Global elliptic estimates on symmetric spaces

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Abstract. The domination properties of elliptic invariant differential operators on symmetric spaces of noncompact type are investigated. Using the relation between parametrices and fundamental solutions on symmetric space we will show that the invariant differential operator applied to a function can be uniformly estimated by function and an elliptic operator of higher order applied to the function in L^p spaces for all $1 \leq p \leq \infty$. As a consequence, by algebraic methods we will give a simple unifying proof that derivatives of a function can be uniformly estimated by function and its Laplacian.

0 Introduction

In this paper we will investigate the question of the domination of the (invariant) differential operator by the elliptic operator of higher order on symmetric spaces of noncompact type. We are also interested in the estimates of the derivative of a function by the function and its Laplacian in L^p spaces for $1 \leq p \leq \infty$. Similar estimates are known for $p = \infty$ for manifolds with bounded curvature ([1]). On the other hand estimates for $1 < p < \infty$ are well known for Euclidean spaces ([6]) and are closely related to the question of the continuity of pseudo-differential operators of order zero, which hold only locally on general manifolds. We will give a unifying algebraic proof for global estimates in L^p-space for all $1 \leq p \leq \infty$ on symmetric spaces of the noncompact type.

In Section 2 we discuss local integrability properties of distributional kernels of pseudo differential operators on manifolds and fix the notation. It will be applied to establish a relation between parametrices and fundamental solutions of elliptic operators on symmetric space in Section 3 (Lemma 3). Based on this lemma, we will prove the domination property for invariant differential operators (Theorem 1). Then, reformulating the problem in Lie group terms, we will show how the invariance condition can be dropped for the first order differential operators reducing it to a problem on Lie group (Theorem 2). We will give a simple proof of it in the last section. The general theory of the second order differential operators on Lie groups can be found in [5], [7].

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1 Local properties on manifolds

In this section we will fix the notation and establish simple integrability properties necessary for the sequel. Let $P \in \Psi^m(\mathbb{R}^n)$ be an elliptic properly supported pseudo differential operator of positive order m. Let $Q \in \Psi^{-m}(\mathbb{R}^n)$ denote a left parametrix for P, i.e. a pseudo differential operator satisfying

$$QP = I + R \tag{1}$$

in the space of pseudo differential operators, with $R \in \Psi^{-\infty}(\mathbb{R}^n)$ a smoothing operator. Let K denote a Schwartz distributional kernel for Q. This means that $K \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ and for all $\phi, \psi \in \mathcal{D}(\mathbb{R}^n)$ holds

$$\langle Q\phi, \psi \rangle = \langle K, \phi \otimes \psi \rangle.$$

We can write it formally as

$$Q\phi(x) = \int_{\mathbb{R}^n} K(x,y)\phi(y)dy. \tag{2}$$

The singularities of $K(x,y)$ are well known (see, for example, [6]):

Lemma 1 *Let $Q \in \Psi^{-k}(\mathbb{R}^n)$ be a properly supported pseudo differential operator of negative order. Then its distributional kernel can be identified with function $K(x,y)$, which is smooth outside the diagonal $x = y$ and has integrable singularities at the diagonal. Formula (2) then makes sense.*

Let M be a smooth Riemannian manifold. Let $P \in \Psi^m(M)$ be a properly supported pseudo differential operator on M of order m. This means that its distributional kernel K is smooth in the complement of the diagonal and its behaviour at the diagonal locally looks like behaviour in $\mathbb{R}^n$, namely if (U, χ) is a chart in M, then the restriction of K to $U \times U$ is equal to the pullback by $\chi \times \chi$ of the distributional kernel of an element of $\Psi^m(\mathbb{R}^n)$. Note, that the existence of the Riemannian structure on M induces isomorphisms of line bundles of smooth densities on M and smooth functions by fixing the Riemannian measure. This fact essentially simplifies the calculus of pseudo differential operators on M.

The principal symbol σ of P is an invariantly defined function on $T^*M \backslash 0$. We assume P to be elliptic, which means that σ is nowhere vanishing on $T^*M \backslash 0$. Ellipticity implies existence of parametrix, an operator $Q \in \Psi^{-m}(M)$ satisfying

$$QP = I + R \tag{3}$$

with $R \in \Psi^{-\infty}(M)$. In a sequel, writing

$$(Qu)(x) = \int_M K(x,y)u(y)dm(y),$$

we view Q as a singular integral operator with kernel K. In analogy with Lemma 1 we have

Lemma 2 *Let $Q \in \Psi^{-k}(M)$ be a properly supported pseudo differential operator of negative order. Then its distributional kernel can be identified with function $K(x,y)$, which is smooth outside the diagonal $x=y$ and has integrable singularities at the diagonal.*

Proof. We must only show the integrability of the singularities. Using cutoff functions we see, that in a chart $U \times U$, $K|_{U\times U} = (\chi \times \chi)^* \tilde{K}$, $\tilde{K}$ being the distributional kernel of an element of $\Psi^{-k}(\mathbb{R}^n)$. We have

$$\int_U K(x,y)dm(y) = \int_{\chi(U)} \tilde{K}(x, \chi^{-1}p)g(p)dp,$$

where $g(p)$ is the Jacobian of χ, therefore locally bounded. Application of Lemma 1 finishes the proof.

2 Symmetric space

Let $M = G/H$ be a symmetric space of the noncompact type. This means, that it can be viewed as a quotient $M = G/H$, where G is a connected semisimple Lie group with trivial center and H is its maximal compact subgroup. The Riemannian structure on M is supposed to be invariant under the left action of G.

We will describe first a relation between parametrix and fundamental solutions of P. This will lead to some integrability properties of fundamental solutions, which will be applied for getting uniform L^p estimates. Let P be an elliptic differential operator of order m on M. Then, as in Section 2, there exist a parametrix for P, namely a pseudo differential operator $Q \in \Psi^{-m}(M)$, such that equation (3) is satisfied. Note also, that P is automatically proper supported being a local operator.

Assume now that P is G–left invariant. Then, according to [3], there exist a fundamental solution for P, namely a distribution $K \in \mathcal{D}'(M)$, such that

$$PK = \delta, \tag{4}$$

δ being a delta function at the origin p of M.

Let $\alpha \in \mathcal{D}(M)$ be a test function, such that $\alpha(x) = 1$ for x in a small neighborhood of p. Then equality (4) implies the existence of $\beta \in \mathcal{D}(M)$, such that

$$P(\alpha K) = \delta + \beta. \tag{5}$$

In fact, taking $\beta = P(\alpha K) - \delta$ one readily verifies $\beta \in \mathcal{D}(M)$.

The only singularity of K occurs at the point p due to the ellipticity of P and we will be interested in the integrability properties of αK. An application of formula (3) to αK yields

$$\alpha K + R(\alpha K) = QP(\alpha K) = Q\delta + Q\beta, \tag{6}$$

the last equality due to (5). We have $\beta \in \mathcal{D}(M)$ and $R \in \Psi^{-\infty}(M)$ implying $R(\alpha K), Q\beta \in \mathcal{C}^\infty(M)$. The operators Q and R are properly supported, therefore all the functions in (6) are compactly supported. This means that the integrability properties of αK are equivalent to the integrability properties of $Q\delta$, the latter being equal to the distributional kernel of Q.

Let $D \in \Psi^k(M)$ be properly supported, $k < m$. The application of D to (6) and our arguments above imply

$$D(\alpha K) = DQ\delta + \psi, \tag{7}$$

with $\psi \in \mathcal{D}(M)$. Now, the operator DQ is of a negative order $k - m$ and Lemma 2 implies the integrability of of its integral kernel at p, the latter being equal to $DQ\delta(x)$. Equality (7) implies the integrability of $D(\alpha K)$. Thus, we have proved the following

Lemma 3 *Let P be an invariant elliptic differential operator of order m on M and K its fundamental solution at p. Then, for every $D \in \Psi^k(M)$, $k < m$, DK is locally integrable $DK \in L^1_{loc}(M)$.*

We will apply this lemma to two cases, D being an invariant differential operator and P being the Laplace operator on M equipped with a Riemannian structure. The space of G–left invariant differential operators of order k on M will be denoted $\mathbb{D}^k(M)$.

Theorem 1 *Let $P \in \mathbb{D}^m(M)$ be elliptic, $D \in \mathbb{D}^k(M)$, $0 < k < m$ and $1 \leq p \leq \infty$. Then there exist constants A, B, such that for every $u \in L^p(M)$ satisfying $Pu \in L^p(M)$, we have $Du \in L^p(M)$ and*

$$||Du||_p \leq A||Pu||_p + B||u||_p. \tag{8}$$

If $p = \infty$, then Du is continuous.

Let X be a smooth vector field on M. X is called *bounded* if there exist a constant C such that $||X_x|| \leq C$ for every point $x \in M$, where $||\cdot|| = \langle \cdot, \cdot \rangle^{1/2}$ is the Riemannian norm on T_xM, corresponding to the Riemannian structure.

Theorem 2 *Let M be a Riemannian symmetric space and Δ the associated Laplace operator. Let $1 \leq p \leq \infty$ and X a smooth bounded vector field on M. Then there exist constants A, B, such that for every $u \in L^p(M)$ satisfying $\Delta u \in L^p(M)$, we have the derivative of u with respect to X is L^p–integrable, $Xu \in L^p(M)$ and*

$$||Xu||_p \leq A||\Delta u||_p + B||u||_p. \tag{9}$$

If $p = \infty$, then Xu is continuous. Moreover, A and B can be chosen independently over the set of the smooth vector fields X bounded by 1.

First, convolving (5) with $u \in \mathcal{D}'(M)$ we conclude that

$$u = u * \delta = u * P(\alpha K) - u * \beta. \tag{10}$$

Let $\mathbb{Z}(G)$ denote the center of an algebra of left invariant differential operators on G. Let $\pi : G \to M = G/H$ be the canonical projection and $\mathfrak{g}$ denote Lie algebra of G. Then $d\pi : \mathfrak{g} \to T_pM$ can be extended to an algebra of left invariant differential operators on G with the property $d\pi(\mathbb{Z}(G)) = \mathbb{D}(M)$, (cf.[3]). This means that $P \in \mathbb{D}^m(M)$ is an image of a bi-invariant operator on G and, in particular, commutes with left and right convolution. Therefore, (10) implies

$$u = Pu * \alpha K - u * \beta. \tag{11}$$

Proof of Theorem 1. An application of D to equality (11) together with an argument above imply

$$Du = Pu * D(\alpha K) - u * D\beta.$$

By Lemma 3 we have $D(\alpha K) \in L^1(M)$ and Young inequality ([4]) yields estimate (8). In case $p = \infty$ we have convolutions of the type $L^\infty * L^1$, which give continuous functions. This completes the proof.

3 Proof of Theorem 2

For the proof of Theorem 2 we will need some notation and auxiliary results. We start with constructing an invariant Riemannian structure on G, making the projection π a Riemannian submersion. In the sequel $l_gh = gh$, $r_gh = hg$, $g, h \in G$ will denote the left and right group action on G. The induced actions of G on M will be denoted by the same letters. The adjoint representation will be denoted by $\mathrm{Ad} : G \to GL(T_eG)$.

Lemma 4 *There exists a G–left and H–right invariant Riemannian structure on G, such that the canonical projection $\pi : G \to M$ is a Riemannian submersion, i.e. a submersion, for which horizontal lift of vector fields preserves Riemannian norms.*

Proof. Let $\langle \cdot, \cdot \rangle_0$ be an arbitrary inner product on T_eG. Then one can define $\mathrm{Ad}(H)$–invariant inner product $(\cdot, \cdot)_0$ on T_eG by taking

$$(X, Y)_0 := \int_{\mathrm{Ad}(H)} \langle A(X), A(Y) \rangle_0 d\mu(A), \tag{12}$$

where μ is Haar measure on a compact set $\mathrm{Ad}(H)$. The tangent space T_pM to M at $p = \pi(e)$ can be identified with the quotient T_eG/T_eH via an isomorphism $i([X]) = d\pi(X)$, where $[X] \in T_eG/T_eH$ is an equivalence class of $X \in T_eG$. For every $g \in G$ let $K_g = \mathrm{Ker}\ d_g\pi$ ($K_g \cong T_gH \cong T_eH$). Let N_g be the orthogonal complement to K_g with respect to $(\cdot, \cdot)_0$: $T_gG = K_g \bigoplus N_g$. Define an inner product $(\cdot, \cdot)_{N_e}$ on N_e for vectors $\bar{X}, \bar{Y} \in N_e$ by

$$(\bar{X}, \bar{Y})_{N_e} := \langle X, Y \rangle_{M_p}, \tag{13}$$

where $\langle \cdot, \cdot \rangle_{M_p}$ is restriction to T_pM of the given invariant Riemannian structure on M and $X = d\pi(\bar{X}), Y = d\pi(\bar{Y}) \in T_pM$. Vectors $\bar{X}$ and $\bar{Y}$ are uniquely defined, $d\pi$ being an isomorphism of N_e and T_pM, and they are called the *horizontal lifts* of X and Y. The desired Riemannian structure on G can now be constructed by applying dl_g to

$$\langle X, Y\rangle := (X|_{K_e}, Y|_{K_e})_0 + (X|_{N_e}, Y|_{N_e})_{N_e}, \tag{14}$$

where $X, Y \in T_eG$, $X|_{K_e}, Y|_{K_e}$ and $X|_{N_e}, Y|_{N_e}$ are projections of X, Y on K_e and N_e respectively. Inner product (14) is clearly $\mathrm{Ad}(H)$–invariant, the expansion is therefore G–left and H–right invariant. It follows immediately from formulas (13) and (14) that all $d_g\pi$ are partial isometries (isometries from N_g to $T_{\pi(g)}M$). This completes the proof.

The following properties of the pullback $\sharp$ will be necessary.

Lemma 5 (i) *Let $\phi \in \mathcal{C}_c(G)$ be a continuous compactly supported function on G. Then for $x = \pi(g)$ the function $\phi^\flat(x)$ is correctly defined by*

$$\phi^\flat(x) = \int_H \phi(gh)dh,$$

where dh is the normalized Haar measure on H. The function $\phi^\flat$ is continuous and compactly supported on M: $\phi^\flat \in \mathcal{C}_c(M)$. Moreover, mapping $\phi \to \phi^\flat$ is linear surjective from $\mathcal{C}_c(G)$ to $\mathcal{C}_c(M)$ and from $\mathcal{D}(G)$ to $\mathcal{D}(M)$.

(ii) *The transpose of $\phi \to \phi^\flat$ defined by*

$$\langle T^\sharp, \phi\rangle = \langle T, \phi^\flat\rangle$$

is injective mapping from $\mathcal{D}'(M) \to \mathcal{D}'(G)$.

(iii) *Let $S \in \mathcal{D}'(G)$. Then $R_hS = S$ for all $h \in H$ if and only if there exists $T \in \mathcal{D}'(M)$, such that $S = T^\sharp$. For $T_1, T_2 \in \mathcal{D}'(M)$ the convolution products on G and M are related by*

$$T_1^\sharp * T_2^\sharp = (T_1 * T_2)^\sharp.$$

(iv) *Let Y be a horizontal lift of a vector field X on M and let $T \in \mathcal{D}'(M)$ be a distribution on M. Then $Y(T^\sharp) = (XT)^\sharp$.*

The proof is left to the reader.

Proof of Theorem 2. The pullback of formula (11) now reads

$$u^\sharp = (\Delta u)^\sharp * (\alpha K)^\sharp - u^\sharp * \beta^\sharp. \tag{15}$$

Let X be a smooth vector field on M, bounded by one: $||X_x|| \leq 1$. Let Y denote the horizontal lift of X:

1. $d_g\pi(Y_g) = X_x$, where $x = \pi(g)$.
2. $Y_g \in N_g = (\mathrm{Ker}\ d_g\pi)^\perp$.

It is not difficult to see that Y is smooth. Let $Y_1, \ldots, Y_N$ be an orthonormal basis of the Lie algebra $\mathfrak{g}$, such that $(Y_1)_g, \ldots, (Y_n)_g \in N_g$ for all $g \in G$. Vector field Y can be decomposed with respect to the basis $Y_1, \ldots, Y_n$ at every point $g \in G$:

$$Y_g = \sum_{i=1}^{n} a_i(g) Y_{i,g} \in N_g \subset T_g G, \tag{16}$$

where $Y_{i,g} = (Y_i)_g = d_e l_g (Y_i)_e$ are values at g of the left invariant vector fields Y_i. Note, that such decomposition is pointwise because Y need not be left invariant in general, we use that $Y_g \in N_g$ and the fact that $Y_{1,g}, \ldots, Y_{n,g}$ constitute a basis for a linear space N_g. It also has a global character and functions $a_1, \ldots, a_n$ are smooth due to the smoothness of Y and $Y_1, \ldots, Y_n$.

The norm of Y_g at $T_g G$ is $||Y_g||^2 = \sum_{i=1}^n |a_i(g)|^2$. In view of Lemma 4, $||Y_g|| = ||X_x|| \leq 1$. In particular, $|a_i(g)| \leq 1$ for all $g \in G$. Now we differentiate $u^\sharp$ in (15) with respect to the basis vector fields Y_i and the left invariance of Y_i yield:

$$Y_i u^\sharp = (\Delta u)^\sharp * Y_i (\alpha K)^\sharp + u^\sharp * Y_i \beta^\sharp. \tag{17}$$

Obviously $Y_i \beta^\sharp \in \mathcal{D}(G) \subset L^1(G)$. In view of Lemma 3 the compactly supported distribution αK and its derivatives are integrable and so are their pullbacks, the pullback mapping being an isometry of L^p spaces. Let $A_i = ||Y_i (\alpha K)^\sharp||_1$ and $B_i = ||Y_i \beta^\sharp||_1$. Application of Young inequality [4, Cor.20.14] to (17) yields

$$||Y_i u^\sharp||_p \leq A_i ||(\Delta u)^\sharp||_p + B_i ||u^\sharp||_p.$$

Decomposition (16) together with bounds on a_i and equalities $||(\Delta u)^\sharp||_p = ||\Delta u||_p$ and $||u^\sharp||_p = ||u||_p$ imply

$$||Y u^\sharp||_p \leq A ||\Delta u||_p + B ||u||_p$$

with $A = \sum_{i=1}^n A_i$ and $B = \sum_{i=1}^n B_i$. By Lemma 5 we have $||Y u^\sharp||_p = ||X u||_p$, establishing inequality (9). In case $p = \infty$, formulas (17) and (16) imply the continuity of $Y u^\sharp$. By Lemma 5, $(Xu)^\sharp = Y u^\sharp$ is continuous. The continuity of Xu now follows from the fact that M is equipped with quotient topology, i.e. the strongest topology, for which π is a continuous mapping. This finishes the proof of Theorem 2.

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