On Equilibria in Continuum Games

Erik J. Balder, Mathematical Institute, University of Utrecht, the Netherlands *

January, 1997

A substantial generalization of the usual continuum game model is obtained by the introduction of a new *feeble* topology on the set of action profiles, which no longer presupposes their integrability. Another improvement of the model concerns a reduction of the usual quasi-concavity condition. Moreover, new light is shed on a serious inconsistency in the usual model for continuum games with non-ordered preferences, exposed in Balder (1996c).

*Budapestlaan 6, P.O. Box 80.010, 3508 TA Utrecht, the Netherlands, tel: +31-30-2531458, fax: +31-30-2518394, email: `balder@math.ruu.nl`

1 Introduction

In his influential paper on the existence of Cournot-Nash equilibria, Schmeidler (1973) introduced a continuum game model which supposes the action sets to be integrably bounded from the outset. It is clear that this restriction was only made for the sake of analytical tractability, because the weak quotient topology $\sigma(L^1, L^\infty)$ in Schmeidler (1973) plays a crucial role in the application of a Kakutani-type fixed point result. For the same technical reason, such integrable boundedness was retained in all the subsequent literature, even though no natural reasons can be offered for the integrability of action profiles. The purpose of this paper is to generalize Schmeidler's model to such a degree that integrable boundedness is no longer required of the action sets. We do this by formulating a new topology for the action profiles, the *feeble topology*. This topology, which coincides on the set of action profiles with the weak prequotient topology $\sigma(\mathcal{L}^1, \mathcal{L}^\infty)$ when the action sets are integrably bounded, is very closely related to the narrow topology for the *mixed* action profiles. As a consequence of non-integrability of the profiles, the action space can also be much more general than is customary at present. Our main existence result, Theorem 2.1, generalizes and unifies the extensions of Schmeidler's first principal result in Schmeidler (1973), as given in Theorems 7.1, 7.8, 7.11 and 7.13 of Khan (1985); see Corollary 2.1. It also generalizes the unified treatment of Schmeidler's two principal results in pseudogame form, as given by Ichiishi (1983); see Corollary 2.3. Also, it is indicated how both Theorem 2.1 itself (even though it is presented as a pure equilibrium existence result) and Theorem 3.3.1, an auxiliary mixed equilibrium existence result, can be converted into an extension of the principal mixed equilibrium existence result of Theorem 2.1 in Balder (1995a). Further, in terms of the feeble topology, we present new evidence of a debilitating inconsistency, recently exposed in Balder (1996c), that mars the literature on continuum games with non-ordered preferences. Another mathematical novelty introduced in this paper is a non-Hausdorff version of a well-known existence result for quasi-variational inequalities; such non-Hausdorffness serves to give existence of Cournot-Nash equilibria in terms of the original action profiles, and not just their quotients.

2 Main results

In this section we formulate a continuum pseudogame Γ and our main equilibrium existence results for these games. Let $(T, \mathcal{T}, \mu)$ be an abstract finite measure space; here T is the set of all *players* (or player's *types*). Interesting choices for T could be: (a) T is a finite or countable set (then $\mathcal{T} = 2^T$), (b) T is a continuum, such as the unit interval (equipped with Lebesgue σ-algebra and measure), or (c) a mixture of (a) and (b) – e.g., see Corollary 2.3 below. We make the following assumption in the main part of this paper. In Remark 3.3.2 it is demonstrated that without this technically quite helpful assumption the main equilibrium existence results of this paper, that is, Theorems 2.1 and 3.3.1, continue to hold.

Assumption 2.1 *The measure space $(T, \mathcal{T}, \mu)$ is complete and separable.*

Let the *action space* S be a Hausdorff locally convex topological vector space that is a Suslin space for its topology (Dellacherie and Meyer (1975), Schwartz ((1973)). Examples of such spaces include separable Banach spaces, equipped with their norm or weak topology, duals of separable Banach spaces, equipped with their weak star topology, separable Fréchet spaces, such as $\mathcal{C}(\mathbb{R})$, equipped with the compact-open topology, or the space of all bounded, signed measures on a separable metric and complete space. The topological dual of S is denoted by S'. For each $t \in T$ let $S_t \subset S$ be the *action set* of player t. We suppose that the following holds, where the multifunction $\Sigma : T \mapsto 2^S$ is defined by setting $\Sigma(t) := S_t$.

Assumption 2.2 *For every t in T the set S_t is compact and convex, and the graph*

$$D := \{(t, s) \in T \times S : s \in S_t\},$$

of the multifunction Σ belongs to $\mathcal{T} \times \mathcal{B}(S)$.

As with other assumptions still to follow, we assume that any assumption, once presented, continues to hold in the entire paper, unless the contrary is mentioned explicitly. Observe that Assumption 2.2 places no integrability conditions whatsoever on the multifunction Σ. As usual, $\mathcal{B}(S)$ stands for the Borel σ-algebra on S, i.e., the σ-algebra generated by the open subsets of S. By $\mathcal{D}$ we denote the σ-algebra $D \cap (\mathcal{T} \times \mathcal{B}(S))$, that is, the trace σ-algebra of $\mathcal{T} \times \mathcal{B}(S)$ on D. Let $\mathcal{S}_\Sigma$ be the set of all measurable functions $f : T \to S$ for which $f(t) \in S_t$ for a.e. t in T. Mathematically speaking, $\mathcal{S}_\Sigma$ is the set of all measurable a.e.-selections of the multifunction Σ; in terms of the game Γ, $\mathcal{S}_\Sigma$ forms the set of all canonical *(pure) action profiles* of Γ. It is useful to keep in mind that in the present framework a function $f : T \to S$ is measurable (of course, by this we mean measurability with respect to $\mathcal{T}$ and $\mathcal{B}(S)$) if and only if f is *scalarly* measurable (that is, $t \mapsto < f(t), s' >$ is $\mathcal{T}$-measurable for every $s' \in S'$). This holds because the Borel σ-algebra $\mathcal{B}(S)$ on the Suslin space S is already generated by the $\sigma(S, S')$-open sets of S, by Corollary 2 of Theorem II.10 of Schwartz (1973). Assumption 2.2 implies that there exists a measurable selection of Σ (apply the von Neumann-Aumann measurable selection theorem (Theorem III.22 of Castaing and Valadier (1977)); hence, the set $\mathcal{S}_\Sigma$ is nonempty. Let $\mathcal{G}_{AC,\Sigma}$ be the vector space of all $\mathcal{D}$-measurable functions $g : D \to \mathbb{R}$ such that $g(t, \cdot)$ is affine and continuous on S_t for every t in T and such that $\sup_{s \in S_t} |g(t, s)| \leq \phi(t)$ for all t in T for some function $\phi \in \mathcal{L}^1_{\mathbb{R}}$. The *feeble topology* on $\mathcal{S}_\Sigma$ is defined as the weakest topology for which all integral functionals

$$J_g : f \mapsto \int_T g(t, f(t)) \mu(dt),\ g \in \mathcal{G}_{AC,\Sigma},$$

are continuous; $\mathcal{S}_\Sigma$ will be equipped with this topology from now on. The following remark calls attention to the fact that the feeble topology, which is a novel feature of this paper, is the natural generalization of the usual relative weak topology on $\mathcal{L}^1_\Sigma$.

Remark 2.1 (*a*) *The standard situation considered in the continuum game literature obtains when S is a separable Banach space, equipped with the weak topology $\sigma(S, S')$, and when, in addition to Assumption 2.2, the multifunction Σ is integrably bounded; i.e., there exists $\phi \in \mathcal{L}^1_{\mathbb{R}}$ such that $\sup_{s \in S_t} \|s\| \leq \phi(t)$ for every $t \in T$. Here $\|\cdot\|$ stands for the norm on S. In this situation S is a Suslin locally convex topological vector space, and $\mathcal{S}_\Sigma$ is nothing but the prequotient space $\mathcal{L}^1_\Sigma$, that is to say, the set of all integrable a.e.-selectors of Σ. Also, the feeble topology is then the relative (prequotient) weak topology $\sigma(\mathcal{L}^1_S, \mathcal{L}^\infty_{S'})$ on $\mathcal{L}^1_\Sigma$: Indeed, since Σ is integrably bounded, all functions $(t, s) \mapsto 1_A(t) < s, s' >: D \to \mathbb{R}$, $A \in \mathcal{T}$, $s' \in S'$, belong to $\mathcal{G}_{AC,\Sigma}$, and, in the converse direction, it is a well known fact that the integral functionals J_g, $g \in \mathcal{G}_{AC,\Sigma}$, are sequentially weakly continuous on the space $\mathcal{L}^1_\Sigma$ (which is semimetrizable by Assumption 2.1). For an instant proof of this fact the reader is invited to combine Corollary 2.2 and Theorem 3.3 of Balder (1990). We should observe, however, that, less generally (e.g., see the proof of Corollary 2.1 below), all such models in the literature on the subject work with the quotient space L^1_Σ instead of the prequotient $\mathcal{L}^1_\Sigma$ as their set of action profiles. We finish by observing that, if the weak topology on the Banach space S is replaced by the norm-topology, then $\mathcal{S}_\Sigma$ is still the prequotient space $\mathcal{L}^1_\Sigma$ and the class $\mathcal{G}_{AC,\Sigma}$ is the same as before (this follows from standard facts involving lower semicontinuous convex functions and the Hahn-Banach theorem, similar to the well known fact that for affine functions on S weak continuity and norm-continuity are the same). Thus, the feeble topology is then again the weak topology on $\mathcal{L}^1_\Sigma$.*

(*b*) *Another situation used on some occasions (e.g., cf. Khan (1985)) is as follows: S is the dual of a separable Banach space R, and, in addition to Assumption 2.2, the multifunction Σ is uniformly bounded by a compact, i.e., there exists a $\sigma(S, R)$-compact set K such that $S_t \subset K$ for all $t \in T$. In this situation S is a Suslin locally convex topological vector space for the weak star topology $\sigma(S, R)$, and $\mathcal{S}_\Sigma$ is obviously the prequotient space $\mathcal{L}^\infty_\Sigma$, i.e., the set of all bounded and measurable a.e.-selectors of Σ. The feeble topology is then the relative weak star topology $\sigma(\mathcal{L}^\infty_S, \mathcal{L}^1_R)$ on $\mathcal{L}^\infty_\Sigma$. This is seen by the fact that, on the one hand, all functions $(t, s) \mapsto < s, \ell(t) >: D \to \mathbb{R}$, $\ell \in \mathcal{L}^1_R$, belong to $\mathcal{G}_{AC,\Sigma}$ and, on the other hand, the integral functionals J_g, $g \in \mathcal{G}_{AC,\Sigma}$, are weak star continuous.*

Let $Y := \mathbb{R}^m \times \mathcal{S}_\Sigma(T \backslash T_0)$, where $m \in \mathbb{N}$ is a given, fixed number, $T_0 \in \mathcal{T}$ a given, fixed subset of the *nonatomic* part T^{na} of the measure space $(T, \mathcal{T}, \mu)$, and where $\mathcal{S}_\Sigma(T \backslash T_0)$ stands for the set of all measurable functions $f : T\backslash T_0 \to S$ for which $f(t) \in S_t$ for a.e. t in $T\backslash T_0$. Following Balder (1995a) – but more concretely – we introduce an *externality mapping* $d : \mathcal{S}_\Sigma \to Y$ in the next assumption.

Assumption 2.3 *The externality mapping d is of the form*

$$d(f) := ((\int_{T_0} g_i(t, f(t))\mu(dt))_{i=1}^m, f\mid_{T\backslash T_0}),$$

where $g_1, \cdots, g_m$ are Carathéodory functions on $D \cap (T_0 \times S)$. [1]

Above, $f\mid_{T\backslash T_0}$ stands for the restriction of $f \in \mathcal{S}_\Sigma$ to $T\backslash T_0$. Interesting choices for T_0 are (a) $T_0 := \emptyset$, when under Assumption 2.3 d simply amounts to the identity mapping, or (b) $T_0 := T^{na}$, when one could consider $g_i(t,s) :=$ i-th coordinate function in the setup of Remark 2.1(a) if the action space S is additionally finite-dimensional. Observe that when $(T, \mathcal{T}, \mu)$ is nonatomic, this amounts to $T_0 := T^{na} = T$ and the familiar choice $d(f) := \int_T f\, d\mu$.

We continue by introducing a social feasibility feature of Γ (incidentally, this is why Γ is frequently referred to as a pseudogame; cf. Ichiishi (1983)). Let $A : T \times Y \to 2^S$ be a given multifunction; an action profile $f \in \mathcal{S}_\Sigma$ is said to be *socially feasible* if $f(t) \in A(t, d(f))$ for a.e. t in T. In this way, (almost) each player $t \in T$ is forced to take not only his/her own action into consideration (which must belong to S_t), but also the actions of the other players. Often we write A_t instead of $A(t, \cdot)$.

Assumption 2.4 *The multifunction $A : T \times Y \to 2^S$ has nonempty closed values and satisfies*

$$A(t, d(f)) \subset S_t \text{ for every } (t, f) \text{ in } T \times \mathcal{S}_\Sigma.$$

Also, for every t in T the multifunction $A_t : Y \to 2^{S_t}$ is continuous, and the graph of A, given by

$$\{(t, s, y) \in D \times Y : s \in A(t, y)\},$$

belongs to $\mathcal{D} \times \mathcal{B}(Y)$.

Next, we introduce the payoff functions of the game Γ. For each player t let $U_t : S_t \times Y \to [-\infty, +\infty]$ be player t's payoff function. Given the action profile $f \in \mathcal{S}_\Sigma$, player t's payoff amounts to $U_t(s, d(f))$ if he/she replaces the profile-prescribed action $f(t)$ by the action $s \in S_t$. Let $U : D \times Y \to [-\infty, +\infty]$ be the function given by $U(t, s, y) := U_t(s, y)$.

Assumption 2.5 *For every $t \in T$ the function U_t is continuous on $S_t \times Y$ and for every $y \in Y$ the function $U(\cdot, \cdot, y)$ is $\mathcal{D}$-measurable.*

Assumption 2.6 *For every $(t, y) \in (T\backslash T_0) \times Y$ the set* $\mathrm{argmax}_{s \in A_t(y))} U_t(s, y)$ *is convex.*

The latter assumption is very mild. Most certainly it is fulfilled under standard conditions which demand that $A_t(y)$ is convex and $U_t(\cdot, y)$ is quasi-concave on $A_t(y)$ for every t and y, but it allows for nonstandard situations as well: e.g., think of a non-quasi-concave U_t for which the argmax set figuring in Assumption 2.6 is a singleton for each t (see Example 2.2 below). Observe that in the nonatomic case, already considered above (i.e., $T_0 = T^{na} = T$), Assumption 2.6 holds vacuously.

Theorem 2.1 (equilibrium existence result) *Under the above assumptions the pseudogame $\Gamma := (T, \Sigma, U, A)$ has a socially feasible Nash equilibrium in pure action profiles, i.e., there exists $f^* \in \mathcal{S}_\Sigma$ such that*

$$f^*(t) \in \mathrm{argmax}_{s \in A_t(d(f^*))} U_t(s, d(f^*)) \text{ for a.e. } t \text{ in } T.$$

Observe that the equilibrium solution f^* of the theorem is both socially feasible and has the property that almost every player achieves maximum payoff under the profile f^*.

[1] I.e., each $g_i : D \cap (T_0 \times S) \to \mathbb{R}$ is $\mathcal{D}$-measurable, with $g_i(t, \cdot)$ continuous on S_t for every $t \in T$ and with $\sup_{s \in S_t} |g_i(t, s)| \le \phi(t)$ for some integrable $\phi : T_0 \to \mathbb{R}$; cf. section 3.2.

Corollary 2.1 (Theorems 7.1, 7.8, 7.11 of Khan (1985)) *Let $(S, \|\cdot\|)$ be a separable Banach space, equipped with the weak topology $\sigma(S, S')$. Let S_t be weakly compact, convex and nonempty for every $t \in T$. Also, let $\Sigma : t \mapsto S_t$ have a measurable graph D and be integrably bounded (cf. Remark 2.1(a)). Let $u : D \times L^1_\Sigma \to [-\infty, +\infty]$ be such that $u_t := u(t, \cdot, \cdot)$ is continuous on $S_t \times L^1_\Sigma$ for every $t \in T$, $u(\cdot, \cdot, x)$ is $\mathcal{D}$-measurable on D for every $x \in L^1_\Sigma$, and $u(t, \cdot, x)$ is quasi-concave for every $(t, x) \in T \times L^1_\Sigma$. Here L^1_Σ is equipped with the relative weak topology $\sigma(L^1_S, L^\infty_{S'})$. Then there exists $x^* \in L^1_\Sigma$ such that*

$$x^*(t) \in \operatorname{argmax}_{s \in S_t} u_t(s, x^*) \text{ for a.e. } t \text{ in } T.$$

PROOF. Take $T_0 := \emptyset$; so d is the identity on $\mathcal{S}_\Sigma$. Observe that $\mathcal{S}_\Sigma = \mathcal{L}^1_\Sigma$ by Remark 2.1. Let π be the usual quotient mapping from the prequotient space $\mathcal{L}^1_S$ into L^1_S and set $U(t, s, f) := u(t, s, \pi(f))$ and $A_t(f) := S_t$. Then the assumptions of Theorem 2.1 all hold. Given f^*, as guaranteed to exist by Theorem 2.1, we then set $x^* := \pi(f^*)$ to find an equilibrium solution in the present context. Q.E.D.

Observe that Theorem 7.1 of Khan (1985) has an additional uniform inclusion for the action sets S_t, $t \in T$, which we have dropped altogether. The fact that Theorem 2.1 completely generalizes Theorem 7.1 of Khan (1985), solves a question left open in Balder (1995a), p. 89. In the same way we can show that Theorem 2.1 generalizes Theorem 7.13 of Khan (1985), by taking on $\mathcal{S}_\Sigma$ the (relative) weak star topology instead of the weak topology; see Remark 2.1(b). As a new application in the above standard context, consider the version of Corollary 2.1 in which the weak topology $\sigma(S, S')$ is systematically replaced by the norm topology:

Corollary 2.2 *Let $(S, \|\cdot\|)$ be a separable Banach space, equipped with the norm topology. Let S_t be norm-compact, convex and nonempty for every $t \in T$. Also, let $\Sigma : t \mapsto S_t$ have an measurable graph D and be integrably bounded (cf. Remark 2.1(a)). Let $u : D \times L^1_\Sigma \to [-\infty, +\infty]$ be such that $u_t := u(t, \cdot, \cdot)$ is continuous on $S_t \times L^1_\Sigma$ for every $t \in T$, $u(\cdot, \cdot, x)$ is $\mathcal{D}$-measurable on D for every $x \in L^1_\Sigma$, and $u(t, \cdot, x)$ is quasi-concave for every $(t, x) \in T \times L^1_\Sigma$. Here L^1_Σ is equipped with the relative weak topology $\sigma(L^1_S, L^\infty_{S'})$. Then there exists $x^* \in L^1_\Sigma$ such that*

$$x^*(t) \in \operatorname{argmax}_{s \in S_t} u_t(s, x^*) \text{ for a.e. } t \text{ in } T.$$

The proof is virtually a replica of the one given for the previous corollary, in view of what was concluded in Remark 2.1(a). In comparison with the previous corollary, the compactness assumption for the action sets has become stronger, whereas the continuity condition for the payoff functions is weakened. The next corollary of Theorem 2.1 captures Theorem 4.7.3, the main continuum pseudogame existence result by Ichiishi (1983), which combines the two results of the original paper of Schmeidler (1973). In Ichiishi (1983), just as in the present paper, this result is stated in terms of the prequotient space $\mathcal{L}^1_\Sigma$, but it should be noticed that the mathematical apparatus of Ichiishi (1983), which is built on Hausdorff spaces, does not go beyond the quotient space L^1_Σ and hence does not seem capable of fully supporting such a result.

Corollary 2.3 (Ichiishi (1983), Theorem 4.7.3) *Let T be the union (or direct sum) of a set C and a singleton-atom b, where $(C, \mathcal{C}, \nu)$ is a finite nonatomic measure space. Let S be the direct sum of $\mathbb{R}^l$ and a compact convex subset Z of a Hausdorff locally convex space. Let S_t be weakly compact, convex and nonempty for every $t \in C$, such that $\Sigma : t \mapsto S_t$ has a measurable graph D and is integrably bounded on C. Let $u : D \times \int_C \Sigma \times Z \to [-\infty, +\infty]$ be such that u_t is continuous on $S_t \times \int_C \Sigma \times Z$ for every $t \in C$ and such that $u(\cdot, \cdot, y, z)$ is $\mathcal{D}$-measurable for every $(y, z) \in \int_C \Sigma \times Z$. Also, let $u_b : Z \times \int_C \Sigma \times Z \to [-\infty, +\infty]$ be continuous and such that $u_b(\cdot, y, z)$ is quasi-concave on Z for every $(y, z) \in \int_C \Sigma \times Z$. Further, let $F : C \times \int_C \Sigma \times Z \to 2^{\mathbb{R}^l}$ have a measurable graph and be such that F_t is continuous for every $t \in C$ with nonempty closed values, and let $G : \int_C \Sigma \times Z \to 2^Z$ also have a measurable graph, be continuous and have nonempty closed convex values. Here $\int_C \Sigma$ is the usual integral $\{\int_C f \, d\nu : f \in \mathcal{L}^1_\Sigma\}$ of the multifunction Σ. Then there exists a pair $(f^*, z^*) \in \mathcal{L}^1_\Sigma \times Z$ such that $f^*(t) \in \operatorname{argmax}_{s \in F(t, \int_C f^*, z^*)} u_t(s, \int_C f^* d\nu, z^*)$ for a.e. t in C and $z^* \in \operatorname{argmax}_{z \in G(\int_C f^*, z^*)} u_b(z, \int_C f^* d\nu, z^*)$.*

PROOF. We take $T_0 := C$; observe already that no Suslin condition is required for Z, since $T \backslash T_0$ is the singleton $\{b\}$, upon which all measurability considerations are trivial. We define g_i in $\mathcal{G}_{bb,\Sigma}$ by $g_i(t,s) :=$ i-th coordinate of s, $i = 1, \cdots, l$. For the externality mapping this gives $d(f) := (\int_C f, f(b))$. We substitute $U_t(s, d(f)) := u_t(s, \int_C f, f(b))$ and $A_t(d(f)) := F_t(\int_C f, f(b))$ for "continuum players" $t \in C$, and $U_b(s, d(f)) := u_b(s, \int_C f, f(b))$, $A_b(d(f)) := G(\int_C f, f(b))$ for the "atomic player" b. Then all assumptions of Theorem 2.1 are easily seen to hold. For $f^* \in \mathcal{S}_\Sigma = \mathcal{L}^1_{Sigma}$, as guaranteed to exist by Theorem 2.1, we then set $z^* := f^*(b)$. Q.E.D.

The following paradigmatic examples describe trivial continuum game equilibrium existence problems, in which each player can take only one action. Existence of a Nash equilibrium profile in these examples is a trivial matter, but neither the two corollaries above, nor the literature which they generalize can deal directly with this problem. However, Theorem 2.1 applies in both instances:

Example 2.1 *Let $f : T \to S$ be a measurable, non-integrable function. Consider the case where S_t is the singleton $\{f(t)\}$ for each $t \in T$, and where $A_t \equiv S_t$ and $U \equiv 0$. The standard continuum game literature on existence is unable to deal with this situation, because $\Sigma : t \mapsto \{f(t)\}$ is not integrably bounded – cf. Remark 2.1(a). Nevertheless, Theorem 2.1 applies, since Assumption 2.2 evidently holds and for $T_0 := \emptyset$ the other assumptions hold trivially.*

Example 2.2 *For arbitrary $(T, \mathcal{T}, \mu)$ we consider the case where S_t is the interval $[-1/2, +2]$ for all t, where $A_t \equiv S_t$ and $U_t(s,y) := -(1-s^2)^2$ for each t in T. The standard continuum game literature on existence is unable to deal with this situation (directly), because $U_t(s,y)$ is not quasiconcave in s. However, Theorem 2.1 applies here: Assumption 2.2 clearly holds, as do Assumptions 2.5 and 2.6 (observe that* $\text{argmax}_{s \in [-1/2,2]} U_t(s,y)$ *is the singleton $\{1\}$, which is a convex set), and for $T_0 := \emptyset$ the other assumptions hold trivially.*

If in the last example S_t is taken to be $[-2, +2]$ for all t, then Theorem 2.1 does not apply, since Assumption 2.6 no longer holds. In contrast to Theorem 2.1, however, the related mixed equilibrium existence result Theorem 3.3.1 still applies in that situation, and it leads to the desired (but trivial) existence result by an *ad hoc* purification argument. This underlines the fact that mixed equilibrium existence results are more fundamental than their pure counterparts, a fact known at least for finite games since von Neumann.

3 Proofs

Roughly speaking, the proof of Theorem 2.1 consists of the following stages: (1) formulation of a mixed version of the pseudogame Γ, (2) obtaining the existence of a mixed equilibrium profile δ^* as the solution of a quasi-variational inequality in mixed profiles (Theorem 3.3.1), (3) purification of δ^*. As for (1), we shall see in section 3.2 that, via the barycentric mapping $\delta \mapsto \text{bar}\,\delta : \mathcal{R}_\Sigma \to \mathcal{S}_\Sigma$, the feeble topology is strongly related to the narrow topology for transition probabilities; cf. Balder (1988, 1995b). The latter topology plays a crucial role in stage (2), which hinges on the application of an abstract existence result for quasi-variational inequalities in a non-Hausdorff space (Corollary 3.1.1). Section 3.1 serves to derive this abstract existence result from Ky Fan's well-known inequality. In stage (3) the mixed equilibrium profile of Theorem 3.3.1 is converted into an equilibrium of the desired pure type, on the one hand by taking pointwise barycenters (i.e., expectations) of $\delta^*(t)$ for players t in $T \backslash T_0$ and on the other hand by aggregated Lyapunov-type purification for players in T_0.

3.1 Quasi-variational inequalities on a non-Hausdorff space

To enable the use of the *prequotient* space $\mathcal{S}_\Sigma$, the proof of Theorem 2.1 is based on an application, in a *non-Hausdorff* context, of Corollary 3.1.1. The ancillary Theorem 3.1.2, which we derive first, is an existence result for a quasi-variational inequality, whose counterpart is well-known in a Hausdorff vector space context; see Theorem 9.13 of Aubin (1993) or Theorem 3.1 of Balder (1996b). Below we recall Ky Fan's inequality (Ky Fan (1961), Lemma 1); we point out in particular that this result remains valid in a non-Hausdorff setting, because, as already observed in Ding and Tan (1990), pp. 500-501, the proof of Ky Fan (1961) does not require the Hausdorff property.

Theorem 3.1.1 (Ky Fan's inequality) *Let C be a compact convex and nonempty subset of a topological vector space (possibly non-Hausdorff). Let $\pi : C \times C \to [-\infty, +\infty]$ be such that*

$$\pi(\cdot, y) \text{ is lower semicontinuous for every } y \in C,$$

$$\pi(x, \cdot) \text{ is quasiconcave for every } x \in C,$$

$$\pi(x, x) \leq 0 \text{ for every } x \in C.$$

Then there exists $x^* \in C$ such that $\pi(x^*, y) \leq 0$ for all $y \in C$.

With the aid of Theorem 3.1.1 we now prove an existence result for a quasi-variational inequality on a *non-Hausdorff* vector space, similar to Theorem 9.13 of Aubin (1993), the proof of which we mimick in a non-Hausdorff way. Let E be a locally convex topological vector space (possibly non-Hausdorff); the topological dual of E is denoted by E'.

Theorem 3.1.2 *Let $C \subset E$ be compact, convex and nonempty. Let $\pi : C \times C \to \mathbb{R}$ be such that*

$$\pi(\cdot, y) \text{ is lower semicontinuous for every } y \in C,$$

$$\pi(x, \cdot) \text{ is concave for every } x \in C,$$

$$\pi(x, x) \leq 0 \text{ for every } x \in C.$$

Also, let $F : C \to 2^C$ be a multifunction with convex and nonempty values, such that

$$\sigma(F(\cdot), x') : x \mapsto \sup_{y \in F(x)} < y, x' > \text{ is upper semicontinuous on } C \text{ for every } x' \in E',$$

$$\alpha : x \mapsto \sup_{y \in F(x)} \pi(x, y) \text{ is lower semicontinuous.}$$

Then there exists $x^* \in C$ such that $x^* \in F(x^*)$ and $\pi(x^*, y) \leq 0$ for all $y \in F(x^*)$.

PROOF. Suppose that for every $x \in C$ one either has (1) $x \notin F(x)$ or (2) $\alpha(x) > 0$. By the Hahn-Banach theorem, which continues to hold in the present non-Hausdorff setup (Edwards (1995), Corollary 2.2.3), possibility (1) implies the existence of $x' \in E'$ such that $x \in V(x') := \{z \in C :< z, x' >> \sigma(F(z), x')\}$. Thus, C is covered by the open sets $V_0 := \{x \in C : \alpha(x) > 0\}$ and $V(x')$, $x' \in E'$. By compactness of C there exists a finite subset $\{x'_1, \cdots, x'_n\}$ of E' such that V_0 and the $V(x'_i)$, $1 \leq i \leq n$ also cover C. By point (5) on p. 23 of Edwards (1995), a reference which carefully avoids making unnecessary Hausdorff assumptions, there exists a continuous partition $\{c_0, c_1, \cdots, c_n\}$ of unity which is subordinate to the cover of the $n+1$ sets mentioned above (observe that C is of course paracompact). We now define $\bar{\pi} : C \times C \to \mathbb{R}$ by

$$\bar{\pi}(x, y) := c_0(x)\pi(x, y) + \sum_{i=1}^{n} c_i(x) < x - y, x'_i >,$$

which is lower continuous in x, concave in y, and meets $\bar{\pi}(x, x) \leq 0$. By Ky Fan's inequality (Theorem 3.1.1), this implies existence of $\bar{x} \in C$ such that $\bar{\pi}(\bar{x}, y) \leq 0$ for all $y \in C$. If $\alpha(\bar{x}) > 0$, then there exists $y \in F(\bar{x})$ with $\pi(\bar{x}, y) > 0$. This causes $\bar{\pi}(\bar{x}, y) > 0$, which cannot be (observe that $< \bar{x}, x'_i >> \sigma(F(\bar{x}), x'_i) \geq < y, x'_i >$ whenever $c_i(\bar{x}) > 0$). On the other hand, if $\alpha(\bar{x}) \leq 0$, then $c_0(\bar{x}) = 0$ and there exists at least one i, $1 \leq i \leq n$, such that $c_i(\bar{x}) > 0$. Again we find an impossibility: now $\bar{\pi}(\bar{x}, y) > 0$ for any $y \in F(\bar{x})$ (use the same observation as above). This brings the desired *reductio ad absurdum*. Q.E.D.

Corollary 3.1.1 *Let $C \subset E$ be compact, convex and nonempty. Let $b : C \times C \to \mathbb{R}$ be such that*

$$b(\cdot,\cdot) \text{ is upper semicontinuous,}$$

$$b(\cdot, y) \text{ is continuous for every } y \in C,$$

$$b(x,\cdot) \text{ is concave for every } x \in C.$$

Also, let $F : C \to 2^C$ be a multifunction with convex and nonempty values, such that

$$\sigma(F(\cdot), x') : x \mapsto \sup_{y \in F(x)} < x, x' > \text{ is upper semicontinuous on } C \text{ for every } x' \in E',$$

$$a : x \mapsto \sup_{y \in F(x)} b(x,y) \text{ is lower semicontinuous.}$$

Then there exists $x^* \in C$ such that $x^* \in F(x^*)$ and $b(x^*, x^*) \geq b(x^*, y)$ for all $y \in F(x^*)$.

This result follows immediately from applying Theorem 3.1.2 to $\pi(x,y) := b(x,y) - b(x,x)$.

3.2 On the feeble and narrow topologies

This subsection establishes some general facts about the narrow topology for transition probabilities and its connection with the feeble topology. The Hausdorff locally convex space S is completely regular; hence, its points are separated by $\mathcal{C}_b(S)$, the set of all bounded continuous functions on S. Since S is also Suslin, the points of S are already separated by a countable subset (c_i) of $\mathcal{C}_b(S)$ (apply Lemma III.32 of Castaing and Valadier (1977)). It is easy to see that

$$d_S(s,z) := \sum_{i=1}^{\infty} 2^{-i} \frac{|c_i(s) - c_i(z)|}{\sup_S |c_i|}$$

defines a *weak metric* on S that is weaker than the original topology. Hence, on compact subsets of S the original and d_S-topology coincide. It is important to observe that the Borel σ-algebra corresponding to d_S coincides with $\mathcal{B}(S)$, because S is Suslin (apply Corollary 2 of Theorem II.10 of Schwartz (1973)).

Let $M_1^+(S)$ be the set of all probability measures on $(S, \mathcal{B}(S))$. Recall from Proposition 26.3 of Choquet (1969) that for every compact convex subset K of S and every $\nu \in M_1^+(S)$, $\nu(K) = 1$, there exists a *barycenter* (or representant) bar ν of ν; this is a point in K that is uniquely determined by

$$< \text{bar } \nu, s' >= \int_K < s, s' > \nu(ds) \text{ for all } s' \in S'. \tag{3.1}$$

Recall also from Dellacherie and Meyer (1975) that the *classical narrow topology* on $M_1^+(S)$ is defined as the coarsest topology for which all mappings

$$\nu \mapsto \int_S c(s)\nu(ds), c \in \mathcal{C}_b(S),$$

are continuous. Recall further that a *transition probability* (alias *Young measure*) from T into S can be defined as a $\mathcal{T}$-measurable function $\delta : T \to M_1^+(S)$, where $M_1^+(S)$ is equipped with the Borel σ-algebra corresponding to the classical narrow topology (since (S, d_S) is certainly separable and metrizable, it is not hard to see that this definition is equivalent to the one given in section III.2 of Neveu (1965)). Let $\mathcal{R}_S$ be the set of all transition probabilities from T into S and let $\mathcal{R}_\Sigma$ be the set of all $\delta \in \mathcal{R}_S$ such that $\delta(t)(S_t) = 1$ for a.e. t in T. The elements from $\mathcal{R}_\Sigma$ will be referred to as *mixed action profiles.* Assumption 2.2 implies that to every mixed action profile $\delta \in \mathcal{R}_\Sigma$ there corresponds a pure action profile bar $\delta \in \mathcal{S}_\Sigma$, defined as follows: Let $\bar{f}$ be some arbitrary, fixed element of $\mathcal{S}_\Sigma$. Let N be the null set of those t in T for which $\delta(t)(S_t) < 1$. Define (bar $\delta)(t)$:= bar $\delta(t)$ if $t \in T \backslash N$ and (bar $\delta)(t) := \bar{f}(t)$ otherwise. Then scalar measurability of bar δ follows from the fact that for every $s' \in S'$ the function $t \mapsto <$ bar $\delta(t), s' >= \int_S \ell(t,s)\delta(t)(ds)$ is measurable by section III.2 of

Neveu (1965). Here $\ell(t,s) :=< s, s' >$ if $(t,s) \in D$ and $\ell(t,s) := 0$ if $(t,s) \in (T \times S)\backslash D$. Thus, bar δ is also measurable with respect to $\mathcal{T}$ and $\mathcal{B}(S)$, in view of an earlier observation. Since bar $\delta(t) \in S_t$ for all $t \in T\backslash N$ by Assumption 2.2, it follows that bar δ belongs to $\mathcal{S}_\Sigma$.

Recall from Balder (1988) that the *narrow topology* (alias *Young measure topology*) on $\mathcal{R}_S$ is the coarsest topology for which all mappings

$$\delta \mapsto \int_A [\int_S c(s)\delta(t)(ds)]\mu(dt), A \in \mathcal{T}, c \in \mathcal{C}_b(S),$$

are continuous; the narrow topology on $\mathcal{R}_\Sigma$ is of course defined by relativization. Equivalently, the narrow topology on $\mathcal{R}_\Sigma$ is the coarsest topology for which all mappings

$$I_g : \delta \mapsto \int_T [\int_S g(t,s)\delta(t)(ds)]\mu(dt),\ g \in \mathcal{G}_{bb,\Sigma}, \tag{3.2}$$

are lower semicontinuous; here $\mathcal{G}_{bb,\Sigma}$ is the set of all *normal integrands* on D, i.e., the set of all $\mathcal{D}$-measurable functions $g : D \to \mathbb{R}$ such that $g(t,\cdot)$ is lower semicontinuous on S_t for every $t \in T$ and $\inf_{s\in S_t} g(t,s) \geq \phi(t)$ for some $\phi \in \mathcal{L}^1_{\mathbb{R}}$. This follows by Theorem 2.2.(c) in Balder (1988) (observe that for $g \in \mathcal{G}_{bb,\Sigma}$ the function $g' : T \times S \to (-\infty, +\infty]$, defined by $g'(t,s) := g(t,s)$ if $(t,s) \in D$ and $g'(t,s) := +\infty$ if $(t,s) \in (T \times S)\backslash D$, is a normal integrand on $T \times S$). Hence, equivalently (*bis*), the narrow topology on $\mathcal{R}_\Sigma$ is the coarsest topology for which all mappings I_g, $g \in \mathcal{G}_{C,\Sigma}$, are continuous, where $\mathcal{G}_{C,\Sigma}$ is the set of all *Carathéodory integrands* on D, i.e., the set of all $\mathcal{D}$-measurable functions $g : D \to \mathbb{R}$ such that $g(t,\cdot)$ is continuous on S_t for every $t \in T$ and $\sup_{s\in S_t} |g(t,s)| \leq \phi(t)$ for some $\phi \in \mathcal{L}^1_{\mathbb{R}}$. This follows simply by observing that on the one hand $\mathcal{G}_{C,\Sigma}$ is the intersection of $\mathcal{G}_{bb,\Sigma}$ and $-\mathcal{G}_{bb,\Sigma}$, and that on the other hand $\mathcal{G}_{C,\Sigma}$ contains all functions $g : D \to \mathbb{R}$ of the form $g(t,s) := 1_A(t)c(s)$ for $A \in \mathcal{T}$ and $c \in \mathcal{C}_b(S)$. More fundamentally, a similar equivalence holds on $\mathcal{R}_S$ itself Theorem 2.2(b) of Balder (1988): the narrow topology is the coarsest topology for which all mappings I_g, $g \in \mathcal{G}_C$, are continuous, where $\mathcal{G}_C$ is the set of all Carathéodory integrands on $T \times S$. The definition of the narrow topology on $\mathcal{R}_S$ extends in an obvious way to $\mathcal{M}_S$, the vector space spanned by $\mathcal{R}_S$, and we shall continue to refer to this as the narrow topology. Since $\mathcal{G}_C$ is a linear space, it is identifiable with the topological dual $\mathcal{M}'_S$ (apply Proposition 22.4 of Choquet (1969)).

Proposition 3.2.1 *The set $\mathcal{R}_\Sigma$ is a compact, convex and nonempty subset of the seminormed space $\mathcal{M}_S$.*

Proof. By Assumption 2.1 (which is essential for this result to hold), the σ-algebra $\mathcal{T}$ is generated by some (countable) sequence (A_j) in $\mathcal{T}$. Since (S, d_S) is obviously a metric Suslin space, it follows that the points of $M^+_1(S)$ can be separated by a countable collection (c'_i) in $\mathcal{C}_b(S)$ (in fact, this already follows from the fact that S is separable and metric). Define for any δ in $\mathcal{M}_S$

$$p_\mathcal{M}(\delta) := \sum_{i=1}^\infty \sum_{j=1}^\infty 2^{-i-j} \frac{|\int_{A_j}\int_S c'_i(s)\delta(t)(ds)\mu(dt)|}{\mu(A_j)\sup_S |c'_i|}.$$

Then $p_\mathcal{M}$ is a seminorm on $\mathcal{M}_S$, whose topology obviously coincides with the narrow topology on $\mathcal{R}_S \supset \mathcal{R}_\Sigma$. Convexity of $\mathcal{R}_\Sigma$ is trivial, and narrow compactness of $\mathcal{R}_\Sigma$ follows by the results in Balder (1988): Set $h(t,s) := 0$ if $s \in S_t$ and $h(t,s) := +\infty$ if $s \notin S_t$; then h is $\mathcal{T} \times \mathcal{B}(S)$-measurable and $h(t,\cdot)$ is inf-compact on S for every $t \in T$. Hence, $\mathcal{R}_\Sigma$ is compact for the narrow topology by Theorem 2.3(b) of Balder (1988), [2] since it is the set of all $\delta \in \mathcal{R}_S$ for which $I_h(\delta) \leq 0$, where I_h is as defined in (3.2). Finally, it was already observed that $\mathcal{S}_\Sigma$ is nonempty and contains some f. Then ϵ_f belongs to $\mathcal{R}_\Sigma$, which is therefore nonempty. Q.E.D.

In the above proof the following notation was used: for any pure action profile $f \in \mathcal{S}_\Sigma$, $\epsilon_f \in \mathcal{R}_\Sigma$ stands for the canonical mixed action profile given by

$$\epsilon_f(t) := \text{Dirac measure at } f(t).$$

[2] In Balder (1989) it was proven that this result remains valid in the present Suslin space context; see also Theorem 5.1 in Balder (1990) and Theorem 5.5 in Balder (1995a).

Proposition 3.2.2 *The mapping $\delta \mapsto \text{bar } \delta$ from $\mathcal{R}_\Sigma$, equipped with the narrow topology, into $\mathcal{S}_\Sigma$, equipped with the feeble topology, is continuous.*

PROOF. Simply observe that for any g in $\mathcal{G}_{AC,\Sigma}$, the class of integrands defining the feeble topology on $\mathcal{S}_\Sigma$, we have by (3.1)

$$\int_T [\int_{S_t} g(t, \text{bar } \delta(t))]\mu(dt) = \int_T [\int_{S_t} g(t,s)\delta(t)(ds)]\mu(dt) = I_g(\delta)$$

for all $\delta \in \mathcal{R}_\Sigma$, where the right hand side is narrowly continuous in δ because of the obvious inclusion $\mathcal{G}_{AC,\Sigma} \subset \mathcal{G}_{C,\Sigma}$. Q.E.D.

The following result generalizes what is called Diestel's theorem in Theorem 3.1 of Yannelis (1991) (cf. Diestel (1977)).

Proposition 3.2.3 *$\mathcal{S}_\Sigma$ is semimetrizable and compact.*

PROOF. Compactness follows immediately by the previous results: If (f_n) is a sequence in $\mathcal{S}_\Sigma$, then the corresponding sequence (ϵ_{f_n}) in $\mathcal{R}_\Sigma$ has a subsequence $(\epsilon_{f_{n_k}})$ that converges narrowly to some $\delta \in \mathcal{R}_\Sigma$ (Proposition 3.2.1). By Proposition 3.2.2, it follows that (f_{n_k}) feebly converges to $f := \text{bar } \delta$. To prove semimetrizability we argue as follows. Because S is Hausdorff locally convex, S' separates the points of S (Hahn-Banach theorem). Since S is also Suslin, it follows by Lemma III.32 of Castaing and Valadier (1977) that S' contains a countable sequence (s'_i) which still separates the points of S. For each i,j define $g_{i,j} : D \to \mathbb{R}$ by $g_{i,j}(t,s) := 1_{A_j}(t)\frac{<s,s'_i>}{\mu(A_j)\sup_{z\in S_t}|<z,s'_i>|}$, where (A_j) is the countable collection generating the σ-algebra $\mathcal{T}$, as introduced in the proof of Proposition 3.2.1. Then we claim that

$$d_{\mathcal{S}_\Sigma}(f,f') := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} 2^{-i-j}|\int_T [g_{i,j}(t,f(t)) - g_{i,j}(t,f'(t))]\mu(dt)|$$

metrizes the feeble topology. First, observe that the above integrals are well-defined, in view of the definition of the $g_{i,j}$. For by Assumption 2.2 and Theorem III.30 of Castaing and Valadier (1977), the multifunction Σ has a Castaing representation: There exists a sequence (σ_n) of measurable selections of Σ such that $S_t = \text{cl } \{\sigma_n(t)\}$ for all $t \in T$, and this gives $\sup_{s\in S_t}|<s,s'_i>| = \sup_n|<\sigma_n(t), s'_i>|$, hence measurability. Clearly, $d_{\mathcal{S}_\Sigma}$-convergence is implied by feeble convergence, since each function $g_{i,j}$ belongs to $\mathcal{G}_{AC,\Sigma}$. Conversely, suppose that a sequence (f_n) converges in $d_{\mathcal{S}_\Sigma}$ to f_0 in $\mathcal{S}_\Sigma$, without giving feeble convergence as well. Then for some $\tilde{g} \in \mathcal{G}_{AC,\Sigma}$, $\epsilon > 0$ and some subsequence (f_{n_k}) we have $|\int_T[\tilde{g}(t,f_{n_k}(t)) - \tilde{g}(t,f_0(t))]\mu(dt)| \geq \epsilon$. By Proposition 3.2.1, the corresponding sequence $(\epsilon_{f_{n_k}})$ in $\mathcal{R}_\Sigma$ has a subsequence $(\epsilon_{f_{n_p}})$ that narrowly converges to some $\delta_* \in \mathcal{R}_\Sigma$. By Proposition 3.2.2 this implies $f_{n_p} = \text{bar } \epsilon_{f_{n_p}} \to f_* := \text{bar } \delta_*$ in the feeble topology. But now $\int_T g_{i,j}(t,f_*(t))\mu(dt) = \int_T g_{i,j}(t,f_0(t))\mu(dt)$ for all i,j, and this implies easily $f_*(t) = f_0(t)$ for a.e. t in T (recall that (A_j) generates $\mathcal{T}$ and that (s'_i) separates the points of S). Since also $\int_T \tilde{g}(t,f_{n_p}(t))\mu(dt) \to \int_T \tilde{g}(t,f_*(t))\mu(dt) = \int_T \tilde{g}(t,f_0(t))\mu(dt)$, a contradiction has been reached. This proves the claim. Q.E.D.

It is rather interesting to observe that, in the opposite direction, the feeble topology on $\mathcal{S}_\Sigma$ leads to the narrow topology on $\mathcal{R}_\Sigma$. This goes as follows: given a metric Suslin space Z, it follows by a slight expansion of Theorem III.60 in Dellacherie and Meyer (1975) that $S := M(Z)$, the space of all bounded, signed measures on Z, is a Hausdorff locally convex topological vector space that is Suslin for the *classical* narrow topology (defined just as above, but now on $M(Z)$). Let $Z_t \subset Z$ be compact convex and nonempty for every $t \in T$, such that the graph of $\Omega : t \mapsto Z_t$ is measurable. For $t \in T$, define S_t to be the set of all probability measures s in $M_1^+(Z)$ such that $s(Z_t) = 1$. Then S_t is compact for the classical narrow topology by Theorem III.60 in Dellacherie and Meyer (1975) and the multifunction $t \mapsto S_t$ is easily seen to have a measurable graph by Theorem IV.12 of Castaing and Valadier (1977). In short, we have the starting situation of Assumption 2.2, but now with Ω instead of Σ. The set $\mathcal{S}_\Omega$ is easily seen to be identical to $\mathcal{R}_\Sigma$ (recall that scalar measurability is enough, since S was seen to be Suslin, and observe that the dual space of S can be identified with $\mathcal{C}_b(Z)$, by Proposition 22.4 of Choquet (1969)). Also, the feeble topology on $\mathcal{S}_\Omega$ coincides with the narrow topology on $\mathcal{R}_\Sigma$, since for any $g \in \mathcal{G}_{C,\Omega}$ the function $\tilde{g}$, defined by

$$\tilde{g}(t,s) := \int_{Z_t} g(t,z)s(dz),$$

belongs to $\mathcal{G}_{AC,\Sigma}$.

3.3 Proof of the main result

The proof of Theorem 2.1 consists of an application of Corollary 3.1.1 in the setting of section 3.2, followed by a standard purification argument. First, we follow Balder (1995a) in defining a *mixed* version $e : \mathcal{R}_\Sigma \to Y$ of the externality mapping d. Namely, we define $e(\delta) := (e_1(\delta), e_2(\delta))$, where

$$e_1(\delta) := (\int_{T_0} [\int_S g_i(t,s)\delta(t)(ds)]\mu(dt))_{i=1}^m, \;\; e_2(\delta) := \text{bar } \delta \mid_{T \setminus T_0} .$$

By Assumption 2.3 and the previously established facts about the barycentric mapping, this is well-defined. First, we prove the following mixed version of Theorem 2.1. By an argument similar to Remark 3.3.2, one can show that its Assumption 2.1 can be lifted.

Theorem 3.3.1 (mixed equilibrium existence result) *Under the above Assumptions 2.1–2.5 the pseudogame $\Gamma := (T, \Sigma, U, A)$ has a socially feasible Nash equilibrium in mixed action profiles, i.e., there exists $\delta^* \in \mathcal{R}_\Sigma$ such that*

$$\delta^*(t)(\text{argmax}_{s \in A_t(e(\delta^*))} U_t(s, e(\delta^*))) = 1 \textit{ for a.e. } t \textit{ in } T,$$

Lemma 3.3.1 *The mapping $e : \mathcal{R}_\Sigma \to Y$ is continuous.*

PROOF. Since each g_i is a Carathéodory function on $D \cap (T_0 \times S)$, the function $g'_i : D \to \mathbb{R}$, defined by $g'_i(t,s) := g_i(t,s)$ if $(t,s) \in D \cap (T_0 \times S)$ and $g'_i(t,s) := 0$ if $(t,s) \in D \cap (T \setminus T_0 \times S)$, belongs to $\mathcal{G}_{C,\Sigma}$. Hence, the m-vector function that forms the first component of e is narrowly continuous. The continuity of the second, barycentric component of e is an immediate consequence of Proposition 3.2.2. Q.E.D.

We now define a function $b : \mathcal{R}_\Sigma \times \mathcal{R}_\Sigma \to \mathbb{R}$ by

$$b(\delta, \eta) := \int_T [\int_S \tilde{U}(t,s,e(\delta))\eta(t)(ds)]\mu(dt).$$

Here $\tilde{U} := \arctan U$ is used to ensure boundedness. Quite similar forms were employed in Balder (1991,1995,1996a) for more abstract and *a priori* given mixed externality mappings e_t, $t \in T$, whose role is now taken by the mixed externality (e_1, e_2) defined above.

Lemma 3.3.2 *The function $b : \mathcal{R}_\Sigma \times \mathcal{R}_\Sigma \to \mathbb{R}$ has the following properties:*

$$b(\cdot,\cdot) \textit{ is continuous,}$$

$$b(\delta, \cdot) \textit{ is affine for every } \delta \in \mathcal{R}_\Sigma.$$

PROOF. Since $\mathcal{R}_\Sigma$ is semimetrizable (Proposition 3.2.1), sequential arguments suffice. Let (δ_k, η_k) converge narrowly to $(\delta_\infty, \eta_\infty)$ in $\mathcal{R}_\Sigma \times \mathcal{R}_\Sigma$. Define a function $g : T \times \hat{\mathbb{N}} \times S \to \{0, +\infty\}$ by

$$g(t,k,s) := \begin{cases} -\tilde{U}_t(s, e(\delta_k)) & \text{if } s \in S_t, \\ +\infty & \text{otherwise.} \end{cases}$$

Here $\hat{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ (Alexandrov compactification). By Assumptions 2.2, 2.5 and Lemma 3.3.1 it follows that $g(t,\cdot,\cdot)$ is lower semicontinuous on $\hat{\mathbb{N}} \times S$ for every $t \in T$. Also, g is $\mathcal{T} \times \mathcal{B}(\hat{\mathbb{N}} \times S)$-measurable. To see this, note first that, by the discrete nature of $\hat{\mathbb{N}}$, this claim amounts to the $\mathcal{T} \times \mathcal{B}(S)$-measurability of $g(\cdot, k, \cdot)$ for every $k \in \hat{\mathbb{N}}$. The latter follows directly from Assumptions 2.2 and 2.5: for any $\alpha \in \mathbb{R}$ the set of all $(t,s) \in T \times S$ with $g(t,k,s) \leq \alpha$ equals the set of all $(t,s) \in D$ with $\tilde{U}(t,s,e(\delta_k)) \geq -\alpha$. These two properties of g imply that g is a normal integrand on $T \times (\hat{\mathbb{N}} \times S)$; by Theorem 2.2(a) of Balder (1988) it then follows that the integral functional I_g is lower semicontinuous on $\mathcal{R}_{\hat{\mathbb{N}} \times S}$. Now by Theorem 2.5 of Balder (1988) the sequence of tensor products $(\epsilon_k \otimes \eta_k)$ in $\mathcal{R}_{\hat{\mathbb{N}} \times S}$ converges narrowly to $\epsilon_\infty \otimes \eta_\infty$. Combining the above, we find that $\limsup_k b(\delta_k, \eta_k) = -\liminf_k I_g(\epsilon_k \otimes \delta_k) \leq -I_g(\epsilon_\infty \otimes \delta_\infty) = b(\delta_\infty, \eta_\infty)$. Therefore, b is upper semicontinuous. Lower semicontinuity of b follows immediately by repeating the above argument for $\tilde{U}$ replaced by $-\tilde{U}$. Finally, the affinity of $b(\delta, \cdot)$ is trivial. Q.E.D.

Next, we let $F : \mathcal{R}_\Sigma \to 2^{\mathcal{R}_\Sigma}$ be the multifunction defined by

$$F(\delta) := \{\eta \in \mathcal{R}_\Sigma : \eta(t)(A(t, e(\delta))) = 1 \text{ for a.e. } t \text{ in } T\}.$$

Lemma 3.3.3 *(i) For every $g \in \mathcal{G}_C$ the functional $\delta \mapsto \sup_{\eta \in F(\delta)} I_g(\eta)$ from $\mathcal{R}_\Sigma$ into $\mathbb{R}$ is continuous.*

(ii) The function $a : \mathcal{R}_\Sigma \to \mathbb{R}$, defined by

$$a(\delta) := \sup_{\eta \in F(\delta)} b(\delta, \eta),$$

is continuous; in fact, one has

$$a(\delta) = \int_T \sup_{s \in A_t(e(\delta))} \tilde{U}_t(s, e(\delta)) \mu(dt),$$

where the function $\delta \mapsto \sup_{s \in A_t(e(\delta))} \tilde{U}_t(s, e(\delta))$ is continuous for every $t \in T$.

PROOF. (i) By what are essentially measurable selection arguments it follows that

$$\sup_{\eta \in F(\delta)} I_g(\eta) = \int_T \sup_{s \in A_t(e(\delta))} g(t,s) \mu(dt),$$

and that $t \mapsto \sup_{s \in A_t(e(\delta))} g(t,s)$ is integrable; e.g., see Proposition 3 of Balder (1991) or Theorem 3 of Balder (1996a). Also, for every $t \in T$ the function $\delta \mapsto \sup_{s \in A_t(e(\delta))} g(t,s)$ is continuous by Lemma 3.3.1, Assumptions 2.2, 2.4 and Berge's theorem (Aubin (1993), p. 391, Khan (1985), Theorem 2.2). Hence, the desired narrow continuity of $\sup_{\eta \in F(\cdot)} I_g(\eta)$ follows by the dominated convergence theorem, in view of the fact that sequential arguments suffice (Proposition 3.2.1).

(ii) By the same measurable selection arguments as used in the proof of part (i), we obtain

$$a(\delta) = \int_T \sup_{s \in A_t(e(\delta))} \tilde{U}_t(s, \text{bar } \delta) \mu(dt),$$

and after that the reasoning continues just as in the proof of part (i), since it is easy to prove, à la Berge, that for each t in T the function $\delta \mapsto \sup_{s \in A_t(e(\delta))} \tilde{U}_t(s, e(\delta))$ is continuous, in view of Lemma 3.3.1 and Assumptions 2.2, 2.4 and 2.5. Q.E.D.

PROOF OF THEOREM 3.3.1. We apply Corollary 3.1.1 to $E := \mathcal{M}_S$, $C := \mathcal{R}_\Sigma$ and F, a and b as introduced in the lemmas above. Then the conditions of Corollary 3.1.1 hold by Proposition 3.2.1 and Lemmas 3.3.2, 3.3.3. Hence, there exists a mixed action profile $\delta^* \in \mathcal{R}_\Sigma$ such that $\delta^*(t)(A_t(e(\delta^*))) = 1$ for a.e. t and

$$\int_T [\int_{S_t} \tilde{U}_t(s, e(\delta^*)) \delta^*(t)(ds)] \mu(dt) = \int_T \sup_{s \in A_t(e(\delta^*))} \tilde{U}_t(s, e(\delta^*)) \mu(dt),$$

which gives immediately that

$$\delta^*(t)(\mathrm{argmax}_{s\in A_t(e(\delta^*))}\tilde{U}_t(s,e(\delta^*))) = 1 \text{ for a.e. } t \text{ in } T.$$

By monotonicity of the arctangent function this proves that δ^* is the desired mixed equilibrium profile for Γ. Q.E.D.

PROOF OF THEOREM 2.1. First, we assume in addition validity of Assumption 2.1. Let δ^* be the mixed equilibrium profile whose existence is ensured by Theorem 3.3.1. Purification by means of Lyapunov's theorem (apply Lemma 3.4.1 of Balder (1995)) gives existence of a function $f_* \in \mathcal{S}_\Sigma(T_0)$ such that

$$\int_{T_0} g_i(t,f_*(t))\mu(dt) = \int_{T_0}[\int_{S_t} g_i(t,s)\delta^*(t)(ds)]\mu(dt),\, i=1,\cdots,m+2,$$

where $g_{m+j}(t,s) := (-1)^j\tilde{U}(t,s,e(\delta^*))$, $j=1,2$. The first m identities imply $e_1(\delta^*) = d_1(f_*)$. Define now $f^* : T \mapsto S$ by setting $f^*(t) := f_*(t)$ for $t \in T_0$ and $f^*(t) :=$ bar $\delta^*(t)$; then f^* belongs to $\mathcal{S}_\Sigma$ by the above, (3.1) and Assumption 2.2. Obviously, $e_2(\delta^*) = d_2(f^*)$, whence now $e(\delta^*) = d(f^*)$. As a consequence, the final two identities above come down to

$$\int_{T_0} \tilde{U}(t,f_*(t),d(f^*))\mu(dt) = \int_{T_0}[\int_{S_t} \tilde{U}(t,s,e(\delta^*))\delta^*(t)(ds)]\mu(dt).$$

By Theorem 3.3.1 and the identity $e(\delta^*) = d(f^*)$, the right-hand side equals $\sup_{s\in A_t(d(f^*))}\tilde{U}(t,s,d(f^*))$. This clearly implies

$$\int_{T_0}[\sup_{s\in A_t(d(f^*))}\tilde{U}(t,s,d(f^*)) - \tilde{U}(t,f^*(t),d(f^*))]\mu(dt) = 0,$$

where the integrand is nonnegative; hence that integrand must be zero a.e. Therefore,

$$f^*(t) \in \mathrm{argmax}_{s\in A_t(d(f^*))}U_t(s,d(f^*)) \text{ for a.e. } t \text{ in } T_0.$$

Finally, Assumptions 2.4 – 2.6 imply that the set $\mathrm{argmax}_{s\in A_t(d(f_*))}U_t(s,d(f_*))$ is compact and convex for every $t \in T\backslash T_0$. Hence, by combining Theorem 3.3.1, (3.1) and the identity $e(\delta^*) = d(f^*)$, we find

$$f^*(t) \in \mathrm{argmax}_{s\in A_t(d(f^*))}U_t(s,d(f^*)) \text{ for a.e. } t \text{ in } T\backslash T_0.$$

Together with the similar statement above on the part T_0, this demonstrates that f^* is a pure equilibrium profile for Γ. In combination with Remark 3.3.2 this finishes the proof. Q.E.D.

Remark 3.3.1 *If the social feasibility aspect of Γ, caused by the multifunction A, is lifted by setting $A_t \equiv S_t$, it can easily be seen from the proof of Theorem 3.3.1 that the continuity of U_t required in Assumption 2.5 can be weakened in the way of Assumptions 2.4 and 2.6 in Balder (1995a): It is then sufficient to require continuity of $U_t(s,\cdot)$ for every $(t,s) \in D$ and upper semicontinuity of $U_t(\cdot,\cdot)$. It can then further be seen that the application of Corollary 3.1.1 simplifies into a direct application of Ky Fan's inequality Theorem 3.1.1. In this way, Theorem 3.3.1 can easily be turned into a generalization of the main mixed equilibrium result in Theorem 2.1 of Balder (1995a), since the only essential property required of the mixed externality mapping e in the proof of Theorem 3.3.1 is continuity as given by Lemma 3.3.1. The improvement of Theorem 2.1 in Balder (1995a) then consists of the possibility to work with the externality space $\tilde{Y} := \mathcal{R}_\Sigma$ and a mixed externality mapping $\tilde{e} : \mathcal{R}_\Sigma \rightarrow \tilde{Y}$ which is nothing but the identity on $\mathcal{R}_\Sigma$.* [3] *It is rather interesting that the power of Theorem 2.1 is such that, on its own accord, it implies the same improvement of Theorem 2.1 in Balder (1995a). This can be seen by working out the comments at the end of section 3.2.*

[3] I am indebted to Sylvain Sorin (Paris) for a stimulating question about the possibility to phrase the externality mapping in this way.

Remark 3.3.2 (removal of Assumption 2.1) *In the proof of Theorem 2.1 we dealt with a function $U : D \times Y \to [-\infty, +\infty]$ satisfying Assumption 2.5. Let us define $\hat{U} : T \times S \times Y \to [-\infty, +\infty]$ by $\hat{U} := U$ on $D \times Y$ and $\hat{U} := -\infty$ elsewhere. It is clear from Assumptions 2.2 and 2.5 that $\hat{U}$ is $\mathcal{T} \times \mathcal{B}(S \times Y)$-measurable. By Castaing and Valadier (1977), p. 78, there exists a countably generated sub-σ-algebra $\mathcal{A}$ of $\mathcal{T}$ such that the same $\hat{U}$ is also $\mathcal{A} \times \mathcal{B}(S \times Y)$-measurable. Moreover, this result can be sharpened as follows (see the Appendix of Valadier (1984)): $\mathcal{A}$ can be chosen in such a way that $(T^{na}, \mathcal{A} \cap T^{na}, \mu(\cdot \cap T^{na}))$ is nonatomic. Also the graphs of the multifunctions Σ and A can be treated in this way, for we can identify those graphs with their indicator functions, and this gives us two more $\mathcal{T} \times \mathcal{B}(S \times Y)$-measurable functions to which the above result from Castaing and Valadier (1977) and Valadier (1984) applies. Combining these three countably generated sub-σ-algebras we obtain a single countably generated sub-σ-algebra $\mathcal{B}$, with respect to which the measurability requirements of Assumptions 2.2, 2.4, 2.5 continue to hold. Then all assumptions needed for Theorem 2.1 continue to hold* a fortiori *if we replace $\mathcal{S}_\Sigma$ by its subset consisting of all $\mathcal{B}$-measurable pure action profiles. This shows that without loss of generality $(T, \mathcal{T}, \mu)$ can be supposed to be separable. Also, since S is metrizable Suslin when equipped with d_S (and* a fortiori *separable metric), it follows from a well-known modification argument involving "step functions" (Dellacherie and Meyer (1975), Theorem I.17), and completion (Dellacherie and Meyer (1975), Remark II.32), that we may assume without loss of generality that $(T, \mathcal{T}, \mu)$ is a complete measure space. Indeed, in a first stage all measurability properties continue to hold* a fortiori *with respect to the μ-completion $\mathcal{T}_\mu$ of $\mathcal{T}$. Thereafter, once the equilibrium profile has been found to exist in the larger setup, the results cited above allow the construction of a modification which only involves changes on null sets, is measurable with respect to the original σ-algebra $\mathcal{T}$ and still obeys the requirements for an equilibrium for Γ. In sum, the preceding observations show that Assumption 2.1 can always be made to hold by restriction from the outset to the completion of a suitable sub-σ-algebra of $\mathcal{T}$.*

The method discussed in the preceding remark comes from Balder (1995b), where K-*convergence* for Young measures, an intrinsically sequential, nontopological mode of convergence, which constitutes a sharpening of sequential narrow convergence, is studied extensively.

Remark 3.3.3 *The above remark can also be used to justify a gap in the proof of Lemma 4.2(i), p. 92, of Balder (1995a), where the dominated convergence theorem is invoked in a context with generalized sequences. The "free" introduction of the separability Assumption 2.1, discussed in the preceding remark, guarantees semimetrizability of $\mathcal{R}_\Sigma$. Hence, sequential arguments can indeed be used in p. 92 of Balder (1995a), which justifies the use of the dominated convergence theorem there.*

4 Comments on related non-ordered preference models

A considerable part of the literature on continuum games is devoted to existence results for models with non-ordered preferences; e.g., see Kim, Prikry and Yannelis (1989), Khan (1985), Khan and Papageorgiou (1987a,b), Khan and Vohra (1984), Yannelis (1987,1991). These are presented as continuum analogues of the seminal existence results by Gale and Mas-Colell (1974) and Shafer and Sonnenschein (1975), which have a finite set of players. However, as was recently demonstrated in Balder (1996c), all such continuum analogies suffer from a serious inconsistency: Under weak versions of the usual open lower section condition for the (strict) preference multifunction P and the usual nonreflexivity condition for P (these conditions stem from Gale and Mas-Colell (1974) and Shafer and Sonnenschein (1975)), the preference multifunction P can essentially only have empty values on the nonatomic part of the measure space of players.

Here we shall demonstrate such inconsistency by an argument that is quite different from the proof given in Balder (1996c). Moreover, the present exposition carries a little further (see Remark 2.1), since we continue to use the feeble topology on $\mathcal{S}_\Sigma$. In contrast to Balder (1996c), the present argument is based on a denseness result that is very closely related to the purification by nonatomicity used in the proof of Theorem 2.1. We continue to work under the Assumptions 2.1 and 2.2, but, just as in Remark 3.3.2, one can show that Assumption 2.1 may be removed.

Theorem 4.1 (inconsistency result) *Suppose that $(T, \mathcal{T}, \mu)$ is nonatomic.* [4] *Let $P : T \times \mathcal{S}_\Sigma \to 2^S$ be a multifunction with measurable graph such that*

$$P(t, f) \subset S_t \text{ for every } (t, f) \in T \times \mathcal{S}_\Sigma,$$

for every $t \in T$

$$O_t := \{(s, f) \in S \times \mathcal{S}_\Sigma : s \in P(t, f)\} \text{ is an open subset of } S_t \times \mathcal{S}_\Sigma \text{ (open lower section condition)}$$

and for every $f \in \mathcal{S}_\Sigma$

$$f(t) \notin P(t, f) \text{ for a.e. } t \text{ in } T \text{ (nonreflexivity)} .$$

Then for every $f \in \mathcal{S}_\Sigma$

$$P(t, f) = \emptyset \text{ for a.e. } t \text{ in } T.$$

Proof. Assume that we had some $f \in \mathcal{S}_\Sigma$ for which $P(t, f) \neq \emptyset$ for all t in some set C with $\mu(C) > 0$. Then by the von Neumann-Aumann measurable selection theorem there would exist a measurable function $\bar{f} : C \to S$ such that $\bar{f}(t) \in P(t, f) \subset S_t$ for a.e. t in C. Outside C, we simply set $\bar{f} := f$. For arbitrary $\alpha \in (0, 1]$, define $\delta_\alpha \in \mathcal{R}_\Sigma$ by $\delta_\alpha(t) := \alpha\epsilon_{\bar{f}}(t) + (1-\alpha)\epsilon_f(t)$. By Corollary 3 of Balder (1984), a denseness result that is an immediate spinoff of Lemma 3.4.1 of Balder (1995a), the purification result used earlier, there exists a sequence (f_n) in $\mathcal{S}_\Sigma$ such that (ϵ_{f_n}) narrowly converges to δ_α. Hence, (f_n) converges feebly to $f_\infty :=$ bar $\delta_\alpha = \alpha\bar{f} + (1-\alpha)f$ (Proposition 3.2.2). By the same lower semicontinuity argument as used in the proof of Lemma 3.3.2, it follows that $\liminf_n \int_T \ell(t, n, f_n(t))\mu(dt) \geq \int_T [\int_S \ell(t, \infty, s)\delta_\alpha(t)(ds)]\mu(dt)$, where $\ell(t, n, s) := 0$ if $(s, f_n) \notin O_t$ and $:= +\infty$ otherwise. By the nonreflexivity condition, the left-hand side of the above inequality equals zero, and this implies $\delta_\alpha(t)(S_t \backslash P(t, \text{bar } \delta_\alpha)) = 1$ for a.e. t in T. In turn, this implies $\bar{f}(t) \in S_t \backslash P(t, \text{bar } \delta_\alpha)$ for a.e. t in T (note that $\bar{f}(t)$ is in the support of $\delta_\alpha(t)$), which amounts to $(\bar{f}(t), \alpha\bar{f} + (1-\alpha)f)$ belonging to the complement of the open set O_t. By letting α go to zero, it follows that the complement of O_t contains $(\bar{f}(t), f)$ for a.e. t in T, and in particular of course for a.e. t in C. This contradicts the definition of the selection $\bar{f}$. Q.E.D.

References

Aubin, J.-P., *Optima and Equilibria,* Springer-Verlag, Berlin, 1993.

Balder, E.J., 1984, A general denseness result for relaxed control theory. *Bull. Austral. Math. Soc.* **30**, 463-475.

Balder, E.J., 1988, Generalized equilibrium results for games with incomplete information. *Math. Oper. Res.* **13**, 265-276.

Balder, E.J., 1989, On Prohorov's theorem for transition probabilities. *Sém. Analyse Convexe* **19**, 9.1-9.11.

Balder, E.J., 1990, New sequential compactness results for spaces of scalarly integrable functions. *J. Math. Anal. Appl.* **151**, 1-16.

Balder, E.J., 1991, On Cournot-Nash equilibrium distributions for games with differential information and discontinuous payoffs. *Econ. Theory* **1**, 339-354.

Balder, E.J., 1995a, A unifying approach to existence of Nash equilibria. *Intern. J. Game Theory* **24**, 79-94.

Balder, E.J., 1995b, *Lectures on Young Measures.* Cahiers du Centre de Recherche de Mathématiques de la Décision (CEREMADE) **9517**, Université Paris-Dauphine, Paris.

[4] This assumption is only made to simplify the proof. One can repeat the entire proof for the nonatomic part T^{na} of $(T, \mathcal{T}, \mu)$.

Balder, E.J., 1996a, Comments on the existence of equilibrium distributions. *J. Math. Econ.* **25**, 307-323.

Balder, E.J., 1996b, Towards an existence theory for Cournot-Nash-Walras equilibria, submitted for publication.

Balder, E.J., 1996c, Inconsistency of the usual conditions for Nash equilibrium existence in continuum economies without ordered preferences, submitted for publication.

Castaing, C. and Valadier, M., 1977, *Convex Analysis and Measurable Multifunctions.* Lecture Notes in Math. **580**, Springer-Verlag, Berlin.

Choquet, G., 1969, *Lectures on Analysis.* Benjamin, Reading, Mass.

Dellacherie, C., and Meyer, P.-A., 1975, *Probabilités et Potentiel.* Hermann, Paris, 1975 (English translation: North-Holland, Amsterdam, 1978).

Diestel, J., 1977, Remarks on weak compactness in $L_1(\mu, X)$. *Glasgow Math. J.* **18**, 87-91.

Ding, X.P., and Tan, K.-K., 1990, Generalized variational inequalities and generalized quasi-variational inequalities. *J. Math. Anal. Appl.* **148**, 497-508.

Edwards, R.E., 1995, *Functional Analysis.* Dover Publications, New York.

Gale, D., and Mas-Colell, A., 1978, An equilibrium existence result for a general model without ordered preferences. *J. Math. Econ.* **2** (1974), 9-15. Erratum, *ibid.* **6**, 297-298.

Ichiishi, T., 1983, *Game Theory for Economic Analysis.* Academic Press.

Khan, M.A., 1985, On extensions of the Cournot-Nash theorem, in: *Advances in Equilibrium Theory* (C.D. Aliprantis, O. Burkinshaw and N.J. Rothman, eds.), Lecture Notes in Economics and Math. Systems **244**, Springer-Verlag, pp. 79-106.

Khan, M.A., and Papageorgiou, N.S., 1987, On Cournot-Nash equilibria in generalized qualitative games with an atomless measure space of agents. *Proc. Amer. Math. Soc.* **100**, 505-510.

Khan, M.A., and Papageorgiou, N.S., 1987, On Cournot-Nash equilibria in generalized qualitative games with a continuum of players. *Nonlinear Anal.* **11**, 741-755.

Khan, M.A., and Vohra, R., 1984, Equilibrium in abstract economies without ordered preferences and with a measure space of agents. *J. Math. Econ.* **13**, 133-142.

Kim, T., Prikry, K., and Yannelis, N.C., 1989, Equilibria in abstract economies with a measure space of agents and with an infinite dimensional strategy space. *J. Approx. Theory* **56**, 256-266.

Fan, Ky, 1961, A generalization of Tychonoff's fixed point theorem. *Math. Ann.* **142**, 305-310.

Mosco, U., 1975, Implicit variational problems and quasi-variational inequalities. In: *Nonlinear Operators and the Calculus of Variations* (Gossez, J.P., et al., eds.) Lecture Notes in Math. **543**, Springer-Verlag, Berlin.

Neveu, J., 1965, *Mathematical Foundations of the Calculus of Probability.* Holden-Day, San Fransisco.

Schmeidler, D., 1973, Equilibrium points of non-atomic games. *J. Statist. Phys.* **7**, 295-300.

Schwartz, L., 1973, *Radon Measures.* Oxford University Press, London.

Shafer, W., and Sonnenschein, H., 1975, Equilibrium in abstract economies without ordered preferences. *J. Math. Econ.* **2**, 345-348.

Valadier, M., 1984, Some bang-bang theorems. in: *Multifunctions and Integrands* (G. Salinetti, ed.) Lecture Notes in Math. **1091**, Springer-Verlag, Berlin, pp. 225-234.

Yannelis, N.C., 1987, Equilibria in noncooperative models of competition. *J. Econ. Theory* **41**, 96-111.

Yannelis, N.C, 1991, Integration of Banach-valued correspondences. in: *Equilibrium Theory in Infinite Dimensional Spaces* (M.A. Khan and N.C. Yannelis, eds.), Springer-Verlag, pp. 2-35.