\ & \qquad\qquad (i - 1 < u\beta^{-\frac{1}{3}} \le i, i \in \mathbb{N}) \end{aligned} \tag{8.22}$$

(put $\tau_\beta^{(1)}(0) = \tau_\beta^{(1)}(1)$ and compare with HH Eq. (3.3)). We see from (8.17) and (8.20) that (A3)(iii) is implied by the following lemma.

Lemma 14

$$\begin{array}{ll} (i) & \tilde{\tau}_\beta^{(1)} - \overline{\tau}_\beta^{(1)} \to^{L^2} 0 \;\; as\; \beta \downarrow 0 \\ (ii) & F_\beta(\overline{\tau}_\beta^{(1)}) - F_\beta(\tilde{\tau}_\beta^{(1)}) \to 0 \;\; as\; \beta \downarrow 0 \\ (iii) & \liminf_{\beta \downarrow 0} F(\tilde{\tau}_\beta^{(1)}) > -\infty. \end{array}$$

Proof. The estimates in HH Lemmas 9-10 for $(\lambda^{(0)}(\beta), \overline{\tau}_\beta^{(0)})$ carry over to $(\lambda^{(1)}(\beta), \overline{\tau}_\beta^{(1)})$ because they only use the eigenvalue/eigenvector relation. Hence, Lemma 14 will be proved once we check that the following estimates in HH Lemma 11 carry over as well. In the following, we use C as a generic positive constant.

Lemma 15 *For small $\beta > 0$,*

$$\begin{array}{llll} (i) & \sum_{i\in\mathbb{N}} i^2[\tau_\beta^{(1)}(i)]^2 \le C\beta^{-\frac{2}{3}}, & (ii) & \sum_{i\in\mathbb{N}} i\Delta[\tau_\beta^{(1)}(i)]^2 \le C\beta^{\frac{1}{3}}, \\ (iii) & \tau_\beta^{(1)}(0)^2 \le C\beta^{\frac{1}{3}}\log\frac{1}{\beta}, & (iv) & \|\Delta\tau_\beta^{(1)}\|_{l^2}^2 \le C\beta^{\frac{2}{3}}\log\frac{1}{\beta}. \end{array}$$

Proof. (i) We give here a shorter proof as for Lemma 11(i) in HH. First note that

$$\frac{1}{C} \le \beta^{-\frac{1}{3}}\left[\lambda^{(1)}(\beta) - 1\right] \le C \text{ for all } \beta \text{ small enough.} \tag{8.23}$$

Indeed, the upper bound is implied by $\lambda^{(1)}(\beta) \le \lambda^{(0)}(\beta)$ and Proposition 3(i) for $l = 0$, the lower bound can be obtained from Lemma 13 by an approximate testfunction, namely $x = x_a^{(1)} - \langle x_a^{(1)}, \overline{\tau}_\beta^{(0)}\rangle_{L^2}\overline{\tau}_\beta^{(0)}$, using (8.15) and HH Lemma 5-8.

Fix $\beta > 0$ so small that $\lambda^{(1)}(\beta) > 0$. Use the eigenvector property of $\tau_\beta^{(1)}$ and use $1 + \frac{1}{2}\beta i^2 \le e^{\frac{1}{2}\beta(i+j-1)^2}$ for $i, j \in \mathbb{N}$ to estimate

$$\begin{aligned} \lambda^{(1)}(\beta)\left(1 + \frac{\beta}{2}\sum_{i\in\mathbb{N}} i^2\tau_\beta^{(1)}(i)^2\right) &= \sum_{i,j\in\mathbb{N}} \left(1 + \frac{\beta}{2}i^2\right)\tau_\beta^{(1)}(i)A_{a\beta^{\frac{2}{3}},\beta}(i,j)\tau_\beta^{(1)}(j) \\ \le \sum_{i,j\in\mathbb{N}} |\tau_\beta^{(1)}(i)|A_{a\beta^{\frac{2}{3}},\frac{1}{2}\beta}(i,j)|\tau_\beta^{(1)}(j)| &\le \lambda(a\beta^{\frac{2}{3}}, \tfrac{1}{2}\beta), \end{aligned} \tag{8.24}$$

where the last inequality uses the Rayleigh formula.

Now subtract $\lambda^{(1)}(\beta)$ on both sides of (8.24) and divide by $\frac{1}{2}\beta^{\frac{1}{3}}\lambda^{(1)}(\beta)$ to arrive at

$$\beta^{\frac{2}{3}}\sum_{i\in\mathbb{N}} i^2\tau_\beta^{(1)}(i)^2 \le \frac{2}{\lambda^{(1)}(\beta)}\beta^{-\frac{1}{3}}\left[\lambda(a\beta^{\frac{2}{3}}, \frac{1}{2}\beta) - \lambda^{(1)}(\beta)\right]. \tag{8.25}$$

Use Proposition 3(i) for $l = 0$ and (8.23) to see that the r.h.s. of (8.25) is bounded as $\beta \downarrow 0$.

(ii) In the proof of HH Lemma 11(ii), Step 4 is an equality for $\tau_\beta^{(1)}$ and Step 5 should be proved with absolute value signs (since $\tau_\beta^{(1)}$ is not nonnegative). This causes no further problems.

(iii) The proof of HH Lemma 11(iii) only uses HH Lemma 11(i-ii) and therefore remains valid with the help of Lemma 15 (i-ii).

(iv) In the proof of HH Lemma 11(iv), Step 7 is again an equality. In Step 8, again absolute value signs have to be introduced.

This completes the proof of Lemma 15. □

Lemma 15 completes the proof of Lemma 14. $\square$

Lemma 14 finishes Step 2. $\square$

STEP 3 *Proof of Proposition 3(i) for $l \geq 2$.*

Proof. The extension of the proof to $l \geq 2$ is made via induction on l. We describe the main line, the details are left to the reader.

Indeed, using the Rayleigh representation of $\lambda^{(l)}(\beta)$ for $l \geq 2$ gives

$$\beta^{-\frac{1}{3}}\Big[\lambda^{(l)}(\beta)-1\Big] = \max_{\substack{x\in L^2, \|x\|_{L^2}=1 \\ \langle x, \overline{\tau}_\beta^{(j)}\rangle_{L^2}=0, j=0,\dots,l-1}} F_\beta(x). \tag{8.26}$$

The β-dependence of the set can be removed as in Lemma 13. Using the induction hypothesis and the bounds in Lemma 15 for $\overline{\tau}_\beta^{(l)}$ instead of $\tau_\beta^{(1)}$ then allows to deduce the assertion for l from the one for $l-1$ as we derived the one for $l=1$ from the one for $l=0$ in Steps 1-2. $\square$

8.4 Proof of Proposition 3(ii): uniform convergence of $\overline{\tau_\beta}$

In this subsection we prove uniform convergence of $\overline{\tau}_\beta$ by applying the Arzela-Ascoli theorem. The proof is divided into five steps. Fix $a \in \mathbb{R}$ satisfying $|\rho(a)| < 1$ and recall (8.21).

STEP 1 $\overline{\tau}_\beta(0) \leq 1 + \beta^{-\frac{1}{3}}\|\Delta\tau_\beta\|_{l^2}$.

Proof. Pick $k \in \mathbb{N}, k \leq \beta^{-\frac{1}{3}}$ such that $\tau_\beta(k) \leq \beta^{\frac{1}{6}}$. This is possible, since $\|\tau_\beta\|_{l^2} = 1$. Write

$$\overline{\tau}_\beta(0) = \beta^{-\frac{1}{6}}\tau_\beta(1) = \beta^{-\frac{1}{6}}\tau_\beta(k) - \beta^{-\frac{1}{6}}\sum_{i=1}^{k}\Delta\tau_\beta(i) \leq 1 + \beta^{-\frac{1}{6}}\sum_{i=1}^{k}|\Delta\tau_\beta(i)|. \tag{8.27}$$

Now use the Cauchy-Schwarz inequality. $\square$

STEP 2 $\limsup_{\beta\downarrow 0}\overline{\tau}_\beta(0) < \infty$.

Proof. HH Eq. (3.35) says that

$$\begin{aligned}\|\Delta\tau_\beta\|_{l^2}^2 \quad &\leq \quad \tfrac{2}{\lambda(\beta)}\sum_{(i,j)\in\mathbb{N}^2\setminus\{(1,1)\}}\Big[1-e^{e_\beta(i-1,j)-e_\beta(i,j)}\Big]\,\tau_\beta(i)A_\beta(i,j)\tau_\beta(j) \\ &\quad -\tau_\beta^2(1)\left[1-\tfrac{2}{\lambda(\beta)}A_\beta(1,1)\right],\end{aligned} \tag{8.28}$$

where e_β is the exponent in (0.5)

$$e_\beta(i,j) = a\beta^{\frac{2}{3}}(i+j-1) - \beta(i+j-1)^2. \tag{8.29}$$

Use $1 - e^t \le t$ for all $t \in \mathbb{R}$ and $e_\beta(i-1,j) - e_\beta(i,j) \ge -a\beta^{\frac{2}{3}}$ and (0.5-0.6) and (0.14) to estimate in (8.28)

$$\|\Delta\tau_\beta\|_{l^2}^2 \le 2|a|\beta^{\frac{2}{3}} - \tau_\beta^2(1)\frac{\lambda(\beta) - e^{a\beta^{\frac{2}{3}}-\beta}}{\lambda(\beta)} = \beta^{\frac{2}{3}}\left(2|a| - \overline{\tau}_\beta^2(0)\rho(a)(1+o(1))\right). \tag{8.30}$$

Substitute this in Step 1 and use the triangle inequality to obtain

$$\overline{\tau}_\beta(0) \le 1 + \sqrt{2|a|} + \overline{\tau}_\beta(0)\sqrt{|\rho(a)|}(1+o(1)). \tag{8.31}$$

Now use that $|\rho(a)| < 1$ to get the claim. □

STEP 3 $\limsup_{\beta\downarrow 0}\beta^{-\frac{1}{3}}\|\Delta\tau_\beta\|_{l^2} < \infty$.

Proof. This is an easy consequence of (8.30) and Step 2. □

STEP 4 *As $\beta \downarrow 0$, $\overline{\tau}_\beta$ converges uniformly to x_a on $[0,N]$ for all $N > 0$.*

Proof. Define $\widehat{\tau}_\beta : \mathbb{R}_0^+ \to \mathbb{R}^+$ to be the scaled linear interpolation of $\{\tau_\beta(i)\}_{i\in\mathbb{N}}$, i.e.,

$$\widehat{\tau}_\beta(u) = \overline{\tau}_\beta(u) + \beta^{-\frac{1}{6}}(u\beta^{-\frac{1}{3}} - i)\Delta\tau_\beta(i-1) \qquad (i-1 < u\beta^{-\frac{1}{3}} \le i, i \in \mathbb{N}). \tag{8.32}$$

With the help of Step 3, one obtains that

$$\|\widehat{\tau}_\beta - \overline{\tau}_\beta\|_\infty \le \beta^{-\frac{1}{6}}\|\Delta\tau\|_{l^2} \to 0 \qquad (\beta \downarrow 0). \tag{8.33}$$

Similarly as the proof of Step 1, one obtains that

$$|\widehat{\tau}_\beta(u) - \widehat{\tau}_\beta(v)| = \left|\int_v^u \overline{\Delta\tau_\beta}(s)\,ds\right| \le |u-v|^{\frac{1}{2}}\beta^{-\frac{1}{3}}\|\Delta\tau_\beta\|_{l^2} \tag{8.34}$$

using the Cauchy-Schwarz inequality. Then use the Arzela-Ascoli theorem and Steps 2-3 to see that $\{\widehat{\tau}_\beta\}_{\beta\in\mathbb{R}^+}$ is relatively compact in the uniform norm on $[0,N]$. Use (8.33) and Proposition 3(i) for $l = 0$ to finish the proof. □

STEP 5 *Conclusion of the proof of Proposition 3(ii).*

Proof. Let $\varepsilon > 0$ be given and choose N so large that $\max\{\overline{\tau}_\beta(v), x_a(v)\} < \frac{\varepsilon}{2}$ for all $v \ge N$ and all $\beta \in (0, \frac{1}{2})$, say. This is possible by HH Lemma 12. From Step 4 we have $\sup_{v\in[0,N]}|\overline{\tau}_\beta(v) - x_a(v)| < \varepsilon$ for sufficiently small β. Thus, for those β, we have $\|\overline{\tau}_\beta - x_a\|_\infty \le \varepsilon$. □

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